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Abstract : In numerous applications (Biology, Finance, Internet Traffic, Oceanography,...) data are observed at random times and a graph of an estimation of the spectral density may be relevant for characterizing phenomena and explaining. By using a wavelet analysis, one derives a nonparametric estimator of the spectral density of a Gaussian process with stationary increments (also stationary Gaussian process) from the observation of one path at random discrete times. For any positive frequency, this estimator is proved to satisfy a central limit theorem with a convergence rate depending on the roughness of the process and the order moment of duration between times of observation. In the case of stationary Gaussian processes, one can compare this estimator with estimators based on the empirical periodogram. Both estimators reach the same optimal rate of convergence, but the estimator based on wavelet analysis converges for a different class of random times. One gives also numerical examples and application to biological and financial data.

Keywords: Biological and financial data; Fractional Brownian motion; Gaussian processes observed at random times; Nonparametric estimation; Spectral density; Wavelet analysis.

1 Introduction

Consider first a Gaussian process $X = \{X(t), t \in I\!\!R\}$ with zero mean and stationary increments, but results will be extended in case where a polynomial trend is added to such processes. Therefore X can be written following harmonizable representations (see for instance Cramèr and Leadbetter, 1967),

$$X(t) = \int_{I\!\!R} \left(e^{it\xi} - 1\right) f^{1/2}(\xi)\, dW(\xi), \qquad \text{for all } \ t \in I\!\!R, \tag{1}$$

where $W(dx)$ is a complex Brownian measure such that $W(dx) = \overline{W(-dx)}$ and $I\!\!E|W(dx)|^2 = dx$, and f is a Borelian positive even function so-called the spectral density of X and is such that

$$\int_{I\!\!R} \left(1 \wedge |\xi|^2\right) f(\xi)\, d\xi < \infty. \tag{2}$$

In the sequel, f will be supposed to satisfy also Assumption F defined below but the conditions are weak and the class of processes that can be considered is general. As a particular case, if X is a stationary processes, one will still denote f the spectral density such that

$$X(t) = \int_{I\!\!R} e^{it\xi} f^{1/2}(\xi)\, dW(\xi), \qquad \text{for all } \ t \in I\!\!R. \tag{3}$$

Even if their definition are different, in the sequel f will denote as well the spectral density of a process having stationary increments or a stationary process (see the explanation in Proposition 2.1). Define also the σ-algebra $\mathcal{F}_X$ generated by the process X, *i.e.*

$$\mathcal{F}_X := \sigma\big\{X(t),\, t \in I\!\!R\big\}. \tag{4}$$

A path of such a process X on the interval $[0, T_n]$ at the discrete times t_i for $i = 0, 1, \ldots, n$ is observed, *i.e.*

$$\big(X(t_0), X(t_1), \ldots, X(t_n)\big) \text{ is known, with } 0 = t_0 < t_1 < \cdots < t_n = T_n.$$

A unified frame of irregular observed times, grouping deterministic and stochastic ones, will be considered. Let us assume first that there exists a sequence of positive real numbers $(\delta_n)_{n\in I\!\!N}$ and a sequence of random variables (r.v. in the sequel) $(L_k)_{k\in I\!\!N}$ (which could be deterministic real numbers) such that

$$\forall k \in \{0, 1, \ldots, n-1\}, \quad t_{k+1} - t_k \quad := \quad \delta_n\, L_k, \quad \text{and} \quad \delta_n \underset{n\to\infty}{\longrightarrow} 0. \tag{5}$$

It is clear that $T_n = \delta_n\big(L_0 + \ldots + L_{n-1}\big)$. For Z a r.v. and $\alpha \in (0, \infty)$, denote $\|Z\|_\alpha := \big(I\!\!E(|Z|^\alpha)\big)^{1/\alpha}$ if $I\!\!E(|Z|^\alpha) < \infty$. Now, assume that there exists $s \in [1, \infty]$ such that

Assumption S(s) *$(L_k)_{k\in I\!\!N}$ is a sequel of independent positive random variables, independent to $\mathcal{F}_X$, such that there exist $0 < m_s < M_s < \infty$ satisfying*

- *if $s < \infty$, $m_s \le \|L_k\|_s \le M_s$ for all $k \in I\!\!N$.*
- *if $s = \infty$, $m_\infty \le L_k \le M_\infty$ for all $k \in I\!\!N$.*

Before going further, let us give a detailed example: the heart rate variability. Cardiologists are interested in the behavior of its spectral density on various frequencies bands $(\omega_k,\, \omega_{k+1})$, $k = 1, \ldots, K$ with usually both frequencies bands $(0.04\, Hz,\, 0.15\, Hz)$ and $(0.15\, Hz,\, 0.5\, Hz)$ corresponding respectively to the orthosympathic nervous system and the parasympathic one, see [28]. The spectral density follows different power laws on the different frequencies bands, *i.e.* $f(\xi) = \sigma_i \cdot |\xi|^{-\beta_i}$ when $\xi \in (\omega_i, \omega_{i+1})$. Finally, according to the type of activity or the period of the day, one notices a variation of these parameters. We send back to Section 3, which brings to light different power laws during the working hours and the hours of sleeps, *i.e.* $(\beta_1^{work}, \beta_2^{work}) \neq (\beta_1^{sleep}, \beta_2^{sleep})$. In the example above both frequencies bands seem fixed. In other examples, the frequency of cut between the various bands associated with various power laws must be determined and constitutes the parameter of interest. We refer to our work on biomechanicals data [4] and in the study of the financial data (Section 3). These examples show the concrete character and interesting perspectives of a non-parametric estimation of the spectral density. In the applications, signals are observed at discrete times mostly irregularly spaced and random. This type of observation can be met in health, physics, mechanics, oceanography and in these cases the times of observation depend on the measuring instrument, thus of a hazard independent from that of the process X. In this context, the hypothesis of independence of durations $(L_k)_{k\in I\!\!N}$ and $\mathcal{F}_X$ is completely realistic. The only case where this assumption seems restrictive concerns financial data. However it is until this day always made, see for instance, Aït-Sahalia and Mykland (2008), Hayashi and Yoshida (2005) or Engle and Russel (1998).

To our knowledge, estimating the spectral density of a Gaussian process with stationary increments on finite bands of frequencies from observation at discrete times is a new problem. Recall that the spectral density $f(\xi) = \sigma^2|\xi|^{-(2H+1)}$ corresponds to a fBm with Hurst index H. However, most of the statistical studies on the fBm or its generalizations concern the estimation of the local regularity (linked to the behavior of the spectral density at $+\infty$) or that of the long memory (linked to the behavior of the spectral density in the neighborhood of 0) from the observation of a path at deterministic and regularly spaced discrete times, see for instance Dahlaus (1989), Gloter and Hoffmann (2007), Moulines *et al.* (2007) or the book edited by Doukhan, Oppenheim and Taqqu (2003). Begyn (2005) seems to be the only reference concerning the estimation of H under irregular (but deterministic) observation times. On the other hand, the estimation of the spectral density of stationary Gaussian processes is a well known problem corresponding numerous practical applications, see Shapiro and Silverman (1960), Dalhaus (1989), Parzen (1983). The used methods are based on the periodogram $I_T^X(\xi) = (2\pi T)^{-1} \left| \int_0^T e^{-i\xi t} X(t)\, dt \right|^2$. However, the observation at regularly spaced times $t_i = i\Delta$ induces *aliasing*, that is $\lim_{T\to\infty} \mathbb{E} I_T(\xi) = f(\xi)$ but $\lim_{N\to\infty} \mathbb{E} J_N(\xi) = \sum_{k\in\mathbf{Z}} f(\xi + 2k\pi\Delta^{-1})$ where $J_N(\xi) := (2\pi N\Delta)^{-1} \left| \Delta \sum_{k=1}^{N} e^{-i\xi k\Delta} X(k\Delta) \right|^2$ denotes the empirical periodogram. To avoid *aliasing*, one uses random sampling, then the empirical periodogram becomes asymptotically unbiased, and by using a spectral window one can deduce an estimator of the spectral density and a Central Limit Theorem (CLT) with a rate of convergence in $T^{-2/5}$, see Masry (1978 a and b) or Lii and Masry (1994). These results are obtained for random sampling verifying very specific conditions that we will call in the sequel

Masry's conditions: the process of observation times $(t_i)_{i=1,\dots,n}$ is stationary, independent of X, with known mean intensity β and density of covariance $c(u)$ and verify the condition $\beta^2 + c(u) > 0$ a.e. (Masry, 1978).

When the trajectory is not sampled but observed at random times not chosen by the experimenter, we then have to verify that the family (t_i) satisfies Masry's conditions and to estimate the mean intensity β and the density of the covariance function $c(u)$.

We chose another approach: a wavelet analysis. This approach was introduced for the fBm by Flandrin (1992), and popularized by many authors, see for e.g. Abry, Flandrin, Veitch, Taqqu (2002), to estimate the parametric behavior of a power law spectral density when $\ln|\xi| \to \infty$ or $\ln|\xi| \to -\infty$. In this work, we show that the wavelet analysis is also an interesting tool to estimate the spectral density on finite bands of frequencies for Gaussian processes with stationary increments (or a stationary Gaussian processes) when one observes one path at random times. Let us underline that the wavelet analysis in Abry *et al.* (2002) is based on the empirical variance of the wavelet coefficients and thus is different from that proposed by Lehr and Lii (1997) or Goa, Ahn and Heyde (2002) who consider respectively the wavelet decomposition of the estimator derived from the empirical periodogram and the periodogram of the Haar wavelet transform of the process. In both cases, discrete times observation should satisfy Masry's conditions to avoid *aliasing*, this is not any more the case for the wavelet analysis *à la* Abry, Flandrin. We obtain then a non-parametric estimator and a CLT with the same rate of convergence in $T^{-2/5}$ than for the periodogram, but for a class of observation different from the Masry's one which allows non-stationnary or regularly spaced times. This method plainly uses the time-frequency localization of the wavelet:
- in frequency, to build a nonparametric estimator of the spectral density with continuous time observation.
- in time, to bound the error of approximation of the wavelet coefficient with discrete time observation. Indeed, one uses that $\psi(t)$ decreases faster than $(1+|t|)^{-3}$ and $\psi \in L^1(\mathbb{R})$ to bound the cutting error and the discretization error, see lemma 4.5.

The remainder of the paper is organized as follow: Section 2 is devoted to the wavelet analysis of X and the CLT satisfied by the estimator of f. This estimator is applied to generated data and real data in Section 3. Section 4 contains the proofs.

2 Main results

Let $\psi : \mathbb{R} \to \mathbb{C}$ be a function so-called the "mother" wavelet, and denote $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi\, x} f(x)\, dx$ the Fourier transform of $f \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$. Let $(m, r) \in [1, \infty) \times \mathbb{R}_+$ and the family of assumptions on ψ:

Assumption $\mathbf{W}(m, r)$ *$\psi : \mathbb{R} \mapsto \mathbb{C}$ is a differentiable function satisfying:*
• *$\forall n \in \mathbb{N}$, $\int_{\mathbb{R}} |t^n \psi(t)|\, dt < \infty$ if $n \leq m + 1$ and $\int_{\mathbb{R}} t^n \psi(t)\, dt = 0$ if $n \leq m$;*
• *$\exists C_\psi > 0$ such that $\forall \xi \in \mathbb{R}$, $\big(1 + |\xi|\big)^r \left(\big|\widehat{\psi}(\xi)\big| + \big|\widehat{\psi}'(\xi)\big|\right) \leq C_\psi$.*

The first condition of $\mathrm{W}(m, r)$ implies that $\widehat{\psi}(\xi)$ has a zero of order $(m+1)$ at zero and is m times continuously differentiable. These conditions are mild and are satisfied by many famous wavelets (Daubechies, Lemarié-Meyer,...). It is also not mandatory to choose ψ to be a "mother" wavelet associated to a multiresolution analysis of $\mathbb{L}^2(\mathbb{R})$ and the whole theory can be developed without resorting to this assumption.

Let $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$, and define $d_X(a, b)$ the wavelet coefficient of the process X for the scale a and the shift b, such that

$$d_X(a, b) := \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi\left(\frac{t - b}{a}\right) X(t)\, dt$$

This family of wavelet coefficients satisfies the following property:

Proposition 2.1 *Let ψ satisfy Assumption $W(1, 0)$ and X be a Gaussian process defined by (1) or (3) with a spectral density f satisfying (2). Then,*

$$d_X(a, b) = \sqrt{a} \int_{\mathbb{R}} e^{ib\xi}\, \overline{\widehat{\psi}}(a\xi)\, f^{1/2}(\xi)\, dW(\xi) \text{ for all } (a, b) \in \mathbb{R}_+^* \times \mathbb{R}, \tag{1}$$

and, for $a > 0$, $(d_X(a, b))_{b \in \mathbb{R}}$ is a stationary centered Gaussian process with

$$\mathbb{E}\big|d_X(a, b)\big|^2 = \mathcal{I}_1(a) := a \int_{\mathbb{R}} |\widehat{\psi}(au)|^2 f(u)\, du \text{ for all } b \in \mathbb{R}. \tag{2}$$

The proof of this proposition is grouped with all the other proofs in Section 4. A straightforward computation of $\mathcal{I}_1(a)$ is not available from $\big(X(t_0), \ldots, X(t_n)\big)$ for two reasons:

1. on one hand, $d_X(a, b)$ is defined with a Lebesgue integral and cannot be directly computed from data. As in Gloter and Hoffmann (2007), an approximation formula will be considered for computing wavelet coefficients. Thus, for $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$ we define an empirical wavelet coefficient by

$$e_X(a, b) := \frac{1}{\sqrt{a}} \sum_{i=0}^{n-1} \Big(\int_{t_i}^{t_{i+1}} \psi\Big(\frac{t - b}{a}\Big)\, dt\Big)\, X\big(t_i\big). \tag{3}$$

2. on the other hand, a sample mean of $\big|d_X(a, b)\big|^2$ instead on $\mathbb{E}\big|d_X(a, b)\big|^2$ is only computable. Thus, define the sample estimator of $\mathcal{I}_1(a)$ by

$$J_n(a) := \frac{1}{n+1} \sum_{k=0}^{n} \big|e_X(a, c_k)\big|^2, \tag{4}$$

where $(c_k)_k$ is a family of increasing real numbers (so-called shifts). In this paper, we will consider a uniform repartition of shifts, *i.e.*

$$c_k = T_n^\rho + k\, \frac{T_n - 2\, T_n^\rho}{n} \quad \text{with } \rho \in (3/4, 1). \tag{5}$$

In this example $(c_k)_{1\leq k\leq n}$ is depending on T_n since shifts could be r.v. depending on random times $(t_1,\ldots,t_n)$. Another choices of $(c_k)_k$ are possible (for instance $c_k = t_k$) but we have not been able to find an optimal choice and simulations do not show differences between these choices. Remark that the terms T_n^ρ are necessary to avoid border effects. Now additional conditions on f have to be considered:

Assumption F *f is an even function, differentiable on $[0,\infty)$ except for a finite number K of real numbers $\omega_0 = 0 < \omega_1 < \cdots < \omega_K$, but f admits left and right limits in ω_k, with a derivative f' (defined on all open intervals (ω_k,ω_{k+1}) with $\omega_{K+1} = \infty$ by convention) such that*

$$\int_{\mathbb{R}} \big(1\wedge|\xi|^3\big)\cdot|f'(\xi)|\,d\xi < \infty. \tag{6}$$

Moreover, there exist $C_0, C_0' > 0$ and $H \in (0,1)$, such that $\forall |x| \geq \omega_K$

$$f(x) \leq C_0\,|x|^{-(2H+1)} \qquad \text{and} \qquad f'(x) \leq C_0'\,|x|^{-(2H+2)}. \tag{7}$$

Here there are several examples of processes having a spectral density f satisfying Assumption F:

Examples : 1. A smooth Gaussian process having stationary increments;
2. A fractional Brownian motion with Hurst parameter $H \in (0,1)$ satisfying $\mathbb{E}X(1) = \sigma^2$ is such that $f(\xi) = \sigma^2 H\Gamma(2H)\sin(\pi H)\,|\xi|^{-(2H+1)}/\pi$; 3. In Bardet and Bertrand (2007a), the family of multiscale fractional Brownian motions is introduced for which $f(\xi) = \sigma_i^2\,|\xi|^{-(2H_i+1)}$ for $|\xi| \in [\omega_k,\omega_{k+1}[$ where $\omega_0 = 0 < \omega_1 < \cdots < \omega_K < \omega_{K+1} = +\infty$, $H_0 < 1$, $0 < H_K < 1$ and $(\sigma_i,H_i) \in \mathbb{R}_+\times\mathbb{R}$ for $i = 1,\ldots,K-1$. Then Condition (2) and Assumption F are checked with $H = H_K$.
4. A stationary process with a bounded spectral density such as Ornstein-Uhlenbeck process.

It is possible to establish a CLT satisfied by $J_n(a)$ which is computed from the observed trajectory $(X(t_0),\ldots,X(t_n))$.

Theorem 2.1 *Let X be a Gaussian process defined by (1) or (3) with a spectral density f satisfying (2) and Assumption F, ψ satisfying Assumption $W(1,3)$ and $(c_k)_k$ defining by (5). Under Assumption S(s) with $2+2H \leq s \leq \infty$ and if δ_n is such that*

$$n\,\delta_n^{1+\theta(s,H)} \underset{n\to\infty}{\longrightarrow} 0 \quad \textit{with} \quad \theta(s,H) := \begin{cases} \frac{(H+1)(s-1)}{s+H+1} & \textit{if } 2+2H \leq s < \frac12 + \frac{1}{2H} \\ \frac{2H(s-1)}{2H+1} & \textit{if } \frac12+\frac{1}{2H} \leq s < \frac32 + \frac{1}{2H} \\ \frac{(H+1)(s-1)}{s+H} & \textit{if } \frac32 + \frac{1}{2H} \leq s \leq 2+\frac1H \\ 1 & \textit{if } s \geq 2+\frac1H \end{cases}$$

$(1/3 < \theta(s,H) \leq 2)$, then $\forall a \in [a_{min},\,a_{max}]$,

$$\sqrt{\mathbb{E}\,T_n}\,\big(J_n(a) - \mathcal{I}_1(a)\big) \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \mathcal{N}\Big(0\,,\,4\pi\,a^2\int_{\mathbb{R}}\big|\widehat{\psi}(az)\big|^4 f^2(z)\,dz\Big). \tag{8}$$

From the computation of the variance of T_n, the convergence rate of the CLT (8) is $(n\delta_n)^{1/2}$. Therefore, when H is unknown, Theorem 2.1 always holds when $s \geq 4$ and $n\,\delta_n^{2-\frac1s} = O(1)$ and its convergence rate is $n^{\frac{s-1}{4s-2}}$ and $o\big(n^{1/4}\big)$ when $s = \infty$. When H is known, this results can be a little improved and the convergence rate is $o\big(n^{\frac12\frac{\theta(s,H)}{1+\theta(s,H)}}\big)$.

The CLT 2.1 can be used to prove a CLT satisfied by an estimator of f. Indeed, let us consider a family $(\psi_\lambda)_{\lambda\in\mathbb{R}_+^*}$ such as

$$\psi_\lambda(x) := \frac{1}{\sqrt{\lambda}}\,e^{i\,x}\,\psi\big(\frac{x}{\lambda}\big) \quad \forall x \in \mathbb{R} \qquad \Longrightarrow \quad \widehat{\psi}_\lambda(\xi) = \sqrt{\lambda}\,\widehat{\psi}\big(\lambda(\xi-1)\big) \quad \forall \xi \in \mathbb{R},$$

and ψ satisfying $\widehat{\psi}(\xi) = 0$ for $|\xi| \geq \Lambda$ with $\Lambda > 0$. Note that for all $\lambda \geq \Lambda$, ψ_λ satisfies Assumption W(m,r) when ψ satisfies Assumption W(m,r). Now, $\mathcal{I}_\lambda(a) := \int_{I\!R} |\widehat{\psi}_\lambda(u)|^2 f(u/a)\, du \to f(1/a)\, \|\psi\|^2_{\mathcal{L}^2}$ when $\lambda \to \infty$ under weak hypothesis. Therefore for $0 < \xi$, let us define $\widehat{f}_n^{(\lambda)}(\xi) := J_n^{(\lambda)}(1/\xi)/\, \|\psi\|^2_{\mathcal{L}^2}$, where $J_n^{(\lambda)}$ denotes J_n when ψ is replaced by ψ_λ, and

$$\widehat{f}_n^{(\lambda)}(\xi) \quad := \quad \frac{\xi}{\|\psi\|^2_{\mathcal{L}^2}}\, \frac{1}{n+1} \sum_{k=0}^{n} \Big| \sum_{i=0}^{n-1} X(t_i) \int_{t_i}^{t_{i+1}} \psi_\lambda\big(\xi(t-c_k)\big)\, dt \Big|^2,$$

with $c_k = T_n^\rho + \dfrac{k}{n}\,(T_n - 2\,T_n^\rho)$. By using an appropriated choice of a sequence (ψ_{λ_n}), one obtains:

Corollary 2.1 *Let X be a Gaussian process defined by (1) or (3) where the spectral density f is a twice continuously differentiable function on $I\!R^*$ satisfying (2) and Assumption F. Under Assumption S(s) and W$(1,5)$ and if ψ satisfies $\widehat{\psi}(\xi) = 0$ for $|\xi| \geq \Lambda$ with $\Lambda > 0$, then $\forall \xi > 0$,*

$$\sqrt{\frac{T_n}{\lambda_n}}\, \big(\widehat{f}_n^{(\lambda_n)}(\xi) - f(\xi)\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\Big(0\,,\, \frac{4\,\pi}{\xi}\, f^2(\xi)\, \frac{\int_{I\!R} \big|\widehat{\psi}(u)\big|^4 du}{\big(\int_{I\!R} \big|\widehat{\psi}(u)\big|^2 du\big)^2}\Big), \tag{9}$$

if $\delta_n = C_\delta\, n^{-d}$, $\lambda_n = C_\lambda\, n^{d'}$, where $0 < \frac{1}{5}\,(1-d) < d' < \frac{1}{2}\,(1-d) < 1$ and

- *if $s = \infty$, when $d > (2+H)^{-1}$;*
- *if $\max\big(2+2H\,,\, \frac{3}{2} + \frac{1}{2H}\big) \leq s < \infty$, when $(s-1-(sH+2s-1)d) < d'(H+1)$;*
- *if $2+2H \leq s < \max\big(2+2H\,,\, \frac{3}{2} + \frac{1}{2H}\big)$, when $(s-1)(2-d(3H+2)) < d'(H+1)$.*

Remark 2.1 *The rate of convergence of the parametric estimator is $T_n^{-1/2}$.The optimal rate of convergence for the nonparametric estimator is $T_n^{-2/5}$, it is obtained by equaling the bias and the variance with $\lambda_n = T_n^{1/5}$. This is the same rate of convergence than for the periodogram in continuous time (Parzen, 1983) or with a random sampling satisfying Masry's conditions (Lii and Masry, 1994).*

Moreover, under W$(m,5)$, $\int t^n \psi(t) dt = 0$ for all $n \leq m$ and any wavelet coefficients of any polynomial function with degree less or equal to m are vanished. Therefore, the estimator $\widehat{f}_n^{(\lambda_n)}$ is robust since

Corollary 2.2 *Under Assumption W$(m,5)$ with $m \in I\!N^*$, Proposition 2.1 holds when a polynomial trend with degree less or equal to m is added to X.*

The following Table 1 summarizes the "optimal" choices of d' (in order to maximize the convergence rate of $\widehat{f}_n$) following several cases.

	H known δ_n fixed	H known δ_n non-fixed	H unknown δ_n fixed	H unknown δ_n non-fixed
Choice of d	$d\ \big(> \frac{5s+4H-1}{5s(H+2)-6-H}\big)$	$\frac{5s+4H-1}{5s(H+2)-6-H} + \kappa$	$d\ \big(\geq \frac{1}{2} + \frac{1}{5s-3}\big)$	$\frac{1}{2} + \frac{1}{5s-3}$
Choice of d'	$\frac{1-d}{5} + \kappa$	$\frac{(s-1)(H+1)}{5s(H+2)-6-H}$	$\frac{1-d}{5} + \kappa$	$\frac{s-1}{2(5s-3)} + \kappa$
Convergence rate	$n^{\frac{2}{5}(1-d)-\frac{\kappa}{2}}$	$n^{\frac{2(s-1)(H+1)}{5s(H+2)-6-H}-\frac{\kappa}{2}}$	$n^{\frac{2}{5}(1-d)-\frac{\kappa}{2}}$	$n^{\frac{s-1}{5s-3}-\frac{\kappa}{2}}$

Table 1: *Optimal choices of d' (and therefore (λ_n)) and convergence rate of $\widehat{f}_n$ (the case $s = \infty$ is obtained as the limit of ratios) with $0 < \kappa$ arbitrary small.*

3 Numerical experiments

For the numerical applications, one has chosen:

1. ψ is chosen such as $\widehat{\psi}(\xi) = \exp\big(-(|\xi| \cdot (5-|\xi|))^{-1}\big)\mathbf{1}_{|\xi|\leq 5}(\xi)$ which satisfies Assumption W(m,r) for any (m,r) (and $\widehat{\psi}(\xi)=0$ for $|\xi|\geq 5$).

2. $\delta_n = n^{-0.6}$ for insuring the convergence of $\widehat{f}_n^{(\lambda_n)}(\xi)$ for any $H\in(0,1)$ and $s\geq 3$.

3. $\lambda_n = n^{d'}$ with $1/6 < d' < 1/2$. However, admissibility condition on wavelets (ψ_{λ_n}) requires that $n^{d'} \geq \Lambda = 5$. Moreover, for removing the bias term, d' has to be chosen large enough following n. Thus, after numerous simulations, we have chosen $d' = \log(15)/\log(n)$.

3.1 Estimation of the spectral density of a fractional Brownian motion observed at random times

For a standard ($I\!\!E X^2(1) = 1$) fBm with Hurst parameter H, $f(\xi) = C(H)\,|\xi|^{-2H-1}d\xi$ with $C(H) = \big(H\Gamma(2H)\sin(\pi H)\big)/\pi$. Four different kind of random times are considered:

1. **(T1)**: non-random uniform sampling, such that $L_k = 1$ for all $k\in I\!\!N^*$;

2. **(T2)**: exponential random times, such that $I\!\!E L_k = 1$ for all $k \in I\!\!N^*$;

3. **(T3)**: random times such that for for all $k\in I\!\!N^*$, the cumulative distribution function of L_k is $F_{L_k}(x) = (1-x^{-4})\mathbf{1}_{x\geq 1} \Longrightarrow I\!\!E L_k^p < \infty$ for all $p<4$ and $I\!\!E L_k^4 = \infty$;

4. **(T4)**: random times such that for for all $k\in I\!\!N^*$, the cumulative distribution function of L_k is $F_{L_k}(x) = (1-x^{-2})\mathbf{1}_{x\geq 1} \Longrightarrow I\!\!E L_k^p < \infty$ for all $p<2$ and $I\!\!E L_k^2 = \infty$.

An example of such estimation of the spectral density for $H=0.2$, $N=50000$ and random times T2 is presented in Figure 1. The results of simulations are also provided in the Table 2.

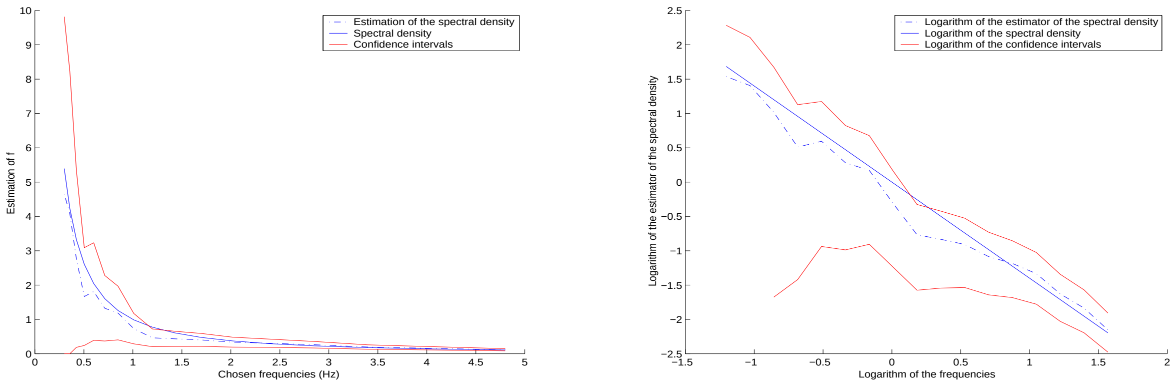

Figure 1: *An example of the estimation of the spectral density (left) and its logarithm (right) of a FBM observed at exponential random times (T2) with confidence intervals ($H=0.2$, $N=50000$).*

Comments on simulation results: 1. the larger N the more accurate the estimator of f except for case of random times T4 (which is a case not included in conditions of Proposition 2.1); 2. The results are similar for T1 and T2, a little less accurate for T3; 3. the smaller H the more accurate the estimator of f.

			$H = 0.2$	$H = 0.5$	$H = 0.8$
	T1	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.47	0.65	0.77
		$\widehat{MISE}$ on $[0.3, 5]$	2.53	13.50	80.89
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.65	0.67	0.75
$N = 10^3$		$\widehat{MISE}$ on $[0.3, 5]$	3.64	10.65	39.85
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.42	0.72	1.20
		$\widehat{MISE}$ on $[0.3, 5]$	2.48	7.83	55.20
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	1.03	3.34	2.44
		$\widehat{MISE}$ on $[0.3, 5]$	6.07	84.05	144.40

			$H = 0.2$	$H = 0.5$	$H = 0.8$
	T1	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.35	0.37	0.79
		$\widehat{MISE}$ on $[0.3, 5]$	0.95	3.90	57.19
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.45	0.47	0.29
$N = 10^4$		$\widehat{MISE}$ on $[0.3, 5]$	1.04	3.17	16.26
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.47	0.46	0.95
		$\widehat{MISE}$ on $[0.3, 5]$	1.20	4.91	26.6
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.61	0.61	1.74
		$\widehat{MISE}$ on $[0.3, 5]$	2.74	9.55	49.55

			$H = 0.2$	$H = 0.5$	$H = 0.8$
	T1	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.36	0.30	0.40
		$\widehat{MISE}$ on $[0.3, 5]$	0.81	2.60	10.77
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.21	0.22	0.31
$N = 5 \cdot 10^4$		$\widehat{MISE}$ on $[0.3, 5]$	1.07	2.07	7.65
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.34	0.26	0.48
		$\widehat{MISE}$ on $[0.3, 5]$	0.74	3.17	13.3
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.40	0.56	2.59
		$\widehat{MISE}$ on $[0.3, 5]$	1.02	5.69	41.41

Table 2: *Consistency of the estimator $\widehat{f}_N$ in the case of paths of FBM observed at random times (*50 *independent replications are generated in each case).*

3.2 Estimation of the spectral density of the stationary Ornstein-Uhlenbeck process

Here, instead of FBM which is a process having stationary increments, we consider the stationary Ornstein-Uhlenbeck process which is a Gaussian stationary process with covariance $r(t) := \exp(-\alpha|t|)$ and therefore with spectral density $f(\xi) := \alpha\big(\pi(\alpha^2 + \xi^2)\big)^{-1}$. In such case, since this spectral density is an analytic function, there exists more accurate nonparametric estimator (see for instance, Ibragimov, 2004). However, to our knowledge, the case of paths observed at random times is not considered is this literature. The results of simulations are provided in the Table 3.

Comments on simulation results: 1. the larger N the more accurate the estimator of f for all choice of random time; 2. The results are similar for T1, T2, T3 and a little less accurate for T4; 3. surprisingly, the case $\alpha = 1$ is not clearly better than $\alpha = 0.1$ despite the fact that the larger α the less correlated the process.

3.3 Estimation of the spectral density of heart inter-beat series

Heart inter-beats of several patients have been recorded during 24h (see an example in Figure 3.3). These data have been kindly furnished by professor Alain Chamoux and Gil Boudet (Faculty of Medicine, University of Auvergne, Clermont-Ferrand). We decompose these data in 3 temporal zones following the activity:

- Quiet activities ($t \in [1, 28000]$ in seconds);
- Intensive activities ($t \in [28000, 51400]$ in seconds);
- Sleep ($t \in [60000, 83400]$ in seconds).

Applying the spectral density estimator on those 3 sub-data and plotting its log-log representation for frequencies in $[0.02, 1]$ Hz, we observe that:

- in zone "Sleep" (see Figure 2), only one regression line could be computed for frequencies in $[0.04, 0.5]$ Hz which is the usual spectral interval considered by specialists; in this zone $\widehat{H} \simeq 0.99$;

			$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$
$N = 10^3$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.51	0.22	0.020
		$\widehat{MISE}$ on $[0.3, 5]$	0.022	0.014	0.00067
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.30	0.30	0.021
		$\widehat{MISE}$ on $[0.3, 5]$	0.010	0.024	0.0010
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.36	0.23	0.018
		$\widehat{MISE}$ on $[0.3, 5]$	0.00052	0.015	0.00052
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.28	0.23	0.032
		$\widehat{MISE}$ on $[0.3, 5]$	0.016	0.016	0.0045

			$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$
$N = 10^4$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.20	0.18	0.017
		$\widehat{MISE}$ on $[0.3, 5]$	0.0033	0.0088	0.00031
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.14	0.18	0.019
		$\widehat{MISE}$ on $[0.3, 5]$	0.0032	0.0092	0.00036
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.17	0.18	0.016
		$\widehat{MISE}$ on $[0.3, 5]$	0.0027	0.011	0.00032
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.18	0.13	0.024
		$\widehat{MISE}$ on $[0.3, 5]$	0.0058	0.0095	0.00037

			$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$
$N = 5 \cdot 10^4$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.14	0.10	0.012
		$\widehat{MISE}$ on $[0.3, 5]$	0.0016	0.0045	0.00015
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.26	0.13	0.011
		$\widehat{MISE}$ on $[0.3, 5]$	0.012	0.0055	0.00014
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.18	0.14	0.012
		$\widehat{MISE}$ on $[0.3, 5]$	0.0023	0.0049	0.00017
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$	0.16	0.16	0.017
		$\widehat{MISE}$ on $[0.3, 5]$	0.0084	0.034	0.00019

Table 3: *Consistency of $\widehat{f}_N$ in the case of paths of stationary Ornstein-Uhlenbeck process observed at random times (*50 *independent replications are generated in each case).*

- in zone "Quiet activities" (respectively "Intensive activities", (see Figure 3), two regression lines could be drawn for frequencies in $[0.04, 0.5]$ Hz, distinguishing the orthosympathic and the parasympathic spectral domains. Using an algorithm computing the "best" two regression lines (see for instance Bardet and Bertrand, 2007b), one obtains that $H \simeq 1.34$ (respectively $H \simeq 1.44$) in the orthosympathic domain which is $[0.04, 0.09]$ Hz (respectively $[0.04, 0.11]$ Hz) and $H \simeq 0.89$ (respectively $H \simeq 0.79$) in the parasympathic domain which is $[0.09, 0.5]$ Hz (respectively $[0.11, 0.5]$ Hz).

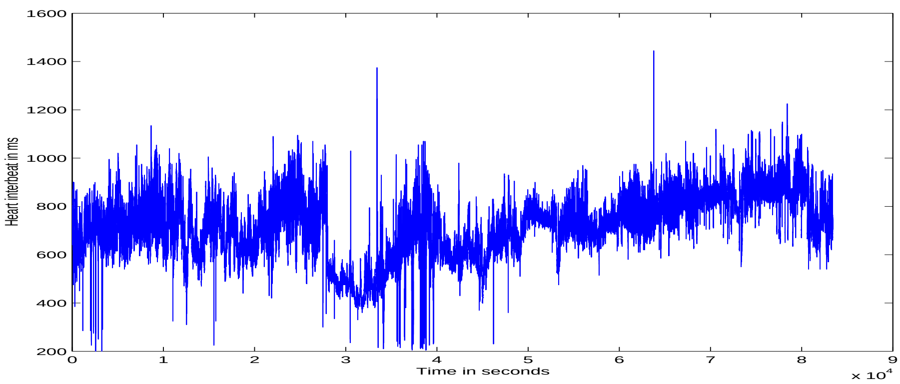

Figure 2: *An example of heart inter-beats during* 24*h*

3.4 Estimation of the spectral density of log-return of a share

One considers the price of share Total during a day at Paris (see Figure 4). These data has been kindly furnished by Crédit Agricole Cheuvreux, CALYON (Paris). Applying the spectral density estimator and

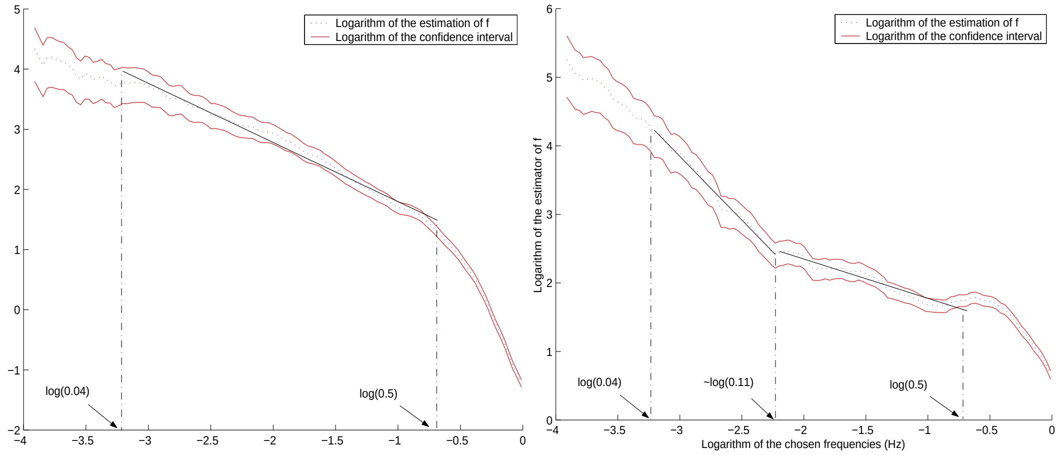

Figure 3: *Log-log representation of the spectral density estimator during "Sleep" zone (left) and "Intensive activities" zone (right)*

plotting its log-log representation, we observe that durations fit an exponential law with mean 11 seconds and that the spectral density is linear in for frequencies smaller than 0.008 Hz and has an erratic behavior at higher frequencies. The critical frequency corresponds to a time lag of 125 seconds and could be interpreted as the frontier between events and regularity. Remark that for high frequencies a Gaussian distribution is not appropriated.

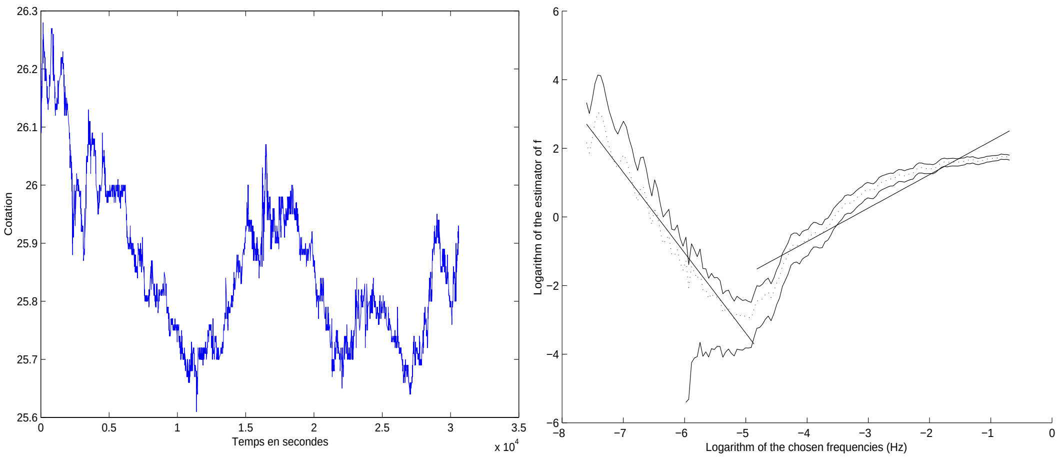

Figure 4: *An example of quotation of a share during a day, i.e.* 8.5 *hours (left), and the log-log representation of the spectral density estimator (right)*

4 Proofs

4.1 Proofs of useful lemmas and Proposition 2.1

In the sequel, the following lemmas will be useful:

Lemma 4.1 *Let X be a Gaussian process defined by (1) with a spectral density function f satisfying (2).*

Then there exists $C_0 > 0$ such that

$$|\mathbb{E}\left(X(t_1)X(t_2)\right)| \quad \le \quad C_0(1+|t_1|)(1+|t_2|) \quad \textit{for all } (t_1,t_2) \in \mathbb{R}^2. \tag{1}$$

Proof. For all $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}X^2(t) \quad &= \quad \int_{\mathbb{R}} \left|e^{it\xi}-1\right|^2 f(\xi)\,d\xi \le 2\int_0^1 |t\xi|^2 f(\xi)\,d\xi + 8\int_1^\infty f(\xi)\,d\xi \\ &\le \quad (2t^2+8)\times \int_0^\infty \left(1\wedge|\xi|^2\right) f(\xi)\,d\xi.\end{aligned}$$

This implies $\mathbb{E}\big(X(t)^2\big) \le C_0\,(1+|t|^2)$ where $C_0 = 4\int_{\mathbb{R}}\left(1\wedge|\xi|^2\right)f(\xi)\,d\xi$. Then, by using Cauchy-Schwartz inequality, one deduces (1). ■

Proof. [of Proposition 2.1] Firstly, one can show that for all $a > 0$ and $b \in \mathbb{R}$, $\mathbb{E}d_X^2(a,b) < \infty$. This induces that $d_X(a,b)$ is well defined. Indeed, one has

$$\begin{aligned}\mathbb{E}d_X^2(a,b) &= \frac{1}{a}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi\left(\frac{t_1-b}{a}\right)\psi\left(\frac{t_2-b}{a}\right)\mathbb{E}\left(X(t_1)X(t_2)\right)dt_1dt_2 \\ &\le \frac{C_0}{a}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\psi\left(\frac{t_1-b}{a}\right)\right|\left|\psi\left(\frac{t_2-b}{a}\right)\right|(1+|t_1|)(1+|t_2|)dt_1dt_2 \\ &\le a\,C_0\left(\int_{\mathbb{R}}|\psi(u)|\,(1+|b|+|a\,u|)du\right)^2 < \infty,\end{aligned}$$

where we have used successively the bound (1), the change of variable $u=(t-b)/a$ and the first condition of Assumption W$(1,0)$. Next, one turns to the proof of the formula (1). It is obvious that $\displaystyle\int_{\mathbb{R}}\left|\psi\Big(\frac{t-b}{a}\Big)\right|dt<\infty$ and $\displaystyle\int_{\mathbb{R}}\left|\big(e^{it\xi}-1\big)f^{1/2}(\xi)\right|dW(\xi)<\infty$ since the condition (2) holds. From the Fubini Theorem for stochastic integral (see [16, Lemma 4.1, p. 116]),

$$\begin{aligned}d_X(a,b) = \int_{\mathbb{R}}\psi\Big(\frac{t-b}{a}\Big)X(t)dt &= \int_{\mathbb{R}}\psi\Big(\frac{t-b}{a}\Big)\Big[\int_{\mathbb{R}}(e^{it\xi}-1)f^{1/2}(\xi)dW(\xi)\Big]dt \\ &= \int_{\mathbb{R}}\Big[\int_{\mathbb{R}}(e^{it\xi}-1)\psi\Big(\frac{t-b}{a}\Big)dt\Big]f^{1/2}(\xi)dW(\xi).\end{aligned}$$

But $\int_{\mathbb{R}}\left(e^{it\xi}-1\right)\psi\left(\frac{t-b}{a}\right)dt = a\,e^{ib\xi}\int_{\mathbb{R}}e^{iau\xi}\,\psi(u)\,du = a\,e^{ib\xi}\,\overline{\widehat{\psi}}(a\xi)$ for all $(a,b)\in]0,\infty[\times\mathbb{R}$, which implies (1) and $d_X(a,b)$ is a Gaussian centered r.v. with variance $\mathcal{I}_1(a)$. Moreover, for all $a>0$ and $(b_1,b_2)\in\mathbb{R}^2$,

$$\mathbb{E}\big(d_X(a,b_1)\,\overline{d_X(a,b_2)}\big) = a\int_{\mathbb{R}} e^{i(b_1-b_2)\xi}\left|\widehat{\psi}(a\xi)\right|^2 f(\xi)\,d\xi. \tag{2}$$

Thus for a given $a>0$, $\mathbb{E}\big(d_X(a,b_1)\,d_X(a,b_2)\big)$ is only depending on (b_1-b_2) which induces that $(d_X(a,b))_{b\in\mathbb{R}}$ is a stationary process. ■

From formula (1), it is clear that $\forall (a_1,a_2)\in[a_{min},a_{max}]^2$, $\forall b_1,b_2,\theta\in\mathbb{R}$,

$$\begin{aligned}\mathbb{E}\big(d_X(a_1,b_1),\overline{d_X(a_2,b_2)}\big) \quad &= \quad \sqrt{a_1a_2}\cdot\gamma(b_2-b_1,a_1,a_2) \\ \gamma(\theta,a_1,a_2) : \quad &= \quad \int_{\mathbb{R}} e^{i\theta\xi}\,\widehat{\psi}(a_1\xi)\,\overline{\widehat{\psi}(a_2\xi)}\,f(\xi)\,d\xi.\end{aligned} \tag{3}$$

Lemma 4.2 *Let ψ verify Assumption $W(1,2)$ and f be an even function satisfying (2) and Assumption F.*

1. *There exists $C>0$ depending on ψ, f and a_{max} such that $\forall (a_1,a_2)\in[a_{min},a_{max}]^2$, $\left|\gamma(\theta,a_1,a_2)\right| < C\left(1\wedge|\theta|^{-1}\right)$ for all $\theta\in\mathbb{R}$.*

2. *The function γ is derivable with respect to θ and there exists $C'>0$ depending on ψ, f, a_{min} and a_{max} such that $\forall (a_1,a_2)\in[a_{min},a_{max}]^2$, $|\gamma'(\theta,a_1,a_2)|:=\big|\dfrac{\partial \gamma}{\partial\theta}(\theta,a_1,a_2)\big|\le C'\,(1\wedge|\theta|^{-1})$ for all $\theta\in\mathbb{R}$.*

Proof. **[of Lemma 4.2]** From Assumption W$(1,1/2)$, $\exists c>0$ such that

$$\big|\widehat{\psi}(\xi)\big|\le c\left(1\wedge|\xi|^2\right)\quad\text{for all}\ \ \xi\in\mathbb{R}. \tag{4}$$

Indeed, from one hand, $\big|\widehat{\psi}(\xi)\big|\le\|\psi\|_{L^1(\mathbb{R})}<\infty$. From the other hand, $\psi\in W(1,1/2)$ implies that $\widehat{\psi}$ is twice continuously differentiable and $\widehat{\psi}(0)=\widehat{\psi}'(0)=0$. From Taylor-Lagrange Formula, for all $\xi\in\mathbb{R}^*$, there exists $\xi_0\in\mathbb{R}$ with $|\xi_0|\le|\xi|$ such that $\widehat{\psi}(\xi)=\dfrac{1}{2}\cdot\xi^2\times\widehat{\psi}''(\xi_0)$. This induces $\big|\widehat{\psi}(\xi)\big|\le\dfrac{1}{2}\,|\xi|^2\times\Big(\displaystyle\int_{\mathbb{R}}t^2\cdot|\psi(t)|\,dt\Big)$ providing the second bound of (4).

To show the first item, inequality (4) implies that

$$\begin{aligned}\int_{\mathbb{R}}\Big|\overline{\widehat{\psi}}(a\xi)\Big|^2\,f(\xi)\,d\xi &\le c^2\Big(\int_{|\xi|\le 1}|a\xi|^4 f(\xi)\,d\xi+\int_{|\xi|>1}f(\xi)\,d\xi\Big)\\ &\le c^2\big(1\vee a^4\big)\int_{\mathbb{R}}\big(1\wedge\xi^2\big)f(\xi)\,d\xi<\infty,\end{aligned}$$

with $C>0$ depending on ψ, f and a_{max}. From Cauchy-Schwarz Inequality,

$$\gamma(\theta,a_1,a_2)\le c^2\big(1\vee a_1^2\big)\big(1\vee a_2^2\big)\int_{\mathbb{R}}\big(1\wedge\xi^2\big)f(\xi)\,d\xi.$$

Moreover, with $f(\omega_k^+)$ and $f(\omega_k^-)$) the right and left limit of f at ω_k, for all $1\le k\le K-1$, $\theta\in\mathbb{R}^*$ and $(a_1,a_2)\in[a_{min},a_{max}]^2$,

$$\begin{aligned}&\int_{\omega_k}^{\omega_{k+1}}e^{i\theta\xi}\overline{\widehat{\psi}}(a_1\xi)\,\widehat{\psi}(a_2\xi)\,f(\xi)\,dx\\ &=\frac{1}{i\theta}\Big(e^{i\theta\omega_{k+1}}f(\omega_{k+1}^-)\overline{\widehat{\psi}}(a_1\omega_{k+1})\widehat{\psi}(a_2\omega_{k+1})-e^{i\theta\omega_k}f(\omega_k^+)\overline{\widehat{\psi}}(a_1\omega_k)\widehat{\psi}(a_2\omega_k)\Big)\\ &-\int_{\omega_k}^{\omega_{k+1}}\frac{e^{i\theta\xi}}{i\theta}\Big[f'(\xi)\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}(a_2\xi)+f(\xi)\Big(a_1\overline{\widehat{\psi}'}(a_1\xi)\widehat{\psi}(a_2\xi)+a_2\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}'(a_2\xi)\Big)\Big]\,d\xi.\end{aligned}$$

The same result remains in force for $k=0$ and $k=K$. Indeed, by using (4) combined with Assumption F, one deduces that $\forall\theta\in\mathbb{R}$, $\forall(a_1,a_2)\in[a_{min},a_{max}]^2$,

$$\lim_{\xi\to 0}e^{i\theta\xi}\,f(\xi)\,\overline{\widehat{\psi}}(a_1\xi)\,\widehat{\psi}(a_2\xi)=0\qquad\text{and}\qquad\lim_{\xi\to\infty}e^{i\theta\xi}\,f(\xi)\,\overline{\widehat{\psi}}(a_1\xi)\,\widehat{\psi}(a_2\xi)=0.$$

Thus, by summing up and using Assumption F, $\forall\theta\in\mathbb{R}$, $\forall(a_1,a_2)\in[a_{min},a_{max}]^2$,

$$\begin{aligned}&\gamma(\theta,a_1,a_2)\\ &=-\frac{1}{i\theta}\sum_{k=1}^{K}\Big(e^{i\theta\omega_k}\overline{\widehat{\psi}}(a_1\omega_k)\widehat{\psi}(a_2\omega_k)-e^{-i\theta\omega_k}\overline{\widehat{\psi}}(-a_1\omega_k)\widehat{\psi}(-a_2\omega_k)\Big)\big(f(\omega_k^+)-f(\omega_k^-)\big)\\ &-\frac{1}{i\theta}\int_{\mathbb{R}}e^{i\theta\xi}\Big[f'(\xi)\overline{\widehat{\psi}}(a_1\xi)\,\widehat{\psi}(a_2\xi)+f(\xi)\Big(a_1\overline{\widehat{\psi}'}(a_1\xi)\widehat{\psi}(a_2\xi)+a_2\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}'(a_2\xi)\Big)\Big]d\xi\end{aligned}$$

since the integral of the r.h.s. of the previous equality is well defined. Then,

$$\begin{aligned}|\gamma(\theta,a_1,a_2)| &\le \frac{1}{|\theta|}\Big(2c\sum_{k=1}^{K}\big|f(\omega_k^+)\ -\ f(\omega_k^-)\big|\\ &+\int_{\mathbb{R}}\Big[\big|f'(\xi)\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}(a_2\xi)\big|+|f(\xi)|\big(|a_1|\big|\overline{\widehat{\psi}'}(a_1\xi)\widehat{\psi}(a_2\xi)\big|+|a_2|\,\big|\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}'(a_2\xi)\big|\big)\Big]d\xi\Big).\end{aligned}$$

It remains to show the convergence of the previous integral. Using the same trick as in Formula (4),under Assumption W$(1,1/2)$, $\big|\widehat{\psi}'(\xi)\big| \leq c'\,\big(1\wedge|\xi|\big)$ with c' depending on ψ and a_{max}. So, for all $(a_1,a_2)\in[a_{min},a_{max}]^2$

$$\begin{aligned}
&\int_{I\!\!R}\Big[\big|f'(\xi)\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}(a_2\xi)\big|+|f(\xi)|\big(|a_1|\big|\overline{\widehat{\psi}'}(a_1\xi)\widehat{\psi}(a_2\xi)\big|+|a_2|\,\big|\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}'(a_2\xi)\big|\big)\Big]d\xi\\
&\quad\leq c(c+c')|a_1a_2|\int_{|\xi|\leq 1}|a_1a_2f'(\xi)\xi^4|+(|a_1|+|a_2|)|f(\xi)\xi^3|\,d\xi\\
&\qquad\qquad+c^2\int_{|\xi|>1}|f'(\xi)|+(|a_1|+|a_2|)|f(\xi)|\,d\xi\\
&\quad\leq C\int_{I\!\!R}\Big[\big(1\wedge|\xi|^4\big)\cdot|f'(\xi)|+\big(1\wedge|\xi|^3\big)\cdot|f(\xi)|\Big]\,d\xi<\infty
\end{aligned}$$

where $C>0$ and this completes the proof of the first item.

Eventually, one proves the second item. The differentiability is obvious and

$$\varphi'(\theta,a_1,a_2) \;=\; i\int_{I\!\!R}e^{i\theta\xi}\,\xi\,\overline{\widehat{\psi}}(a_1\xi)\,\widehat{\psi}(a_2\xi)\,f(\xi)\,d\xi.$$

Assumption W$(1,1/2)$ implies that $\forall a\in[a_{min},a_{max}]$, $\quad|a\xi|^{1/2}|\widehat{\psi}(a\xi)|\leq C_\psi$ for all $\xi\in I\!\!R$. Combined with (4), this induces that $\forall a\in[a_{min},a_{max}]$, $\theta\in I\!\!R$,

$$\begin{aligned}
|\varphi'(\theta,a_1,a_2)| &\leq \int_{I\!\!R}|\xi|\,|\widehat{\psi}(a_1\xi)|\,|\widehat{\psi}(a_2\xi)|\,f(\xi)\,d\xi\\
&\leq c^2(a_1\,a_2)^2\int_{|\xi|\leq1}|\xi|^5\,f(\xi)\,d\xi\;+\;\frac{C_\psi^2}{\sqrt{a_1\,a_2}}\int_{|\xi|>1}f(\xi)\,d\xi\\
&\leq C,
\end{aligned}$$

with $C>0$ only depending on ψ, f, a_{min} and a_{max}. Using the same arguments as for the first item, $\forall\theta\in I\!\!R^*$, $(a_1,a_2)\in[a_{min},a_{max}]^2$,

$$\begin{aligned}
\gamma'(\theta,a_1,a_2) &= -\frac{1}{\theta}\sum_{k=1}^{K}\Big(e^{i\theta\omega_k}\omega_k\overline{\widehat{\psi}}(a_1\omega_k)\widehat{\psi}(a_2\omega_k)+\\
&\qquad +e^{-i\theta\omega_k}\omega_k\overline{\widehat{\psi}}(-a_1\omega_k)\widehat{\psi}(-a_2\omega_k)\Big)\big(f(\omega_k^+)-f(\omega_k^-)\big)\\
&- \frac{1}{\theta}\int_{I\!\!R}e^{i\theta\xi}\Big[f(\xi)\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}(a_2\xi)+\xi f'(\xi)\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}(a_2\xi)+\\
&\qquad +\xi f(\xi)\Big((a_1\,\overline{\widehat{\psi}'}(a_1\xi)\widehat{\psi}(a_2\xi)+a_2\overline{\widehat{\psi}}(a_1\xi)\widehat{\psi}'(a_2\xi)\Big)\Big]d\xi
\end{aligned}$$

and therefore $|\gamma'(\theta,a_1,a_2)|\leq\dfrac{C}{|\theta|}$, with $C>0$ only depending on ψ, a_{min} and a_{max}. ■

4.2 Proofs of Proposition 4.1 and 4.2

Since $\mathcal{I}_1(a)$ is obviously defined from $\big|d_X(a,b)\big|^2$, we begin with the study of

$$I_n(a) \;:=\; \frac{1}{n+1}\sum_{k=0}^{n}\big|d_X(a,c_k)\big|^2,\quad\text{for } a>0 \text{ and } n\in I\!\!N^*. \tag{5}$$

For $n\in I\!\!N^*$ and $a\in[a_{min},\,a_{max}]$, define also:

$$S_n^2(a):=\frac{2\,a^2}{(n+1)^2}\sum_{k=0}^{n}\sum_{\ell=0}^{n}\Big|\int_{I\!\!R}e^{i\big(I\!\!E(c_k-c_\ell)\big)\xi}\,\big|\widehat{\psi}(a\xi)\big|^2f(\xi)\,d\xi\Big|^2. \tag{6}$$

Proposition 4.1 *Let X be a Gaussian process defined by (1) or (3) with a spectral density f satisfying (2), ψ satisfy Assumption $W(1,1)$. Then if $(c_k)_k$ is a family of real numbers such that $c_1 < c_2 < \ldots < c_n$, $n\,\max_{1\le k\le n}\{c_{k+1}-c_k\} \underset{n\to\infty}{\longrightarrow} \infty$ and $\exists C''>0$ satisfying $\forall n\in \mathbb{N}^*$*

$$\max_{1\le k\le n}\{c_{k+1}-c_k\} \le C'' \min_{1\le k\le n}\{c_{k+1}-c_k\} < \infty$$

then $\forall a \in [a_{min},\, a_{max}]$,

$$\frac{1}{S_n(a)}\big(I_n(a)-\mathcal{I}_1(a)\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1). \tag{7}$$

Moreover, there exist $0<C_m<C_M$ not depending on n such that $\forall n\in N^*$,

$$C_m \le S_n(a)\big(n \max_{1\le k\le n}\{c_{k+1}-c_k\}\big)^{1/2} \le C_M. \tag{8}$$

The proof of Proposition 4.1 relies on Lemma 4.2 and the following Lemma which is a Lindeberg CLT (see a proof in Istas and Lang, 1997):

Lemma 4.3 *Let $(Y_{N,i})_{1\le i\le N,\, N\in\mathbb{N}^*}$ be a triangular array of zero-mean Gaussian r.v. Let $S_N^2 := var\,(V_N)$ with $V_N := \sum_{i=1}^N Y_{N,i}^2$ and $\beta_N := \max_{1\le i\le N}\sum_{j=1}^N |cov\;(Y_{N,i},Y_{N,j})\,|$. If $\displaystyle\lim_{N\to\infty}\frac{\beta_N}{S_N}=0$, then $S_N^{-1}(V_N-\mathbb{E}(V_N))$ converges weakly to a standard Gaussian random variable.*

Proof. [**of Proposition 4.1**]
Consider $Y_{n,i}=(n+1)^{-1/2}\,d_X(a,c_i)$ for $i=0,\ldots,n$ and

$$\left\{\begin{array}{l} \beta_n=(n+1)^{-1}\,\max_{1\le i\le n}\Big\{\sum_{j=0}^n\big|\mathrm{cov}\,\big(d_X(a,c_i),d_X(a,c_j)\big)\big|\Big\}, \\ S_n^2=(n+1)^{-2}\,\sum_{i=0}^n\sum_{j=0}^n \mathrm{cov}\,\big(d_X^2(a,c_i),d_X^2(a,c_j)\big)\end{array}\right. .$$

But, by using Formula (3), $\forall (a,a_1,a_2)\in(0,\infty[^3$, $(b_1,b_2)\in\mathbb{R}^2$

$$\begin{array}{rcl}\mathrm{cov}\,\big(d_X(a,b_1),d_X(a,b_2)\big) &=& a\,\gamma\big(b_1-b_2,a,a\big)\\ \mathrm{cov}\,\big(d_X^2(a_1,b_1),\,d_X^2(a_2,b_2)\big) &=& 2\,(a_1a_2)\,\gamma^2(b_1-b_2,a_1,a_2),\end{array}$$

since variables $d_X(a,b)$ are zero-mean Gaussian r.v. Therefore,

$$\left\{\begin{array}{l} \beta_N=a\,(n+1)^{-1}\,\max_{0\le i\le n}\Big\{\sum_{j=0}^n\big|\gamma\big(c_i-c_j,a,a\big)\big|\Big\} \\ S_n^2=2\,a^2\,(n+1)^{-2}\,\sum_{i=0}^n\sum_{j=0}^n\gamma^2\big(c_i-c_j,a,a\big)\end{array}\right. .$$

Let p and q be such that $1/p+1/q=1$ with $(p,q)\in(1,\infty)^2$. Then the Hölder Inequality implies that

$$\beta_n \;\le\; C\,a\cdot n^{1/q-1}\times\max_{0\le i\le n}\Big\{\Big(\sum_{j=0}^n\big|\gamma\big(c_i-c_j,a,a\big)\big|^p\Big)^{1/p}\Big\}.$$

Lemma 4.2 i) implies that for every $1\le i\le n$, for n large enough,

$$\begin{aligned}\sum_{j=0}^n\big|\gamma\big(c_i-c_j,a,a\big)\big|^p &\le C\Big(\#\,\{0\le j\le n,\,\big|c_i-c_j\big|\le 1\}+\sum_{j=0;\,|c_i-c_j|>1}^n\big|c_i-c_j\big|^{-p}\Big)\\ &\le C\Big(\frac{2}{\min_{0\le k\le n-1}\big|c_{k+1}-c_k\big|}+\sum_{j=0;\,|c_i-c_j|>1}^n\Big[\big|i-j\big|\min_{0\le k\le n-1}\big|c_{k+1}-c_k\big|\Big]^{-p}\Big)\\ &\le 2C\big(\min_{0\le k\le n-1}\big|c_{k+1}-c_k\big|\big)^{-1}\Big(1+\big(\min_{0\le k\le n-1}\big|c_{k+1}-c_k\big|\big)^{1-p}\sum_{\ell\ge(\min_{0\le k\le n-1}|c_{k+1}-c_k|)^{-1}}|\ell|^{-p}\Big).\end{aligned} \tag{9}$$

Since $p>1$, $\sum_{\ell=1}^{\infty}|\ell|^{-p}<\infty$ is finite and thus

$$\sum_{\ell\geq (\min_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|)^{-1}} |\ell|^{-p}\leq \frac{1}{p-1}\left(\min_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{p-1}.$$

Therefore,

$$\beta_n \quad\leq\quad C\,a\cdot\Big\{n\times \min_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\Big\}^{-1/p}, \tag{10}$$

with $C>0$ depending only on ψ, a_{min}, a_{max} and p. Now, a lower bound for S_n^2 is required. $\forall a\in[a_{min},a_{max}]$, $\theta\in I\!R\mapsto\gamma(\theta,a,a)$ is a continuous map and $\gamma(0,a,a)=\int_{I\!R}\left|\widehat{\psi}(a\xi)\right|^2 f(\xi)\,d\xi>0$. Therefore, for all $a\in[a_{min},a_{max}]$, there exists $\theta_a>0$ such that $\gamma(\theta,a,a)\geq\frac{1}{2}\cdot\gamma(0,a,a)$ when $|\theta|\leq\theta_a$. Then,

$$\begin{aligned} S_n^2 &\geq C_1'\,a^2\,n^{-2}\,\gamma^2(0,a,a)\,\#\left\{0\leq i,j\leq n,\ \left|c_i-c_j\right|\leq\theta_a\right\}\\ &\geq C_1'\,a^2\,n^{-2}\,\gamma^2(0,a,a)\,\#\Big\{0\leq i,j\leq n,\ |i-j|\max_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\leq\theta_a\Big\}\\ &\geq C_1'\,a^2\,\gamma^2(0,a,a)\,\theta_a\left(n\max_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{-1}. \end{aligned}\tag{11}$$

Thus, for n large enough, from (11) and(10),

$$\frac{\beta_n}{S_n}\leq C\cdot n^{1/2-1/p}\left(\max_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{1/2}\left(\min_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{-1/p}.$$

Therefore $\beta_n/S_n\leq C\left(n\max_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{1/2-1/p}$ with $C>0$. Next for any $p\in(1,2)$, $\lim_{n\to\infty}\left(n\max_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{1/2-}$ thus, $\lim_{n\to\infty}\beta_n/S_n=0$ and the assumptions of Lemma 4.3 are fulfilled.

Finally, (11) and Assumptions imply $S_n(a)^2\geq C_M\left(n\max_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{-1}$ with $C_M>0$. Moreover, using the bound (9) for $p=2$,

$$\sum_{j=0}^{n}\gamma^2\big(c_i-c_j,a,a\big)\leq C\left(\min_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{-1}$$

$$\Longrightarrow\ S_n^2\leq C'\,a^2\,n^{-2}\sum_{i=0}^{n}\sum_{j=0}^{n}\gamma^2\big(c_i-c_j,a,a\big)\ \leq\ C_m\,\frac{1}{n}\left(\max_{0\leq k\leq n-1}\left|c_{k+1}-c_k\right|\right)^{-1}.$$

Therefore, inequalities (8) are proved. ■

Proposition 4.2 *Let X be a Gaussian process defined by (1) or (3) with a spectral density f satisfying (2), ψ satisfy Assumption $W(1,1)$. Then if $(c_k)_k$ is a family of r.v. independent to $\mathcal{F}_X$ such that $c_k=c_1+\frac{k}{n}(c_n-c_0)$, with $n^{-1}\,I\!E(c_n-c_0)\underset{n\to\infty}{\longrightarrow}0$ and $var\,(c_n-c_0)\underset{n\to\infty}{\longrightarrow}0$, then (7) holds with*

$$\lim_{n\to\infty}\big(I\!E(c_n-c_0)\big)\,S_n^2(a)=4\pi\,a^2\int_{I\!R}\left|\widehat{\psi}(az)\right|^4 f^2(z)\,dz. \tag{12}$$

Remark 4.1 *For $(c_k)_k$ satisfying (5), under Assumption S(2), Proposition 4.2 holds when $n^{1/2}\,\delta_n\underset{n\to\infty}{\longrightarrow}0$ because $I\!E|T_n-I\!ET_n|^2\leq n\,\delta_n^2\,\max_{1\leq k\leq n}I\!EL_k$.*

Proof. **[of Proposition 4.2]**
(c_k) is a sequence of r.v. independent to $\mathcal{F}_X$. Therefore, $(d_X(a,c_k))_k$ as the same distribution than $(d_X(a,c_k-$

$c_0))_k$ (stationarity of the sequence), and we can only consider here the case: $c_k = k\tau_n/n$ with $\tau_n := c_n - c_0$. Define

$$I'_n(a) \quad := \quad \frac{1}{n}\sum_{k=0}^{n} d_X^2\big(a, \mathbb{E}c_k\big).$$

It is clear that $(\mathbb{E}c_k)_{1\le k\le n}$ is a deterministic sequence and therefore

$$\frac{I'_n(a) - \mathcal{I}_1(a)}{S_n(a)} \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \mathcal{N}(0,1). \tag{13}$$

Nowadays, one has to check that the error $I'_n(a) - I_n(a)$ is negligible before $S_n(a)$ in norm $L^2(\Omega)$. But

$$S_n(a) \ge C_M \cdot \big(n \max_{0\le k\le n-1} \big|\mathbb{E}\big(c_{k+1} - c_k\big)\big|\big)^{-1/2} \ge C \times \big(\mathbb{E}\tau_n\big)^{-1/2}.$$

Therefore, it suffices to prove that $\lim\limits_{n\to\infty} \mathbb{E}\tau_n \times \mathbb{E}\big[(I'_n(a) - I_n(a))^2\big] = 0$. Since the r.v. c_k are independent on $\mathcal{F}_X$, one gets

$$\begin{aligned}
&\mathbb{E}\big[(I'_n(a) - I_n(a))^2\big] = \mathbb{E}\Big[\mathbb{E}\big[(I'_n(a) - I_n(a))^2 \mid \mathcal{F}_X\big]\Big] \\
&= \frac{1}{(n+1)^2}\sum_{k,k'=0}^{n} \mathbb{E}\Big[\mathbb{E}\big[\big(d_X^2\big(a,\mathbb{E}c_k\big) - d_X^2\big(a,c_k\big)\big)\big(d_X^2\big(a,\mathbb{E}c_{k'}\big) - d_X^2\big(a,c_{k'}\big)\big)\big|\mathcal{F}_X\big]\Big] \\
&= \frac{2a^2}{(n+1)^2}\sum_{k,k'=0}^{n} \mathbb{E}\Big[\gamma^2\big(\mathbb{E}c_k - \mathbb{E}c_{k'}, a, a\big) - \gamma^2\big(\mathbb{E}c_k - c_{k'}, a, a\big) \\
&\qquad\qquad -\gamma^2\big(c_k - \mathbb{E}c_{k'}, a, a\big) + \gamma^2\big(c_k - c_{k'}, a, a\big)\Big].
\end{aligned}$$

Next, from Taylor expansions,

$$\begin{aligned}
&\gamma^2\big(\mathbb{E}c_k - c_{k'}, a, a\big) = \gamma^2\big(\mathbb{E}c_k - \mathbb{E}c_{k'}, a, a\big) + 2\big(\mathbb{E}c_{k'} - c_{k'}\big) \times \cdots \\
&\int_0^1 \gamma\big(\mathbb{E}c_k - \mathbb{E}c_{k'} + \lambda\,(\mathbb{E}c_{k'} - c_{k'}), a, a\big)\gamma'\big(\mathbb{E}c_k - \mathbb{E}c_{k'} + \lambda\,(\mathbb{E}c_{k'} - c_{k'}), a, a\big)d\lambda \\
&\gamma^2\big(c_k - \mathbb{E}c_{k'}, a, a\big) = \gamma^2\big(c_k - c_{k'}, a, a\big) + 2\big(c_{k'} - \mathbb{E}c_{k'}\big) \times \cdots \\
&\int_0^1 \gamma\big(c_k - c_{k'} + \lambda\,(\mathbb{E}c_{k'} - c_{k'}), a, a\big)\,\gamma'\big(c_k - c_{k'} + \lambda\,(\mathbb{E}c_{k'} - c_{k'}), a, a\big)\,d\lambda.
\end{aligned}$$

From Lemma 4.2, $\exists C > 0$ such that $\big|\gamma(\theta, a, a)\gamma'(\theta, a, a)\big| \le C \times \big(1 \wedge \theta^{-2}\big)$ for all $\theta \in \mathbb{R}$. One can deduce that

$$\begin{aligned}
&\Big|\gamma^2\big(\mathbb{E}c_k - \mathbb{E}c_{k'}, a, a\big) - \gamma^2\big(\mathbb{E}c_k - c_{k'}, a, a\big) - \gamma^2\big(c_k - \mathbb{E}c_{k'}, a, a\big) + \gamma^2\big(c_k - c_{k'}, a, a\big)\Big| \\
&\qquad \le C\,\big|c_{k'} - \mathbb{E}c_{k'}\big| \times \int_0^1 \big(1 \wedge \theta_{1,k,k'}^{-2}(\lambda)\big) + \big(1 \wedge \theta_{2,k,k'}^{-2}(\lambda)\big)\,d\lambda
\end{aligned}$$

with $\theta_{1,k,k'}(\lambda) = \mathbb{E}(c_k - c_{k'}) + \lambda(\mathbb{E}c_{k'} - c_{k'})$ and $\theta_{2,k,k'}(\lambda) = c_k - c_{k'} + \lambda(\mathbb{E}c_{k'} - c_{k'})$.
Then,

$$\mathbb{E}\big[(I'_n(a) - I_n(a))^2\big] \le 2a^2 \times (\mathfrak{Er}_1 + \mathfrak{Er}_2)\,, \tag{14}$$

where, for $i = 1, 2$, $\mathfrak{Er}_i := \displaystyle\int_0^1 \mathbb{E}\Big[\frac{1}{(n+1)^2}\sum_{k,k'=0}^{n} \big|c_{k'} - \mathbb{E}c_{k'}\big| \times \big(1 \wedge \theta_{i,k,k'}^{-2}(\lambda)\big)\Big]\,d\lambda$. Thus $\theta_{1,k,k'}(\lambda) = \delta'_n\big((k - k') - \lambda k' z_n\big)$ with $\delta'_n := \dfrac{\mathbb{E}\tau_n}{n}$ and $z_n := \dfrac{\tau_n - \mathbb{E}\tau_n}{\mathbb{E}\tau_n}$. Then, using $\delta'_n \underset{n\to\infty}{\longrightarrow} 0$, for n large enough,

$$\mathfrak{Er}_1 \quad = \quad \int_0^1 \mathbb{E}\Big[\frac{C}{(n+1)^2}\sum_{k,k'=0}^{n} \big|(k'\delta'_n)\,z_n\big| \times \Big(1 \wedge \big[\delta'_n\big((k - k') - \lambda k' z_n\big)\big]^{-2}\Big)\Big]\,d\lambda$$

$$\leq \int_0^1 \mathbb{E}\Big[\frac{C}{(\mathbb{E}\tau_n)^2}\int_0^{\mathbb{E}\tau_n}\int_0^{\mathbb{E}\tau_n} dxdy|y\, z_n| \times \Big(1 \wedge \big[(x-y)-\lambda y\, z_n\big]^{-2}\Big)\Big]\, d\lambda.$$

But, for all $\lambda \in (0,1)$, one has

$$\begin{aligned}
&\frac{1}{(\mathbb{E}\tau_n)^2}\int_0^{\mathbb{E}\tau_n}\int_0^{\mathbb{E}\tau_n} |y| \times \big(1 \wedge [(x-y)-\lambda y\, z_n]^{-2}\big)dx\, dy \\
&\qquad = \mathbb{E}\tau_n \int_0^1\int_0^1 |v| \times \big(1 \wedge (\mathbb{E}\tau_n)^{-2}\, [(u-v)-\lambda v\, z_n]^{-2}\big)du\, dv \\
&\qquad \leq 2\mathbb{E}\tau_n \int_0^2\int_0^\infty \big(1 \wedge (\mathbb{E}\tau_n)^{-2}s^{-2}\big)\, ds\, dt \leq 4.
\end{aligned}$$

Therefore $\mathfrak{Er}_1 \leq 4\mathbb{E}|z_n|$. Now, using the same method for $\mathfrak{Er}_2$, one obtains,

$$\begin{aligned}
\mathbb{E}\tau_n \times \mathbb{E}\big[(I_n'(a)-I_n(a))^2\big] \quad &\leq \quad C\, \mathbb{E}\tau_n \times \mathbb{E}|z_n| \\
&\leq \quad C\big(\mathrm{var}\,(c_n-c_0)\big)^{1/2} \\
&\underset{n\to\infty}{\longrightarrow} \quad 0
\end{aligned}$$

from assumptions and therefore the CLT (7) holds.

Now the asymptotic expansion (12) can be proved. Consider first the deterministic case and

$$\begin{aligned}
S_n^2(a) &= \frac{2\,a^2}{(n+1)^2}\int_{\mathbb{R}^2} \big|\widehat{\psi}(a\xi)\big|^2 f(\xi)\big|\widehat{\psi}(a\xi')\big|^2 f(\xi')d\xi d\xi' \sum_{k,k'=0}^n e^{i(k-k')\frac{(c_n-c_0)}{n}(\xi-\xi')} \\
&= \frac{2\,a^2}{(n+1)^2}\int_{\mathbb{R}^2} \big|\widehat{\psi}(a\xi)\big|^2 f(\xi)\big|\widehat{\psi}(a\xi')\big|^2 f(\xi')\frac{\sin^2\big(\frac{c_n-c_0}{2}(\xi-\xi')\big)}{\sin^2\big(\frac{c_n-c_0}{2n}\,(\xi-\xi')\big)}d\xi d\xi' \\
&= \frac{16a^2}{c_n-c_0}\int_{\mathbb{R}_+^2}\big|\widehat{\psi}(az')\big|^2 f(z')\Big|\widehat{\psi}\Big(a\big(z'+\frac{2z}{c_n-c_0}\big)\Big)\Big|^2 f\Big(z'+\frac{2z}{c_n-c_0}\Big)\frac{\sin^2(z)}{n^2\sin^2\big(\frac{z}{n}\big)}dzdz'.
\end{aligned}$$

Let us define $h_n(x) := \dfrac{\sin(x)}{n\sin\big(\frac{x}{n}\big)}$ and $h(x) := \dfrac{\sin x}{x}$. For all $(z,z') \in \mathbb{R}^2$,

$$\Big|\widehat{\psi}\Big(a\big(z'+\frac{2z}{c_n-c_0}\big)\Big)\Big|^2 f\Big(z'+\frac{2z}{c_n-c_0}\Big) \underset{n\to\infty}{\longrightarrow} \big|\widehat{\psi}(az')\big|^2 f(z') \quad \text{and} \quad h_n^2(z) \underset{n\to\infty}{\longrightarrow} h^2(z).$$

However Lebesgue Theorem cannot be applied. Denote $\nu(x) := |\psi(ax)|^2 f(x)$ for $x>0$. From Assumptions F and W$(1,3)$, ν is a differentiable function in $(0,\infty)$ and $\exists C>0$, $\forall z', x>0$, $|\nu'(z'+x)| \leq C\,|\nu'(z')|$. Then,

$$\Big|\nu\big(z'+\frac{2z}{c_n-c_0}\big)-\nu(z')\Big| \quad \leq \quad \frac{2z}{c_n-c_0}\, C\,|\nu'(z')|.$$

Moreover, $|h_n(z')| \leq 1$ for all $z' \in \mathbb{R}$, and

$$\begin{aligned}
\int_{-n}^n h_n^2(z)\, dz &= \frac{1}{n^2}\int_{-n}^n \sum_{k,k'=1}^n e^{2i(k-k')\frac{z}{n}}\, dz \\
&= \frac{2}{n}\sum_{1\leq k'<k\leq n} \frac{\sin\big(2\,(k-k')\big)}{(k-k')} + \frac{1}{n^2}\sum_{k=0}^n 2\,n \\
&= \frac{2}{n}\left\{\sum_{k=0}^n\, (n-k)\,\frac{\sin\,(2\,k)}{k}\right\} + 2
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\int_{-n}^n h_n^2(z)\, dz = 2\left\{\sum_{k=0}^\infty \frac{\sin\,(2\,k)}{k}\right\} + 2 = \pi$$

since $\frac{2}{n}\sum_{1\le k\le n} k\frac{\sin\big(2k\big)}{k} \le 4\frac{\log n}{n} \underset{n\to\infty}{\longrightarrow} 0$ and from Dirichlet Theorem, $x-\pi = -2\sum_{n\ge 1}\frac{\sin(nx)}{n}$ for all $x\in(0,2\pi)$. Now, for $z'\ge 0$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}_+}\nu\big(z'+\frac{2z}{c_n-c_0}\big)h_n^2(z)dz-\nu(z')\int_0^n h_n^2(z)dz\Big| \\
&\quad\le \frac{2}{c_n-c_0}\int_0^n z h_n^2(z)dz+\int_n^\infty \nu\big(z'+\frac{2z}{c_n-c_0}\big)h_n^2(z)dz \\
&\quad\le \frac{2}{c_n-c_0}\int_0^n z\,\frac{4\sin^2(z)}{z^2}\,dz+\int_n^\infty \nu\big(z'+\frac{2z}{c_n-c_0}\big)\,dz \\
&\quad\le 8\,\frac{4+\log(n)}{c_n-c_0}+C\,f(z')\int_n^\infty \frac{C_\psi}{\big(1+\big(z'+\frac{2z}{c_n-c_0}\big)\big)^{2r}}\,dz \\
&\quad\le 8\,\frac{4+\log(n)}{c_n-c_0}+C'\,f(z')\,n\,\delta_n^{2r}
\end{aligned}$$

Finally, with $n\,\delta_n^{2r} \underset{n\to\infty}{\longrightarrow} 0$ when $r=3$, one deduces that for all $z'\ge 0$,

$$\lim_{n\to\infty}\int_{\mathbb{R}_+}\nu\big(z'+\frac{2z}{c_n-c_0}\big)\,h_n^2(z)\,dz=\lim_{n\to\infty}\nu(z')\int_0^n h_n^2(z)\,dz=\frac{\pi}{2}\,\nu(z').$$

Therefore,

$$\int_{\mathbb{R}}\big|\widehat{\psi}(az')\big|^2 f(z')\int_{\mathbb{R}}\Big|\widehat{\psi}\Big(a\big(z'+\frac{2z}{c_n-c_0}\big)\Big)\Big|^2 f\Big(z'+\frac{2z}{c_n-c_0}\Big)\,\frac{\sin^2(z)}{n^2\,\sin^2\big(\frac{z}{n}\big)}\,dz\,dz' \underset{n\to\infty}{\longrightarrow} \pi\int_{\mathbb{R}}\big|\widehat{\psi}(az')\big|^4 f^2(z')\,dz',$$

providing the asymptotic behavior of S_n^2. The proof is similar in the stochastic case with c_n-c_0 replaced by $\mathbb{E}(c_n-c_0)$. ■

4.3 Proof of Theorem 2.1

The proof of Theorem 2.1 uses the following lemmas:

Lemma 4.4 *Let X be a Gaussian process defined by (1) with a spectral density f satisfying (2) and Assumption F. Let us define,*

$$R(t,u,t',u'):=\mathbb{E}\Big[(X(t+u)-X(t))\cdot(X(t'+u')-X(t'))\mid\mathcal{F}_X\Big],$$

for $(t,t')\in\mathbb{R}^2$, $(u,u')\in\mathbb{R}_+^2$. Then $\exists C_f>0$ depending only on the spectral density f such that for all $(u,u',t,t')\in\mathbb{R}_+^2\times\mathbb{R}^2$, with $\beta=\big(t'-t+u'-u\big)$,

$$\big|R(t,2u,t',2u')\big|\le C_f\Big(u\,u'\Big)+\Big((u\,u')^H\wedge\big((u\,u')|\beta|^{2H-2}+(u\,u')^{H+1/2}|\beta|^{-1}\big)\Big).$$

Proof. To begin with, remark that for all $(t,t',u,u')\in\mathbb{R}^4$,

$$\begin{aligned}
R(t,2u,t',2u')&=\int_{\mathbb{R}}(e^{-i(t+2u)\xi}-e^{-it\xi})(e^{i(t'+2u')\xi}-e^{it'\xi})\,f(\xi)\,d\xi \\
&=\int_{\mathbb{R}}(e^{-iu\xi}-e^{iu\xi})(e^{iu'\xi}-e^{-iu'\xi})\,e^{i\xi(t'-t)+i\xi(u'-u)}\;f(\xi)\,d\xi \\
&=8\int_0^\infty \sin(u\xi)\cdot\sin(u'\xi)\cdot\cos\big(\xi(t'-t+u'-u)\big)\,f(\xi)\,d\xi \\
&=8\,(I_1+I_2)
\end{aligned}$$

with $I_1 := \int_0^{\omega_K} \cdots d\xi$ and $I_2 = \int_{\omega_K}^{\infty} \cdots d\xi$. From one hand, with $|\sin a| \leq |a|$,

$$I_1 \quad \leq \quad u\,u' \int_0^{\omega_K} \xi^2\, f(\xi)\, d\xi \;\leq\; C\,u\,u',$$

where the last inequality follows from (2). From the other hand,

$$I_2 = \frac{1}{\sqrt{u\,u'}} \int_{\omega_K \sqrt{u\,u'}}^{\infty} \sin\big(\frac{u}{\sqrt{u\,u'}}\xi\big)\big(\frac{u'}{\sqrt{u\,u'}}\xi\big) \cos\big(\frac{\beta}{\sqrt{u\,u'}}\xi\big) f\big(\frac{1}{\sqrt{u\,u'}}\xi\big) d\xi.$$

Then, with Assumption F combined with $|\cos a| \leq 1$ and $|\sin a| \leq (1 \wedge |a|)$,

$$\begin{aligned} |I_2| \quad &\leq \quad (u\,u')^H \Big(\int_0^1 x^2\, x^{-(2H+1)}\, dx \,+\, \int_1^{\infty} x^{-(2H+1)}\, dx \Big) \\ &\leq \quad C\,(u\,u')^H, \end{aligned} \tag{15}$$

since $H \in (0,\,1)$. It remains to prove $|I_2| \leq C\big(uu'\beta^{2H-2} + \beta^{-1}(uu')^{H+1/2} + uu'\big)$ with $C > 0$. First, with an integration by parts,

$$\begin{aligned} I_2 = \beta^{-1}\Big(&\Big[\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big) f\Big(\frac{x}{\sqrt{u\,u'}}\Big)\Big]_{\omega_K \sqrt{u\,u'}}^{\infty} \\ &- \int_{\omega_K \sqrt{u\,u'}}^{\infty} \sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big) \frac{\partial}{\partial x}\Big(\sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\, f\Big(\frac{x}{\sqrt{u\,u'}}\Big)\Big) dx\Big), \end{aligned}$$

where Assumption F insures the convergence of bracket term at ∞. But $\big|\beta^{-1}\, \sin(b\omega_K)\, \sin(u\omega_K)\, \sin(u'\omega_K)\, f(\omega_K^+)\big| \leq C\,u\,u'$ since $|\sin a| \;\leq\; |a|$. Thus,

$$|I_2| \quad \leq I_3 + I_4 \,+\, C\,u\,u'$$

with, using again Assumption F, $|\cos a| \;\leq\; 1$,

$$\begin{aligned} I_3 &= \beta^{-1} \int_{\sqrt{u\,u'}\omega_K}^{\infty} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big)\Big| \Big| \frac{u}{\sqrt{u\,u'}} \cos\Big(\frac{u}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big) \\ &\qquad + \frac{u'}{\sqrt{u\,u'}} \cos\Big(\frac{u'}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big)\Big| \Big|f\Big(\frac{x}{\sqrt{u\,u'}}\Big)\Big| dx \\ &= \frac{C}{\beta}(u\,u')^{H+1/2} \int_0^{\infty} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big)\Big| \Big(\Big|\sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\Big| + \Big|\sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big)\Big|\Big) x^{-(2H+1)} dx \end{aligned}$$

and

$$\begin{aligned} I_4 &= \frac{1}{\beta\sqrt{u\,u'}} \int_{\delta\omega_K}^{\infty} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big) f'\Big(\frac{x}{\sqrt{u\,u'}}\Big)\Big| dx \\ &= \frac{C}{\beta}(u\,u')^{H+1/2} \int_0^{\infty} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\Big| x^{-(2H+2)} dx. \end{aligned}$$

Both those integrals can be decomposed as $\int_0^{\sqrt{u\,u'}/\beta} \cdots + \int_{\sqrt{u\,u'}/\beta}^{1} \cdots + \int_1^{\infty} \cdots$. Using $|\sin a| \leq (|a| \wedge 1)$, with $C > 0$ denoting a constant which may vary from one line to the other,

$$\begin{aligned} I_{31} &= \int_0^{\sqrt{u\,u'}/\beta} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big)\Big| \Big(\Big|\sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\Big| + \Big|\sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big)\Big|\Big) x^{-(2H+1)} dx \\ &\leq \frac{2b}{\sqrt{u\,u'}} \int_0^{\sqrt{u\,u'}/\beta} x^2\, x^{-(2H+1)} dx \leq C\,\beta^{2H-1} (u\,u')^{1/2-H}, \\ I_{41} &= \int_0^{\sqrt{u\,u'}/\beta} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big) \sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\Big| x^{-(2H+2)} dx \\ &\leq \frac{\beta}{\sqrt{u\,u'}} \int_0^{\sqrt{u\,u'}/\beta} x^3\, x^{-(2H+2)} dx \leq C\,\beta^{2H-1} (u\,u')^{1/2-H}, \end{aligned}$$

$$
\begin{aligned}
I_{32} &= \int_{\sqrt{u\,u'}/\beta}^{1} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big)\Big|\Big(\Big|\sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\Big| + \Big|\sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big)\Big|\Big)x^{-(2H+1)}dx \\
&\leq 2\int_{\sqrt{u\,u'}/\beta}^{1} x\,x^{-(2H+1)}dx \leq C\big(1+\beta^{2H-1}(u\,u')^{1/2-H}\big), \\
I_{42} &= \int_{\sqrt{u\,u'}/\beta}^{1} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big)\sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big)\sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\Big|x^{-(2H+2)}dx \\
&\leq \int_{\sqrt{u\,u'}/\beta}^{1} x^2\,x^{-(2H+2)}dx \leq C\big(1+\beta^{2H-1}(u\,u')^{1/2-H}\big), \\
I_{33} &= \int_{1}^{\infty} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big)\Big|\Big(\Big|\sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\Big| + \Big|\sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big)\Big|\Big)x^{-(2H+1)}dx \\
&\leq 2\int_{1}^{\infty} x^{-(2H+1)}dx \;\leq\; \frac{1}{H}, \\
I_{43} &= \int_{1}^{\infty} \Big|\sin\Big(\frac{\beta}{\sqrt{u\,u'}}x\Big)\sin\Big(\frac{u}{\sqrt{u\,u'}}x\Big)\sin\Big(\frac{u'}{\sqrt{u\,u'}}x\Big)\Big|x^{-(2H+2)}dx \\
&\leq \int_{1}^{\infty} x^{-(2H+2)}dx \;\leq\; \frac{1}{2H+1}.
\end{aligned}
$$

This implies that it exists $C>0$ such that,

$$
\begin{aligned}
I_3 + I_4 &\leq C\,\beta^{-1}\,(u\,u')^{1/2+H}\big(1\,+\,\beta^{2H-1}(u\,u')^{1/2-H}\big) \\
&\leq C\big(\beta^{-1}(u\,u')^{1/2+H} + (u\,u')\beta^{2H-2}\big) \\
\Longrightarrow \quad |I_2| &\leq C\big(\beta^{-1}(u\,u')^{1/2+H} + (u\,u')\beta^{2H-2} + (u\,u')\big).
\end{aligned}
$$

Combined with (15), this completes the proof of Lemma 4.4. ■

Next, let us define the error of discretization of the wavelet coefficients by

$$\varepsilon(a,b):=\varepsilon_{1,n}(a,b)+\varepsilon_{2,n}(a,b)+\varepsilon_{3,n}(a,b):=d_X(a,b)-e_X(a,b), \tag{16}$$

$$
\text{with}\;\left\{\begin{array}{l}
\varepsilon_{1,n}(a,b):=a^{-1/2}\Big(\int_0^{T_n}\psi\big(\frac{t-b}{a}\big)X(t)dt-\sum_{i=0}^{n-1}X(t_i)\int_{t_i}^{t_{i+1}}\psi\big(\frac{t-b}{a}\big)dt\Big)\\
\varepsilon_{2,n}(a,b):=a^{-1/2}\int_{T_n}^{\infty}\psi\big(\frac{t-b}{a}\big)X(t)dt\\
\varepsilon_{3,n}(a,b):=a^{-1/2}\int_{-\infty}^{0}\psi\big(\frac{t-b}{a}\big)X(t)dt
\end{array}\right. .
$$

The following lemmas give bounds on $I\!E\big|\varepsilon_{i,n}(a,k)\big|^2$ for $i=1,2,3$.

Lemma 4.5 *Let X be a Gaussian process defined by (1) with a spectral density f satisfying (2) and Assumption F. Assume also Assumptions $W(1,3)$ and $(S(s))$. Then, with C_f defined in Lemma 4.4, if b is a r.v. independent on $\mathcal{F}_X$ such that $T_n^\rho \leq b \leq T_n - T_n^\rho$ with $\rho > 1/2$, for $i=2,3$,*

$$
\begin{aligned}
&I\!E\big(\big|\varepsilon_{i,n}(a,b)\big|^2 \mid \mathcal{F}_X\big) \leq C_f C_\psi^2\big(a^5 T_n^{2-4\rho}\mathbf{1}_{T_n\geq 1} + a\big(1+\frac{a}{2}\big)^2\mathbf{1}_{T_n<1}\big), \qquad (17)\\
\textit{and}\quad &I\!E\big(\big|\varepsilon_{1,n}(a,b)\big|^2 \mid \mathcal{F}_X\big) \leq C_f\,v_{1,n}(a)
\end{aligned}
$$

where

- *if $s=\infty$, $v_{1,n}(a) \leq \|\psi\|_{\mathcal{L}^1}\delta_n^{1+H}\Big(a^{2-H}\big(\|\psi\|_{\mathcal{L}^1} + \dfrac{1}{H}\|\psi\|_\infty\big) + \dfrac{2}{H}\|\psi\|_\infty\delta_n^H + a\big\|\psi\big\|_{\mathcal{L}^1}\delta_n^{1-H}\Big)$;*
- *if $s<\infty$, $v_{1,n}(a) \leq \dfrac{a^{1-2/p_1}\big\|\psi\big\|^2_{\mathcal{L}^{q_1}}}{(p_1+1)^{2/p_1}}\Big(\displaystyle\sum_{i=0}^{n-1}L_i^{p_1+1}\Big)^{2/p_1}\delta_n^{2+2/p_1}$*

$$+\frac{2\|\psi\|_\infty\|\psi\|_{\mathcal{L}^{q_2}}}{H\,a^{1/p_2}\big(1+p_2(1+2H)\big)^{1/p_2}}\Big(\sum_{i=0}^{n-1}L_i^{1+p_2(1+2H)}\Big)^{1/p_2}\delta_n^{1+2H+1/p_2}$$
$$+\frac{\|\psi\|^2_{\mathcal{L}^{q_3}}+\frac{\|\psi\|_{\mathcal{L}^{q_3}}\|\psi\|_\infty}{1-q_3(1-H)}}{a^{H-2+2/p_3}\big(1+p_3(1+H)/2\big)^{2/p_3}}\Big(\sum_{i=0}^{n-1}L_i^{1+p_3(1+H)/2}\Big)^{2/p_3}\delta_n^{1+H+2/p_3},$$

for all $(p_1,p_2,p_3)\in[1,\infty)^2\times(1/H,\infty)$ *with* $\frac{1}{p_j}+\frac{1}{q_j}=1$ *for* $i=1,2,3$.

Proof. (1) Bound of $I\!E\big(\big|\varepsilon_{1,n}(a,b)\big|^2\mid\mathcal{F}_X\big)$. To begin with,

$$\begin{aligned}&I\!E\big(\big|\varepsilon_{1,n}(a,b)\big|^2\mid\mathcal{F}_X\big)\\&=\frac{1}{a}\sum_{i,j=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}\psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}I\!E\Big(\big(X(t)-X(t_i)\big)\big(X(t')-X(t_j)\big)|\mathcal{F}_X\Big)dtdt'\\&=\frac{1}{a}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}\psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}R\big(t_i,t-t_i,t_j,t'-t_j\big)dtdt'.\end{aligned}$$

Lemma 4.4, with $2u=t-t_i$ and $2u'=t'-t_j$, implies

$$I\!E\big(\big|\varepsilon_{1,n}(a,b)\big|^2\mid\mathcal{F}_X\big)\quad\le\quad C_f a^{-1}\big(S_1+S_2\big),\quad\text{with}$$

- $S_1\le\displaystyle\sum_{i,j=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}u\,u'\psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}dtdt'$
- $S_2\le\displaystyle\sum_{i,j=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}\Big((u\,u')^H\wedge\big((u\,u')|\beta|^{2H-2}+(u\,u')^{H+1/2}|\beta|^{-1}\big)\Big)\psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}dtdt'.$

But, $S_1\le\Big(\int_0^{T_n}\chi(t)\,\Big|\psi\Big(\frac{t-b}{a}\Big)\Big|\,dt\Big)^2$ where

$$\chi(t)\quad:=\quad\sum_{i=0}^{n-1}|t-t_i|\mathbf{1}_{[t_i,t_{i+1}]}(t).\tag{18}$$

From Hölder Inequality, $S_1\le\big\|\chi\big\|^2_{\mathcal{L}^p}\big\|\psi\Big(\frac{t-b}{a}\Big)\big\|^2_{\mathcal{L}^q}$ for $(p,q)\in[1,\infty]^2$ with $1/p+1/q=1$. Since $\big\|\psi\Big(\frac{t-b}{a}\Big)\big\|_{\mathcal{L}^q}=a^{1/q}\big\|\psi\big\|_{\mathcal{L}^q}$, from Minkosvski Inequality,

$$\begin{aligned}\big\|\chi\big\|_{\mathcal{L}^p}&=\Big(\int_{I\!R}\chi^p(t)\,dt\Big)^{1/p}\\&\le\Big(\int_{I\!R}\Big(\sum_{i=0}^{n-1}\mathbf{1}_{[t_i,t_{i+1}]}(t)\Big)^{1/q}\sum_{i=0}^{n-1}|t-t_i|^p\,dt\Big)^{1/p}\\&\le\Big(\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|t-t_i|^pdt\Big)^{1/p}\le(p+1)^{-1/p}\Big(\sum_{i=0}^{n-1}L_i^{p+1}\Big)^{1/p}\delta_n^{1+1/p},\end{aligned}$$

for $p<\infty$. It follows $S_1\le a^{2/q}\big\|\psi\big\|^2_{\mathcal{L}^q}\times\big\|\chi\big\|^2_{\mathcal{L}^p}$. If $s=\infty$, one can choose $p=\infty$ and $\big\|\chi\big\|_\infty=\delta_n\max_{1\le i\le n}(\|L_i\|_\infty)$. Then,

$$S_1\quad\le\quad a^2\,M_\infty\,\big\|\psi\big\|^2_{L^1}\,\delta_n^2.\tag{19}$$

If $s<\infty$, one can deduce that for all $1\le p_1<\infty$,

$$S_1\quad\le\quad\frac{a^{2-2/p_1}\,\big\|\psi\big\|^2_{\mathcal{L}^{q_1}}}{(p_1+1)^{2/p_1}}\,\Big(\sum_{i=0}^{n-1}L_i^{p_1+1}\Big)^{2/p_1}\,\delta_n^{2+2/p_1}.\tag{20}$$

Next, in order to bound S_2, one uses twice the inequality $(x \wedge y) \leq x^\alpha y^{1-\alpha}$ which is valid for all $x, y \geq 0$ and $0 \leq \alpha \leq 1$. Thus,

$$\left\{ \begin{array}{lcll} (u\,u')^H \wedge (u\,u')|\beta|^{2H-2} & \leq & (u\,u')^{(1+H)/2)}|\beta|^{-(1-H)} & \text{with } \alpha_1 = 1/2 \\ (u\,u')^H \wedge (u\,u')^{(H+1/2)}|\beta|^{-1} & \leq & (u\,u')^{(1+H)/2)}|\beta|^{-(1-H)} & \text{with } \alpha_2 = H \end{array} \right. .$$

Therefore $S_2 \leq S_{21} + S_{22}$ with

$$S_{21} = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}(u\,u')^{(1+H)/2}|\beta|^{-(1-H)}dtdt'$$
$$S_{22} = 2\sum_{0\leq i<j\leq n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}(uu')^{(1+H)/2}|\beta|^{-(1-H)}dtdt'.$$

On the one hand, when $i = j$ then $b = \frac{1}{2}(t' - t)$ and

$$S_{21} = \frac{1}{2}\int_0^{T_n} \chi(t)^{(1+H)/2} g(t)\Big|\psi\Big(\frac{t-b}{a}\Big)\Big|dt$$

where the functions χ and g are respectively defined by (18) and

$$g(t) := \sum_{i=0}^{n-1} \mathbf{1}_{[t_i,t_{i+1}]}(t)\Big(\int_{t_i}^{t_{i+1}} |t'-t_i|^{(1+H)/2}|t-t'|^{-(1-H)}\Big|\psi\Big(\frac{t-b}{a}\Big)\Big|dt'\Big).$$

Next, by using $\psi \in \mathcal{L}^\infty(I\!\!R)$, $(1-H) \in (0,1)$ and $L_i = t_{i+1} - t_i$, one gets

$$g(t) \leq \sum_{i=0}^{n-1} \mathbf{1}_{[t_i,t_{i+1}]}(t)\Big(\|\psi\|_\infty\,|t_{i+1}-t_i|^{(1+H)/2}\int_{t_i}^{t_{i+1}} |t-t'|^{-(1-H)}dt'\Big)$$
$$\leq \frac{2}{H}\|\psi\|_\infty\Big(\sum_{i=0}^{n-1} \mathbf{1}_{[t_i,t_{i+1}]}(t)L_i^{(1+3H)/2}\Big).$$

Let $\widetilde{\chi}(t) := \sum_{i=0}^{n-1} \mathbf{1}_{[t_i,t_{i+1}]}(t)L_i^{(1+3H)/2}|t-t_i|^{(1+H)/2}$. By using Hölder inequality for all $(p,q) \in [1,\infty]^2$ with $1/p + 1/q = 1$,

$$S_{21} \quad \leq \quad \frac{2}{H}\,a^{1/q}\|\psi\|_\infty\|\psi\|_{\mathcal{L}^q}\|\widetilde{\chi}\|_{\mathcal{L}^p}.$$

If $s = \infty$, one can fix $p = \infty$ and $\|\widetilde{\chi}\|_\infty \leq \delta_n^{1+2H}$ and after

$$S_{21} \quad \leq \quad \frac{2}{H}\,a \times \|\psi\|_\infty\,\|\psi\|_{\mathcal{L}^1} \times \delta_n^{1+2H}. \tag{21}$$

If $s < \infty$, $\|\widetilde{\chi}\|_{\mathcal{L}^p} \leq \big(1 + p(1+2H)\big)^{-1/p}\Big(\sum_{i=0}^{n-1} L_i^{1+p(1+2H)}\Big)^{1/p}\delta_n^{1+2H+1/p}$ and after, for all $1 \leq p_2 < \infty$,

$$S_{21} \leq \frac{2a^{1/q_2}\|\psi\|_\infty\|\psi\|_{\mathcal{L}^{q_2}}}{H\big(1+p_2(1+2H)\big)^{1/p_2}}\Big(\sum_{i=0}^{n-1} L_i^{1+p_2(1+2H)}\Big)^{1/p_2}\delta_n^{1+2H+1/p_2}. \tag{22}$$

From the other hand, since $\beta = \frac{1}{2}\big((t'+t_j) - (t+t_i)\big) \geq \frac{1}{2}(t'-t) \geq 0$ for $i < j$ and $t \in [t_i, t_{i+1}]$, $t' \in [t_j, t_{j+1}]$, and with $1 - H < 1$,

$$S_{22} \leq \sum_{0\leq i<j\leq n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}(uu')^{(1+H)/2}|t-t'|^{H-1}dt\,dt'$$
$$\leq \int_0^{T_n}\int_0^{T_n} \chi(t)^{(1+H)/2}\chi(t')^{(1+H)/2}\psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}|t-t'|^{H-1}dtdt'$$

$$\leq \Big\|\chi(t)^{(1+H)/2}\chi(t')^{(1+H)/2}\Big\|_{\mathcal{L}^p(\mathbb{R}^2)}\Big\|\psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}|t-t'|^{H-1}\Big\|_{\mathcal{L}^q(\mathbb{R}^2)}$$

for any $(p,q)\in[1,\infty]^2$ with $1/p+1/q=1$. But for all $p\geq 2$

$$\Big\|\chi(t)^{(1+H)/2}\chi(t')^{(1+H)/2}\Big\|_{\mathcal{L}^p(\mathbb{R}^2)}= \Big(1+p(1+H)/2\Big)^{-2/p}\Big(\sum_{i=0}^{n-1}L_i^{1+p(1+H)/2}\Big)^{2/p}$$

Next, with $u=\big(t-c_k\big)/a$ and $v=\big(t'-c_k\big)/a$, one gets

$$\begin{aligned}
&\Big\|\psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}|t-t'|^{H-1}\Big\|^q_{\mathcal{L}^q(\mathbb{R}^2)}\leq a^{2+q(1-H)}\int_{\mathbb{R}^2}\frac{|\psi(u)\psi(v)|^q}{|u-v|^{q(1-H)}}dudv\\
&\leq a^{2+q(1-H)}\Big(\int_{\mathbb{R}^2,|u-v|\geq 1}|\psi(u)\psi(v)|^q dudv+\int_{\mathbb{R}^2,|u-v|<1}\frac{|\psi(u)\psi(v)|^q}{|u-v|^{q(1-H)}}dudv\Big)\\
&\leq a^{2+q(1-H)}\Big(\|\psi\|^{2q}_{\mathcal{L}^q}+\ \|\psi\|^q_\infty\,\|\psi\|^q_{\mathcal{L}^q}\int_0^1 s^{-q(1-H)}ds\Big).
\end{aligned}$$

The last integral is equal to $\big(1-q(1-H)\big)^{-1}$ when $p>1/H$. Thus, $\forall p_3>1/H$,

$$\frac{S_{22}}{\delta_n^{1+H+2/p_3}}\leq\frac{\|\psi\|_{\mathcal{L}^{q_3}}\big(\|\psi\|^{q_3}_{\mathcal{L}^{q_3}}+\frac{\|\psi\|^{q_3}_\infty}{1-q_3(1-H)}\big)^{1/q_3}}{a^{H-3+2/p_3}\big(1+p_3(1+H)/2\big)^{2/p_3}}\Big(\sum_{i=0}^{n-1}L_i^{1+p_3(1+H)/2}\Big)^{2/p_3}. \tag{23}$$

If $s=\infty$, one can fix $p=\infty$ and $q=1$, and thus

$$S_{22}\quad\leq\quad a^{3-H}\,\|\psi\|_{\mathcal{L}^1}\,\big(\|\psi\|_{\mathcal{L}^1}+\frac{1}{H}\|\psi\|_\infty\big)\,\delta_n^{1+H}. \tag{24}$$

Finally by summing up (19), (21) and (24) if $s=\infty$, and by summing up (20), (22) and (23) if $s<\infty$, one gets the bounds of $v_{1,n}(a)$.

(2) Bound of $\mathbb{E}\big(\big|\varepsilon_{2,n}(a,b)\big|^2\mid\mathcal{F}_X\big)$. Since T_n is independent on $\mathcal{F}_X$,

$$\begin{aligned}
\mathbb{E}\big(\big|\varepsilon_{2,n}(a,b)\big|^2\mid\mathcal{F}_X\big)&=\frac{1}{a}\int_{T_n}^\infty\int_{T_n}^\infty\psi\Big(\frac{t-b}{a}\Big)\overline{\psi\Big(\frac{t'-b}{a}\Big)}\mathbb{E}\big(X(t)X(t')\big)dtdt'\\
&\leq\frac{C_f}{a}\Big(\int_{T_n}^\infty\Big|\psi\Big(\frac{t-b}{a}\Big)\Big|\,\big(1+|t|\big)dt\Big)^2.
\end{aligned}$$

from Lemma 4.1. But, according to Assumption W$(1,3)$,

$$\mathbb{E}\big(\varepsilon^2_{2,n}(a,b)\mid\mathcal{F}_X\big)\quad\leq\quad C_f\,C_\psi^2\,a^{-1}\Big(\int_{T_n}^\infty\big(1+t\big)\big(1+\big(t-b\big)/a\big)^{-3}dt\Big)^2.$$

If $T_n\geq 1$, then $1+\big(t-b\big)/a\leq 1+\big(t-T_n+T_n^\rho\big)/a$ for all $t\geq T_n$ and with the change of variable $v=\big(t-\ T_n+T_n^\rho\big)/T_n$,

$$\begin{aligned}
\int_{T_n}^\infty\frac{\big(1+t\big)}{\big(1+\big(t-b\big)/a\big)^3}\,dt&\quad\leq\quad T_n\,a^3\int_{T_n^{\rho-1}}^\infty\frac{\big(1+\ v\,T_n+\ T_n-T_n^\rho\big)}{\big(a+\ v\,T_n\big)^3}\,dv\\
&\quad\leq\quad T_n^{-1}\,a^3\int_{T_n^{\rho-1}}^\infty\frac{(v+2)}{v^3}\,dv\ =\ a^3\,\big[T_n^{1-2\rho}+T_n^{-\rho}\big]
\end{aligned}$$

If $T_n\leq 1$, by using $b\leq T_n$,

$$\int_{T_n}^\infty\frac{\big(1+t\big)}{\big(1+\big(t-b\big)/a\big)^3}\,dt\ \leq\ \int_0^\infty\frac{\big(1+v+T_n\big)}{\big(1+v/a\big)^3}\,dv\leq a+\frac{1}{2}\,a^2.$$

Eventually, one deduces (17).

(3) Bound of $\mathbb{E}\big(\big|\varepsilon_{3,n}(a,b)\big|^2 \mid \mathcal{F}_X\big)$. Find a bound for $\mathbb{E}\big(\big|\varepsilon_{3,n}(a,b)\big|^2 \mid \mathcal{F}_X\big)$ follows the same steps than for bounding $\mathbb{E}\big(\big|\varepsilon_{2,n}(a,b)\big|^2 \mid \mathcal{F}_X\big)$. ■

Lemma 4.6 *Under assumptions of Lemma 4.5 and if $s \geq 2H+2$ and*

$$n\,\delta_n^{1+\frac{(H+1)(s-1)}{\theta(s)+H+1}} \underset{n\to\infty}{\longrightarrow} 0,$$

with $\theta(s) := s\,\mathbf{1}_{s<\frac12+\frac{1}{2H}} + \big(\frac12+\frac{1}{2H}\big)\mathbf{1}_{\frac12+\frac{1}{2H}\leq s<\frac12+\frac{1}{2H}} + (s-1)\,\mathbf{1}_{s\geq\frac32+\frac{1}{2H}}$, *then for all $a \in [a_{min},\, a_{max}]$, $1 \leq k \leq n$,*

$$\mathbb{E}\big(\big|\varepsilon_n(a,c_k)\big|^2\big) \leq v_n(a), \quad \text{and} \quad (n\,\delta_n)\,v_n(a) \underset{n\to\infty}{\longrightarrow} 0. \tag{25}$$

Proof. With $(x+y+z)^2 \leq 3\,(x^2+y^2+z^2)$ for all real numbers x, y, z,

$$\mathbb{E}\big(\big|\varepsilon_n(a,c_k)\big|^2\big) \;\leq\; 3\,C_f\,\mathbb{E}v_{1,n}(a) + 6\,C_f\,\mathbb{E}v_{2,n}(a)$$

where $v_{1,n}(a)$ and $v_{2,n}(a)$ have been defined in Lemma 4.5.

• If $s=\infty$, from Assumption S(s), $m_\infty\,\big(n\,\delta_n\big) \leq T_n \Longrightarrow T_n \underset{n\to\infty}{\longrightarrow} \infty$. Thus,

$$(n\,\delta_n)\,\mathbb{E}\big(\big|\varepsilon_n(a,c_k)\big|^2\big) \quad\leq\quad C_1\,\big(n\,\delta_n^{2+H}\big) + C_2\,\big(n\,\delta_n\big)^{3-4\rho}$$

which converges to zero as soon as $n\,\delta_n^{2+H} \to 0$ and $\rho > 3/4$.

• If $1<s<\infty$, from Lemma 4.5, with $(p_1,p_2,p_3) \in [1,\infty)^2 \times (1/H,\infty)$, an optimal choice of p_1,p_2,p_3 will depend on s. Hence, since $\mathbb{E}\left(|Z|^\alpha\right) \leq (\mathbb{E}|Z|)^\alpha$ for any r.v. Z and $\alpha \in [0,1]$ from Jensen Inequality,

1. if $3 \leq s$, with $1+p_1 = s$,
$$\mathbb{E}\Big(\sum_{i=0}^{n-1} L_i^{p_1+1}\Big)^{\frac{2}{p_1}}\delta_n^{2+\frac{2}{p_1}} \leq M_s^{\frac{2}{s-1}}\cdot n^{\frac{2}{s-1}}\,\delta_n^{\frac{2s}{s-1}};$$
2. if $2+2H \leq s$, with $1+p_2(1+2H) = s$,
$$\mathbb{E}\Big(\sum_{i=0}^{n-1} L_i^{1+p_2(1+2H)}\Big)^{\frac{1}{p_2}}\delta_n^{1+2H+\frac{1}{p_2}} \leq M_s^{\frac{1+2H}{s-1}}\cdot n^{\frac{1+2H}{s-1}}\,\delta_n^{\frac{(1+2H)s}{s-1}};$$
3. if $\max\big(2+H, \frac32+\frac{1}{2H}\big) \leq s$, with $1+\dfrac12 p_3(1+H) = s$,
$$\mathbb{E}\Big(\sum_{i=0}^{n-1} L_i^{1+p_3(1+H)/2}\Big)^{\frac{2}{p_3}}\delta_n^{1+H+\frac{2}{p_3}} \leq M_s^{\frac{1+H}{s-1}}\cdot n^{\frac{1+H}{s-1}}\,\delta_n^{\frac{(1+H)s}{s-1}}.$$

However, the inequalities 1. and 3. may be extended respectively to $2<s\leq 3$ and $2+H<s<\frac32+\frac{1}{2H}$ using a more sharp inequality which is $\mathbb{E}\big(\sum|x_i|\big)^{\alpha\beta} \leq \mathbb{E}\big(\sum|x_i|^\beta\big)^\alpha \leq n^\alpha\big(\mathbb{E}(|x_i|^\beta)\big)^\alpha$ when $(\alpha,\beta) \in (0,1]^2$:

1'. if $2<s\leq 3$, with $\alpha\beta = \frac{2}{p_1}$, $\alpha = 1$ and $\beta = s-2$,
$$\mathbb{E}\Big(\sum_{i=0}^{n-1} L_i^{1+\frac{2}{\alpha\beta}}\Big)^{\alpha\beta}\delta_n^{2+\alpha\beta} \leq M_s^s\cdot n\delta_n^s;$$
3'. if $1+3H<s\leq\frac{H+1}{2H}$, with $\alpha\beta = \frac{2}{p_3}$, $\alpha = \frac{H+1}{s}$ and $\beta = s-\frac{H+1}{\alpha}$,
$$\mathbb{E}\Big(\sum_{i=0}^{n-1} L_i^{1+\frac{1+H}{\alpha\beta}}\Big)^{\alpha\beta}\delta_n^{1+H+\alpha\beta} \leq M_s^{\frac{H+1}{s}}\cdot\big(n\delta_n^s\big)^{\frac{H+1}{s}};$$
3". if $\frac{H+1}{2H}<s\leq\frac{3H+1}{2H}$, with $\alpha\beta = \frac{2}{p_3}$, $\alpha = 2H$ and $\beta = s-\frac{H+1}{2H}$,
$$\mathbb{E}\Big(\sum_{i=0}^{n-1} L_i^{1+\frac{1+H}{\alpha\beta}}\Big)^{\alpha\beta}\delta_n^{1+H+\alpha\beta} \leq M_s^{2H}\cdot\big(n\delta_n^s\big)^{2H}.$$

We finally obtain for n large enough and using $n\,\delta_n \underset{n\to\infty}{\longrightarrow} \infty$ and $n\,\delta_n^{2+H} \underset{n\to\infty}{\longrightarrow} 0$ (necessary condition for $s=\infty$),

- $I\!\!E v_{1,n}(a) \le C\,\big(n\delta_n^s\big)^{\frac{H+1}{s}}$, for $2+2H \le s < \max\big(2+2H\,,\,\frac{1}{2}+\frac{1}{2H}\big)$;
- $I\!\!E v_{1,n}(a) \le C\,\big(n\delta_n^s\big)^{2H}$, for $\max\big(2+2H\,,\,\frac{1}{2}+\frac{1}{2H}\big) \le s \le \max\big(2+2H\,,\,\frac{3}{2}+\frac{1}{2H}\big)$;
- $I\!\!E v_{1,n}(a) \le C\,\big(n\delta_n^s\big)^{\frac{1+H}{s-1}}$, for $s \ge \max\big(2+2H\,,\,\frac{3}{2}+\frac{1}{2H}\big)$

(therefore, the first inequality is only possible when $H<1/4$ and the second one when $H<(\sqrt{17}-1)/8$). Those three inequalities may be reduced to only one:

$$I\!\!E v_{1,n}(a) \le C\,\big(n\delta_n^s\big)^{\frac{H+1}{\theta(s)}} \quad \text{for all} \quad s \ge 2H+2, \tag{26}$$

with $\theta(s) = s\,\mathbf{1}_{s<\frac{1}{2}+\frac{1}{2H}} + \big(\frac{1}{2}+\frac{1}{2H}\big)\mathbf{1}_{\frac{1}{2}+\frac{1}{2H}\le s<\frac{1}{2}+\frac{1}{2H}} + (s-1)\,\mathbf{1}_{s\ge\frac{3}{2}+\frac{1}{2H}}$. Hence, $(n\delta_n) I\!\!E v_{1,n}(a) \underset{n\to\infty}{\longrightarrow} 0$ when $s \ge 2H+2$ and

$$n\,\delta_n^{1+\frac{(H+1)(s-1)}{\theta(s)+H+1}} \underset{n\to\infty}{\longrightarrow} 0.$$

To finish the proof of Lemma 4.6 it remains to show $(n\delta_n) I\!\!E v_{2,n}(a) \underset{n\to\infty}{\longrightarrow} 0$. From (17) it follows that $I\!\!E v_{2,n}(a) \le C\displaystyle\int_0^\infty g(x)\,f_n(x)\,dx$ where f_n is the probability distribution function of T_n and $g(x) = \mathbf{1}_{(x<1)} + \mathbf{1}_{(x\ge 1)}\,x^{2-4\rho}$. Since $\rho > 3/4$, $g(x) \le 1$ for all $x>0$ and g is a non increasing map,

$$\begin{aligned}
\int_0^\infty g(x)\,f_n(x)\,dx &\le \int_0^{\frac{1}{2}m_s\,n\,\delta_n} f_n(x)\,dx + g\big(\frac{1}{2}\,m_s\,n\,\delta_n\big)\int_{\frac{1}{2}m_s\,n\,\delta_n}^\infty f_n(x)\,dx \\
&\le I\!\!P\big(T_n \le \frac{1}{2}\,m_s\,n\,\delta_n\big) + g\big(\frac{1}{2}\,m_s\,n\,\delta_n\big) \\
&\le I\!\!P\big(|T_n - I\!\!E[T_n]| \ge I\!\!E[T_n] - \frac{1}{2}\,m_s\,n\,\delta_n\big) + \big(\frac{1}{2}\,m_s\,n\,\delta_n\big)^{2-4\rho}, \\
&\le 4\,\frac{M_s}{m_s}\,\frac{n\,\delta_n^2}{n^2\,\delta_n^2} + \big(\frac{1}{2}\,m_s\,n\,\delta_n\big)^{2-4\rho},
\end{aligned}$$

from Bienaymé-Chebyshev Inequality since $s \ge 2$ and var $(T_n) \le M_s n\delta_n^2$ from the independence of $(L_i)_{i\in I\!\!N}$. Therefore $(n\,\delta_n)\,I\!\!E v_{2,n}(a) \underset{n\to\infty}{\longrightarrow} 0$. ■

Proof. [Theorem 2.1] Following the same method that in the end of the proof of Bardet and Bertrand (2007b), one obtains

$$I\!\!E\,|I_n(a) - J_n(a)| \le C\,v_n(a)^{1/2}, \tag{27}$$

and from this, Lemmas 4.6 and Slutsky Lemma, the proof is achieved. ■

4.4 Proof of Proposition 2.1

Proof. It is obvious that

$$\begin{aligned}
\mathcal{I}_\lambda(a) &= \int_{I\!\!R} \big|\widehat{\psi}_\lambda(\xi)\big|^2 f(\xi/a)d\xi = \lambda\int_{I\!\!R} \big|\widehat{\psi}\big(\lambda(\xi-1)\big)\big|^2 f(\xi/a)d\xi \\
&= \int_{I\!\!R} \big|\widehat{\psi}(v)\big|^2 f\Big(\frac{1}{a} + \frac{v}{a\lambda}\Big)dv.
\end{aligned}$$

Then, from a usual Taylor expansion, and since $\widehat{\psi}$ is supposed to be an even function supported in $[-\Lambda,\Lambda]$,

$$\Big|\mathcal{I}_\lambda(a) - \|\widehat{\psi}\|^2_{\mathcal{L}^2(I\!\!R)}\,f(1/a)\Big| \le \frac{1}{2a^2\lambda^2}\Big(\sup_{-\Lambda/\lambda\le h}\Big\{\Big|f''\big(\frac{1+h}{a}\big)\Big|\Big\}\int_{-\Lambda}^{\Lambda} v^2\big|\widehat{\psi}(v)\big|^2 dv\Big).$$

For $\lambda > 2\Lambda$, then $\sup_{-\Lambda/\lambda \leq h} \Big\{ \Big| f''\big(\frac{1+h}{a}\big)\Big| \Big\} \leq \sup_{x>1/2a} \big\{ \big| f''(x) \big| \big\} < \infty$. Therefore, since ψ satisfies Assumption W$(1,5)$, there exists $C>0$ such that,

$$\Big| \mathcal{I}_\lambda(a) - \|\widehat{\psi}\|^2_{\mathcal{L}^2(I\!\!R)} \, f(1/a) \Big| \leq C \, \frac{1}{\lambda^2}. \tag{28}$$

Let us denote $I_n^{(\lambda)}(a)$ (respectively $\mathcal{I}_\lambda(a)$, $\beta_n^{(\lambda)}$ and $S_n^{(\lambda)}(a)$) instead on $I_n(a)$ (resp. $\mathcal{I}_1(a)$, β_n and $S_n(a)$) when ψ is replaced by ψ_λ. Firstly,

$$\begin{aligned} \frac{1}{\lambda}\Big(4\pi\, a^2 \int_{I\!\!R} \big|\widehat{\psi_\lambda}(az)\big|^4 f^2(z)\, dz \Big) &= 4\pi\, a \int_{I\!\!R} \big|\widehat{\psi}(u)\big|^4 f^2\Big(\frac{1}{a}+\frac{u}{a\lambda}\Big)\, du \\ &\underset{\lambda\to\infty}{\longrightarrow} 4\pi\, a\, f^2(1/a) \int_{I\!\!R} \big|\widehat{\psi}(u)\big|^4\, du, \end{aligned}$$

from Lebesgue Theorem. Hence, if (λ_n) is a sequence such that $\lambda_n \underset{n\to\infty}{\longrightarrow} \infty$,

$$\frac{I\!\!E(T_n)}{\lambda_n} \big(S_n^{(\lambda_n)}(a)\big)^2 \underset{n\to\infty}{\longrightarrow} 4\pi\, a\, f^2(1/a) \int_{I\!\!R} \big|\widehat{\psi}(u)\big|^4\, du. \tag{29}$$

Secondly, from the proof of Proposition 4.1 and inequalities (10) and (11), there exists $C>0$ not depending on n and λ,

$$\beta_n^{(\lambda)}/S_n^{(\lambda)} \leq C\, \mathcal{I}_\lambda^{-1}(a) \, \big(n \max_{1\leq k\leq n} \big|c_{k+1}-c_k\big|\big)^{1/2-1/q} \quad \text{for all } q \in (1,2).$$

Thus, since $\mathcal{I}_\lambda(a)$ is bounded, $\beta_n^{(\lambda_n)}/S_n^{(\lambda_n)} \underset{n\to\infty}{\longrightarrow} 0$ and Proposition 4.1 becomes:

$$\frac{I_n^{(\lambda_n)}(a) - \mathcal{I}_{\lambda_n}(a)}{S_n^{(\lambda_n)}(a)} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

Finally, using (28) and (29), on deduces that for all $a>0$,

$$\sqrt{\frac{I\!\!E T_n}{\lambda_n}} \big(I_n^{(\lambda_n)}(a) - \|\psi\|^2_{\mathcal{L}^2(I\!\!R)} f(1/a)\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\Big(0, 4\pi a f^2(1/a) \int_{I\!\!R} \big|\widehat{\psi}(u)\big|^4 du\Big),$$

when $(\lambda_n)_n$ is such that $\sqrt{\frac{I\!\!E T_n}{\lambda_n}} \frac{1}{\lambda_n^2} \underset{n\to\infty}{\longrightarrow} 0$, *i.e.* when $\lambda_n^{-5}\, n\, \delta_n \underset{n\to\infty}{\longrightarrow} 0$. Since also $\lambda_n^{-1} n \delta_n \underset{n\to\infty}{\longrightarrow} \infty$ (to obtain a consistent estimator), then

$$\frac{1-d}{5} < d' < 1-d. \tag{30}$$

Moreover, Proposition 4.2 has also to be checked. In its proof, $I\!\!E \tau_n$ has to be replaced by $I\!\!E \tau_n/\lambda_n$ and since the bounds $C\,(1 \wedge |\theta|^{-1})$ in Lemma 4.2 have to be replaced by $C/\lambda_n^2\,(1\wedge|\theta|^{-1})$, then condition $n\delta_n^2 \underset{n\to\infty}{\longrightarrow} 0$ has to be replaced by $n\delta_n^2/\lambda_n^5 \underset{n\to\infty}{\longrightarrow} 0$, that is $d' > \dfrac{1-2d'}{5}$ which is satisfied when (30) is satisfied.

It remains to control $\varepsilon_n^2(a, c_k)$ with Lemma 4.5 and 4.6. For all $1 \leq q \leq \infty$, with $1/\infty = 0$ by convention,

$$\big\|\psi_\lambda\big\|_{\mathcal{L}^q} = \lambda^{(2-q)/2q} \, \big\|\psi\big\|_{\mathcal{L}^q} \quad \text{and} \quad \big\|\widehat{\psi}_\lambda\big\|_{\mathcal{L}^q} = \lambda^{(q-2)/2q}\, \big\|\widehat{\psi}\big\|_{\mathcal{L}^q}.$$

Then, using the choice of (p_1,p_2,p_3) obtained in Lemma 4.6, Lemma 4.5 becomes (with $\lambda_n \underset{n\to\infty}{\longrightarrow} \infty$):

- if $s=\infty$, $\; v_{1,n}(a) \leq C\, \lambda_n\, \delta_n^{1+H}$;
- if $\max\big(2+2H\,,\, \frac{3}{2}+\frac{1}{2H}\big) \leq s < \infty$,
$$v_{1,n}(a) \leq C\Big(\lambda_n^{\frac{s-3}{s-1}} \big(n\,\delta_n^s\big)^{\frac{2}{s-1}} + \lambda_n^{-\frac{2H+1}{s-1}} \big(n\,\delta_n^s\big)^{\frac{2H+1}{s-1}} + \lambda_n^{\frac{s-2-H}{s-1}} \big(n\,\delta_n^s\big)^{\frac{H+1}{s-1}} \Big)$$

$$\Longrightarrow \quad v_{1,n}(a) \leq C' \, \lambda_n^{\frac{s-2-H}{s-1}} \, \big(n \, \delta_n^s\big)^{\frac{H+1}{s-1}} \quad \text{since } 0 < H < 1 \text{ and } s \geq 3;$$

- if $2+2H \leq s < \max\big(2+2H\,,\,\frac{3}{2}+\frac{1}{2H}\big)$,

$$\Longrightarrow \quad v_{1,n}(a) \leq C' \, \lambda_n^{\frac{s-2-H}{s-1}} \, n \, \delta_n^{1+3H};$$

Condition (25) is now $\dfrac{n\,\delta_n}{\lambda_n}\, v_{1,n}(a) \underset{n\to\infty}{\longrightarrow} 0$ and then, conditions required on d and d' are:

$$\bullet \, \text{if } s=\infty, \;\; d > (2+H)^{-1}; \tag{31}$$

$$\bullet \, \text{if } \max\big(2+2H\,,\,\tfrac{3}{2}+\tfrac{1}{2H}\big) \leq s < \infty, \;\; d' > \frac{s+H-d(s(H+2)-1)}{H+1}; \tag{32}$$

$$\bullet \, \text{if } 2+2H \leq s < \max\big(2+2H\,,\,\tfrac{3}{2}+\tfrac{1}{2H}\big), \;\; d' > \frac{(s-1)(2-d(3H+2))}{H+1}. \tag{33}$$

Finally, for $b \leq T_n - T_n^\rho$, with ψ satisfying Assumption W$(1,5)$:

$$\begin{aligned}
I\!E\big(\varepsilon_{2,n}^{\lambda,2}(a,b) \mid \mathcal{F}_X\big) &= a^{-1} \int_{T_n}^\infty \int_{T_n}^\infty \psi_\lambda\Big(\frac{t-b}{a}\Big) \psi_\lambda\Big(\frac{t'-b}{a}\Big) I\!E\big(X(t)X(t')\big) dt dt' \\
&\leq C_f \, (a\,\lambda_n)^{-1} \Big(\int_{T_n}^\infty \Big| \psi\Big(\frac{t-b}{a\lambda_n}\Big)\Big| \, (1+|t|)\, dt \Big)^2 \\
&\leq C_f \, (a^3 \lambda_n^3) \Big(\int_{T_n^\rho/a\lambda_n}^\infty |\psi(u)| \, u \, du \Big)^2 \\
&\leq \frac{1}{9} \, C_f \, C_\psi \, (a^3 \lambda_n^3) \Big(\Big[u^{-3}\Big]_{T_n^\rho/a\lambda_n}^\infty \Big)^2 \leq \frac{1}{9} \, C_f \, C_\psi \, a^9 \, \lambda_n^9 \, T_n^{-6\rho}.
\end{aligned}$$

Therefore the CLT holds when $\lambda_n^9 \, (n\,\delta_n)^{1-6\rho} \underset{n\to\infty}{\longrightarrow} 0$, *i.e.* $9d' + (1-d)(6\rho-1) < 0$,

$$\Longrightarrow \quad d' \leq \frac{1}{2}\,(1-d) \;\; \text{since } \rho \in (3/4, 1). \tag{34}$$

Combining conditions (30), (31), (32), (33) and (34) on d and d', one deduces:

- if $s = \infty$, $d > (2+H)^{-1}$ and $\dfrac{1-d}{5} < d' \leq \dfrac{1-d}{2}$;
- if $\max\big(2+2H\,,\,\frac{3}{2}+\frac{1}{2H}\big) \leq s < \infty$,
  $$\max\Big(\frac{s+H-(s(H+2)-1)d}{H+1}\,,\,\frac{1-d}{5}\Big) < d' \leq \frac{1-d}{2};$$
- if $2+2H \leq s < \max\big(2+2H\,,\,\frac{3}{2}+\frac{1}{2H}\big)$,
  $$\max\Big(\frac{(s-1)(2-d(3H+2))}{H+1}\,,\,\frac{1-d}{5}\Big) < d' \leq \frac{1-d}{2}.$$

■

References

[1] Abry, P., Flandrin, P., Taqqu, M.S. and Veitch, D. (2003). Self-similarity and long-range dependence through the wavelet lens, in *Long-range Dependence: Theory and Applications*, P. Doukhan, G. Oppenheim and M.S. Taqqu editors, Birkhäuser.

[2] Aït-Sahalia, Y. & Mykland, P.A. (2008). An analysis of Hansen-Scheinkman moment estimators for discretely and randomly sampled diffusions, *Journal of Econometrics*, In Press.

[3] Bardet, J.M. and Bertrand, P. (2007a), "Definition, properties and wavelet analysis of multiscale fractional Brownian motion", *Fractals*, 15, 73-87.

[4] Bardet, J.M. and Bertrand, P. (2007b), "Identification of the multiscale fractional Brownian motion with biomechanical applications", *J. Time Ser. Anal.*, 28, 1-52.

[5] Bardet J.M., Lang G., Moulines E. and Soulier P. (2000). Wavelet estimator of long-range dependent processes. *Stat. Inference Stoch. Process.*, 3, 85-99.

[6] Begyn, A. (2005). Quadratic Variations along Irregular Subdivisions for Gaussian Processes. *Electron. J. Probab.*, 10, 691-717.

[7] Cramér, H. and Leadbetter, M. R. (1967). *Stationary and related stochastic processes. Sample function properties and their applications.* Wiley and Sons.

[8] Engle, R. F.; Russell, J. R. (1998) Autoregressive conditional duration: a new model for irregularly spaced transaction data. *Econometrica* 66 , no. 5, 1127-1162.

[9] Flandrin, P. (1992). Wavelet analysis and synthesis of fractional Brownian motion. *IEEE Trans. on Inform. Theory*, 38, 910-917.

[10] Dahlhaus R. (1989). Efficient parameter estimation for self-similar processes. *Ann. Statist.* 17, 1749-1766.

[11] Frisch, U., (1995). *Turbulence, the legacy of A.N. Kolmogorov*, Cambridge University Press.

[12] Gao, J.; Anh, V.; Heyde, C. (2002) Statistical estimation of non-stationary Gaussian processes with long-range dependence and intermittency. *Stochastic Process. Appl.* **99**, no. 2, 295–321

[13] Gloter, A. and Hoffmann, M. (2007). "Estimation of the Hurst parameter from discrete noisy data", *Ann Stat.* , 35, 5, 1947-1974.

[14] Hayashi, T. & Yoshida, N. (2005). On covariance estimation of non-synchronously observed diffusion processes. *Bernoulli*, 11, 359-379.

[15] Ibragimov, I (2004). Estimation of analytic spectral density of Gaussian stationary processes. In *Parametric and semiparametric models with applications to reliability, survival analysis, and quality of life*, 419-443, Birkhuser, Boston.

[16] Ikeda, N. and Watanabe, S. (1989), *Stochastic Differential Equations and Diffusion Processes*, 2nd edition, North Holland.

[17] Istas, J. and Lang, G. (1997). Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Ann. Inst. Poincaré*, 33, 407-436.

[18] Lehr, M. and Lii, K.-S. (1997). Wavelet spectral density estimation under irregular sampling. Proceedings of *Conference Record of the Thirty-First Asilomar Conference on Signals, Systems & Computers*.

[19] Lii, K.S. and Masry, E. (1994) Spectral estimation of continuous-time stationary processes from random sampling. *Stochastic Process. Appl.* **52**, no. 1, 39–64.

[20] Masry, E. (1978a), Poisson sampling and spectral estimation of continuous-time processes. *IEEE Trans. Inform. Theory* IT-24 173-183.

[21] Masry, E. (1978b), E. Alias-free sampling: an alternative conceptualization and its applications, *IEEE Trans. Infrom. Theory* IT-24 317-324.

[22] Meyer, Y. (1990), Ondelettes et opérateurs 1. Ondelettes. Hermann. Paris.

[23] Moulines, E., Roueff, F. and Taqqu, M.S. (2007), On the spectral density of the wavelet coefficients of long memory time series with application to the log-regression estimation of the memory parameter, *J. Time Ser. Anal.*, 28, 155-187.

[24] Papanicolaou, G. and Sølna, K., (2002). Wavelet Based Estimation of Local Kolmogorov Turbulence, , in *Long-range Dependence: Theory and Applications*, P. Doukhan, G. Oppenheim and M.S. Taqqu editors, Birkhäuser, p. 473-506

[25] Parzen, E. (1983), Time Series Analysis of irregularly observed data, Springer.

[26] Samorodnitsky, G. and Taqqu M.S. (1994), *Stable non-Gaussian Random Processes*, Chapman and Hall.

[27] Shapiro,H.S. and Silverman, R.A. (1960), Alias-free sampling of random noise. J.SIAM 8 (2), 225-248

[28] Task force of the European Soc. Cardiology and the North American Society of Pacing and Electrophysiology (1996), Heart rate variability. Standards of measurement, physiological interpretation, and clinical use. *Circulation* 93 (5), 1043-1065.