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WAVELET ANALYSIS OF A CONTINUOUS-TIME  
GAUSSIAN PROCESS OBSERVED AT RANDOM TIMES  
AND ITS APPLICATION TO THE ESTIMATION OF THE  
SPECTRAL DENSITY

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In numerous applications (finance, internet traffic, biology,...) data are observed at random times. From a wavelet analysis, one derives a nonparametric estimator of the spectral density of a Gaussian process with stationary increments (also stationary Gaussian process) observed at random times.

**1. Introduction.** The aim of this paper is a wavelet analysis of a continuous-time Gaussian process which is discretely observed at random times. Gaussian centered processes with stationary increments are considered, since this class contains as a paradigmatic example fractional Brownian motions (fBm). Moreover these processes admit an harmonizable representation and are characterized by their spectral density function, called  $f$  in all the sequel. The spectral density of a fBm follows a power law, that is  $f(\xi) = C |\xi|^{-(2H+1)}$  where  $H$  is called the Hurst parameter and  $C > 0$ .

During the two last decades, a lot of statistical studies have been made on fBm or generalizations of fBm (see for instance Doukhan *et al.*, 2003). What is the novelty of this paper? Two points are the core of this work: 1) A nonparametric estimation of the spectral density on a finite band of frequencies is constructed and studied. 2) The process is observed at random times. Both these points justify the use of wavelet analysis.

To our knowledge, there are very few studies in both these directions,

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mainly Gao, Anh and Heyde (2002) and Lii and Masry (1994) for the first point and only Begyn (2005) for the second one (but with irregular deterministic observation times). This is a sharp contrast with the large literature devoted to the estimation of the Hurst parameter of a fBm or its generalizations observed at deterministic regularly spaced discrete times.

Stress that the motivation of this work is not a theoretical refinement, but modelling applications. As Engle and Russell (1999) has so rightly pointed out *"with the rapid development in computing power and storage capacity, data are being collected and analyzed at ever higher frequencies. For many types of data, the ultimate in high frequency data collection has been reached and every transaction is recorded. This limit has been reached for financial market [...] Transaction data inherently arrive in irregular time intervals [...] treated as random variables."* Afterward, this limit has also been reached for biological signals as heart rate obtained by Holter monitoring of ECG and EEG, EMG or DNA sequences. As a corollary, asymptotic estimation of the spectral density when the frequencies goes to infinity is irrelevant.

From the other hand, in some applications the behavior of the spectral density on a finite band can provide relevant information, see for instance, Collins and de Lucas (1993) or Bardet and Bertrand (2007 b) for modelling postural gait with a spectral density following two different power laws on two finite bands of frequencies. Similarly in turbulence and other fields of physics one wish to determine the inertial range, that is the maximal finite band of frequencies on which the spectral density follows a spectral law, see Frisch (1995) or Papanicolaou and Sølna (2002). We guess this approach is relevant in many other applications.

The remainder of the paper is organized as follow: Section 2 is devoted to mathematical description. In Section 3 the main result, a central limit theorem (CLT) for the estimators of the variance of wavelet coefficients, is provided. In Section 4, an application is studied: a nonparametric estimation of the spectral density. This estimator is applied to generated data in Section 5 and to real data in Section 6. Section 7 contains the proofs.

**2. Description of the problem.** Gaussian processes  $\{X(t), t \in \mathbb{R}\}$  with zero mean and stationary increments are considered. Such processes can also be written following harmonizable representations (see for instance Cramèr and Leadbetter, 1967),

$$(1) \quad X(t) = \int_{\mathbb{R}} (e^{it\xi} - 1) f^{1/2}(\xi) dW(\xi), \quad \text{for all } t \in \mathbb{R},$$

where

- $W(dx)$  is a complex Brownian measure, with adapted real and imaginary part such that the Wiener integral is real valued;
- $f$  is a Borelian positive even function so-called the spectral density of  $X$  and is such that

$$(2) \quad \int_{\mathbb{R}} (1 \wedge |\xi|^2) f(\xi) d\xi < \infty.$$

In the sequel,  $f$  will be supposed to satisfy also Assumption F defined below. As a particular case, if  $\{X(t), t \in \mathbb{R}\}$  is a stationary processes, one will still denote  $f$  the spectral density such that

$$(3) \quad X(t) = \int_{\mathbb{R}} e^{it\xi} f^{1/2}(\xi) dW(\xi), \quad \text{for all } t \in \mathbb{R}.$$

Even if their definition are different, in the sequel  $f$  will denote as well the spectral density of a process having stationary increments or a stationary process (see the explanation in Proposition 2.1).

Define also the  $\sigma$ -algebra  $\mathcal{F}_X$  generated by the process  $X$ , *i.e.*

$$(4) \quad \mathcal{F}_X := \sigma\{X(t), t \in \mathbb{R}\}.$$

In the remainder of this paper, a statistical study of these processes based on wavelet analysis is proposed. To begin with, let us specify the statistical framework. A path of such a process  $X$  on the interval  $[0, T_n]$  at the discrete times  $t_i$  for  $i = 0, 1, \dots, n$  is observed, *i.e.*

$(X(t_0), X(t_1), \dots, X(t_n))$  is known, with  $0 = t_0 < t_1 < \dots < t_n = T_n$ .

A unified frame of irregular observed times, grouping deterministic and stochastic ones, will be considered. Let us assume first that there exists a sequence of positive real numbers  $(\delta_n)_{n \in \mathbb{N}}$  and a sequence of random variables (r.v. in the sequel)  $(L_k)_{k \in \mathbb{N}}$  (which could be deterministic real numbers) such that

$$(5) \quad \forall k \in \{0, 1, \dots, n-1\}, \quad t_{k+1} - t_k := \delta_n L_k, \quad \text{and} \quad \delta_n \xrightarrow[n \rightarrow \infty]{} 0.$$

It is clear that  $T_n = \delta_n(L_0 + \dots + L_{n-1})$ . For  $Z$  a r.v. and  $\alpha \in (0, \infty)$ , denote  $\|Z\|_\alpha := (\mathbb{E}(|Z|^\alpha))^{1/\alpha}$  if  $\mathbb{E}(|Z|^\alpha) < \infty$ . Now, assume that there exists  $s \in [1, \infty]$  such that

**S**( $s$ )  $(L_k)_{k \in \mathbb{N}}$  is a sequel of independent positive random variables, independent to  $\mathcal{F}_X$ , such that there exist  $0 < m_s < M_s < \infty$  satisfying

- if  $s < \infty$ ,  $m_s \leq \|L_k\|_s \leq M_s$  for all  $k \in \mathbb{N}$ .
- if  $s = \infty$ ,  $m_\infty \leq L_k \leq M_\infty$  for all  $k \in \mathbb{N}$ .

Remark that the hypothesis “ $(L_k)_{k \in \mathbb{N}}$  independent to  $\mathcal{F}_X$ ” can be a little restrictive, especially in Finance, but this condition always seems supposed in the literature (see for instance, Aït-Sahalia and Mykland, 2008).

The aim of this paper consists in the wavelet analysis of  $(X(t_0), \dots, X(t_n))$  in order to estimate the spectral density  $f$  or at least some parameters relative to  $f$ . This method was already applied with in view the estimation of (respectively) the long-memory or the self-similarity parameter of (respectively) stationary processes or processes having stationary increments, see the seminal paper of Flandrin, 1992 and its developments by Abry *et al.*, 1998, Abry *et al.*, 2001, Bardet *et al.*, 2000, Moulines *et al.*, 2007 or Gloter and Hoffmann, 2007. Note that in all these papers times series (*i.e.*  $t_i = i$ ) were considered. This method was also applied in Bardet and Bertrand (2007a, 2007b), for the special cases of Gaussian continuous time processes regularly sampled with  $t_i = i \Delta_n$  and  $\Delta_n \rightarrow 0$ .

But what the wavelet analysis consists in? Let  $\psi : \mathbb{R} \rightarrow \mathcal{C}$  be a function so-called the “mother” wavelet, satisfying some conditions (denoted  $W(m, r)$ )

specified in Section 3, let  $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$ , and define  $d_X(a, b)$  the wavelet coefficient of the process  $X$  for the scale  $a$  and the shift  $b$ , such that

$$d_X(a, b) := \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right) X(t) dt$$

This family of wavelet coefficients satisfies the following property:

**Proposition 2.1** *Let  $\psi$  satisfy Assumption  $W(1, 0)$  and  $X$  be a Gaussian process defined by (1) or (3) with a spectral density  $f$  satisfying (2). Then,*

$$(6) \quad d_X(a, b) = \sqrt{a} \int_{\mathbb{R}} e^{ib\xi} \overline{\psi}(a\xi) f^{1/2}(\xi) dW(\xi) \text{ for all } (a, b) \in \mathbb{R}_+^* \times \mathbb{R},$$

and, for  $a > 0$ ,  $(d_X(a, b))_{b \in \mathbb{R}}$  is a stationary centered Gaussian process with

$$(7) \quad \mathbb{E}|d_X(a, b)|^2 = \mathcal{I}_1(a) := a \int_{\mathbb{R}} |\widehat{\psi}(au)|^2 f(u) du \text{ for all } b \in \mathbb{R}.$$

The proof of this proposition is grouped with all the other proofs in Section 7. Note that this proposition is valid even if  $\psi$  takes complex values.  $\mathcal{I}_1(a)$  is depending on  $f$ ,  $a$  and  $\psi$ . By considering varying cases of  $a$  and  $\psi$ , it is possible to estimate  $f$  (see Section 4). However, a straightforward computation of  $\mathcal{I}_1(a)$  is not available from  $(X(t_0), \dots, X(t_n))$  for two reasons:

1. on one hand,  $d_X(a, b)$  is defined with a Lebesgue integral and cannot be directly computed from data. Since the process  $X$  will be supposed to have a.s. continuous paths but with a regularity  $\alpha_X < 1$  a.s., an approximation formula will be considered for computing wavelet coefficients. Thus, for  $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$  we define an empirical wavelet coefficient by

$$(8) \quad e_X(a, b) := \frac{1}{\sqrt{a}} \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \psi\left(\frac{t-b}{a}\right) dt \right) X(t_i).$$

2. on the other hand, a sample mean of  $|d_X(a, b)|^2$  instead on  $\mathbb{E}|d_X(a, b)|^2$  is only computable. Thus, define the sample estimator of  $\mathcal{I}_1(a)$  by

$$(9) \quad J_n(a) := \frac{1}{n+1} \sum_{k=0}^n |e_X(a, c_k)|^2,$$

where  $(c_k)_k$  is a family of increasing real numbers (so-called shifts). In this paper, we will consider a uniform repartition of shifts, *i.e.*

$$(10) \quad c_k = T_n^\rho + k \frac{T_n - 2T_n^\rho}{n} \quad \text{with } \rho \in (3/4, 1).$$

In this example  $(c_k)_{1 \leq k \leq n}$  is depending on  $T_n$  since shifts could be r.v. depending on random times  $(t_1, \dots, t_n)$ . Another choices of  $(c_k)_k$  are possible (for instance  $c_k = t_k$ ) but we have not been able to find an optimal choice and simulations do not show differences between these choices. Remark that the terms  $T_n^\rho$  are necessary to avoid border effects.

**3. Estimation of the variance of wavelet coefficients.** The aim of this section is the estimation of  $\mathcal{I}_1(a)$  for  $0 < a_{min} \leq a \leq a_{max} < \infty$  using the estimator  $J_n(a)$  defined in (9). Let us consider first the following assumptions on  $f$  and  $\psi$ .

**Assumption F** *f is an even function, differentiable on  $[0, \infty)$  except for a finite number  $K$  of real numbers  $\omega_0 = 0 < \omega_1 < \dots < \omega_K$ , but  $f$  admits left and right limits in  $\omega_k$ , with a derivative  $f'$  (defined on all open intervals  $(\omega_k, \omega_{k+1})$  with  $\omega_{K+1} = \infty$  by convention) such that*

$$(11) \quad \int_{\mathbb{R}} (1 \wedge |\xi|^3) \cdot |f'(\xi)| d\xi < \infty.$$

Moreover, there exist  $C_0, C'_0 > 0$  and  $H \in (0, 1)$ , such that  $\forall |x| \geq \omega_K$

$$(12) \quad f(x) \leq C_0 |x|^{-(2H+1)} \quad \text{and} \quad f'(x) \leq C'_0 |x|^{-(2H+2)}.$$

Here there are several examples of processes having a spectral density  $f$  satisfying Assumption F:

- Example 1** • *An almost surely smooth Gaussian process having stationary increments;*
- *A fractional Brownian motion with Hurst parameter  $H \in (0, 1)$  satisfying  $\mathbb{E}X(1) = \sigma^2$  is such that  $f(\xi) = \sigma^2 H \Gamma(2H) \sin(\pi H) |\xi|^{-(2H+1)}/\pi$ ;*
  - *In Bardet and Bertrand (2007a), due to biomechanical applications, the family of multiscale fractional Brownian motions is introduced for*

which  $f(\xi) = \sigma_i^2 |\xi|^{-(2H_i+1)}$  for  $|\xi| \in [\omega_k, \omega_{k+1}[$  where  $\omega_0 = 0 < \omega_1 < \dots < \omega_K < \omega_{K+1} = +\infty$ ,  $H_0 < 1$ ,  $0 < H_K < 1$  and  $(\sigma_i, H_i) \in \mathbb{R}_+ \times \mathbb{R}$  for  $i = 1, \dots, K-1$ . Then Condition (2) and Assumption F are checked with  $H = H_K$ .

- A stationary process with a bounded spectral density such as Ornstein-Uhlenbeck process.

Denote  $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx$  the Fourier transform of  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Next, let  $(m, r) \in [1, \infty) \times \mathbb{R}_+$  and the family of assumptions on  $\psi$ :

**Assumption W**( $m, r$ )  $\psi : \mathbb{R} \mapsto \mathcal{C}$  is a function satisfying:

- $\forall n \in \mathbb{N}$ ,  $\int_{\mathbb{R}} |t^n \psi(t)| dt < \infty$  if  $n \leq m+1$  and  $\int_{\mathbb{R}} t^n \psi(t) dt = 0$  if  $n \leq m$ ;
- $\exists C_\psi > 0$  such that  $\forall \xi \in \mathbb{R}$ ,  $(1 + |\xi|)^r (|\widehat{\psi}(\xi)| + |\widehat{\psi}'(\xi)|) \leq C_\psi$ .

The first condition of W( $m, r$ ) implies that  $\widehat{\psi}(\xi)$  has a zero of order ( $m+1$ ) at zero and is  $m$  times continuously differentiable. These conditions are mild and are satisfied by many famous wavelets (Daubechies, Lemarié-Meyer,...). It is also not mandatory to choose  $\psi$  to be a “mother” wavelet associated to a multiresolution analysis of  $L^2(\mathbb{R})$  and the whole theory can be developed without resorting to this assumption: the choice of  $\psi$  is then large.

Now, it is possible to establish a CLT satisfied by  $J_n(a)$  which is computed from the observed trajectory  $(X(t_0), \dots, X(t_n))$ .

**Theorem 3.1** *Let  $X$  be a Gaussian process defined by (1) or (3) with a spectral density  $f$  satisfying (2) and Assumption F,  $\psi$  satisfying Assumption W(1, 3) and  $(c_k)_k$  defining by (10). Under Assumption S( $s$ ) with  $2 + 2H \leq s \leq \infty$  and if  $\delta_n$  is such that*

$$n \delta_n^{1+\theta(s,H)} \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{with} \quad \theta(s, H) := \begin{cases} \frac{(H+1)(s-1)}{s+H+1} & \text{if } 2 + 2H \leq s < \frac{1}{2} + \frac{1}{2H} \\ \frac{2H(s-1)}{2H+1} & \text{if } \frac{1}{2} + \frac{1}{2H} \leq s < \frac{3}{2} + \frac{1}{2H} \\ \frac{(H+1)(s-1)}{s+H} & \text{if } \frac{3}{2} + \frac{1}{2H} \leq s \leq 2 + \frac{1}{H} \\ 1 & \text{if } s \geq 2 + \frac{1}{H} \end{cases}$$

( $1/3 < \theta(s, H) \leq 2$ ), then  $\forall a \in [a_{min}, a_{max}]$ ,

$$(13) \quad \sqrt{\mathbb{E} T_n} (J_n(a) - \mathcal{I}_1(a)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, 4\pi a^2 \int_{\mathbb{R}} |\widehat{\psi}(az)|^4 f^2(z) dz\right).$$



From the computation of the variance of  $T_n$ , the convergence rate of the CLT (13) is  $(n\delta_n)^{1/2}$ . Therefore, when  $H$  is unknown, Theorem 3.1 always holds when  $s \geq 4$  and  $n\delta_n^{2-\frac{1}{s}} = O(1)$  and its convergence rate is  $n^{\frac{s-1}{4s-2}}$  and  $o(n^{1/4})$  when  $s = \infty$ . When  $H$  is known, this results can be a little improved and the convergence rate is  $o(n^{\frac{1}{2} \frac{\theta(s,H)}{1+\theta(s,H)}})$ .

**4. A nonparametric estimator of the spectral density.** Let us consider a family  $(\psi_\lambda)_{\lambda \in \mathbb{R}_+^*}$  such as

$$\psi_\lambda(x) := \frac{1}{\sqrt{\lambda}} e^{ix} \psi\left(\frac{x}{\lambda}\right) \quad \forall x \in \mathbb{R} \quad \implies \quad \widehat{\psi}_\lambda(\xi) = \sqrt{\lambda} \widehat{\psi}(\lambda(\xi-1)) \quad \forall \xi \in \mathbb{R},$$

and  $\psi$  satisfying  $\widehat{\psi}(\xi) = 0$  for  $|\xi| \geq \Lambda$  with  $\Lambda > 0$ . Note that for all  $\lambda \geq \Lambda$ ,  $\psi_\lambda$  satisfies Assumption  $W(m, r)$  when  $\psi$  satisfies Assumption  $W(m, r)$ . Now,  $\mathcal{I}_\lambda(a) := \int_{\mathbb{R}} |\widehat{\psi}_\lambda(u)|^2 f(u/a) du \rightarrow f(1/a) \|\psi\|_{\mathcal{L}^2}^2$  when  $\lambda \rightarrow \infty$  under weak hypothesis. Therefore for  $0 < \xi$ , let us define  $\widehat{f}_n^{(\lambda)}(\xi) := J_n^{(\lambda)}(1/\xi) / \|\psi\|_{\mathcal{L}^2}^2$ , where  $J_n^{(\lambda)}$  denotes  $J_n$  when  $\psi$  is replaced by  $\psi_\lambda$ , and

$$\widehat{f}_n^{(\lambda)}(\xi) := \frac{\xi}{\|\psi\|_{\mathcal{L}^2}^2} \frac{1}{n+1} \sum_{k=0}^n \left| \sum_{i=0}^{n-1} X(t_i) \int_{t_i}^{t_{i+1}} \psi_\lambda(\xi(t-c_k)) dt \right|^2,$$

with  $c_k = T_n^\rho + \frac{k}{n} (T_n - 2T_n^\rho)$ . Using an appropriated choice of a sequence  $(\psi_{\lambda_n})$ , one obtains:

**Proposition 4.1** *Under the same assumptions as in Theorem 3.1, with  $\psi$  satisfying  $\widehat{\psi}(\xi) = 0$  for  $|\xi| \geq \Lambda$  with  $\Lambda > 0$  and Assumption  $W(1, 5)$ , and if  $f$  is twice continuously differentiable on  $\mathbb{R}^*$ , then for all  $\xi > 0$ ,*

$$(14) \quad \sqrt{\frac{T_n}{\lambda_n}} (\widehat{f}_n^{(\lambda_n)}(\xi) - f(\xi)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, \frac{4\pi}{\xi} f^2(\xi) \frac{\int_{\mathbb{R}} |\widehat{\psi}(u)|^4 du}{\left(\int_{\mathbb{R}} |\widehat{\psi}(u)|^2 du\right)^2}\right),$$

if  $\delta_n = C_\delta n^{-d}$ ,  $\lambda_n = C_\lambda n^{d'}$ , where  $0 < \frac{1}{5}(1-d) < d' < \frac{1}{2}(1-d) < 1$  and

- if  $s = \infty$ , when  $d > (2+H)^{-1}$ ;
- if  $\max(2+2H, \frac{3}{2} + \frac{1}{2H}) \leq s < \infty$ , when  $(s-1 - (sH+2s-1)d) < d'(H+1)$ ;
- if  $2+2H \leq s < \max(2+2H, \frac{3}{2} + \frac{1}{2H})$ , when  $(s-1)(2-d(3H+2)) < d'(H+1)$ .

The following Table 1 summarizes the ‘‘optimal’’ choices of  $d'$  (in order to maximize the convergence rate of  $\widehat{f}_n$ ) following several cases.

	$H$ known $\delta_n$ fixed	$H$ known $\delta_n$ non-fixed	$H$ unknown $\delta_n$ fixed	$H$ unknown $\delta_n$ non-fixed
Choice of $d$	$d \left( > \frac{5s+4H-1}{5s(H+2)-6-H} \right)$	$\frac{5s+4H-1}{5s(H+2)-6-H} + \kappa$	$d \left( \geq \frac{1}{2} + \frac{1}{5s-3} \right)$	$\frac{1}{2} + \frac{1}{5s-3}$
Choice of $d'$	$\frac{1-d}{5} + \kappa$	$\frac{(s-1)(H+1)}{5s(H+2)-6-H}$	$\frac{1-d}{5} + \kappa$	$\frac{s-1}{2(5s-3)} + \kappa$
Convergence rate	$n^{\frac{2}{5}(1-d)-\frac{\kappa}{2}}$	$n^{\frac{2(s-1)(H+1)}{5s(H+2)-6-H}-\frac{\kappa}{2}}$	$n^{\frac{2}{5}(1-d)-\frac{\kappa}{2}}$	$n^{\frac{s-1}{5s-3}-\frac{\kappa}{2}}$

TABLE 1

Optimal choices of  $d'$  (and therefore  $(\lambda_n)$ ) and convergence rate of  $\widehat{f}_n$  (the case  $s = \infty$  is obtained as the limit of ratios) with  $0 < \kappa$  arbitrary small.

**5. Numerical experiments.** For the numerical applications, one has chosen:

1.  $\psi$  is chosen such as  $\widehat{\psi}(\xi) = \exp(-(|\xi| \cdot (5 - |\xi|))^{-1}) \mathbf{1}_{|\xi| \leq 5}(\xi)$  which satisfies Assumption W( $m, r$ ) for any  $(m, r)$  (and  $\widehat{\psi}(\xi) = 0$  for  $|\xi| \geq 5$ ).
2.  $\delta_n = n^{-0.6}$  for insuring the convergence of  $\widehat{f}_n^{(\lambda_n)}(\xi)$  for any  $H \in (0, 1)$  and  $s \geq 3$ .
3.  $\lambda_n = n^{d'}$  with  $1/6 < d' < 1/2$ . However, admissibility condition on wavelets  $(\psi_{\lambda_n})$  requires that  $n^{d'} \geq \Lambda = 5$ . Moreover, for removing the bias term,  $d'$  has to be chosen large enough following  $n$ . Thus, after numerous simulations, we have chosen  $d' = \log(15)/\log(n)$ .

5.1. *Estimation of the spectral density of a fractional Brownian motion observed at random times.* For a standard ( $\mathbb{E}X^2(1) = 1$ ) fBm with Hurst parameter  $H$ ,  $f(\xi) = C(H) |\xi|^{-2H-1} d\xi$  with  $C(H) = (H\Gamma(2H) \sin(\pi H))/\pi$ . Very large samples of fBm can be generated using the circulant matrix embedding method (see Bardet *et al.*, 2003) which is a very low time consuming method. Four different kind of random times are considered:

1. **(T1)**: non-random uniform sampling, such that  $L_k = 1$  for all  $k \in \mathbb{N}^*$ ;
2. **(T2)**: exponential random times, such that  $\mathbb{E}L_k = 1$  for all  $k \in \mathbb{N}^*$ ;
3. **(T3)**: random times such that for for all  $k \in \mathbb{N}^*$ , the cumulative distribution function of  $L_k$  is  $F_{L_k}(x) = (1 - x^{-4}) \mathbf{1}_{x \geq 1} \implies \mathbb{E}L_k^p < \infty$  for all  $p < 4$  and  $\mathbb{E}L_k^4 = \infty$ ;
4. **(T4)**: random times such that for for all  $k \in \mathbb{N}^*$ , the cumulative distribution function of  $L_k$  is  $F_{L_k}(x) = (1 - x^{-2}) \mathbf{1}_{x \geq 1} \implies \mathbb{E}L_k^p < \infty$  for all  $p < 2$  and  $\mathbb{E}L_k^2 = \infty$ .

An example of such estimation of the spectral density for  $H = 0.2$ ,  $N = 50000$  and random times T2 is presented in Figure (5.1). The results of simulations are also provided in the Table (5.1).

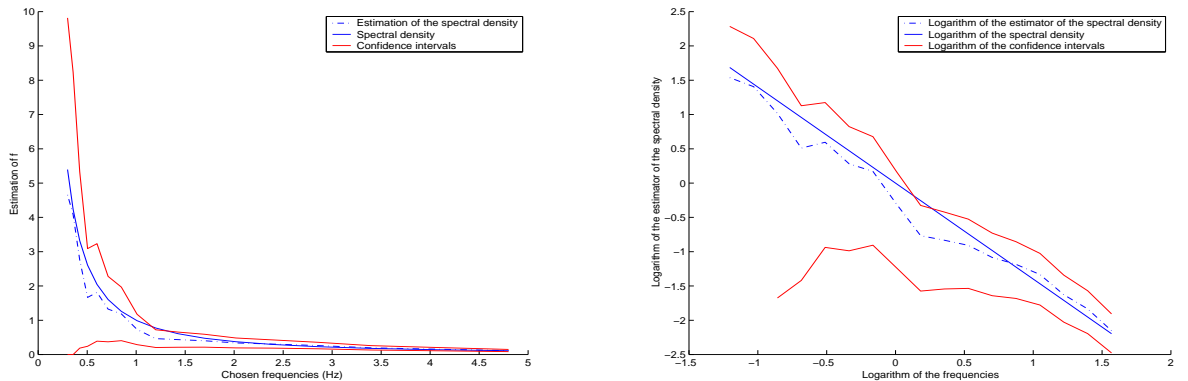


FIG 1. An example of the estimation of the spectral density (left) and its logarithm (right) of a FBM observed at exponential random times ( $T_2$ ) with confidence intervals ( $H = 0.2$ ,  $N = 50000$ ).

Comments on simulation results: 1. the larger  $N$  the more accurate the estimator of  $f$  except for case of random times T4 (which is a case not included in conditions of Proposition 4.1); 2. The results are similar for T1 and T2, a little less accurate for T3; 3. the smaller  $H$  the more accurate the estimator of  $f$ .

5.2. *Estimation of the spectral density of the stationary Ornstein-Uhlenbeck process.* Here, instead of FBM which is a process having stationary increments, we consider the stationary Ornstein-Uhlenbeck process which is a Gaussian stationary process with covariance  $r(t) := \exp(-\alpha|t|)$  and therefore with spectral density  $f(\xi) := \alpha(\pi(\alpha^2 + \xi^2))^{-1}$ . In such case, since this spectral density is an analytic function, there exists more accurate non-parametric estimator (see for instance, Ibragimov, 2004). However, to our knowledge, the case of paths observed at random times is not considered in this literature. The results of simulations are provided in the Table (5.2).

		$H = 0.2$	$H = 0.5$	$H = 0.8$	
$N = 10^3$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.47	0.65	0.77
		$\widehat{MISE}$ on $[0.3, 5]$	2.53	13.50	80.89
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.65	0.67	0.75
		$\widehat{MISE}$ on $[0.3, 5]$	3.64	10.65	39.85
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.42	0.72	1.20
		$\widehat{MISE}$ on $[0.3, 5]$	2.48	7.83	55.20
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	1.03	3.34	2.44
		$\widehat{MISE}$ on $[0.3, 5]$	6.07	84.05	144.40

		$H = 0.2$	$H = 0.5$	$H = 0.8$	
$N = 10^4$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.35	0.37	0.79
		$\widehat{MISE}$ on $[0.3, 5]$	0.95	3.90	57.19
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.45	0.47	0.29
		$\widehat{MISE}$ on $[0.3, 5]$	1.04	3.17	16.26
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.47	0.46	0.95
		$\widehat{MISE}$ on $[0.3, 5]$	1.20	4.91	26.6
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.61	0.61	1.74
		$\widehat{MISE}$ on $[0.3, 5]$	2.74	9.55	49.55

		$H = 0.2$	$H = 0.5$	$H = 0.8$	
$N = 5 \cdot 10^4$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.36	0.30	0.40
		$\widehat{MISE}$ on $[0.3, 5]$	0.81	2.60	10.77
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.21	0.22	0.31
		$\widehat{MISE}$ on $[0.3, 5]$	1.07	2.07	7.65
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.34	0.26	0.48
		$\widehat{MISE}$ on $[0.3, 5]$	0.74	3.17	13.3
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(1)$	0.40	0.56	2.59
		$\widehat{MISE}$ on $[0.3, 5]$	1.02	5.69	41.41

TABLE 2

Consistency of the estimator  $\widehat{f}_N$  in the case of paths of FBM observed at random times (50 independent replications are generated in each case).

Comments on simulation results: 1. the larger  $N$  the more accurate the estimator of  $f$  for all choice of random time; 2. The results are similar for T1, T2, T3 and a little less accurate for T4; 3. surprisingly, the case  $\alpha = 1$  is not clearly better than  $\alpha = 0.1$  despite the fact that the larger  $\alpha$  the less correlated the process.

## 6. Application to real data.

6.1. *Estimation of the spectral density of heart rates.* Heart inter-beats of several patients have been recorded during 24h (see an example in Figure 6.1). These data has been kindly furnished by professor Alain Chamoux and Gil Boudet (Faculty of Medicine, University of Auvergne, Clermont-Ferrand). We decompose these data in 3 temporal zones following the activity:

		$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$	
$N = 10^3$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.51 0.022	0.22 0.014	0.020 0.00067
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.30 0.010	0.30 0.024	0.021 0.0010
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.36 0.00052	0.23 0.015	0.018 0.00052
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.28 0.016	0.23 0.016	0.032 0.0045
<hr/>					
		$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$	
$N = 10^4$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.20 0.0033	0.18 0.0088	0.017 0.00031
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.14 0.0032	0.18 0.0092	0.019 0.00036
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.17 0.0027	0.18 0.011	0.016 0.00032
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.18 0.0058	0.13 0.0095	0.024 0.00037
<hr/>					
		$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$	
$N = 5 \cdot 10^4$	T1	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.14 0.0016	0.10 0.0045	0.012 0.00015
	T2	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.26 0.012	0.13 0.0055	0.011 0.00014
	T3	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.18 0.0023	0.14 0.0049	0.012 0.00017
	T4	$\sqrt{MSE}$ of $\widehat{f}_N(0.3)$ $MISE$ on $[0.3, 5]$	0.16 0.0084	0.16 0.034	0.017 0.00019

TABLE 3

Consistency of  $\widehat{f}_N$  in the case of paths of stationary Ornstein-Uhlenbeck process observed at random times (50 independent replications are generated in each case).

- Quiet activities ( $t \in [1, 28000]$  in seconds);
- Intensive activities ( $t \in [28000, 51400]$  in seconds);
- Sleep ( $t \in [60000, 83400]$  in seconds).

Applying the spectral density estimator on those 3 sub-data and plotting its log-log representation for frequencies in  $[0.02, 1]$  Hz, we observe that:

- in zone “Sleep” (see Figure 6.1), only one regression line could be computed for frequencies in  $[0.04, 0.5]$  Hz which is the usual spectral interval considered by specialists; in this zone  $\widehat{H} \simeq 0.99$ ;
- in zone “Quiet activities” (respectively “Intensive activities”), (see Figure 6.1), two regression lines could be drawn for frequencies in  $[0.04, 0.5]$  Hz, distinguishing the orthosympathic and the parasympathic spectral domains. Using an algorithm computing the “best” two regression lines (see for instance Bardet and Bertrand, 2007, for details), one obtains

that  $H \simeq 1.34$  (respectively  $H \simeq 1.44$ ) in the orthosympathic domain which is  $[0.04, 0.09]$  Hz (respectively  $[0.04, 0.11]$  Hz) and  $H \simeq 0.89$  (respectively  $H \simeq 0.79$ ) in the parasympathic domain which is  $[0.09, 0.5]$  Hz (respectively  $[0.11, 0.5]$  Hz).

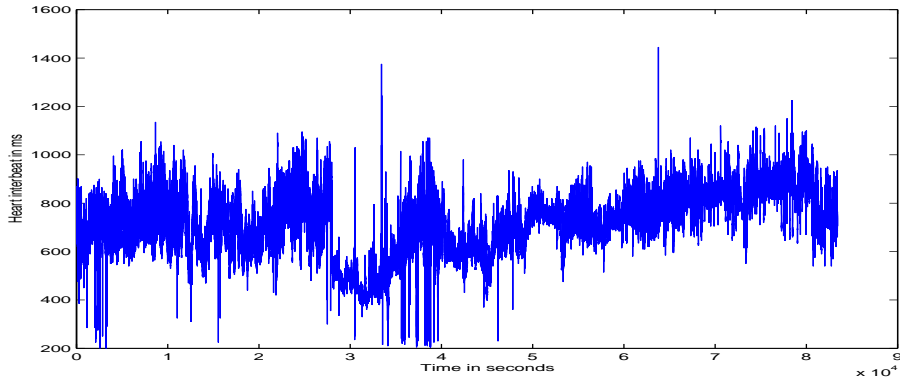


FIG 2. An example of heart inter-beats during 24h

6.2. *Estimation of the spectral density of log-return of a share.* One considers the price of share Total during a day at Paris (see Figure 6.2). These data has been kindly furnished by Cr dit Agricole Cheuvreux, CALYON (Paris). Applying the spectral density estimator and plotting its log-log representation, we observe that durations fit an exponential law with mean 11 seconds and that the spectral density is linear in for frequencies smaller than 0.008 Hz and has an erratic behavior at higher frequencies. The critical frequency corresponds to a time lag of 125 seconds and could be interpreted as the frontier between events and regularity. Remark that for high frequencies a Gaussian distribution is not appropriated.

## 7. Proofs.

7.1. *Proofs of useful lemmas and Proposition 2.1.* In the sequel, the following lemmas will be useful:

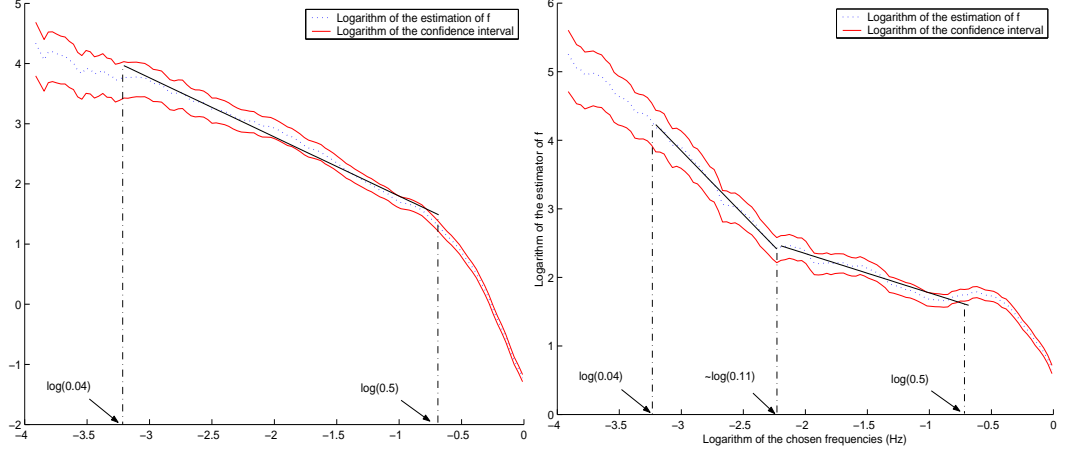


FIG 3. Log-log representation of the spectral density estimator during “Sleep” zone (left) and “Intensive activities” zone (right)

**Lemma 7.1** *Let  $X$  be a Gaussian process defined by (1) with a spectral density function  $f$  satisfying (2). Then there exists  $C_0 > 0$  such that*

$$(15) \quad |\mathbb{E}(X(t_1)X(t_2))| \leq C_0(1 + |t_1|)(1 + |t_2|) \quad \text{for all } (t_1, t_2) \in \mathbb{R}^2.$$

**Proof.** For all  $t \in \mathbb{R}$ , we have

$$\begin{aligned} \mathbb{E}X^2(t) &= \int_{\mathbb{R}} |e^{it\xi} - 1|^2 f(\xi) d\xi \leq 2 \int_0^1 |t\xi|^2 f(\xi) d\xi + 8 \int_1^\infty f(\xi) d\xi \\ &\leq (2t^2 + 8) \times \int_0^\infty (1 \wedge |\xi|^2) f(\xi) d\xi. \end{aligned}$$

This implies  $\mathbb{E}(X(t)^2) \leq C_0(1 + |t|^2)$  where  $C_0 = 4 \int_{\mathbb{R}} (1 \wedge |\xi|^2) f(\xi) d\xi$ . Then, by using Cauchy-Schwartz inequality, one deduces (15). ■

**Proof.** [of Proposition 2.1] Firstly, one can show that for all  $a > 0$  and  $b \in \mathbb{R}$ ,  $\mathbb{E}d_X^2(a, b) < \infty$ . This induces that  $d_X(a, b)$  is well defined. Indeed, one has

$$\begin{aligned} \mathbb{E}d_X^2(a, b) &= \frac{1}{a} \int_{\mathbb{R}} \int_{\mathbb{R}} \psi\left(\frac{t_1 - b}{a}\right) \psi\left(\frac{t_2 - b}{a}\right) \mathbb{E}(X(t_1)X(t_2)) dt_1 dt_2 \\ &\leq \frac{C_0}{a} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \psi\left(\frac{t_1 - b}{a}\right) \right| \left| \psi\left(\frac{t_2 - b}{a}\right) \right| (1 + |t_1|)(1 + |t_2|) dt_1 dt_2 \end{aligned}$$

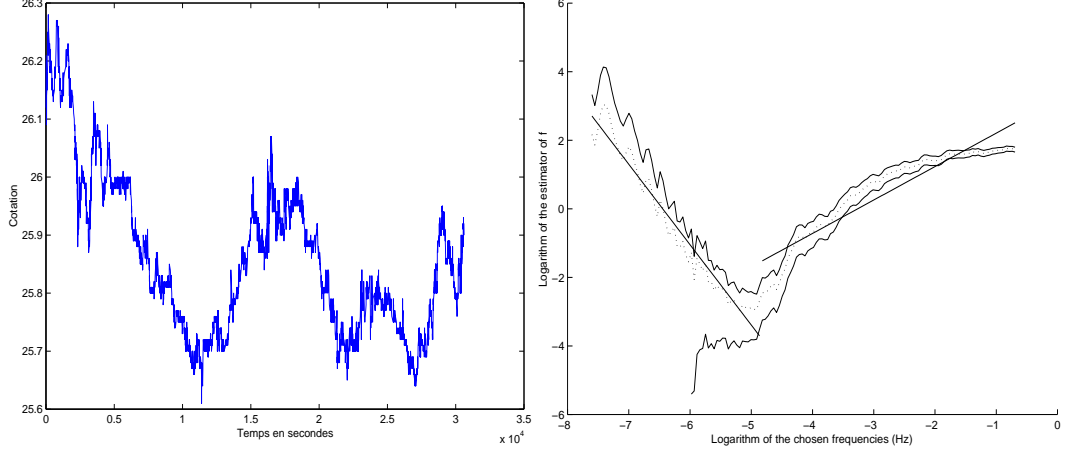


FIG 4. An example of quotation of a share during a day, i.e. 8.5 hours (left), and the log-log representation of the spectral density estimator (right)

$$\leq a C_0 \left( \int_{\mathbb{R}} |\psi(u)| (1 + |b| + |a u|) du \right)^2 < \infty,$$

where we have used successively the bound (15), the change of variable  $u = (t - b)/a$  and the first condition of Assumption  $W(1, 0)$ . Next, one turns to the proof of the formula (6). It is obvious that  $\int_{\mathbb{R}} \left| \psi\left(\frac{t-b}{a}\right) \right| dt < \infty$  and  $\int_{\mathbb{R}} |(e^{it\xi} - 1) f^{1/2}(\xi)| dW(\xi) < \infty$  since the condition (2) holds. From the Fubini Theorem for stochastic integral (see [16, Lemma 4.1, p. 116]),

$$\begin{aligned} d_X(a, b) &= \int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right) X(t) dt = \int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right) \left[ \int_{\mathbb{R}} (e^{it\xi} - 1) f^{1/2}(\xi) dW(\xi) \right] dt \\ &= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} (e^{it\xi} - 1) \psi\left(\frac{t-b}{a}\right) dt \right] f^{1/2}(\xi) dW(\xi). \end{aligned}$$

But  $\int_{\mathbb{R}} (e^{it\xi} - 1) \psi\left(\frac{t-b}{a}\right) dt = a e^{ib\xi} \int_{\mathbb{R}} e^{iau\xi} \psi(u) du = a e^{ib\xi} \widehat{\psi}(a\xi)$  for all  $(a, b) \in ]0, \infty[ \times \mathbb{R}$ , which implies (6) and  $d_X(a, b)$  is a Gaussian centered r.v. with variance  $\mathcal{I}_1(a)$ . Moreover, for all  $a > 0$  and  $(b_1, b_2) \in \mathbb{R}^2$ ,

$$(16) \quad \mathbb{E}(d_X(a, b_1) \overline{d_X(a, b_2)}) = a \int_{\mathbb{R}} e^{i(b_1 - b_2)\xi} |\widehat{\psi}(a\xi)|^2 f(\xi) d\xi.$$



Thus for a given  $a > 0$ ,  $\mathbb{E}(d_X(a, b_1) d_X(a, b_2))$  is only depending on  $(b_1 - b_2)$  which induces that  $(d_X(a, b))_{b \in \mathbb{R}}$  is a stationary process.  $\blacksquare$

From formula (6), it is clear that  $\forall (a_1, a_2) \in [a_{min}, a_{max}]^2, \forall b_1, b_2, \theta \in \mathbb{R}$ ,

$$\begin{aligned} \mathbb{E}(d_X(a_1, b_1), \overline{d_X(a_2, b_2)}) &= \sqrt{a_1 a_2} \cdot \gamma(b_2 - b_1, a_1, a_2) \\ (17) \quad \gamma(\theta, a_1, a_2) &:= \int_{\mathbb{R}} e^{i\theta\xi} \widehat{\psi}(a_1\xi) \overline{\widehat{\psi}(a_2\xi)} f(\xi) d\xi. \end{aligned}$$

**Lemma 7.2** *Let  $\psi$  verify Assumption W(1,2) and  $f$  be an even function satisfying (2) and Assumption F.*

1. *There exists  $C > 0$  depending on  $\psi$ ,  $f$  and  $a_{max}$  such that  $\forall (a_1, a_2) \in [a_{min}, a_{max}]^2, |\gamma(\theta, a_1, a_2)| < C(1 \wedge |\theta|^{-1})$  for all  $\theta \in \mathbb{R}$ .*
2. *The function  $\gamma$  is derivable with respect to  $\theta$  and there exists  $C' > 0$  depending on  $\psi$ ,  $f$ ,  $a_{min}$  and  $a_{max}$  such that  $\forall (a_1, a_2) \in [a_{min}, a_{max}]^2, |\gamma'(\theta, a_1, a_2)| := \left| \frac{\partial \gamma}{\partial \theta}(\theta, a_1, a_2) \right| \leq C'(1 \wedge |\theta|^{-1})$  for all  $\theta \in \mathbb{R}$ .*

**Proof. [of Lemma 7.2]** From Assumption W(1, 1/2),  $\exists c > 0$  such that

$$(18) \quad |\widehat{\psi}(\xi)| \leq c(1 \wedge |\xi|^2) \quad \text{for all } \xi \in \mathbb{R}.$$

Indeed, from one hand,  $|\widehat{\psi}(\xi)| \leq \|\psi\|_{L^1(\mathbb{R})} < \infty$ . From the other hand,  $\psi \in W(1, 1/2)$  implies that  $\widehat{\psi}$  is twice continuously differentiable and  $\widehat{\psi}(0) = \widehat{\psi}'(0) = 0$ . From Taylor-Lagrange Formula, for all  $\xi \in \mathbb{R}^*$ , there exists  $\xi_0 \in \mathbb{R}$  with  $|\xi_0| \leq |\xi|$  such that  $\widehat{\psi}(\xi) = \frac{1}{2} \cdot \xi^2 \times \widehat{\psi}''(\xi_0)$ . This induces  $|\widehat{\psi}(\xi)| \leq \frac{1}{2} |\xi|^2 \times \left( \int_{\mathbb{R}} t^2 \cdot |\psi(t)| dt \right)$  providing the second bound of (18).

To show the first item, inequality (18) implies that

$$\begin{aligned} \int_{\mathbb{R}} \left| \widehat{\psi}(a\xi) \right|^2 f(\xi) d\xi &\leq c^2 \left( \int_{|\xi| \leq 1} |a\xi|^4 f(\xi) d\xi + \int_{|\xi| > 1} f(\xi) d\xi \right) \\ &\leq c^2(1 \vee a^4) \int_{\mathbb{R}} (1 \wedge \xi^2) f(\xi) d\xi < \infty, \end{aligned}$$

with  $C > 0$  depending on  $\psi$ ,  $f$  and  $a_{max}$ . From Cauchy-Schwarz Inequality,

$$\gamma(\theta, a_1, a_2) \leq c^2(1 \vee a_1^2)(1 \vee a_2^2) \int_{\mathbb{R}} (1 \wedge \xi^2) f(\xi) d\xi.$$

Moreover, with  $f(\omega_k^+)$  and  $f(\omega_k^-)$  the right and left limit of  $f$  at  $\omega_k$ , for all  $1 \leq k \leq K-1$ ,  $\theta \in \mathbb{R}^*$  and  $(a_1, a_2) \in [a_{min}, a_{max}]^2$ ,

$$\begin{aligned} & \int_{\omega_k}^{\omega_{k+1}} e^{i\theta\xi} \overline{\widehat{\psi}}(a_1\xi) \widehat{\psi}(a_2\xi) f(\xi) dx \\ &= \frac{1}{i\theta} \left( e^{i\theta\omega_{k+1}} f(\omega_{k+1}^-) \overline{\widehat{\psi}}(a_1\omega_{k+1}) \widehat{\psi}(a_2\omega_{k+1}) - e^{i\theta\omega_k} f(\omega_k^+) \overline{\widehat{\psi}}(a_1\omega_k) \widehat{\psi}(a_2\omega_k) \right) \\ & \quad - \int_{\omega_k}^{\omega_{k+1}} \frac{e^{i\theta\xi}}{i\theta} \left[ f'(\xi) \overline{\widehat{\psi}}(a_1\xi) \widehat{\psi}(a_2\xi) + f(\xi) \left( a_1 \overline{\widehat{\psi}'(a_1\xi)} \widehat{\psi}(a_2\xi) + a_2 \overline{\widehat{\psi}(a_1\xi)} \widehat{\psi}'(a_2\xi) \right) \right] d\xi. \end{aligned}$$

The same result remains in force for  $k = 0$  and  $k = K$ . Indeed, by using (18) combined with Assumption F, one deduces that  $\forall \theta \in \mathbb{R}$ ,  $\forall (a_1, a_2) \in [a_{min}, a_{max}]^2$ ,

$$\lim_{\xi \rightarrow 0} e^{i\theta\xi} f(\xi) \overline{\widehat{\psi}}(a_1\xi) \widehat{\psi}(a_2\xi) = 0 \quad \text{and} \quad \lim_{\xi \rightarrow \infty} e^{i\theta\xi} f(\xi) \overline{\widehat{\psi}}(a_1\xi) \widehat{\psi}(a_2\xi) = 0.$$

Thus, by summing up and using Assumption F,  $\forall \theta \in \mathbb{R}$ ,  $\forall (a_1, a_2) \in [a_{min}, a_{max}]^2$ ,

$$\begin{aligned} & \gamma(\theta, a_1, a_2) \\ &= -\frac{1}{i\theta} \sum_{k=1}^K \left( e^{i\theta\omega_k} \overline{\widehat{\psi}}(a_1\omega_k) \widehat{\psi}(a_2\omega_k) - e^{-i\theta\omega_k} \overline{\widehat{\psi}}(-a_1\omega_k) \widehat{\psi}(-a_2\omega_k) \right) (f(\omega_k^+) - f(\omega_k^-)) \\ & \quad - \frac{1}{i\theta} \int_{\mathbb{R}} e^{i\theta\xi} \left[ f'(\xi) \overline{\widehat{\psi}}(a_1\xi) \widehat{\psi}(a_2\xi) + f(\xi) \left( a_1 \overline{\widehat{\psi}'(a_1\xi)} \widehat{\psi}(a_2\xi) + a_2 \overline{\widehat{\psi}(a_1\xi)} \widehat{\psi}'(a_2\xi) \right) \right] d\xi \end{aligned}$$

since the integral of the r.h.s. of the previous equality is well defined. Then,

$$\begin{aligned} |\gamma(\theta, a_1, a_2)| &\leq \frac{1}{|\theta|} \left( 2c \sum_{k=1}^K |f(\omega_k^+) - f(\omega_k^-)| \right. \\ & \quad \left. + \int_{\mathbb{R}} \left[ |f'(\xi) \overline{\widehat{\psi}}(a_1\xi) \widehat{\psi}(a_2\xi)| + |f(\xi)| (|a_1| |\overline{\widehat{\psi}'(a_1\xi)} \widehat{\psi}(a_2\xi)| + |a_2| |\overline{\widehat{\psi}(a_1\xi)} \widehat{\psi}'(a_2\xi)|) \right] d\xi \right). \end{aligned}$$

It remains to show the convergence of the previous integral. Using the same trick as in Formula (18), under Assumption W(1, 1/2),  $|\widehat{\psi}'(\xi)| \leq c' (1 \wedge |\xi|)$  with  $c'$  depending on  $\psi$  and  $a_{max}$ . So, for all  $(a_1, a_2) \in [a_{min}, a_{max}]^2$

$$\int_{\mathbb{R}} \left[ |f'(\xi) \overline{\widehat{\psi}}(a_1\xi) \widehat{\psi}(a_2\xi)| + |f(\xi)| (|a_1| |\overline{\widehat{\psi}'(a_1\xi)} \widehat{\psi}(a_2\xi)| + |a_2| |\overline{\widehat{\psi}(a_1\xi)} \widehat{\psi}'(a_2\xi)|) \right] d\xi$$

$$\begin{aligned}
&\leq c(c+c')|a_1a_2| \int_{|\xi|\leq 1} |a_1a_2f'(\xi)\xi^4| + (|a_1|+|a_2|)|f(\xi)\xi^3| d\xi \\
&\quad + c^2 \int_{|\xi|>1} |f'(\xi)| + (|a_1|+|a_2|)|f(\xi)| d\xi \\
&\leq C \int_{\mathbb{R}} \left[ (1 \wedge |\xi|^4) \cdot |f'(\xi)| + (1 \wedge |\xi|^3) \cdot |f(\xi)| \right] d\xi < \infty
\end{aligned}$$

where  $C > 0$  and this completes the proof of the first item.

Eventually, one proves the second item. The differentiability is obvious and

$$\varphi'(\theta, a_1, a_2) = i \int_{\mathbb{R}} e^{i\theta\xi} \xi \overline{\widehat{\psi}(a_1\xi)} \widehat{\psi}(a_2\xi) f(\xi) d\xi.$$

Assumption W(1, 1/2) implies that  $\forall a \in [a_{min}, a_{max}]$ ,  $|a\xi|^{1/2}|\widehat{\psi}(a\xi)| \leq C_\psi$  for all  $\xi \in \mathbb{R}$ . Combined with (18), this induces that  $\forall a \in [a_{min}, a_{max}]$ ,  $\theta \in \mathbb{R}$ ,

$$\begin{aligned}
|\varphi'(\theta, a_1, a_2)| &\leq \int_{\mathbb{R}} |\xi| |\widehat{\psi}(a_1\xi)| |\widehat{\psi}(a_2\xi)| f(\xi) d\xi \\
&\leq c^2(a_1 a_2)^2 \int_{|\xi|\leq 1} |\xi|^5 f(\xi) d\xi + \frac{C_\psi^2}{\sqrt{a_1 a_2}} \int_{|\xi|>1} f(\xi) d\xi \\
&\leq C,
\end{aligned}$$

with  $C > 0$  only depending on  $\psi$ ,  $f$ ,  $a_{min}$  and  $a_{max}$ . Using the same arguments as for the first item,  $\forall \theta \in \mathbb{R}^*$ ,  $(a_1, a_2) \in [a_{min}, a_{max}]^2$ ,

$$\begin{aligned}
\gamma'(\theta, a_1, a_2) &= -\frac{1}{\theta} \sum_{k=1}^K \left( e^{i\theta\omega_k} \omega_k \overline{\widehat{\psi}(a_1\omega_k)} \widehat{\psi}(a_2\omega_k) + \right. \\
&\quad \left. + e^{-i\theta\omega_k} \omega_k \overline{\widehat{\psi}(-a_1\omega_k)} \widehat{\psi}(-a_2\omega_k) \right) (f(\omega_k^+) - f(\omega_k^-)) \\
&\quad - \frac{1}{\theta} \int_{\mathbb{R}} e^{i\theta\xi} \left[ f(\xi) \overline{\widehat{\psi}(a_1\xi)} \widehat{\psi}(a_2\xi) + \xi f'(\xi) \overline{\widehat{\psi}(a_1\xi)} \widehat{\psi}(a_2\xi) + \right. \\
&\quad \left. + \xi f(\xi) \left( (a_1 \overline{\widehat{\psi}'(a_1\xi)} \widehat{\psi}(a_2\xi) + a_2 \overline{\widehat{\psi}(a_1\xi)} \widehat{\psi}'(a_2\xi)) \right) \right] d\xi
\end{aligned}$$

and therefore  $|\gamma'(\theta, a_1, a_2)| \leq \frac{C}{|\theta|}$ , with  $C > 0$  only depending on  $\psi$ ,  $a_{min}$  and  $a_{max}$ . ■

7.2. *Proofs of Proposition 7.1 and 7.2.* Since  $\mathcal{I}_1(a)$  is obviously defined from  $|d_X(a, b)|^2$ , we begin with the study of

$$(19) \quad I_n(a) := \frac{1}{n+1} \sum_{k=0}^n |d_X(a, c_k)|^2, \quad \text{for } a > 0 \text{ and } n \in \mathbb{N}^*.$$

For  $n \in \mathbb{N}^*$  and  $a \in [a_{\min}, a_{\max}]$ , define also:

$$(20) \quad S_n^2(a) := \frac{2a^2}{(n+1)^2} \sum_{k=0}^n \sum_{\ell=0}^n \left| \int_{\mathbb{R}} e^{i(\mathbb{E}(c_k - c_\ell))\xi} |\widehat{\psi}(a\xi)|^2 f(\xi) d\xi \right|^2.$$

**Proposition 7.1** *Let  $X$  be a Gaussian process defined by (1) or (3) with a spectral density  $f$  satisfying (2),  $\psi$  satisfy Assumption  $W(1,1)$ . Then if  $(c_k)_k$  is a family of real numbers such that  $c_1 < c_2 < \dots < c_n$ ,  $n \max_{1 \leq k \leq n} \{c_{k+1} - c_k\} \xrightarrow{n \rightarrow \infty} \infty$  and  $\exists C'' > 0$  satisfying  $\forall n \in \mathbb{N}^*$*

$$\max_{1 \leq k \leq n} \{c_{k+1} - c_k\} \leq C'' \min_{1 \leq k \leq n} \{c_{k+1} - c_k\} < \infty$$

then  $\forall a \in [a_{\min}, a_{\max}]$ ,

$$(21) \quad \frac{1}{S_n(a)} (I_n(a) - \mathcal{I}_1(a)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1).$$

Moreover, there exist  $0 < C_m < C_M$  not depending on  $n$  such that  $\forall n \in \mathbb{N}^*$ ,

$$(22) \quad C_m \leq S_n(a) (n \max_{1 \leq k \leq n} \{c_{k+1} - c_k\})^{1/2} \leq C_M.$$

The proof of Proposition 7.1 relies on Lemma 7.2 and the following Lemma which is a Lindeberg CLT (see a proof in Istas and Lang, 1997):

**Lemma 7.3** *Let  $(Y_{N,i})_{1 \leq i \leq N, N \in \mathbb{N}^*}$  be a triangular array of zero-mean Gaussian r.v. Let  $\beta_N := \max_{1 \leq i \leq N} \sum_{j=1}^N |\text{cov}(Y_{N,i}, Y_{N,j})|$  and  $V_N := \sum_{i=1}^N Y_{N,i}^2$ ,  $S_N^2 := \text{var}(V_N)$ . If  $\lim_{N \rightarrow \infty} \frac{\beta_N}{S_N} = 0$ , then  $S_N^{-1}(V_N - \mathbb{E}(V_N))$  converges weakly to a standard Gaussian random variable.*

**Proof. [of Proposition 7.1]**

Consider  $Y_{n,i} = (n+1)^{-1/2} d_X(a, c_i)$  for  $i = 0, \dots, n$  and

$$\begin{cases} \beta_n = (n+1)^{-1} \max_{1 \leq i \leq n} \left\{ \sum_{j=0}^n |\text{cov}(d_X(a, c_i), d_X(a, c_j))| \right\}, \\ S_n^2 = (n+1)^{-2} \sum_{i=0}^n \sum_{j=0}^n \text{cov}(d_X^2(a, c_i), d_X^2(a, c_j)) \end{cases}.$$

But, by using Formula (17),  $\forall (a, a_1, a_2) \in (0, \infty]^3$ ,  $(b_1, b_2) \in \mathbb{R}^2$

$$\begin{aligned} \text{cov}(d_X(a, b_1), d_X(a, b_2)) &= a \gamma(b_1 - b_2, a, a) \\ \text{cov}(d_X^2(a_1, b_1), d_X^2(a_2, b_2)) &= 2(a_1 a_2) \gamma^2(b_1 - b_2, a_1, a_2), \end{aligned}$$

since variables  $d_X(a, b)$  are zero-mean Gaussian r.v. Therefore,

$$\begin{cases} \beta_N = a(n+1)^{-1} \max_{0 \leq i \leq n} \left\{ \sum_{j=0}^n |\gamma(c_i - c_j, a, a)| \right\} \\ S_n^2 = 2a^2(n+1)^{-2} \sum_{i=0}^n \sum_{j=0}^n \gamma^2(c_i - c_j, a, a) \end{cases}.$$

Let  $p$  and  $q$  be such that  $1/p + 1/q = 1$  with  $(p, q) \in (1, \infty)^2$ . Then the Hölder Inequality implies that

$$\beta_n \leq C a \cdot n^{1/q-1} \times \max_{0 \leq i \leq n} \left\{ \left( \sum_{j=0}^n |\gamma(c_i - c_j, a, a)|^p \right)^{1/p} \right\}.$$

Lemma 7.2 i) implies that for every  $1 \leq i \leq n$ , for  $n$  large enough,

$$\begin{aligned} \sum_{j=0}^n |\gamma(c_i - c_j, a, a)|^p &\leq C \left( \#\{0 \leq j \leq n, |c_i - c_j| \leq 1\} + \sum_{j=0; |c_i - c_j| > 1}^n |c_i - c_j|^{-p} \right) \\ &\leq C \left( \frac{2}{\min_{0 \leq k \leq n-1} |c_{k+1} - c_k|} + \sum_{j=0; |c_i - c_j| > 1}^n \left[ |i - j| \min_{0 \leq k \leq n-1} |c_{k+1} - c_k| \right]^{-p} \right) \\ &\stackrel{(23)}{\leq} 2C \left( \min_{0 \leq k \leq n-1} |c_{k+1} - c_k| \right)^{-1} \left( 1 + \left( \min_{0 \leq k \leq n-1} |c_{k+1} - c_k| \right)^{1-p} \sum_{\ell \geq (\min_{0 \leq k \leq n-1} |c_{k+1} - c_k|)^{-1}} |\ell|^{-p} \right). \end{aligned}$$

Since  $p > 1$ ,  $\sum_{\ell=1}^{\infty} |\ell|^{-p} < \infty$  is finite and thus

$$\sum_{\ell \geq (\min_{0 \leq k \leq n-1} |c_{k+1} - c_k|)^{-1}} |\ell|^{-p} \leq \frac{1}{p-1} \left( \min_{0 \leq k \leq n-1} |c_{k+1} - c_k| \right)^{p-1}.$$

Therefore,

$$(24) \quad \beta_n \leq C a \cdot \left\{ n \times \min_{0 \leq k \leq n-1} |c_{k+1} - c_k| \right\}^{-1/p},$$

with  $C > 0$  depending only on  $\psi$ ,  $a_{min}$ ,  $a_{max}$  and  $p$ . Now, a lower bound for  $S_n^2$  is required.  $\forall a \in [a_{min}, a_{max}]$ ,  $\theta \in \mathbb{R} \mapsto \gamma(\theta, a, a)$  is a continuous map

and  $\gamma(0, a, a) = \int_{\mathbb{R}} |\widehat{\psi}(a\xi)|^2 f(\xi) d\xi > 0$ . Therefore, for all  $a \in [a_{min}, a_{max}]$ , there exists  $\theta_a > 0$  such that  $\gamma(\theta, a, a) \geq \frac{1}{2} \cdot \gamma(0, a, a)$  when  $|\theta| \leq \theta_a$ . Then,

$$\begin{aligned}
 S_n^2 &\geq C'_1 a^2 n^{-2} \gamma^2(0, a, a) \# \{0 \leq i, j \leq n, |c_i - c_j| \leq \theta_a\} \\
 &\geq C'_1 a^2 n^{-2} \gamma^2(0, a, a) \# \left\{0 \leq i, j \leq n, |i - j| \max_{0 \leq k \leq n-1} |c_{k+1} - c_k| \leq \theta_a\right\} \\
 (25) \quad &\geq C'_1 a^2 \gamma^2(0, a, a) \theta_a \left(n \max_{0 \leq k \leq n-1} |c_{k+1} - c_k|\right)^{-1}.
 \end{aligned}$$

Thus, for  $n$  large enough, from (25) and (24),

$$\frac{\beta_n}{S_n} \leq C \cdot n^{1/2-1/p} \left(\max_{0 \leq k \leq n-1} |c_{k+1} - c_k|\right)^{1/2} \left(\min_{0 \leq k \leq n-1} |c_{k+1} - c_k|\right)^{-1/p}.$$

Therefore  $\beta_n/S_n \leq C \left(n \max_{0 \leq k \leq n-1} |c_{k+1} - c_k|\right)^{1/2-1/p}$  with  $C > 0$ . Next for any  $p \in (1, 2)$ ,  $\lim_{n \rightarrow \infty} \left(n \max_{0 \leq k \leq n-1} |c_{k+1} - c_k|\right)^{1/2-1/p} = 0$ , thus,  $\lim_{n \rightarrow \infty} \beta_n/S_n = 0$  and the assumptions of Lemma 7.3 are fulfilled.

Finally, (25) and Assumptions imply  $S_n(a)^2 \geq C_M \left(n \max_{0 \leq k \leq n-1} |c_{k+1} - c_k|\right)^{-1}$  with  $C_M > 0$ . Moreover, using the bound (23) for  $p = 2$ ,

$$\begin{aligned}
 &\sum_{j=0}^n \gamma^2(c_i - c_j, a, a) \leq C \left(\min_{0 \leq k \leq n-1} |c_{k+1} - c_k|\right)^{-1} \\
 \implies S_n^2 &\leq C' a^2 n^{-2} \sum_{i=0}^n \sum_{j=0}^n \gamma^2(c_i - c_j, a, a) \leq C_m \frac{1}{n} \left(\max_{0 \leq k \leq n-1} |c_{k+1} - c_k|\right)^{-1}.
 \end{aligned}$$

Therefore, inequalities (22) are proved. ■

**Proposition 7.2** *Let  $X$  be a Gaussian process defined by (1) or (3) with a spectral density  $f$  satisfying (2),  $\psi$  satisfy Assumption  $W(1, 1)$ . Then if  $(c_k)_k$  is a family of r.v. independent to  $\mathcal{F}_X$  such that  $c_k = c_1 + \frac{k}{n}(c_n - c_0)$ , with  $n^{-1} \mathbb{E}(c_n - c_0) \xrightarrow{n \rightarrow \infty} 0$  and  $\text{var}(c_n - c_0) \xrightarrow{n \rightarrow \infty} 0$ , then (21) holds with*

$$(26) \quad \lim_{n \rightarrow \infty} (\mathbb{E}(c_n - c_0)) S_n^2(a) = 4\pi a^2 \int_{\mathbb{R}} |\widehat{\psi}(az)|^4 f^2(z) dz.$$

**Remark 7.1** For  $(c_k)_k$  satisfying (10), under Assumption S(2), Proposition 7.2 holds when  $n^{1/2} \delta_n \xrightarrow{n \rightarrow \infty} 0$  because  $\mathbb{E}|T_n - \mathbb{E}T_n|^2 \leq n \delta_n^2 \max_{1 \leq k \leq n} \mathbb{E}L_k$ .

**Proof. [of Proposition 7.2]**

$(c_k)$  is a sequence of r.v. independent to  $\mathcal{F}_X$ . Therefore,  $(d_X(a, c_k))_k$  as the same distribution than  $(d_X(a, c_k - c_0))_k$  (stationarity of the sequence), and we can only consider here the case:  $c_k = k\tau_n/n$  with  $\tau_n := c_n - c_0$ . Define

$$I'_n(a) := \frac{1}{n} \sum_{k=0}^n d_X^2(a, \mathbb{E}c_k).$$

It is clear that  $(\mathbb{E}c_k)_{1 \leq k \leq n}$  is a deterministic sequence and therefore

$$(27) \quad \frac{I'_n(a) - \mathcal{I}_1(a)}{S_n(a)} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1).$$

Nowadays, one has to check that the error  $I'_n(a) - I_n(a)$  is negligible before  $S_n(a)$  in norm  $L^2(\Omega)$ . But

$$S_n(a) \geq C_M \cdot (n \max_{0 \leq k \leq n-1} |\mathbb{E}(c_{k+1} - c_k)|)^{-1/2} \geq C \times (\mathbb{E}\tau_n)^{-1/2}.$$

Therefore, it suffices to prove that  $\lim_{n \rightarrow \infty} \mathbb{E}\tau_n \times \mathbb{E}[(I'_n(a) - I_n(a))^2] = 0$ . Since the r.v.  $c_k$  are independent on  $\mathcal{F}_X$ , one gets

$$\begin{aligned} \mathbb{E}[(I'_n(a) - I_n(a))^2] &= \mathbb{E}[\mathbb{E}[(I'_n(a) - I_n(a))^2 \mid \mathcal{F}_X]] \\ &= \frac{1}{(n+1)^2} \sum_{k, k'=0}^n \mathbb{E}[\mathbb{E}[(d_X^2(a, \mathbb{E}c_k) - d_X^2(a, c_k))(d_X^2(a, \mathbb{E}c_{k'}) - d_X^2(a, c_{k'})) \mid \mathcal{F}_X]] \\ &= \frac{2a^2}{(n+1)^2} \sum_{k, k'=0}^n \mathbb{E}[\gamma^2(\mathbb{E}c_k - \mathbb{E}c_{k'}, a, a) - \gamma^2(c_k - c_{k'}, a, a) \\ &\quad - \gamma^2(c_k - \mathbb{E}c_{k'}, a, a) + \gamma^2(c_k - c_{k'}, a, a)]. \end{aligned}$$

Next, from Taylor expansions,

$$\begin{aligned} \gamma^2(\mathbb{E}c_k - c_{k'}, a, a) &= \gamma^2(\mathbb{E}c_k - \mathbb{E}c_{k'}, a, a) + 2(\mathbb{E}c_{k'} - c_{k'}) \times \dots \\ &\int_0^1 \gamma(\mathbb{E}c_k - \mathbb{E}c_{k'} + \lambda(\mathbb{E}c_{k'} - c_{k'}), a, a) \gamma'(\mathbb{E}c_k - \mathbb{E}c_{k'} + \lambda(\mathbb{E}c_{k'} - c_{k'}), a, a) d\lambda \\ \gamma^2(c_k - \mathbb{E}c_{k'}, a, a) &= \gamma^2(c_k - c_{k'}, a, a) + 2(c_{k'} - \mathbb{E}c_{k'}) \times \dots \end{aligned}$$

$$\int_0^1 \gamma(c_k - c_{k'} + \lambda(\mathbb{E}c_{k'} - c_{k'}), a, a) \gamma'(c_k - c_{k'} + \lambda(\mathbb{E}c_{k'} - c_{k'}), a, a) d\lambda.$$

From Lemma 7.2,  $\exists C > 0$  such that  $|\gamma(\theta, a, a)\gamma'(\theta, a, a)| \leq C \times (1 \wedge \theta^{-2})$  for all  $\theta \in \mathbb{R}$ . One can deduce that

$$\begin{aligned} & \left| \gamma^2(\mathbb{E}c_k - \mathbb{E}c_{k'}, a, a) - \gamma^2(\mathbb{E}c_k - c_{k'}, a, a) - \gamma^2(c_k - \mathbb{E}c_{k'}, a, a) + \gamma^2(c_k - c_{k'}, a, a) \right| \\ & \leq C |c_{k'} - \mathbb{E}c_{k'}| \times \int_0^1 (1 \wedge \theta_{1,k,k'}^{-2}(\lambda)) + (1 \wedge \theta_{2,k,k'}^{-2}(\lambda)) d\lambda \end{aligned}$$

with  $\theta_{1,k,k'}(\lambda) = \mathbb{E}(c_k - c_{k'}) + \lambda(\mathbb{E}c_{k'} - c_{k'})$  and  $\theta_{2,k,k'}(\lambda) = c_k - c_{k'} + \lambda(\mathbb{E}c_{k'} - c_{k'})$ . Then,

$$(28) \quad \mathbb{E}[(I'_n(a) - I_n(a))^2] \leq 2a^2 \times (\mathfrak{E}\mathfrak{r}_1 + \mathfrak{E}\mathfrak{r}_2),$$

where, for  $i = 1, 2$ ,  $\mathfrak{E}\mathfrak{r}_i := \int_0^1 \mathbb{E} \left[ \frac{1}{(n+1)^2} \sum_{k,k'=0}^n |c_{k'} - \mathbb{E}c_{k'}| \times (1 \wedge \theta_{i,k,k'}^{-2}(\lambda)) \right] d\lambda$ .

Thus  $\theta_{1,k,k'}(\lambda) = \delta'_n((k - k') - \lambda k' z_n)$  with  $\delta'_n := \frac{\mathbb{E}\tau_n}{n}$  and  $z_n := \frac{\tau_n - \mathbb{E}\tau_n}{\mathbb{E}\tau_n}$ .

Then, using  $\delta'_n \xrightarrow[n \rightarrow \infty]{} 0$ , for  $n$  large enough,

$$\begin{aligned} \mathfrak{E}\mathfrak{r}_1 &= \int_0^1 \mathbb{E} \left[ \frac{C}{(n+1)^2} \sum_{k,k'=0}^n |(k' \delta'_n) z_n| \times (1 \wedge [\delta'_n((k - k') - \lambda k' z_n)]^{-2}) \right] d\lambda \\ &\leq \int_0^1 \mathbb{E} \left[ \frac{C}{(\mathbb{E}\tau_n)^2} \int_0^{\mathbb{E}\tau_n} \int_0^{\mathbb{E}\tau_n} dx dy |y z_n| \times (1 \wedge [(x - y) - \lambda y z_n]^{-2}) \right] d\lambda. \end{aligned}$$

But, for all  $\lambda \in (0, 1)$ , one has

$$\begin{aligned} & \frac{1}{(\mathbb{E}\tau_n)^2} \int_0^{\mathbb{E}\tau_n} \int_0^{\mathbb{E}\tau_n} |y| \times (1 \wedge [(x - y) - \lambda y z_n]^{-2}) dx dy \\ &= \mathbb{E}\tau_n \int_0^1 \int_0^1 |v| \times (1 \wedge (\mathbb{E}\tau_n)^{-2} [(u - v) - \lambda v z_n]^{-2}) du dv \\ &\leq 2\mathbb{E}\tau_n \int_0^2 \int_0^\infty (1 \wedge (\mathbb{E}\tau_n)^{-2} s^{-2}) ds dt \leq 4. \end{aligned}$$

Therefore  $\mathfrak{E}\mathfrak{r}_1 \leq 4\mathbb{E}|z_n|$ . Now, using the same method for  $\mathfrak{E}\mathfrak{r}_2$ , one obtains,

$$\begin{aligned} \mathbb{E}\tau_n \times \mathbb{E}[(I'_n(a) - I_n(a))^2] &\leq C \mathbb{E}\tau_n \times \mathbb{E}|z_n| \\ &\leq C (\text{var}(c_n - c_0))^{1/2} \end{aligned}$$



$$\xrightarrow[n \rightarrow \infty]{} 0$$

from assumptions and therefore the CLT (21) holds.

Now the asymptotic expansion (26) can be proved. Consider first the deterministic case and

$$\begin{aligned} S_n^2(a) &= \frac{2a^2}{(n+1)^2} \int_{\mathbb{R}^2} |\widehat{\psi}(a\xi)|^2 f(\xi) |\widehat{\psi}(a\xi')|^2 f(\xi') d\xi d\xi' \sum_{k,k'=0}^n e^{i(k-k')\frac{(c_n-c_0)}{n}(\xi-\xi')} \\ &= \frac{2a^2}{(n+1)^2} \int_{\mathbb{R}^2} |\widehat{\psi}(a\xi)|^2 f(\xi) |\widehat{\psi}(a\xi')|^2 f(\xi') \frac{\sin^2\left(\frac{c_n-c_0}{2}(\xi-\xi')\right)}{\sin^2\left(\frac{c_n-c_0}{2n}(\xi-\xi')\right)} d\xi d\xi' \\ &= \frac{16a^2}{c_n-c_0} \int_{\mathbb{R}_+^2} |\widehat{\psi}(az')|^2 f(z') \left| \widehat{\psi}\left(a\left(z'+\frac{2z}{c_n-c_0}\right)\right) \right|^2 f\left(z'+\frac{2z}{c_n-c_0}\right) \frac{\sin^2(z)}{n^2 \sin^2\left(\frac{z}{n}\right)} dz dz'. \end{aligned}$$

Let us define  $h_n(x) := \frac{\sin(x)}{n \sin\left(\frac{x}{n}\right)}$  and  $h(x) := \frac{\sin x}{x}$ . For all  $(z, z') \in \mathbb{R}^2$ ,

$$\left| \widehat{\psi}\left(a\left(z'+\frac{2z}{c_n-c_0}\right)\right) \right|^2 f\left(z'+\frac{2z}{c_n-c_0}\right) \xrightarrow[n \rightarrow \infty]{} |\widehat{\psi}(az')|^2 f(z') \quad \text{and} \quad h_n^2(z) \xrightarrow[n \rightarrow \infty]{} h^2(z).$$

However Lebesgue Theorem cannot be applied. Denote  $\nu(x) := |\psi(ax)|^2 f(x)$  for  $x > 0$ . From Assumptions F and W(1, 3),  $\nu$  is a differentiable function in  $(0, \infty)$  and  $\exists C > 0, \forall z', x > 0, |\nu'(z'+x)| \leq C |\nu'(z')|$ . Then,

$$\left| \nu\left(z'+\frac{2z}{c_n-c_0}\right) - \nu(z') \right| \leq \frac{2z}{c_n-c_0} C |\nu'(z')|.$$

Moreover,  $|h_n(z')| \leq 1$  for all  $z' \in \mathbb{R}$ , and

$$\begin{aligned} \int_{-n}^n h_n^2(z) dz &= \frac{1}{n^2} \int_{-n}^n \sum_{k,k'=1}^n e^{2i(k-k')\frac{z}{n}} dz \\ &= \frac{2}{n} \sum_{1 \leq k' < k \leq n} \frac{\sin(2(k-k'))}{(k-k')} + \frac{1}{n^2} \sum_{k=0}^n 2n \\ &= \frac{2}{n} \left\{ \sum_{k=0}^n (n-k) \frac{\sin(2k)}{k} \right\} + 2 \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{-n}^n h_n^2(z) dz = 2 \left\{ \sum_{k=0}^{\infty} \frac{\sin(2k)}{k} \right\} + 2 = \pi$$

since  $\frac{2}{n} \sum_{1 \leq k \leq n} k \frac{\sin(2k)}{k} \leq 4 \frac{\log n}{n} \xrightarrow{n \rightarrow \infty} 0$  and from Dirichlet Theorem,  $x - \pi = -2 \sum_{n \geq 1} \frac{\sin(nx)}{n}$  for all  $x \in (0, 2\pi)$ . Now, for  $z' \geq 0$ ,

$$\begin{aligned} & \left| \int_{\mathbb{R}_+} \nu\left(z' + \frac{2z}{c_n - c_0}\right) h_n^2(z) dz - \nu(z') \int_0^n h_n^2(z) dz \right| \\ & \leq \frac{2}{c_n - c_0} \int_0^n z h_n^2(z) dz + \int_n^\infty \nu\left(z' + \frac{2z}{c_n - c_0}\right) h_n^2(z) dz \\ & \leq \frac{2}{c_n - c_0} \int_0^n z \frac{4 \sin^2(z)}{z^2} dz + \int_n^\infty \nu\left(z' + \frac{2z}{c_n - c_0}\right) dz \\ & \leq 8 \frac{4 + \log(n)}{c_n - c_0} + C f(z') \int_n^\infty \frac{C_\psi}{\left(1 + \left(z' + \frac{2z}{c_n - c_0}\right)\right)^{2r}} dz \\ & \leq 8 \frac{4 + \log(n)}{c_n - c_0} + C' f(z') n \delta_n^{2r} \end{aligned}$$

Finally, with  $n \delta_n^{2r} \xrightarrow{n \rightarrow \infty} 0$  when  $r = 3$ , one deduces that for all  $z' \geq 0$ ,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}_+} \nu\left(z' + \frac{2z}{c_n - c_0}\right) h_n^2(z) dz = \lim_{n \rightarrow \infty} \nu(z') \int_0^n h_n^2(z) dz = \frac{\pi}{2} \nu(z').$$

Therefore,

$$\begin{aligned} & \int_{\mathbb{R}} |\widehat{\psi}(az')|^2 f(z') \int_{\mathbb{R}} \left| \widehat{\psi}\left(a\left(z' + \frac{2z}{c_n - c_0}\right)\right) \right|^2 f\left(z' + \frac{2z}{c_n - c_0}\right) \frac{\sin^2(z)}{n^2 \sin^2\left(\frac{z}{n}\right)} dz dz' \\ & \xrightarrow{n \rightarrow \infty} \pi \int_{\mathbb{R}} |\widehat{\psi}(az')|^4 f^2(z') dz', \end{aligned}$$

providing the asymptotic behavior of  $S_n^2$ . The proof is similar in the stochastic case with  $c_n - c_0$  replaced by  $\mathbb{E}(c_n - c_0)$ .  $\blacksquare$

7.3. *Proof of Theorem 3.1.* The proof of Theorem 3.1 uses the following lemmas:

**Lemma 7.4** *Let  $X$  be a Gaussian process defined by (1) with a spectral density  $f$  satisfying (2) and Assumption F. Let us define,*

$$R(t, u, t', u') := \mathbb{E}\left[(X(t+u) - X(t)) \cdot (X(t'+u') - X(t')) \mid \mathcal{F}_X\right],$$

for  $(t, t') \in \mathbb{R}^2$ ,  $(u, u') \in \mathbb{R}_+^2$ . Then  $\exists C_f > 0$  depending only on the spectral density  $f$  such that for all  $(u, u', t, t') \in \mathbb{R}_+^2 \times \mathbb{R}^2$ , with  $\beta = (t' - t + u' - u)$ ,

$$|R(t, 2u, t', 2u')| \leq C_f \left( u u' + \left( (u u')^H \wedge ((u u') |\beta|^{2H-2} + (u u')^{H+1/2} |\beta|^{-1}) \right) \right).$$

**Proof.** To begin with, remark that for all  $(t, t', u, u') \in \mathbb{R}^4$ ,

$$\begin{aligned} R(t, 2u, t', 2u') &= \int_{\mathbb{R}} (e^{-i(t+2u)\xi} - e^{-it\xi})(e^{i(t'+2u')\xi} - e^{it'\xi}) f(\xi) d\xi \\ &= \int_{\mathbb{R}} (e^{-iu\xi} - e^{iu\xi})(e^{iu'\xi} - e^{-iu'\xi}) e^{i\xi(t'-t)+i\xi(u'-u)} f(\xi) d\xi \\ &= 8 \int_0^\infty \sin(u\xi) \cdot \sin(u'\xi) \cdot \cos(\xi(t' - t + u' - u)) f(\xi) d\xi \\ &= 8(I_1 + I_2) \end{aligned}$$

with  $I_1 := \int_0^{\omega_K} \dots d\xi$  and  $I_2 = \int_{\omega_K}^\infty \dots d\xi$ . From one hand, with  $|\sin a| \leq |a|$ ,

$$I_1 \leq uu' \int_0^{\omega_K} \xi^2 f(\xi) d\xi \leq C uu',$$

where the last inequality follows from (2). From the other hand,

$$I_2 = \frac{1}{\sqrt{uu'}} \int_{\omega_K \sqrt{uu'}}^\infty \sin\left(\frac{u}{\sqrt{uu'}}\xi\right) \left(\frac{u'}{\sqrt{uu'}}\xi\right) \cos\left(\frac{\beta}{\sqrt{uu'}}\xi\right) f\left(\frac{1}{\sqrt{uu'}}\xi\right) d\xi.$$

Then, with Assumption F combined with  $|\cos a| \leq 1$  and  $|\sin a| \leq (1 \wedge |a|)$ ,

$$\begin{aligned} |I_2| &\leq (uu')^H \left( \int_0^1 x^2 x^{-(2H+1)} dx + \int_1^\infty x^{-(2H+1)} dx \right) \\ (29) \quad &\leq C (uu')^H, \end{aligned}$$

since  $H \in (0, 1)$ . It remains to prove  $|I_2| \leq C(uu'\beta^{2H-2} + \beta^{-1}(uu')^{H+1/2} + uu')$  with  $C > 0$ . First, with an integration by parts,

$$\begin{aligned} I_2 &= \beta^{-1} \left( \left[ \sin\left(\frac{\beta}{\sqrt{uu'}}x\right) \sin\left(\frac{u}{\sqrt{uu'}}x\right) \sin\left(\frac{u'}{\sqrt{uu'}}x\right) f\left(\frac{x}{\sqrt{uu'}}\right) \right]_{\omega_K \sqrt{uu'}}^\infty \right. \\ &\quad \left. - \int_{\omega_K \sqrt{uu'}}^\infty \sin\left(\frac{\beta}{\sqrt{uu'}}x\right) \frac{\partial}{\partial x} \left( \sin\left(\frac{u}{\sqrt{uu'}}x\right) \sin\left(\frac{u'}{\sqrt{uu'}}x\right) f\left(\frac{x}{\sqrt{uu'}}\right) \right) dx \right), \end{aligned}$$

where Assumption F insures the convergence of bracket term at  $\infty$ . With  $|\sin a| \leq |a|$ , then  $|\beta^{-1} \sin(b\omega_K) \sin(u\omega_K) \sin(u'\omega_K) f(\omega_K^+)| \leq C uu'$ . Thus,

$$|I_2| \leq I_3 + I_4 + C uu'$$

with, using again Assumption F,  $|\cos a| \leq 1$ ,

$$I_3 = \beta^{-1} \int_{\sqrt{uu'}\omega_K}^\infty \left| \sin\left(\frac{\beta}{\sqrt{uu'}}x\right) \right| \frac{u}{\sqrt{uu'}} \cos\left(\frac{u}{\sqrt{uu'}}x\right) \sin\left(\frac{u'}{\sqrt{uu'}}x\right)$$

$$\begin{aligned}
 & + \frac{u'}{\sqrt{u u'}} \cos\left(\frac{u'}{\sqrt{u u'}} x\right) \sin\left(\frac{u}{\sqrt{u u'}} x\right) \left|f\left(\frac{x}{\sqrt{u u'}}\right)\right| dx \\
 & = \frac{C}{\beta} (u u')^{H+1/2} \int_0^\infty \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \right| \left( \left| \sin\left(\frac{u'}{\sqrt{u u'}} x\right) \right| + \left| \sin\left(\frac{u}{\sqrt{u u'}} x\right) \right| \right) x^{-(2H+1)} dx
 \end{aligned}$$

and

$$\begin{aligned}
 I_4 & = \frac{1}{\beta \sqrt{u u'}} \int_{\delta \omega_K}^\infty \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \sin\left(\frac{u}{\sqrt{u u'}} x\right) \sin\left(\frac{u'}{\sqrt{u u'}} x\right) f'\left(\frac{x}{\sqrt{u u'}}\right) \right| dx \\
 & = \frac{C}{\beta} (u u')^{H+1/2} \int_0^\infty \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \sin\left(\frac{u}{\sqrt{u u'}} x\right) \sin\left(\frac{u'}{\sqrt{u u'}} x\right) \right| x^{-(2H+2)} dx.
 \end{aligned}$$

Both those integrals can be decomposed as  $\int_0^{\sqrt{u u'}/\beta} \dots + \int_{\sqrt{u u'}/\beta}^1 \dots + \int_1^\infty \dots$ . Using  $|\sin a| \leq (|a| \wedge 1)$ , with  $C > 0$  denoting a constant which may vary from one line to the other,

$$\begin{aligned}
 I_{31} & = \int_0^{\sqrt{u u'}/\beta} \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \right| \left( \left| \sin\left(\frac{u'}{\sqrt{u u'}} x\right) \right| + \left| \sin\left(\frac{u}{\sqrt{u u'}} x\right) \right| \right) x^{-(2H+1)} dx \\
 & \leq \frac{2b}{\sqrt{u u'}} \int_0^{\sqrt{u u'}/\beta} x^2 x^{-(2H+1)} dx \leq C \beta^{2H-1} (u u')^{1/2-H},
 \end{aligned}$$

$$\begin{aligned}
 I_{41} & = \int_0^{\sqrt{u u'}/\beta} \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \sin\left(\frac{u}{\sqrt{u u'}} x\right) \sin\left(\frac{u'}{\sqrt{u u'}} x\right) \right| x^{-(2H+2)} dx \\
 & \leq \frac{\beta}{\sqrt{u u'}} \int_0^{\sqrt{u u'}/\beta} x^3 x^{-(2H+2)} dx \leq C \beta^{2H-1} (u u')^{1/2-H},
 \end{aligned}$$

$$\begin{aligned}
 I_{32} & = \int_{\sqrt{u u'}/\beta}^1 \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \right| \left( \left| \sin\left(\frac{u'}{\sqrt{u u'}} x\right) \right| + \left| \sin\left(\frac{u}{\sqrt{u u'}} x\right) \right| \right) x^{-(2H+1)} dx \\
 & \leq 2 \int_{\sqrt{u u'}/\beta}^1 x x^{-(2H+1)} dx \leq C(1 + \beta^{2H-1} (u u')^{1/2-H}),
 \end{aligned}$$

$$\begin{aligned}
 I_{42} & = \int_{\sqrt{u u'}/\beta}^1 \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \sin\left(\frac{u}{\sqrt{u u'}} x\right) \sin\left(\frac{u'}{\sqrt{u u'}} x\right) \right| x^{-(2H+2)} dx \\
 & \leq \int_{\sqrt{u u'}/\beta}^1 x^2 x^{-(2H+2)} dx \leq C(1 + \beta^{2H-1} (u u')^{1/2-H}),
 \end{aligned}$$

$$\begin{aligned}
 I_{33} & = \int_1^\infty \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \right| \left( \left| \sin\left(\frac{u'}{\sqrt{u u'}} x\right) \right| + \left| \sin\left(\frac{u}{\sqrt{u u'}} x\right) \right| \right) x^{-(2H+1)} dx \\
 & \leq 2 \int_1^\infty x^{-(2H+1)} dx \leq \frac{1}{H},
 \end{aligned}$$

$$\begin{aligned}
 I_{43} & = \int_1^\infty \left| \sin\left(\frac{\beta}{\sqrt{u u'}} x\right) \sin\left(\frac{u}{\sqrt{u u'}} x\right) \sin\left(\frac{u'}{\sqrt{u u'}} x\right) \right| x^{-(2H+2)} dx \\
 & \leq \int_1^\infty x^{-(2H+2)} dx \leq \frac{1}{2H+1}.
 \end{aligned}$$

This implies that it exists  $C > 0$  such that,

$$\begin{aligned} I_3 + I_4 &\leq C \beta^{-1} (u u')^{1/2+H} (1 + \beta^{2H-1} (u u')^{1/2-H}) \\ &\leq C (\beta^{-1} (u u')^{1/2+H} + (u u') \beta^{2H-2}) \\ \implies |I_2| &\leq C (\beta^{-1} (u u')^{1/2+H} + (u u') \beta^{2H-2} + (u u')). \end{aligned}$$

Combined with (29), this completes the proof of Lemma 7.4.  $\blacksquare$

Next, let us define the error of discretization of the wavelet coefficients by

$$(30) \quad \varepsilon(a, b) := \varepsilon_{1,n}(a, b) + \varepsilon_{2,n}(a, b) + \varepsilon_{3,n}(a, b) := d_X(a, b) - e_X(a, b),$$

$$\text{with } \begin{cases} \varepsilon_{1,n}(a, b) := a^{-1/2} \left( \int_0^{T_n} \psi\left(\frac{t-b}{a}\right) X(t) dt - \sum_{i=0}^{n-1} X(t_i) \int_{t_i}^{t_{i+1}} \psi\left(\frac{t-b}{a}\right) dt \right) \\ \varepsilon_{2,n}(a, b) := a^{-1/2} \int_{T_n}^{\infty} \psi\left(\frac{t-b}{a}\right) X(t) dt \\ \varepsilon_{3,n}(a, b) := a^{-1/2} \int_{-\infty}^0 \psi\left(\frac{t-b}{a}\right) X(t) dt \end{cases}.$$

The following lemmas give bounds on  $\mathbb{E}|\varepsilon_{i,n}(a, k)|^2$  for  $i = 1, 2, 3$ .

**Lemma 7.5** *Let  $X$  be a Gaussian process defined by (1) with a spectral density  $f$  satisfying (2) and Assumption F. Assume also Assumptions W(1, 3) and (S(s)). Then, with  $C_f$  defined in Lemma 7.4, if  $b$  is a r.v. independent on  $\mathcal{F}_X$  such that  $T_n^\rho \leq b \leq T_n - T_n^\rho$  with  $\rho > 1/2$ , for  $i = 2, 3$ ,*

$$(31) \quad \mathbb{E}(|\varepsilon_{i,n}(a, b)|^2 \mid \mathcal{F}_X) \leq C_f C_\psi^2 (a^5 T_n^{2-4\rho} \mathbf{1}_{T_n \geq 1} + a(1 + \frac{a}{2})^2 \mathbf{1}_{T_n < 1}),$$

$$\text{and } \mathbb{E}(|\varepsilon_{1,n}(a, b)|^2 \mid \mathcal{F}_X) \leq C_f v_{1,n}(a)$$

where

- if  $s = \infty$ ,  $v_{1,n}(a) \leq \|\psi\|_{\mathcal{L}^1} \delta_n^{1+H} \left( a^{2-H} (\|\psi\|_{\mathcal{L}^1} + \frac{1}{H} \|\psi\|_{\infty}) + \frac{2}{H} \|\psi\|_{\infty} \delta_n^H + a \|\psi\|_{\mathcal{L}^1} \delta_n^{1-H} \right);$
- if  $s < \infty$ ,  $v_{1,n}(a) \leq \frac{a^{1-2/p_1} \|\psi\|_{\mathcal{L}^{q_1}}^2}{(p_1 + 1)^{2/p_1}} \left( \sum_{i=0}^{n-1} L_i^{p_1+1} \right)^{2/p_1} \delta_n^{2+2/p_1} + \frac{2 \|\psi\|_{\infty} \|\psi\|_{\mathcal{L}^{q_2}}}{H a^{1/p_2} (1 + p_2(1 + 2H))^{1/p_2}} \left( \sum_{i=0}^{n-1} L_i^{1+p_2(1+2H)} \right)^{1/p_2} \delta_n^{1+2H+1/p_2}$

$$+ \frac{\|\psi\|_{\mathcal{L}^{q_3}}^2 + \frac{\|\psi\|_{\mathcal{L}^{q_3}} \|\psi\|_{\infty}}{1-q_3(1-H)}}{a^{H-2+2/p_3}(1+p_3(1+H)/2)^{2/p_3}} \left( \sum_{i=0}^{n-1} L_i^{1+p_3(1+H)/2} \right)^{2/p_3} \delta_n^{1+H+2/p_3},$$

for all  $(p_1, p_2, p_3) \in [1, \infty)^2 \times (1/H, \infty)$  with  $\frac{1}{p_j} + \frac{1}{q_j} = 1$  for  $j = 1, 2, 3$ .

**Proof. (1) Bound of  $\mathbb{E}(|\varepsilon_{1,n}(a, b)|^2 \mid \mathcal{F}_X)$ .** To begin with,

$$\begin{aligned} & \mathbb{E}(|\varepsilon_{1,n}(a, b)|^2 \mid \mathcal{F}_X) \\ &= \frac{1}{a} \sum_{i,j=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} \mathbb{E}\left((X(t)-X(t_i))(X(t')-X(t_j)) \mid \mathcal{F}_X\right) dt dt' \\ &= \frac{1}{a} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} R(t_i, t-t_i, t_j, t'-t_j) dt dt'. \end{aligned}$$

Lemma 7.4, with  $2u = t - t_i$  and  $2u' = t' - t_j$ , implies

$$\mathbb{E}(|\varepsilon_{1,n}(a, b)|^2 \mid \mathcal{F}_X) \leq C_f a^{-1} (S_1 + S_2), \quad \text{with}$$

$$\begin{aligned} \bullet S_1 &\leq \sum_{i,j=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} u u' \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} dt dt' \\ \bullet S_2 &\leq \sum_{i,j=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \left( (u u')^H \wedge ((u u')|\beta|^{2H-2} + (u u')^{H+1/2} |\beta|^{-1}) \right) \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} dt dt'. \end{aligned}$$

But,  $S_1 \leq \left( \int_0^{T_n} \chi(t) \left| \psi\left(\frac{t-b}{a}\right) \right| dt \right)^2$  where

$$(32) \quad \chi(t) := \sum_{i=0}^{n-1} |t - t_i| \mathbf{1}_{[t_i, t_{i+1})}(t).$$

From Hölder Inequality,  $S_1 \leq \|\chi\|_{\mathcal{L}^p}^2 \|\psi\left(\frac{t-b}{a}\right)\|_{\mathcal{L}^q}^2$  for  $(p, q) \in [1, \infty]^2$  with  $1/p + 1/q = 1$ . Since  $\|\psi\left(\frac{t-b}{a}\right)\|_{\mathcal{L}^q} = a^{1/q} \|\psi\|_{\mathcal{L}^q}$ , from Minkosvski Inequality,

$$\begin{aligned} \|\chi\|_{\mathcal{L}^p} &= \left( \int_{\mathbb{R}} \chi^p(t) dt \right)^{1/p} \\ &\leq \left( \int_{\mathbb{R}} \left( \sum_{i=0}^{n-1} \mathbf{1}_{[t_i, t_{i+1})}(t) \right)^{1/q} \sum_{i=0}^{n-1} |t - t_i|^p dt \right)^{1/p} \\ &\leq \left( \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |t - t_i|^p dt \right)^{1/p} \leq (p+1)^{-1/p} \left( \sum_{i=0}^{n-1} L_i^{p+1} \right)^{1/p} \delta_n^{1+1/p}, \end{aligned}$$

for  $p < \infty$ . It follows  $S_1 \leq a^{2/q} \|\psi\|_{\mathcal{L}^q}^2 \times \|\chi\|_{\mathcal{L}^p}^2$ . If  $s = \infty$ , one can choose  $p = \infty$  and  $\|\chi\|_\infty = \delta_n \max_{1 \leq i \leq n} (\|L_i\|_\infty)$ . Then,

$$(33) \quad S_1 \leq a^2 M_\infty \|\psi\|_{L^1}^2 \delta_n^2.$$

If  $s < \infty$ , one can deduce that for all  $1 \leq p_1 < \infty$ ,

$$(34) \quad S_1 \leq \frac{a^{2-2/p_1} \|\psi\|_{\mathcal{L}^{q_1}}^2}{(p_1 + 1)^{2/p_1}} \left( \sum_{i=0}^{n-1} L_i^{p_1+1} \right)^{2/p_1} \delta_n^{2+2/p_1}.$$

Next, in order to bound  $S_2$ , one uses twice the inequality  $(x \wedge y) \leq x^\alpha y^{1-\alpha}$  which is valid for all  $x, y \geq 0$  and  $0 \leq \alpha \leq 1$ . Thus,

$$\begin{cases} (u u')^H \wedge (u u') |\beta|^{2H-2} & \leq (u u')^{(1+H)/2} |\beta|^{-(1-H)} & \text{with } \alpha_1 = 1/2 \\ (u u')^H \wedge (u u')^{(H+1/2)} |\beta|^{-1} & \leq (u u')^{(1+H)/2} |\beta|^{-(1-H)} & \text{with } \alpha_2 = H \end{cases}.$$

Therefore  $S_2 \leq S_{21} + S_{22}$  with

$$\begin{aligned} S_{21} &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} (u u')^{(1+H)/2} |\beta|^{-(1-H)} dt dt' \\ S_{22} &= 2 \sum_{0 \leq i < j \leq n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} (u u')^{(1+H)/2} |\beta|^{-(1-H)} dt dt'. \end{aligned}$$

On the one hand, when  $i = j$  then  $b = \frac{1}{2}(t' - t)$  and

$$S_{21} = \frac{1}{2} \int_0^{T_n} \chi(t)^{(1+H)/2} g(t) \left| \psi\left(\frac{t-b}{a}\right) \right| dt$$

where the functions  $\chi$  and  $g$  are respectively defined by (32) and

$$g(t) := \sum_{i=0}^{n-1} \mathbf{1}_{[t_i, t_{i+1}]}(t) \left( \int_{t_i}^{t_{i+1}} |t' - t_i|^{(1+H)/2} |t - t'|^{-(1-H)} \left| \psi\left(\frac{t-b}{a}\right) \right| dt' \right).$$

Next, by using  $\psi \in \mathcal{L}^\infty(\mathbb{R})$ ,  $(1-H) \in (0, 1)$  and  $L_i = t_{i+1} - t_i$ , one gets

$$\begin{aligned} g(t) &\leq \sum_{i=0}^{n-1} \mathbf{1}_{[t_i, t_{i+1}]}(t) \left( \|\psi\|_\infty |t_{i+1} - t_i|^{(1+H)/2} \int_{t_i}^{t_{i+1}} |t - t'|^{-(1-H)} dt' \right) \\ &\leq \frac{2}{H} \|\psi\|_\infty \left( \sum_{i=0}^{n-1} \mathbf{1}_{[t_i, t_{i+1}]}(t) L_i^{(1+3H)/2} \right). \end{aligned}$$

Let  $\tilde{\chi}(t) := \sum_{i=0}^{n-1} \mathbf{1}_{[t_i, t_{i+1}]}(t) L_i^{(1+3H)/2} |t - t_i|^{(1+H)/2}$ . By using Hölder inequality for all  $(p, q) \in [1, \infty]^2$  with  $1/p + 1/q = 1$ ,

$$S_{21} \leq \frac{2}{H} a^{1/q} \|\psi\|_\infty \|\psi\|_{\mathcal{L}^q} \|\tilde{\chi}\|_{\mathcal{L}^p}.$$

If  $s = \infty$ , one can fix  $p = \infty$  and  $\|\tilde{\chi}\|_\infty \leq \delta_n^{1+2H}$  and after

$$(35) \quad S_{21} \leq \frac{2}{H} a \times \|\psi\|_\infty \|\psi\|_{\mathcal{L}^1} \times \delta_n^{1+2H}.$$

If  $s < \infty$ ,  $\|\tilde{\chi}\|_{\mathcal{L}^p} \leq (1 + p(1 + 2H))^{-1/p} \left( \sum_{i=0}^{n-1} L_i^{1+p(1+2H)} \right)^{1/p} \delta_n^{1+2H+1/p}$  and after, for all  $1 \leq p_2 < \infty$ ,

$$(36) \quad S_{21} \leq \frac{2a^{1/q_2} \|\psi\|_\infty \|\psi\|_{\mathcal{L}^{q_2}}}{H(1 + p_2(1 + 2H))^{1/p_2}} \left( \sum_{i=0}^{n-1} L_i^{1+p_2(1+2H)} \right)^{1/p_2} \delta_n^{1+2H+1/p_2}.$$

From the other hand, since  $\beta = \frac{1}{2}((t' + t_j) - (t + t_i)) \geq \frac{1}{2}(t' - t) \geq 0$  for  $i < j$  and  $t \in [t_i, t_{i+1}]$ ,  $t' \in [t_j, t_{j+1}]$ , and with  $1 - H < 1$ ,

$$\begin{aligned} S_{22} &\leq \sum_{0 \leq i < j \leq n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} (uu')^{(1+H)/2} |t - t'|^{H-1} dt dt' \\ &\leq \int_0^{T_n} \int_0^{T_n} \chi(t)^{(1+H)/2} \chi(t')^{(1+H)/2} \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} |t - t'|^{H-1} dt dt' \\ &\leq \left\| \chi(t)^{(1+H)/2} \chi(t')^{(1+H)/2} \right\|_{\mathcal{L}^p(\mathbb{R}^2)} \left\| \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} |t - t'|^{H-1} \right\|_{\mathcal{L}^q(\mathbb{R}^2)} \end{aligned}$$

for any  $(p, q) \in [1, \infty]^2$  with  $1/p + 1/q = 1$ . But for all  $p \geq 2$

$$\left\| \chi(t)^{(1+H)/2} \chi(t')^{(1+H)/2} \right\|_{\mathcal{L}^p(\mathbb{R}^2)} = \left(1 + p(1 + H)/2\right)^{-2/p} \left( \sum_{i=0}^{n-1} L_i^{1+p(1+H)/2} \right)^{2/p}$$

Next, with  $u = (t - c_k)/a$  and  $v = (t' - c_k)/a$ , one gets

$$\begin{aligned} &\left\| \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} |t - t'|^{H-1} \right\|_{\mathcal{L}^q(\mathbb{R}^2)}^q \leq a^{2+q(1-H)} \int_{\mathbb{R}^2} \frac{|\psi(u)\psi(v)|^q}{|u-v|^{q(1-H)}} du dv \\ &\leq a^{2+q(1-H)} \left( \int_{\mathbb{R}^2, |u-v| \geq 1} |\psi(u)\psi(v)|^q du dv + \int_{\mathbb{R}^2, |u-v| < 1} \frac{|\psi(u)\psi(v)|^q}{|u-v|^{q(1-H)}} du dv \right) \\ &\leq a^{2+q(1-H)} \left( \|\psi\|_{\mathcal{L}^q}^{2q} + \|\psi\|_\infty^q \|\psi\|_{\mathcal{L}^q}^q \int_0^1 s^{-q(1-H)} ds \right). \end{aligned}$$



The last integral is equal to  $(1-q(1-H))^{-1}$  when  $p > 1/H$ . Thus,  $\forall p_3 > 1/H$ ,

$$(37) \quad \frac{S_{22}}{\delta_n^{1+H+2/p_3}} \leq \frac{\|\psi\|_{\mathcal{L}^{q_3}} (\|\psi\|_{\mathcal{L}^{q_3}}^{q_3} + \frac{\|\psi\|_{\infty}^{q_3}}{1-q_3(1-H)})^{1/q_3}}{a^{H-3+2/p_3} (1+p_3(1+H)/2)^{2/p_3}} \left( \sum_{i=0}^{n-1} L_i^{1+p_3(1+H)/2} \right)^{2/p_3}$$

If  $s = \infty$ , one can fix  $p = \infty$  and  $q = 1$ , and thus

$$(38) \quad S_{22} \leq a^{3-H} \|\psi\|_{\mathcal{L}^1} \left( \|\psi\|_{\mathcal{L}^1} + \frac{1}{H} \|\psi\|_{\infty} \right) \delta_n^{1+H}.$$

Finally by summing up (33), (35) and (38) if  $s = \infty$ , and by summing up (34), (36) and (37) if  $s < \infty$ , one gets the bounds of  $v_{1,n}(a)$ .

**(2) Bound of  $\mathbb{E}(|\varepsilon_{2,n}(a,b)|^2 \mid \mathcal{F}_X)$ .** Since  $T_n$  is independent on  $\mathcal{F}_X$ ,

$$\begin{aligned} \mathbb{E}(|\varepsilon_{2,n}(a,b)|^2 \mid \mathcal{F}_X) &= \frac{1}{a} \int_{T_n}^{\infty} \int_{T_n}^{\infty} \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{t'-b}{a}\right)} \mathbb{E}(X(t)X(t')) dt dt' \\ &\leq \frac{C_f}{a} \left( \int_{T_n}^{\infty} \left| \psi\left(\frac{t-b}{a}\right) \right| (1+|t|) dt \right)^2. \end{aligned}$$

from Lemma 7.1. But, according to Assumption W(1,3),

$$\mathbb{E}(\varepsilon_{2,n}^2(a,b) \mid \mathcal{F}_X) \leq C_f C_{\psi}^2 a^{-1} \left( \int_{T_n}^{\infty} (1+t)(1+(t-b)/a)^{-3} dt \right)^2.$$

If  $T_n \geq 1$ , then  $1 + (t-b)/a \leq 1 + (t - T_n + T_n^{\rho})/a$  for all  $t \geq T_n$  and with the change of variable  $v = (t - T_n + T_n^{\rho})/T_n$ ,

$$\begin{aligned} \int_{T_n}^{\infty} \frac{(1+t)}{(1+(t-b)/a)^3} dt &\leq T_n a^3 \int_{T_n^{\rho-1}}^{\infty} \frac{(1+vT_n+T_n-T_n^{\rho})}{(a+vT_n)^3} dv \\ &\leq T_n^{-1} a^3 \int_{T_n^{\rho-1}}^{\infty} \frac{(v+2)}{v^3} dv = a^3 [T_n^{1-2\rho} + T_n^{-\rho}] \end{aligned}$$

If  $T_n \leq 1$ , by using  $b \leq T_n$ ,

$$\int_{T_n}^{\infty} \frac{(1+t)}{(1+(t-b)/a)^3} dt \leq \int_0^{\infty} \frac{(1+v+T_n)}{(1+v/a)^3} dv \leq a + \frac{1}{2} a^2.$$

Eventually, one deduces (31).

**(3) Bound of  $\mathbb{E}(|\varepsilon_{3,n}(a,b)|^2 \mid \mathcal{F}_X)$ .** Find a bound for  $\mathbb{E}(|\varepsilon_{3,n}(a,b)|^2 \mid \mathcal{F}_X)$  follows the same steps than for bounding  $\mathbb{E}(|\varepsilon_{2,n}(a,b)|^2 \mid \mathcal{F}_X)$ . ■

**Lemma 7.6** *Under assumptions of Lemma 7.5 and if  $s \geq 2H + 2$  and*

$$n \delta_n^{1 + \frac{(H+1)(s-1)}{\theta(s)+H+1}} \xrightarrow{n \rightarrow \infty} 0,$$

with  $\theta(s) := s \mathbf{1}_{s < \frac{1}{2} + \frac{1}{2H}} + (\frac{1}{2} + \frac{1}{2H}) \mathbf{1}_{\frac{1}{2} + \frac{1}{2H} \leq s < \frac{3}{2} + \frac{1}{2H}} + (s-1) \mathbf{1}_{s \geq \frac{3}{2} + \frac{1}{2H}}$ , then for all  $a \in [a_{min}, a_{max}]$ ,  $1 \leq k \leq n$ ,

$$(39) \quad \mathbb{E}(|\varepsilon_n(a, c_k)|^2) \leq v_n(a), \quad \text{and} \quad (n \delta_n) v_n(a) \xrightarrow{n \rightarrow \infty} 0.$$

**Proof.** With  $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$  for all real numbers  $x, y, z$ ,

$$\mathbb{E}(|\varepsilon_n(a, c_k)|^2) \leq 3C_f \mathbb{E}v_{1,n}(a) + 6C_f \mathbb{E}v_{2,n}(a)$$

where  $v_{1,n}(a)$  and  $v_{2,n}(a)$  have been defined in Lemma 7.5.

• If  $s = \infty$ , from Assumption S(s),  $m_\infty(n \delta_n) \leq T_n \implies T_n \xrightarrow{n \rightarrow \infty} \infty$ . Thus,

$$(n \delta_n) \mathbb{E}(|\varepsilon_n(a, c_k)|^2) \leq C_1 (n \delta_n^{2+H}) + C_2 (n \delta_n)^{3-4\rho}$$

which converges to zero as soon as  $n \delta_n^{2+H} \rightarrow 0$  and  $\rho > 3/4$ .

• If  $1 < s < \infty$ , from Lemma 7.5, with  $(p_1, p_2, p_3) \in [1, \infty)^2 \times (1/H, \infty)$ , an optimal choice of  $p_1, p_2, p_3$  will depend on  $s$ . Hence, since  $\mathbb{E}(|Z|^\alpha) \leq (\mathbb{E}|Z|)^\alpha$  for any r.v.  $Z$  and  $\alpha \in [0, 1]$  from Jensen Inequality,

1. if  $3 \leq s$ , with  $1 + p_1 = s$ ,

$$\mathbb{E}\left(\sum_{i=0}^{n-1} L_i^{p_1+1}\right)^{\frac{2}{p_1}} \delta_n^{2+\frac{2}{p_1}} \leq M_s^{\frac{2}{s-1}} \cdot n^{\frac{2}{s-1}} \delta_n^{\frac{2s}{s-1}};$$

2. if  $2 + 2H \leq s$ , with  $1 + p_2(1 + 2H) = s$ ,

$$\mathbb{E}\left(\sum_{i=0}^{n-1} L_i^{1+p_2(1+2H)}\right)^{\frac{1}{p_2}} \delta_n^{1+2H+\frac{1}{p_2}} \leq M_s^{\frac{1+2H}{s-1}} \cdot n^{\frac{1+2H}{s-1}} \delta_n^{\frac{(1+2H)s}{s-1}};$$

3. if  $\max(2 + H, \frac{3}{2} + \frac{1}{2H}) \leq s$ , with  $1 + \frac{1}{2}p_3(1 + H) = s$ ,

$$\mathbb{E}\left(\sum_{i=0}^{n-1} L_i^{1+p_3(1+H)/2}\right)^{\frac{2}{p_3}} \delta_n^{1+H+\frac{2}{p_3}} \leq M_s^{\frac{1+H}{s-1}} \cdot n^{\frac{1+H}{s-1}} \delta_n^{\frac{(1+H)s}{s-1}}.$$

However, the inequalities 1. and 3. may be extended respectively to  $2 < s \leq 3$  and  $2 + H < s < \frac{3}{2} + \frac{1}{2H}$  using a more sharp inequality which is  $\mathbb{E}(\sum |x_i|)^{\alpha\beta} \leq \mathbb{E}(\sum |x_i|^\beta)^\alpha \leq n^\alpha (\mathbb{E}(|x_i|^\beta))^\alpha$  when  $(\alpha, \beta) \in (0, 1]^2$ :

1'. if  $2 < s \leq 3$ , with  $\alpha\beta = \frac{2}{p_1}$ ,  $\alpha = 1$  and  $\beta = s - 2$ ,

$$\mathbb{E} \left( \sum_{i=0}^{n-1} L_i^{1+\frac{2}{\alpha\beta}} \right)^{\alpha\beta} \delta_n^{2+\alpha\beta} \leq M_s^s \cdot n \delta_n^s;$$

3'. if  $1 + 3H < s \leq \frac{H+1}{2H}$ , with  $\alpha\beta = \frac{2}{p_3}$ ,  $\alpha = \frac{H+1}{s}$  and  $\beta = s - \frac{H+1}{\alpha}$ ,

$$\mathbb{E} \left( \sum_{i=0}^{n-1} L_i^{1+\frac{1+H}{\alpha\beta}} \right)^{\alpha\beta} \delta_n^{1+H+\alpha\beta} \leq M_s^{\frac{H+1}{s}} \cdot (n \delta_n^s)^{\frac{H+1}{s}};$$

3''. if  $\frac{H+1}{2H} < s \leq \frac{3H+1}{2H}$ , with  $\alpha\beta = \frac{2}{p_3}$ ,  $\alpha = 2H$  and  $\beta = s - \frac{H+1}{2H}$ ,

$$\mathbb{E} \left( \sum_{i=0}^{n-1} L_i^{1+\frac{1+H}{\alpha\beta}} \right)^{\alpha\beta} \delta_n^{1+H+\alpha\beta} \leq M_s^{2H} \cdot (n \delta_n^s)^{2H}.$$

We finally obtain for  $n$  large enough and using  $n \delta_n \xrightarrow[n \rightarrow \infty]{} \infty$  and  $n \delta_n^{2+H} \xrightarrow[n \rightarrow \infty]{} 0$

(necessary condition for  $s = \infty$ ),

- $\mathbb{E} v_{1,n}(a) \leq C (n \delta_n^s)^{\frac{H+1}{s}}$ , for  $2 + 2H \leq s < \max(2 + 2H, \frac{1}{2} + \frac{1}{2H})$ ;
- $\mathbb{E} v_{1,n}(a) \leq C (n \delta_n^s)^{2H}$ , for  $\max(2 + 2H, \frac{1}{2} + \frac{1}{2H}) \leq s \leq \max(2 + 2H, \frac{3}{2} + \frac{1}{2H})$ ;
- $\mathbb{E} v_{1,n}(a) \leq C (n \delta_n^s)^{\frac{1+H}{s-1}}$ , for  $s \geq \max(2 + 2H, \frac{3}{2} + \frac{1}{2H})$

(therefore, the first inequality is only possible when  $H < 1/4$  and the second one when  $H < (\sqrt{17} - 1)/8$ ). Those three inequalities may be reduced to only one:

$$(40) \quad \mathbb{E} v_{1,n}(a) \leq C (n \delta_n^s)^{\frac{H+1}{\theta(s)}} \text{ for all } s \geq 2H + 2,$$

with  $\theta(s) = s \mathbf{1}_{s < \frac{1}{2} + \frac{1}{2H}} + (\frac{1}{2} + \frac{1}{2H}) \mathbf{1}_{\frac{1}{2} + \frac{1}{2H} \leq s < \frac{3}{2} + \frac{1}{2H}} + (s-1) \mathbf{1}_{s \geq \frac{3}{2} + \frac{1}{2H}}$ . Hence,  $(n \delta_n) \mathbb{E} v_{1,n}(a) \xrightarrow[n \rightarrow \infty]{} 0$  when  $s \geq 2H + 2$  and

$$n \delta_n^{1 + \frac{(H+1)(s-1)}{\theta(s)+H+1}} \xrightarrow[n \rightarrow \infty]{} 0.$$

To finish the proof of Lemma 7.6 it remains to show  $(n \delta_n) \mathbb{E} v_{2,n}(a) \xrightarrow[n \rightarrow \infty]{} 0$ .

From (31) it follows that  $\mathbb{E} v_{2,n}(a) \leq C \int_0^\infty g(x) f_n(x) dx$  where  $f_n$  is the probability distribution function of  $T_n$  and  $g(x) = \mathbf{1}_{(x < 1)} + \mathbf{1}_{(x \geq 1)} x^{2-4\rho}$ . Since  $\rho > 3/4$ ,  $g(x) \leq 1$  for all  $x > 0$  and  $g$  is a non increasing map,

$$\int_0^\infty g(x) f_n(x) dx \leq \int_0^{\frac{1}{2} m_s n \delta_n} f_n(x) dx + g(\frac{1}{2} m_s n \delta_n) \int_{\frac{1}{2} m_s n \delta_n}^\infty f_n(x) dx$$

$$\begin{aligned}
 &\leq \mathbb{P}(T_n \leq \frac{1}{2} m_s n \delta_n) + g(\frac{1}{2} m_s n \delta_n) \\
 &\leq \mathbb{P}(|T_n - \mathbb{E}[T_n]| \geq \mathbb{E}[T_n] - \frac{1}{2} m_s n \delta_n) + (\frac{1}{2} m_s n \delta_n)^{2-4\rho}, \\
 &\leq 4 \frac{M_s}{m_s} \frac{n \delta_n^2}{n^2 \delta_n^2} + (\frac{1}{2} m_s n \delta_n)^{2-4\rho},
 \end{aligned}$$

from Bienaymé-Chebyshev Inequality since  $s \geq 2$  and  $\text{var}(T_n) \leq M_s n \delta_n^2$  from the independence of  $(L_i)_{i \in N}$ . Therefore  $(n \delta_n) \mathbb{E}v_{2,n}(a) \xrightarrow[n \rightarrow \infty]{} 0$ . ■

**Lemma 7.7** *Under the same assumptions and notations as in Lemma 7.6, there exists  $C > 0$  such that for every  $a > 0$  and  $n \in \mathbb{N}^*$ ,*

$$(41) \quad \mathbb{E} |I_n(a) - J_n(a)| \leq C v_n(a)^{1/2}.$$

**Proof.** Since  $d = d(a, c_k)$  and  $e = e(a, c_k)$  are Gaussian r.v.,  $|d|^2 - |e|^2$  have finite second order moment and Jensen's inequality implies

$$\begin{aligned}
 \mathbb{E} |I_n(a) - J_n(a)| &= \frac{1}{n+1} \mathbb{E} \left| \sum_{k=0}^n [ |d(a, c_k)|^2 - |e(a, c_k)|^2 ] \right| \\
 &\leq \frac{1}{n+1} \sum_{k=0}^n \sqrt{\mathbb{E} ( |d(a, c_k)|^2 - |e(a, c_k)|^2 )^2}
 \end{aligned}$$

Since  $d$  and  $e$  are jointly Gaussian r.v. with zero means,

$$\mathbb{E}(|d|^2 - |e|^2)^2 = \mathbb{E}|d - e|^2 |d + e|^2 := \mathbb{E}|\varepsilon|^2 |Z|^2,$$

where  $\varepsilon = d - e$  and  $Z = d + e$  are also jointly Gaussian and have mean zero. By using that  $Z = \sigma_2 \sigma_1^{-1} \rho \varepsilon + \xi$ , where  $\sigma_1^2 = \mathbb{E}|\varepsilon|^2$ ,  $\sigma_2^2 = |\mathbb{E}Z|^2$ ,  $\rho = \text{corr}(\varepsilon, Z)$  and where  $\xi$  is independent of  $\varepsilon$  and Gaussian, one can show that

$$\mathbb{E}|\varepsilon|^2 |Z|^2 = (\mathbb{E}|\varepsilon|^4) \frac{\sigma_2^2 \rho^2}{\sigma_1^2} + \mathbb{E}|\varepsilon|^2 \mathbb{E}|\xi|^2 = 3\sigma_1^2 \sigma_2^2 \rho^2 + \sigma_1^2 \sigma_2^2 (1 - \rho^2) \leq 3\sigma_1^2 \sigma_2^2.$$

Since  $\mathbb{E}|\varepsilon|^2 |Z|^2 = 3 \mathbb{E}|\varepsilon|^2 \mathbb{E}|d + e|^2$  and  $|d + e|^2 = |2d - \varepsilon|^2 \leq 8|d|^2 + 2|\varepsilon|^2$ ,

$$\mathbb{E} |I_n(a) - J_n(a)| \leq \frac{\sqrt{6}}{n+1} \sum_{k=0}^n (\mathbb{E}|\varepsilon(a, c_k)|^2)^{1/2} \left( \mathbb{E} [4|d(a, c_k)|^2 + |\varepsilon(a, c_k)|^2] \right)^{1/2}$$

$$\leq C v_n(a)^{1/2} (4\mathcal{I}_1(a) + C_1 v_n(a))^{1/2},$$

from Lemma 7.6 and since  $\mathbb{E}|d(a, c_k)|^2 = \mathcal{I}_1(a)$  for  $k \in \mathcal{N}$ . Assumption W(1,3) and formulas (2) and (7) imply  $\sup_{a \in [a_{\min}, a_{\max}]} |\mathcal{I}_1(a)| < \infty$ . Owing to Lemma 7.5,  $v_n(a)$  is bounded, this finally provides (41).  $\blacksquare$

**Proof.** [Theorem 3.1] From Lemmas 7.6 and 7.7,

$$(42) \quad \lim_{n \rightarrow \infty} (n \delta_n)^{1/2} \mathbb{E} |I_n(a) - J_n(a)| = 0.$$

Now, using Markov inequality, one deduces that for all  $\varepsilon > 0$ ,

$$\mathbb{P} \left( \sqrt{n \delta_n} |J_n(a) - I_n(a)| \geq \varepsilon \right) \leq \frac{\varepsilon}{\sqrt{n \delta_n} C v_n(a)} \xrightarrow{n \rightarrow \infty} 0,$$

that achieves the proof with  $\sqrt{n \delta_n} |J_n(a) - \mathcal{I}_1(a)| \leq \sqrt{n \delta_n} |I_n(a) - I_n(a)| + \sqrt{n \delta_n} |I_n(a) - \mathcal{I}_1(a)|$  and Slutsky Lemma.  $\blacksquare$

7.4. *Proof of Proposition 4.1.* **Proof.** It is obvious that

$$\begin{aligned} \mathcal{I}_\lambda(a) &= \int_{\mathbb{R}} |\widehat{\psi}_\lambda(\xi)|^2 f(\xi/a) d\xi = \lambda \int_{\mathbb{R}} |\widehat{\psi}(\lambda(\xi - 1))|^2 f(\xi/a) d\xi \\ &= \int_{\mathbb{R}} |\widehat{\psi}(v)|^2 f\left(\frac{1}{a} + \frac{v}{a\lambda}\right) dv. \end{aligned}$$

Then, from a usual Taylor expansion, and since  $\widehat{\psi}$  is supposed to be an even function supported in  $[-\Lambda, \Lambda]$ ,

$$\left| \mathcal{I}_\lambda(a) - \|\widehat{\psi}\|_{\mathcal{L}^2(\mathbb{R})}^2 f(1/a) \right| \leq \frac{1}{2a^2\lambda^2} \left( \sup_{-\Lambda/\lambda \leq h} \left\{ \left| f''\left(\frac{1+h}{a}\right) \right| \right\} \int_{-\Lambda}^{\Lambda} v^2 |\widehat{\psi}(v)|^2 dv \right).$$

For  $\lambda > 2\Lambda$ , then  $\sup_{-\Lambda/\lambda \leq h} \left\{ \left| f''\left(\frac{1+h}{a}\right) \right| \right\} \leq \sup_{x > 1/2a} \{|f''(x)|\} < \infty$ . Therefore, since  $\psi$  satisfies Assumption W(1,5), there exists  $C > 0$  such that,

$$(43) \quad \left| \mathcal{I}_\lambda(a) - \|\widehat{\psi}\|_{\mathcal{L}^2(\mathbb{R})}^2 f(1/a) \right| \leq C \frac{1}{\lambda^2}.$$

Let us denote  $I_n^{(\lambda)}(a)$  (respectively  $\mathcal{I}_\lambda(a)$ ,  $\beta_n^{(\lambda)}$  and  $S_n^{(\lambda)}(a)$ ) instead on  $I_n(a)$  (resp.  $\mathcal{I}_1(a)$ ,  $\beta_n$  and  $S_n(a)$ ) when  $\psi$  is replaced by  $\psi_\lambda$ . Firstly,

$$\frac{1}{\lambda} \left( 4\pi a^2 \int_{\mathbb{R}} |\widehat{\psi}_\lambda(az)|^4 f^2(z) dz \right) = 4\pi a \int_{\mathbb{R}} |\widehat{\psi}(u)|^4 f^2\left(\frac{1}{a} + \frac{u}{a\lambda}\right) du$$

$$\xrightarrow{\lambda \rightarrow \infty} 4\pi a f^2(1/a) \int_{\mathbb{R}} |\widehat{\psi}(u)|^4 du,$$

from Lebesgue Theorem. Hence, if  $(\lambda_n)$  is a sequence such that  $\lambda_n \xrightarrow[n \rightarrow \infty]{} \infty$ ,

$$(44) \quad \frac{\mathbb{E}(T_n)}{\lambda_n} (S_n^{(\lambda_n)}(a))^2 \xrightarrow[n \rightarrow \infty]{} 4\pi a f^2(1/a) \int_{\mathbb{R}} |\widehat{\psi}(u)|^4 du.$$

Secondly, from the proof of Proposition 7.1 and inequalities (24) and (25), there exists  $C > 0$  not depending on  $n$  and  $\lambda$ ,

$$\beta_n^{(\lambda)} / S_n^{(\lambda)} \leq C \mathcal{I}_\lambda^{-1}(a) (n \max_{1 \leq k \leq n} |c_{k+1} - c_k|)^{1/2-1/q} \quad \text{for all } q \in (1, 2).$$

Thus, since  $\mathcal{I}_\lambda(a)$  is bounded,  $\beta_n^{(\lambda_n)} / S_n^{(\lambda_n)} \xrightarrow[n \rightarrow \infty]{} 0$  and Proposition 7.1 becomes:

$$\frac{I_n^{(\lambda_n)}(a) - \mathcal{I}_{\lambda_n}(a)}{S_n^{(\lambda_n)}(a)} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1).$$

Finally, using (43) and (44), one deduces that for all  $a > 0$ ,

$$\sqrt{\frac{\mathbb{E}T_n}{\lambda_n}} (I_n^{(\lambda_n)}(a) - \|\psi\|_{\mathcal{L}^2(\mathbb{R})}^2 f(1/a)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, 4\pi a f^2(1/a) \int_{\mathbb{R}} |\widehat{\psi}(u)|^4 du\right),$$

when  $(\lambda_n)_n$  is such that  $\sqrt{\frac{\mathbb{E}T_n}{\lambda_n}} \frac{1}{\lambda_n^2} \xrightarrow[n \rightarrow \infty]{} 0$ , *i.e.* when  $\lambda_n^{-5} n \delta_n \xrightarrow[n \rightarrow \infty]{} 0$ . Since also  $\lambda_n^{-1} n \delta_n \xrightarrow[n \rightarrow \infty]{} \infty$  (to obtain a consistent estimator), then

$$(45) \quad \frac{1-d}{5} < d' < 1-d.$$

Moreover, Proposition 7.2 has also to be checked. In its proof,  $\mathbb{E}\tau_n$  has to be replaced by  $\mathbb{E}\tau_n/\lambda_n$  and since the bounds  $C(1 \wedge |\theta|^{-1})$  in Lemma 7.2 have to be replaced by  $C/\lambda_n^2(1 \wedge |\theta|^{-1})$ , then condition  $n\delta_n^2 \xrightarrow[n \rightarrow \infty]{} 0$  has to

be replaced by  $n\delta_n^2/\lambda_n^5 \xrightarrow[n \rightarrow \infty]{} 0$ , that is  $d' > \frac{1-2d'}{5}$  which is satisfied when

(45) is satisfied.

It remains to control  $\varepsilon_n^2(a, c_k)$  with Lemma 7.5 and 7.6. For all  $1 \leq q \leq \infty$ , with  $1/\infty = 0$  by convention,

$$\|\psi_\lambda\|_{\mathcal{L}^q} = \lambda^{(2-q)/2q} \|\psi\|_{\mathcal{L}^q} \quad \text{and} \quad \|\widehat{\psi}_\lambda\|_{\mathcal{L}^q} = \lambda^{(q-2)/2q} \|\widehat{\psi}\|_{\mathcal{L}^q}.$$

Then, using the choice of  $(p_1, p_2, p_3)$  obtained in Lemma 7.6, Lemma 7.5 becomes (with  $\lambda_n \xrightarrow[n \rightarrow \infty]{} \infty$ ):

- if  $s = \infty$ ,  $v_{1,n}(a) \leq C \lambda_n \delta_n^{1+H}$ ;
- if  $\max(2 + 2H, \frac{3}{2} + \frac{1}{2H}) \leq s < \infty$ ,
 
$$v_{1,n}(a) \leq C \left( \lambda_n^{\frac{s-3}{s-1}} (n \delta_n^s)^{\frac{2}{s-1}} + \lambda_n^{-\frac{2H+1}{s-1}} (n \delta_n^s)^{\frac{2H+1}{s-1}} + \lambda_n^{\frac{s-2-H}{s-1}} (n \delta_n^s)^{\frac{H+1}{s-1}} \right)$$

$$\implies v_{1,n}(a) \leq C' \lambda_n^{\frac{s-2-H}{s-1}} (n \delta_n^s)^{\frac{H+1}{s-1}} \quad \text{since } 0 < H < 1 \text{ and } s \geq 3;$$
- if  $2 + 2H \leq s < \max(2 + 2H, \frac{3}{2} + \frac{1}{2H})$ ,
 
$$\implies v_{1,n}(a) \leq C' \lambda_n^{\frac{s-2-H}{s-1}} n \delta_n^{1+3H};$$

Condition (39) is now  $\frac{n \delta_n}{\lambda_n} v_{1,n}(a) \xrightarrow[n \rightarrow \infty]{} 0$  and then, conditions required on  $d$  and  $d'$  are:

$$(46) \quad \bullet \text{ if } s = \infty, \quad d > (2 + H)^{-1};$$

$$(47) \quad \bullet \text{ if } \max(2 + 2H, \frac{3}{2} + \frac{1}{2H}) \leq s < \infty, \quad d' > \frac{s + H - d(s(H + 2) - 1)}{H + 1};$$

$$(48) \quad \bullet \text{ if } 2 + 2H \leq s < \max(2 + 2H, \frac{3}{2} + \frac{1}{2H}), \quad d' > \frac{(s - 1)(2 - d(3H + 2))}{H + 1}.$$

Finally, for  $b \leq T_n - T_n^\rho$ , with  $\psi$  satisfying Assumption W(1, 5):

$$\begin{aligned} \mathbb{E}(\varepsilon_{2,n}^{\lambda,2}(a, b) \mid \mathcal{F}_X) &= a^{-1} \int_{T_n}^{\infty} \int_{T_n}^{\infty} \psi_\lambda\left(\frac{t-b}{a}\right) \psi_\lambda\left(\frac{t'-b}{a}\right) \mathbb{E}(X(t)X(t')) dt dt' \\ &\leq C_f (a \lambda_n)^{-1} \left( \int_{T_n}^{\infty} \left| \psi\left(\frac{t-b}{a \lambda_n}\right) \right| (1 + |t|) dt \right)^2 \\ &\leq C_f (a^3 \lambda_n^3) \left( \int_{T_n^\rho/a \lambda_n}^{\infty} |\psi(u)| u du \right)^2 \\ &\leq \frac{1}{9} C_f C_\psi (a^3 \lambda_n^3) \left( [u^{-3}]_{T_n^\rho/a \lambda_n}^{\infty} \right)^2 \leq \frac{1}{9} C_f C_\psi a^9 \lambda_n^9 T_n^{-6\rho}. \end{aligned}$$

Therefore the CLT holds when  $\lambda_n^9 (n \delta_n)^{1-6\rho} \xrightarrow[n \rightarrow \infty]{} 0$ , *i.e.*  $9d' + (1-d)(6\rho - 1) < 0$ ,

$$(49) \quad \implies d' \leq \frac{1}{2} (1 - d) \quad \text{since } \rho \in (3/4, 1).$$

Combining conditions (45), (46), (47), (48) and (49) on  $d$  and  $d'$ , one deduces:

- if  $s = \infty$ ,  $d > (2 + H)^{-1}$  and  $\frac{1-d}{5} < d' \leq \frac{1-d}{2}$ ;

- if  $\max(2 + 2H, \frac{3}{2} + \frac{1}{2H}) \leq s < \infty$ ,  

$$\max\left(\frac{s + H - (s(H + 2) - 1)d}{H + 1}, \frac{1 - d}{5}\right) < d' \leq \frac{1 - d}{2};$$
- if  $2 + 2H \leq s < \max(2 + 2H, \frac{3}{2} + \frac{1}{2H})$ ,  

$$\max\left(\frac{(s - 1)(2 - d(3H + 2))}{H + 1}, \frac{1 - d}{5}\right) < d' \leq \frac{1 - d}{2}.$$

■

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