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# Locomotion of deformable bodies in an ideal fluid: Newtonian versus Lagrangian formalisms

Alexandre Munnier\*

April 9, 2008

## Abstract

It is commonly admitted in Classical Mechanics that as long as the system under consideration is monogenic, Newton's laws and Hamilton's (or D'Alembert's) principle both are equivalent in the sense that each one suffices in providing the same governing equations of motion. For more general configurations, in particular when other kind of forces (which do not derive from a scalar potential) entered the scene, the equivalence between Newtonian and Lagrangian formalisms has not been proved so far. This paper is concerned with comparing Newtonian and Lagrangian approaches for determining the governing equations of motion (usually called Euler-Lagrange's equations) for a collection of deformable bodies (also called *swimmers*) immersed in an incompressible fluid whose flow is inviscid and irrotational. The swimmers are assumed to be able to change their shapes under the action of inner forces and are endowed with *thrusters*, what means that they can generate fluid jets by sucking and blowing out fluid through some localized parts of their boundaries. These capabilities (called generically *controls*) may allow them to propel and steer themselves. Our first contribution is to prove that under smoothness assumptions on the fluid-bodies interface, Newtonian and Lagrangian formalisms yield the same equations of motion. However and quit surprisingly this is no longer true for nonsmooth shaped swimmers. The second novelty brought in in this paper is to treat a broad spectrum of problems (which may involve one or several swimmers being able to undergo any kind of deformation) and to display the Euler-Lagrange's equations as a *fully* explicit system of ODE's, involving only the value of the fluid potential function on the boundaries of the swimmers and geometric data of these surfaces. Thus, the resulting equations are very convenient for being used in a numerical scheme for simulations.

**Keywords and Phrases:** Biohydrodynamics, ideal fluid, Lagrangian and Newtonian mechanics, PDE-ODE coupled system, shape sensitivity analysis.

**AMS Subject Classification:** 74F10, 70S05, 76B03, 49Q10.

## 1 Introduction

In the last decade, significant efforts have been done by mathematicians in studying problems relating to the general theme of locomotion in a fluid by shape-changing bodies [9, 20, 34, 13, 21, 22, 19, 30] ; see also [31] for an interesting survey on this topic and further references. The underlying motivation of these works is to

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better understand the impressive swimming capabilities evolved by fish and aquatic mammals. In most of the recent mathematical models of biohydrodynamics, the shapes of the deformable bodies are assigned as a function of time. By virtue of Newton's third law (principle of action-reaction), these shape-changes generate hydrodynamic forces and torques by which the bodies propel and steer themselves.

Among all of the papers referenced above, one essentially can find two methods to derive the governing equations of motion for the fish-like swimming bodies: either authors invoke Newton's laws (and use Newtonian formalism) or they apply Hamilton's principle (and use Lagrangian formalism). Let us describe succinctly these two approaches.

With Newtonian formalism, each one of the bodies is handled as an independent mechanical system subject to exterior hydrodynamic forces and torques expressed by means of the fluid stress tensor. Then, Newton's laws for linear and angular momenta apply and yield a system of ordinary differential equations (ODE's) whose unknowns are the degrees of freedom of the bodies. The dynamics of the fluid, for its part, is commonly driven by Navier-Stokes' or Stokes' equations when it is viscous [8, 24, 30, 12, 13, 1], whereas Euler's equations are used most of time when it is perfect [32, 25, 19, 26, 29]. The coupling between bodies and fluid is realized not only through the hydrodynamic forces but only through kinematics constraints. Indeed, the region occupied by the fluid as well as its velocity along bodies' boundaries depend on the state (position and velocity) of the bodies.

Governing equations of motion can also be obtained by applying Hamilton's principle to the system fluid-bodies in its entirety [32, 25, 19, 26, 6]. Thus, the Lagrangian function is the sum of the kinetic energy of the bodies and the kinetic energy of the fluid minus the potential energy of the buoyant force. Once the Lagrangian function has been set up, the derivation of the equations of motion requires to differentiate it with respect to the so-called *generalized coordinates* corresponding to the degrees of freedom of the system. Within this process, one usually need to differentiate the solution of an initial/boundary value problem (namely, the equations of fluid's dynamics) with respect to parameters relating to the geometry of the fluid's domain. Although quite involved, this task can be carried out using tools of *shape optimization* [27, 28]. In the particular case of a potential flow and of a swimmer alone in a fluid infinitely extended, this difficulty can be overcome by choosing the *impulse* of the body as new unknown of the problem. This momentum-like amount was first introduced by Kirshoff and Tate in [23, chap. VI]. However, this method does not apply any longer when several swimmers are involved or when the fluid is partially or totally confined.

It is classical for monogenic mechanical systems (which are allowed to involve both holonomic and nonholonomic constraints), that either Newton's laws or D'Alembert's (or Hamilton's) principle suffice in providing Euler-Lagrange's governing equations of motion [4, §13 pages 64-67] (or [16, §2.3 pages 44-45]). More recently, the equivalence has been also shown for multibody system in [33]. However, to our knowledge, this equivalence has not been proved so far for more complex coupled *fluid-structure* systems like those we are dealing with. Actually, we shall prove later on by exhibiting a counterexample that there is not equivalence, even for the simple case of the solid alone in an unbounded fluid.

In this article, we will consider a set of submerged deformable bodies self-propelled partly by means of shape-changes and partly by use of thrusters, inside a potential fluid flow. The system fluid-bodies occupies either a bounded, partially bounded or unbounded domain of  $\mathbb{R}^N$  ( $N = 2, 3$ ). Shape-changes as well as the velocity of the fluid blown out by the thrusters are assigned as functions of time.

We will derive a first system of equations of motion by invoking Newton's laws for each body, while classical Euler's equations are assumed to drive the fluid flow. As explain earlier, the coupling between fluid and bodies is given through the hy-

hydrodynamic forces acting on the swimmers and through kinematics constraints for the fluid flow. Next, we will consider the system fluid-bodies in its entirety and we compute the corresponding Lagrangian function. The system being monogenic, for the only exterior force acting on it is the gravity, we can apply Hamilton's principle ; that yields a second system of equations of motion (the thrusters inducing non-holonomic constraints, generalized forces are required). A third *mixed* approach we will also consider consists in invoking D'Alembert's variational principle restricted to the dynamics of the bodies. This process leads to a third system of equations of motion.

Our first contribution is to prove that under smoothness assumptions on the fluid-bodies interface, all of the resulting systems of equations we have obtained coincide. This assertion does no longer hold true for nonsmooth shaped bodies. Indeed, we present a counterexample for which Lagrangian approach leads to well known D'Alembert's paradox (the body moves uniformly forwards inside the fluid with null hydrodynamic resultant force) whereas, by direct computation, we prove that the hydrodynamic forces on the body nearby singular points of its boundary are in fact of infinite intensity and that the resultant force can not be computed.

The second interest of this paper is to present governing equations of motion valid for wide class of problems. In particular, any number of bodies swimming together can be considered, each one being able to undergo any kind of shape-changes. In addition, the fluid region can be partially or completely bounded or of infinite extend. Final equations of motion are given under the form of a *fully* explicit system of ODE's whose unknowns are the degrees of freedom of the problem. To solve these equations, it is required only to compute potential functions on the boundaries of the swimmers. For being solutions of Neumann boundary value problems, these potential functions can be computed by means of boundary integral formula, that is, only in terms of boundary data. As a consequence, from a numerical point of view, the problem can be solved by meshing the fluid's boundary only. This method has already been applied successfully and has yielded numerical simulations that can be downloaded at <http://www.iecn.u-nancy.fr/~munnier/animations.html>.

## Organization of the paper

Next section is roughly devoted to problem's modelling: topological and geometrical data are made precise, shape-changes are properly described as assigned functions of time and the notion of *virtual displacements* is introduced, in order to apply D'Alembert's principle subsequently. Newton's laws are stated for the bodies, fluid's dynamics and coupling conditions are set up. The Lagrangian function relating to the dynamics of the swimmers is also computed. This work is summed up and used and in Section 3, where we display the equations of motion for both Lagrangian and Newtonian models. We state the main results: the first one discusses the equivalence between both models. The second consists in giving the fully explicit form of the equations of motion which are realized as a system of ODE's. Comparison between both models is carried out in Section 4 where the computation of *shape derivatives* is required to achieve this task. In Section 5, we treat the case of a nonsmooth shaped solid surrounded by an infinitely extended fluid. We prove that Newtonian and Lagrangian approaches lead to distinct conclusions. The appendix contains technical results and definitions.

## 2 Setting of the problem

### 2.1 Topological and geometrical data

We denote by  $S(t) := \cup_{i=1}^n S^i(t)$  the region of  $\mathbb{R}^N$  ( $N = 2, 3$ ) occupied by the  $n$  swimmers ( $n \in \mathbb{N}$ ,  $n \geq 1$ ) at time  $t$  and by  $\Omega(t)$  the domain of the surrounding fluid. The boundary  $\partial\Omega(t)$  of the fluid, denoted by  $\Gamma(t)$ , is assumed to be a twice continuously differentiable surface ( $N = 3$ ) or curve ( $N = 2$ ). It can be decomposed into the boundary shared with the swimmers  $\Gamma_2(t) := \cup_{i=1}^n \Gamma_2^i(t) := \cup_{i=1}^n \partial S^i(t)$  and the exterior boundary  $\Gamma_1 := \partial\Omega(t) \setminus \Gamma_2(t)$ . At the time  $t = 0$ , we denote merely  $\Omega := \Omega(0)$ ,  $S_0^i := S^i(0)$ ,  $S_0 := S(0)$ ,  $\Gamma_2^i := \Gamma_2^i(0)$  and  $\Gamma_2 := \Gamma_2(0)$ . We assume that  $\Omega$  is an open and connected set in  $\mathbb{R}^N$ , that  $\Gamma_2$  is compact and that  $\Gamma_1$  is either bounded, either an hyperplane. On  $\Gamma(t)$  and for all  $t \geq 0$ , we denote by  $\mathbf{n}(t, \mathbf{x})$  the unitary normal at the point  $\mathbf{x}$  directed towards the exterior of the fluid. When  $N = 3$ , we introduce the second fundamental form of  $\Gamma(t)$ , denoted by  $\Pi_2(t, \mathbf{x})$ . It is connected to the normal vector  $\mathbf{n}(t, \mathbf{x})$  and to the principal curvatures  $\kappa_1(t, \mathbf{x})$ ,  $\kappa_2(t, \mathbf{x})$  of  $\Gamma(t)$  at the point  $\mathbf{x}$  by the formula:

$$\langle [D_{\mathbf{x}}\mathbf{n}(t, \mathbf{x})], \boldsymbol{\tau}_i(t, \mathbf{x}), \boldsymbol{\tau}_i(t, \mathbf{x}) \rangle = -\langle \Pi_2(t, \mathbf{x}), \boldsymbol{\tau}_i(t, \mathbf{x}), \boldsymbol{\tau}_i(t, \mathbf{x}) \rangle \quad (2.1a)$$

$$= -\kappa_i(t, \mathbf{x})\boldsymbol{\tau}_i(t, \mathbf{x}), \quad i = 1, 2, \quad (2.1b)$$

where  $[D_{\mathbf{x}}\mathbf{n}(t, \mathbf{x})]$  is the Jacobian matrix of  $\mathbf{n}(t, \mathbf{x})$  at the point  $\mathbf{x}$  and  $\boldsymbol{\tau}_i(t, \mathbf{x})$  ( $i = 1, 2$ ) are the so-called principal directions of  $\Gamma(t)$  (see [10, page 129] for precise definitions). The signs of  $\kappa_1(t, \mathbf{x})$  and  $\kappa_2(t, \mathbf{x})$  are chosen such that  $\{\mathbf{n}(t, \mathbf{x}), \boldsymbol{\tau}_1(t, \mathbf{x}), \boldsymbol{\tau}_2(t, \mathbf{x})\}$  be a direct orthogonal basis of  $\mathbb{R}^3$ . We denote by  $\mathcal{H}(t, \mathbf{x}) := \kappa_1(t, \mathbf{x}) + \kappa_2(t, \mathbf{x})$  the mean curvature of  $\Gamma(t)$ . When  $N = 2$ ,  $\mathcal{H}(t, \mathbf{x})$  coincides with  $\kappa(t, \mathbf{x})$ , the curvature of  $\Gamma(t)$  and relation (2.1) turns into:

$$\nabla_{\Gamma}\mathbf{n}(t, \mathbf{x}) \cdot \boldsymbol{\tau}(t, \mathbf{x}) = -\mathcal{H}(t, \mathbf{x})\boldsymbol{\tau}(t, \mathbf{x}),$$

where  $\nabla_{\Gamma}$  is the tangential gradient (or merely the tangential derivative). As for  $N = 3$ , the sign of  $\mathcal{H}(t, \mathbf{x})$  is chosen such that  $\{\mathbf{n}(t, \mathbf{x}), \boldsymbol{\tau}(t, \mathbf{x})\}$  be a direct orthogonal basis of  $\mathbb{R}^2$ .

### 2.2 Kinematics of the deformable bodies

Let  $\mathcal{F}$  be an inertial reference frame. In  $\mathcal{F}$ , the coordinates of the center of mass of the  $i$ -th swimmer are  $\mathbf{h}^i(t) := (h_1^i(t), \dots, h_N^i(t))^T$  at time  $t$ . We fix a frame  $\mathcal{F}^i(t)$  of origin  $\mathbf{h}^i(t)$  to each swimmer and we write merely  $\mathbf{h}_0^i := \mathbf{h}^i(0)$  and  $\mathcal{F}^i(0) := \mathcal{F}_0^i$ . Considering  $M(t)$  a material point of  $S^i(t)$  of coordinates  $\mathbf{x}(t) := (x_1(t), \dots, x_N(t))^T$  in  $\mathcal{F}$  and  $\mathbf{y}(t) := (y_1(t), \dots, y_N(t))^T$  in  $\mathcal{F}^i(t)$ , there exists a rotation matrix  $[R^i(t)] \in SO(N)$  (the rotation group of  $\mathbb{R}^N$ ,  $N = 2, 3$ ) such that  $\mathbf{x}(t) = [R^i(t)]\mathbf{y}(t) + \mathbf{h}^i(t)$ . We can assume, without loss of generality, that  $[R^i(0)] = [\text{Id}]$  (the identity matrix). We assume that the motion of  $M(t)$  in  $\mathcal{F}^i(t)$  is only due to prescribed deformations of the body  $S^i(t)$ . Introducing then the domain  $S_d^i(t) := [R^i(t)]^T(S^i(t) - \mathbf{h}^i(t)) + \mathbf{h}_0^i$ , the deformation of the body from its initial state  $S_0^i$  into  $S_d^i(t)$  is described in  $\mathcal{F}_0^i$  by a given twice continuously differentiable function  $\mathbf{X}_d^i : [0, \infty) \times (S_0^i - \mathbf{h}_0^i) \rightarrow \mathbb{R}^N$  such that for all  $t \geq 0$  the mapping:

$$\begin{aligned} \mathbf{X}_d^i(t, \cdot) : S_0^i - \mathbf{h}_0^i &\rightarrow S_d^i(t) - \mathbf{h}_0^i \\ \mathbf{y}_0 := \mathbf{x}_0 - \mathbf{h}_0^i &\mapsto \mathbf{X}_d^i(t, \mathbf{y}_0) := \mathbf{y}(t), \end{aligned} \quad (2.2)$$

be one-to-one. We set then  $\mathbf{Y}_d^i(t, \cdot) := \mathbf{X}_d^i(t, \cdot)^{-1}$ . The coordinates of  $M(t)$  in the inertial frame  $\mathcal{F}$  read:

$$\mathbf{x}(t) = [R^i(t)]\mathbf{X}_d^i(t, \mathbf{x}_0 - \mathbf{h}_0^i) + \mathbf{h}^i(t), \quad \forall \mathbf{x}_0 \in S_0^i. \quad (2.3)$$

We define likewise, for all  $t \geq 0$ , the mapping related to the rigid motion by:

$$\begin{aligned} \mathbf{X}_r^i(t, \cdot) : S_d^i(t) &\rightarrow S^i(t) \\ \mathbf{x} &\mapsto \mathbf{X}_r^i(t, \mathbf{y}) := [R^i(t)](\mathbf{x} - \mathbf{h}_0^i) + \mathbf{h}^i(t). \end{aligned} \quad (2.4)$$

We set then  $\mathbf{Y}_r^i(t, \cdot) := \mathbf{X}_r^i(t, \cdot)^{-1}$  and we introduce the composition of both functions (2.2) and (2.4) in  $\mathcal{F}$ :

$$\begin{aligned} \mathbf{X}^i : [0, \infty) \times S_0^i &\rightarrow \mathbb{R}^N \\ (t, \mathbf{x}) &\mapsto \mathbf{X}_r^i(t, \mathbf{X}_d^i(t, \mathbf{x} - \mathbf{h}_0^i) + \mathbf{h}_0^i). \end{aligned} \quad (2.5)$$

As previously done, we define also  $\mathbf{Y}^i(t, \cdot) := \mathbf{X}^i(t, \cdot)^{-1}$ . Using the partition  $\cup_{i=1}^n S_0^i$  of  $S_0$ , we set for all  $\mathbf{x} \in S_0$ :  $\mathbf{X}(t, \mathbf{x}) := \mathbf{X}^i(t, \mathbf{x})$  if  $\mathbf{x} \in S_0^i$ . Differentiating (2.3) with respect to  $t$  one obtains:  $\dot{\mathbf{x}}(t) = [\dot{R}^i(t)]\mathbf{X}_d^i(t, \mathbf{x}_0 - \mathbf{h}_0^i) + [R^i(t)]\dot{\mathbf{X}}_d^i(t, \mathbf{x}_0 - \mathbf{h}_0^i) + \dot{\mathbf{h}}^i(t)$ . Combining again this expression with (2.3), it comes:

$$\dot{\mathbf{x}}(t) = [\dot{R}^i(t)][R^i(t)]^T(\mathbf{x}(t) - \mathbf{h}^i(t)) + \dot{\mathbf{h}}^i(t) + [R^i(t)]\dot{\mathbf{X}}_d^i(t, \mathbf{Y}^i(t, \mathbf{x}(t)) - \mathbf{h}_0^i).$$

The matrix  $[\dot{R}^i(t)][R^i(t)]^T$  being skew symmetric, there exists a scalar  $\omega^i(t)$  when  $N = 2$  and a vector  $\boldsymbol{\omega}^i(t)$  when  $N = 3$  such that, for all  $\mathbf{x} \in S^i(t)$ :

$$[\dot{R}^i(t)][R^i(t)]^T(\mathbf{x}(t) - \mathbf{h}^i(t)) = \begin{cases} \omega^i(t)(\mathbf{x}(t) - \mathbf{h}^i(t))^\perp & (N = 2), \\ \boldsymbol{\omega}^i(t) \times (\mathbf{x}(t) - \mathbf{h}^i(t)) & (N = 3), \end{cases}$$

where  $\mathbf{x}^\perp = ((x_1, x_2)^\perp)^T := (-x_2, x_1)^T$  for all  $\mathbf{x} \in \mathbb{R}^2$ . We can now define, for all  $\mathbf{x} \in S(t)$ , the Eulerian velocity of deformation  $\mathbf{v}_d^i(t)$  and the Eulerian rigid velocity  $\mathbf{v}_r^i(t)$  by:

$$\mathbf{v}_d^i(t, \mathbf{x}) := \begin{cases} [R^i(t)]\dot{\mathbf{X}}_d^i(t, \mathbf{Y}^i(t, \mathbf{x}) - \mathbf{h}_0^i) & \text{if } \mathbf{x} \in S^i(t), \\ \mathbf{0} & \text{if } \mathbf{x} \in \cup_{j \neq i} S^j(t), \end{cases} \quad (2.6a)$$

$$\mathbf{v}_r^i(t, \mathbf{x}) := \begin{cases} \omega^i(t)(\mathbf{x} - \mathbf{h}^i(t))^\perp + \dot{\mathbf{h}}^i(t) & \text{if } \mathbf{x} \in S^i(t), N = 2, \\ \boldsymbol{\omega}^i(t) \times (\mathbf{x} - \mathbf{h}^i(t)) + \dot{\mathbf{h}}^i(t) & \text{if } \mathbf{x} \in S^i(t), N = 3, \\ \mathbf{0} & \text{if } \mathbf{x} \in \cup_{j \neq i} S^j(t). \end{cases} \quad (2.6b)$$

The Eulerian velocity in  $S^i(t)$  is therefore:

$$\mathbf{v}^i(t, \mathbf{x}) := \mathbf{v}_r^i(t, \mathbf{x}) + \mathbf{v}_d^i(t, \mathbf{x}), \quad \forall \mathbf{x} \in S^i(t), \forall t \geq 0. \quad (2.7)$$

### 2.3 Generalized coordinates and virtual displacements

Classical results in Solid Mechanics (see for example [16, Chap.4 pages 134-180]) tell us that the location and orientation of the  $i$ -th swimmer, considered as a rigid body, can be described by  $\mathcal{N}_r^i := N(N+1)/2$  *independent* (this term will be made precise later on) real coordinates  $q_{r,1}^i, \dots, q_{r,\mathcal{N}_r^i}^i$  (notice although  $\mathcal{N}_r^i$  actually does not depend on  $i$ , we keep this notation to be consistent with the following). We denote by  $\mathbf{q}_r^i := (q_{r,1}^i, \dots, q_{r,\mathcal{N}_r^i}^i)^T$  and  $\mathbf{q}_r := (\mathbf{q}_r^1, \dots, \mathbf{q}_r^n)^T$  the so-called *generalized* coordinates related to the rigid motion and  $\dot{\mathbf{q}}_r := (\dot{\mathbf{q}}_r^1, \dots, \dot{\mathbf{q}}_r^n)^T$  stand for the *generalized* velocities.

We assume that all of the controls (that are the shape-changes and the thrusters) are governed by a finite number of parameters and we denote for all  $i = 1, \dots, n$ :

- $q_{d,k}^i \in \mathbb{R}$  for all  $k = 1, \dots, \mathcal{N}_d^i$  ( $\mathcal{N}_d^i \in \mathbb{N}$ ), the  $\mathcal{N}_d^i$  parameters related to the deformation of the  $i$ -th swimmer.

- $q_{t,l}^i \in \mathbb{R}$  for all  $l = 1, \dots, \mathcal{N}_t^i$  ( $\mathcal{N}_t^i \in \mathbb{N}$ ), the  $\mathcal{N}_t^i$  parameters related to its thrusters.

Following our rules of notation, we introduce also the integers  $\mathcal{N}_r := \sum_{i=1}^n \mathcal{N}_r^i$ ,  $\mathcal{N}_d := \sum_{i=1}^n \mathcal{N}_d^i$  and  $\mathcal{N}_t := \sum_{i=1}^n \mathcal{N}_t^i$  as well as the vectors  $\mathbf{q}_d^i := (q_{d,1}^i, \dots, q_{d,\mathcal{N}_d^i}^i)^T$ ,  $\mathbf{q}_d := (\mathbf{q}_d^1, \dots, \mathbf{q}_d^n)^T$ ,  $\mathbf{q}_r^i := (q_{r,1}^i, \dots, q_{r,\mathcal{N}_r^i}^i)^T$ ,  $\mathbf{q}_r := (\mathbf{q}_r^1, \dots, \mathbf{q}_r^n)^T$  and the associated velocities  $\dot{\mathbf{q}}_d^i$ ,  $\dot{\mathbf{q}}_d$ ,  $\dot{\mathbf{q}}_r^i$  and  $\dot{\mathbf{q}}_r$ . Eventually, we set  $\mathcal{N} := \mathcal{N}_r + \mathcal{N}_d + \mathcal{N}_t$ ,  $\mathbf{q} := (\mathbf{q}_r, \mathbf{q}_d, \mathbf{q}_t)^T$  and  $\dot{\mathbf{q}} := (\dot{\mathbf{q}}_r, \dot{\mathbf{q}}_d, \dot{\mathbf{q}}_t)^T$ . The components of  $\mathbf{q}$  and  $\dot{\mathbf{q}}$  are hence denoted merely by  $q_i$  and  $\dot{q}_i$  respectively,  $i = 1, \dots, \mathcal{N}$ .

At this point, it is worth pointing out that all of the involved domains depend on time  $t$  only through the *shape* variables  $\mathbf{q}_r$  and  $\mathbf{q}_d$ . Thus, from now on, we will rather denote  $\Omega(\mathbf{q}_r, \mathbf{q}_d)$  the domain of the fluid,  $S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  the domain of the  $i$ -th swimmer and likewise, we set  $S(\mathbf{q}_r, \mathbf{q}_d) := \cup_{i=1}^n S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ ,  $\Gamma(\mathbf{q}_r, \mathbf{q}_d) := \Gamma(t)$ ,  $\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i) := \partial S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  and  $\Gamma_2(\mathbf{q}_r, \mathbf{q}_d) := \cup_{i=1}^n \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ . We adopt also the notation  $\mathbf{X}_d^i(\mathbf{q}_d^i)$ ,  $\mathbf{X}_r^i(\mathbf{q}_r^i)$ ,  $\mathbf{X}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  for the functions (2.2), (2.4) and (2.5) while notation for the Eulerian velocities (2.6) turns respectively into  $\mathbf{v}_d^i(\mathbf{q}_d^i, \dot{\mathbf{q}}_d^i)$ ,  $\mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)$  and  $\mathbf{v}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i)$ . In order to apply d'Alembert's principle, in a first time to

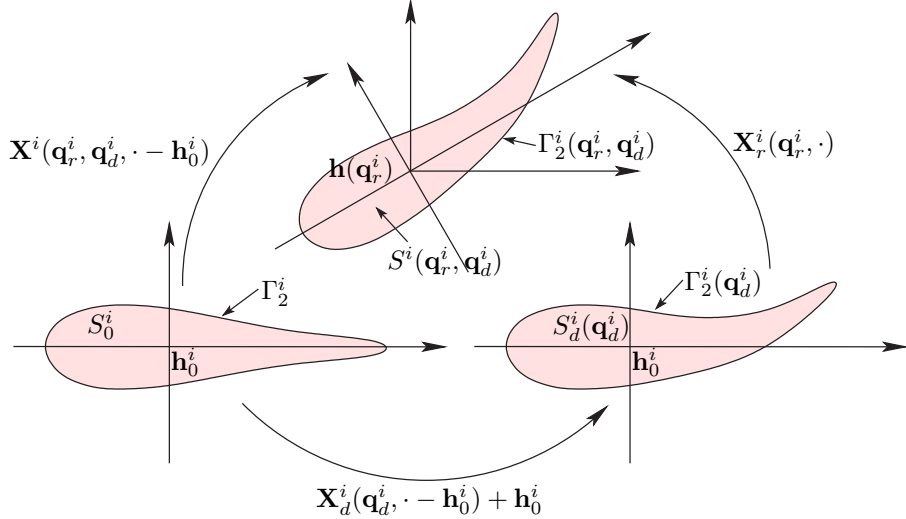


Figure 1: The mappings  $\mathbf{X}_d^i(\mathbf{q}_d^i)$ ,  $\mathbf{X}_r^i(\mathbf{q}_r^i)$  and  $\mathbf{X}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  for the  $i$ -th fish-like deformable body.

the dynamics of the deformable bodies only and later to the full system fluid-bodies, we introduce, like in [16, §1.4 page 16] or in [4, §B. page 97], the notion of virtual rigid displacement:

**Definition 1** For all  $i = 1, \dots, n$ , we call *infinitesimal virtual rigid displacement* (and we write *i. v. r. d.* for short in the sequel) defined in  $S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  any vector field  $\delta\widehat{\mathbf{x}}_r(\mathbf{q}_r^i, \mathbf{x})$  ( $\mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ ) that can be written under the form:

$$\delta\widehat{\mathbf{x}}_r(\mathbf{q}_r^i, \mathbf{x}) = \begin{cases} \delta\widehat{\Omega}(\mathbf{q}_r^i) \times (\mathbf{x} - \mathbf{h}^i(\mathbf{x})) + \delta\widehat{\mathbf{h}}(\mathbf{q}_r^i) & (N = 3), \\ \delta\widehat{\Omega}(\mathbf{q}_r^i)(\mathbf{x} - \mathbf{h}^i(\mathbf{x}))^\perp + \delta\widehat{\mathbf{h}}(\mathbf{q}_r^i) & (N = 2), \end{cases} \quad (2.8)$$

where  $\delta\widehat{\Omega}(\mathbf{q}_r^i)$  and  $\delta\widehat{\mathbf{h}}(\mathbf{q}_r^i)$  are two vectors and  $\delta\widehat{\Omega}(\mathbf{q}_r^i)$  is a scalar. We denote by  $\mathcal{D}_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  the vector space of all the *i. v. r. d.* defined in  $S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ .

Going back to the generalized coordinates, we introduce, for any  $i = 1, \dots, n$  and

any  $j = 1, \dots, \mathcal{N}_r^i$ , the vector field  $\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i)$  defined by:

$$\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x}) := \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{x})), \quad \forall \mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \quad (2.9a)$$

or equivalently, taking into account definition (2.5) of  $\mathbf{X}^i$ , by:

$$\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x}) := \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{Y}_r^i(\mathbf{q}_r^i, \mathbf{x})), \quad \forall \mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i). \quad (2.9b)$$

**Proposition 2.1** For all  $i = 1, \dots, n$  and all  $j = 1, \dots, \mathcal{N}_r^i$ ,  $\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i) \in \mathcal{D}_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ .

Before writing out the proof of this proposition, let us introduce a definition:

**Definition 2** According to Definition 1 and Proposition 2.1 and for any  $i = 1, \dots, n$  and any  $j = 1, \dots, \mathcal{N}_r^i$ , we denote by  $\delta \Omega_j^i(\mathbf{q}_{r,j}^i)$ ,  $\delta \Omega_j^i(\mathbf{q}_{r,j}^i)$  and  $\delta \mathbf{h}_j^i(\mathbf{q}_{r,j}^i)$  the vectors and the scalar such that:

$$\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x}) = \begin{cases} \delta \Omega_j^i(\mathbf{q}_r^i) \times (\mathbf{x} - \mathbf{h}^i) + \delta \mathbf{h}_j^i(\mathbf{q}_r^i), & (N = 3), \\ \delta \Omega_j^i(\mathbf{q}_r^i) (\mathbf{x} - \mathbf{h}^i)^\perp + \delta \mathbf{h}_j^i(\mathbf{q}_r^i), & (N = 2). \end{cases}$$

**Proof of Proposition 2.1:** Definition (2.4) can be rewritten with generalized coordinates as:

$$\mathbf{X}_r^i(\mathbf{q}_r^i, \mathbf{x}) = [R^i(\mathbf{q}_r^i)](\mathbf{x} - \mathbf{h}_0^i) + \mathbf{h}^i(\mathbf{q}_r^i), \quad (2.10)$$

and then definition (2.9b) leads to:

$$\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i) = \frac{\partial [R^i(\mathbf{q}_r^i)]}{\partial q_{r,j}^i} [R^i(\mathbf{q}_r^i)]^T (\mathbf{x} - \mathbf{h}^i(\mathbf{q}_r^i)) + \frac{\partial \mathbf{h}}{\partial q_{r,j}^i}(\mathbf{q}_r^i).$$

The matrix  $(\partial [R^i(\mathbf{q}_r^i)] / \partial q_{r,j}^i) [R^i(\mathbf{q}_r^i)]^T$  being skew symmetric, the expression of  $\delta \mathbf{x}_r^i(\mathbf{q}_r^i)$  matches the forms given in Definition 2. ■

We are now in position to make precise what is an *allowable choice* of generalized coordinates:

**Definition 3** The set of generalized coordinates  $q_{r,j}^i$  is allowable (and in particular, the coordinates are independent) when the set  $\{\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i), j = 1, \dots, \mathcal{N}_r^i\}$  is a basis of  $\mathcal{D}_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ .

When  $N = 2$ , the choice of such of set of coordinates is trivial. When  $N = 3$ , Euler's angles yield a classical local parametrization for the orientation of the bodies (or, in other words, a local chart of the manifold  $SO(3)$ ). We refer for instance to [16, Chap.4 pages 134-180] or to [28] for examples of generalized coordinates that fit with what is required in Definition 3.

We can express  $\mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)$  defined in (2.6), in terms of  $\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i)$  and  $\dot{\mathbf{q}}_r^i$  as follows:

$$\mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i, \mathbf{x}) = \sum_{j=1}^{\mathcal{N}_r^i} \dot{q}_{r,j}^i \delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x}), \quad \forall \mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \quad (2.11)$$

and we introduce also, for all  $i = 1, \dots, n$  and all  $j, k = 1, \dots, \mathcal{N}_r^i$ :

$$\delta \tilde{\mathbf{x}}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x}) := \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{x}), \quad \forall \mathbf{x} \in S_d^i(\mathbf{q}_d^i), \quad (2.12)$$

$$\tilde{\mathbf{v}}_r^i(\mathbf{q}_r^i, \mathbf{x}) := \sum_{j=1}^{\mathcal{N}_r^i} \dot{q}_{r,j}^i \delta \tilde{\mathbf{x}}_{r,j}^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i, \mathbf{x}), \quad \forall \mathbf{x} \in S_d^i(\mathbf{q}_d^i). \quad (2.13)$$



One readily checks that  $\nabla \cdot \delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) = 0$  in  $S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  for any i. v. r. d. and since  $\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  is of class  $C^2$ , we deduce, invoking Stokes' formula, that:

$$\int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i, \mathbf{x}) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \, d\mathbf{x} = 0, \quad \forall \delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) \in \mathcal{D}_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i). \quad (2.14)$$

This condition implies that the volume of the incompressible fluid (when the fluid is confined) is not affected by rigid displacements of the bodies.

We proceed similarly for the velocities of deformation. Thus, we define, for all  $i = 1, \dots, n$  and all  $k = 1, \dots, \mathcal{N}_d^i$ , the infinitesimal virtual displacements of deformation (abbreviated by i. v. d. d. in the sequel):

$$\delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) := [R^i(\mathbf{q}_r^i)] \frac{\partial \mathbf{X}_d^i}{\partial q_{d,k}^i}(\mathbf{q}_d^i, \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) - \mathbf{h}_0^i), \quad \forall \mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i),$$

or equivalently, taking into account definition (2.5), by:

$$\delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) := \frac{\partial \mathbf{X}_d^i}{\partial q_{d,k}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})), \quad \forall \mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i).$$

The i. v. d. d. and the generalized velocities  $\dot{\mathbf{q}}_d^i$  are connected to  $\mathbf{v}_d^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_d^i)$  by the equality:

$$\mathbf{v}_d^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_d^i, \mathbf{x}) = \sum_{k=1}^{\mathcal{N}_d^i} \dot{q}_{d,k}^i \delta \mathbf{x}_{d,k}^i(\mathbf{q}_d^i, \mathbf{x}), \quad \forall \mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i). \quad (2.15)$$

We introduce also, for any  $k = 1, \dots, \mathcal{N}_d^i$ :

$$\delta \widetilde{\mathbf{x}}_{d,k}^i(\mathbf{q}_d^i, \mathbf{x}) := \frac{\partial \mathbf{X}_d^i}{\partial q_{d,k}^i}(\mathbf{q}_d^i, \mathbf{Y}_d^i(\mathbf{q}_d^i, \mathbf{x} - \mathbf{h}_0^i)), \quad \forall \mathbf{x} \in S_d^i(\mathbf{q}_d^i).$$

Remind that the functions of deformation  $\mathbf{X}_d^i$  are given. Hence,  $\delta \widetilde{\mathbf{x}}_{d,k}^i(\mathbf{q}_d^i)$  is also a prescribed quantity which coincides with  $\delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  when expressed in the moving frame  $\mathcal{F}^i(\mathbf{q}_r^i)$ . To ensure that the volume of the fluid does not depend on the shape of the swimmers (i.e. on the parameter  $\mathbf{q}_d$ ), we assume that, for all  $i = 1, \dots, n$  and all  $k = 1, \dots, \mathcal{N}_d^i$ , the i. v. d. d. satisfy:

$$\int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \, d\mathbf{x} = 0. \quad (2.16)$$

This equality means also that one imposes the deformations to preserve the volume of the swimmers.

To model the action of the thrusters, we define on  $\Gamma_2^i$  (the boundary of the bodies at the time  $t = 0$ ) infinitesimal virtual displacements of blown out fluid (we will write i. v. d. b. f. for conciseness) denoted by  $\delta \widetilde{\mathbf{x}}_{t,l}^i$  for all  $l = 1, \dots, \mathcal{N}_t^i$  and we set, for all  $\mathbf{x} \in \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ :

$$\delta \mathbf{x}_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) := [D_{\mathbf{x}} \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})]^{-1} \delta \widetilde{\mathbf{x}}_{t,l}^i(\mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})),$$

where  $[D_{\mathbf{x}} \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})]$  is the Jacobian matrix of  $\mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ . The Eulerian velocity  $\dot{q}_{t,l}^i \delta \mathbf{x}_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})$  is the prescribed velocity of the blown out (or sucked) fluid by the  $l$ -th thruster measured at a point  $\mathbf{x}$  of the boundary of the  $i$ -th body. Summing over all the thrusters of the  $i$ -th body, we define:

$$\mathbf{v}_t^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_t^i, \mathbf{x}) := \sum_{l=1}^{\mathcal{N}_t^i} \dot{q}_{t,l}^i \delta \mathbf{x}_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}), \quad \forall \mathbf{x} \in \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \quad (2.17)$$

Notice that the parameters  $q_{t,l}^i$  do not actually play any role in the modelling. Once the i. v. d. b. f. have been set up, the regime of the thrusters is controlled by the generalized velocities  $\dot{q}_{t,l}^i$  only. The balance of fluid volume that runs through the thrusters being null, we assume in addition that, for all  $i = 1, \dots, n$  and all  $l = 1, \dots, \mathcal{N}_t^i$ :

$$\int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \delta \mathbf{x}_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \, d\mathbf{x} = 0. \quad (2.18)$$

Let us state a last property entering in this section:

**Lemma 2.1** *For any  $i = 1, \dots, n$  and any  $j = 1, \dots, \mathcal{N}_r^i$ , the following identities hold:*

$$\frac{d}{dt} \frac{\partial \tilde{\mathbf{v}}_r^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i, \mathbf{x}) - \frac{\partial \tilde{\mathbf{v}}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i, \mathbf{x}) = \mathbf{0}, \quad \forall \mathbf{x} \in S_d^i(\mathbf{q}_d^i), \quad (2.19)$$

where  $\tilde{\mathbf{v}}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)$  is given in (2.13) and

$$\begin{aligned} \frac{d}{dt}(\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i)) = & \\ & \begin{cases} \frac{d}{dt}(\delta \tilde{\mathbf{x}}_{r,j}^i(\mathbf{q}_r^i)) \circ \mathbf{Y}_r^i(\mathbf{q}_r^i) - \delta \Omega_j(\mathbf{q}_r^i) \times \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) & (N = 3), \\ \frac{d}{dt}(\delta \tilde{\mathbf{x}}_{r,j}^i(\mathbf{q}_r^i)) \circ \mathbf{Y}_r^i(\mathbf{q}_r^i) - \delta \Omega_j(\mathbf{q}_r^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)^\perp & (N = 2), \end{cases} \end{aligned} \quad (2.20)$$

the mapping  $\mathbf{Y}_r^i(\mathbf{q}_r^i)$  being defined just beneath formula (2.4).

**Proof :** According to definitions (2.13) of  $\tilde{\mathbf{v}}_r^i(\mathbf{q}_r^i, \mathbf{x})$  and (2.12) of  $\delta \tilde{\mathbf{x}}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x})$ , we have, on the one hand:

$$\frac{d}{dt} \frac{\partial \tilde{\mathbf{v}}_r^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{x}) = \frac{d}{dt} \delta \tilde{\mathbf{x}}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x}) = \frac{d}{dt} \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{x}), \quad \forall \mathbf{x} \in S_d^i(\mathbf{q}_d^i). \quad (2.21)$$

On the other hand, by virtue of the chain rule, we can compute the time derivative as follows:

$$\frac{d}{dt} \mathbf{X}_r^i(\mathbf{q}_r^i, \mathbf{x}) = \sum_{j=1}^{\mathcal{N}_r^i} \dot{q}_{r,j}^i \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{x}) = \tilde{\mathbf{v}}_r^i(\mathbf{q}_r^i, \mathbf{x}), \quad \forall \mathbf{x} \in S_d^i(\mathbf{q}_d^i),$$

and then

$$\frac{\partial \tilde{\mathbf{v}}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{x}) = \frac{\partial}{\partial q_{r,j}^i} \frac{d}{dt} \mathbf{X}_r^i(\mathbf{q}_r^i, \mathbf{x}), \quad \forall \mathbf{x} \in S_d^i(\mathbf{q}_d^i). \quad (2.22)$$

Identity (2.19) follows since both cross derivatives arising in the right hand side of (2.21) and (2.22) are equal. Concerning the second formula of the lemma, definition (2.9b) leads to:

$$\frac{d}{dt}(\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i)) = \frac{d}{dt} \left( \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{Y}^i(\mathbf{q}_r^i)) \right),$$

what reads also, applying again the chain rule:

$$\begin{aligned} \frac{d}{dt}(\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i)) = & \sum_{j_1=1}^{\mathcal{N}_r^i} \frac{\partial^2 \mathbf{X}_r^i}{\partial q_{r,j_1}^i \partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{Y}^i(\mathbf{q}_r^i)) \dot{q}_{j_1}^i \\ & + \left[ D_{\mathbf{x}} \left( \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j}^i} \right) (\mathbf{q}_r^i, \mathbf{Y}^i(\mathbf{q}_r^i)) \right] \frac{\partial \mathbf{Y}^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i) \dot{q}_{j_1}^i. \end{aligned} \quad (2.23)$$

Proceeding as well with  $\delta\tilde{\mathbf{x}}_{r,j}^i(\mathbf{q}_r^i)$  defined by (2.12), we get:

$$\frac{d}{dt}(\delta\tilde{\mathbf{x}}_{r,j}^i(\mathbf{q}_r^i)) \circ \mathbf{Y}_r^i(\mathbf{q}_r^i) = \sum_{j_1=1}^{\mathcal{N}_r^i} \frac{\partial^2 \mathbf{X}_r^i}{\partial q_{r,j_1}^i \partial q_{r,j}^i}(\mathbf{q}^i, \mathbf{Y}_r^i(\mathbf{q}_r^i)) \dot{q}_{j_1}^i, \quad (2.24)$$

and taking into account identity (2.10), one obtains:

$$\begin{aligned} \left[ D_{\mathbf{x}} \left( \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j}^i} \right) (\mathbf{q}^i, \mathbf{Y}_r^i(\mathbf{q}_r^i)) \right] \frac{\partial \mathbf{Y}_r^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i) \dot{q}_{r,j_1}^i = \\ - \frac{\partial [R^i(\mathbf{q}_r^i)]}{\partial q_{r,j}^i} [R^i(\mathbf{q}_r^i)]^T \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i, \mathbf{Y}_r^i(\mathbf{q}_r^i)) \dot{q}_{r,j_1}^i. \end{aligned} \quad (2.25)$$

Summing over  $j_1$  from 1 to  $\mathcal{N}_r^i$  identities (2.25) and substituting the result and the right hand side of (2.24) into (2.23), we obtain (2.20). ■

Further quantities of less importance but required in the statement of the governing equations of motion are defined in Appendix A.

## 2.4 Lagrangian function of the shape-changing bodies

We denote by  $\rho_S(\mathbf{q}_r, \mathbf{q}_d, \mathbf{x}) > 0$  the density of the bodies at the point  $\mathbf{x} \in S(\mathbf{q}_r, \mathbf{q}_d)$  while  $\rho_S^d(\mathbf{q}_d, \mathbf{x}) > 0$  is the density in  $S_d(\mathbf{q}_d)$ . For all  $\mathbf{x} \in S_0$  we set also  $\rho_S^0(\mathbf{x}) := \rho_S(\mathbf{x}, 0)$ . The law of conservation of mass leads to the identities:

$$\rho_S^d(\mathbf{q}_d^i, \mathbf{x}) = \rho_S^0(\mathbf{Y}_d^i(\mathbf{q}_d^i, \mathbf{x} - \mathbf{h}_0^i) + \mathbf{h}_0^i) |\det[D_{\mathbf{x}} \mathbf{Y}_d^i(\mathbf{q}_d^i, \mathbf{x} - \mathbf{h}_0^i)]|, \quad \forall \mathbf{x} \in S_d^i(\mathbf{q}_d^i), \quad (2.26a)$$

and

$$\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) = \rho_S^0(\mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})) |\det[D_{\mathbf{x}} \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})]|, \quad \forall \mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i). \quad (2.26b)$$

Thus, the density of the deformed body is thoroughly determined by its initial density and by the prescribed deformation. Since  $|\det[D_{\mathbf{x}} \mathbf{X}_r(\mathbf{q}_r, \mathbf{x})]| = 1$  for all  $\mathbf{x} \in S_d(\mathbf{q}_d)$  (see definition (2.4)), we have also:

$$\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) = \rho_S^d(\mathbf{q}_d^i, \mathbf{Y}_r^i(\mathbf{q}_r^i, \mathbf{x})), \quad \forall \mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i). \quad (2.26c)$$

The following amounts are homogeneous to mass and moments of inertia:

**Definition 4** For all  $i = 1, \dots, n$ , all  $j_1, j_2 = 1, \dots, \mathcal{N}_r^i$  and all  $k_1, k_2 = 1, \dots, \mathcal{N}_d^i$ , we define:

$$\begin{aligned} M_{r,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) &:= \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \delta \mathbf{x}_{r,j_1}^i(\mathbf{q}_r^i, \mathbf{x}) \cdot \delta \mathbf{x}_{r,j_2}^i(\mathbf{q}_r^i, \mathbf{x}) \, d\mathbf{x}, \\ M_{d,k_1,k_2}^i(\mathbf{q}_d^i) &:= \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \delta \mathbf{x}_{d,k_1}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \cdot \delta \mathbf{x}_{d,k_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \, d\mathbf{x}, \end{aligned}$$

and we denote by  $[M_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i)]$  and  $[M_d^i(\mathbf{q}_d^i)]$  the  $\mathcal{N}_r^i \times \mathcal{N}_r^i$  and  $\mathcal{N}_d^i \times \mathcal{N}_d^i$  matrices whose entries are  $M_{r,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  and  $M_{d,k_1,k_2}^i(\mathbf{q}_d^i)$  respectively.

Notice that  $[M^i(\mathbf{q}_d^i)]$  does actually not depend on  $\mathbf{q}_r^i$ . Indeed, taking into account (2.26), we can rewrite its entries:

$$M_{d,k_1,k_2}^i(\mathbf{q}_d^i) = \int_{S_0^i} \rho_S^0(\mathbf{x}) \frac{\partial \mathbf{X}_d^i}{\partial q_{d,k_1}^i}(\mathbf{q}_d^i, \mathbf{x} - \mathbf{h}_0^i) \cdot \frac{\partial \mathbf{X}_d^i}{\partial q_{d,k_2}^i}(\mathbf{q}_d^i, \mathbf{x} - \mathbf{h}_0^i) \, d\mathbf{x}.$$

We denote by  $m^i > 0$  the mass of the  $i$ -th bodies and, when  $N = 3$ , by  $[I^i(\mathbf{q}_r^i, \mathbf{q}_d^i)]$  its inertia tensor, classically defined by:

$$[I^i(\mathbf{q}_r^i, \mathbf{q}_d^i)] := \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) (|\mathbf{x} - \mathbf{h}^i|^2 [\text{Id}] - (\mathbf{x} - \mathbf{h}^i) \otimes (\mathbf{x} - \mathbf{h}^i)) \, d\mathbf{x}.$$

We have, equivalently:

$$[I^i(\mathbf{q}_r^i, \mathbf{q}_d^i)] := [R^i(\mathbf{q}_r^i)] [I_d^i(\mathbf{q}_d^i)] [R^i(\mathbf{q}_r^i)]^T, \quad (2.27)$$

where  $[R^i(\mathbf{q}_r^i)]$  is the rotation matrix already encountered in (2.10) and:

$$[I_d^i(\mathbf{q}_d^i)] := \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i, \mathbf{x}) (|\mathbf{x} - \mathbf{h}_0^i|^2 [\text{Id}] - (\mathbf{x} - \mathbf{h}_0^i) \otimes (\mathbf{x} - \mathbf{h}_0^i)) \, d\mathbf{x}.$$

One readily checks that mass and inertia tensors are connected to  $M_{r,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  as follows:

$$M_{r,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) = \delta \Omega_{j_1}^i(\mathbf{q}_r^i) \cdot [I^i(\mathbf{q}_r^i, \mathbf{q}_d^i)] \delta \Omega_{j_2}^i(\mathbf{q}_r^i) + m^i \delta \mathbf{h}_{j_1}^i(\mathbf{q}_r^i) \cdot \delta \mathbf{h}_{j_2}^i(\mathbf{q}_r^i), \quad (2.28)$$

the vectors  $\delta \Omega_{j_k}^i(\mathbf{q}_r^i)$  and  $\delta \mathbf{h}_{j_k}^i(\mathbf{q}_r^i)$  ( $k = 1, 2$ ) being given in Definition 2. In the sequel, to write the equations of motion, we shall also require:

$$[J^i(\mathbf{q}_r^i, \mathbf{q}_d^i)] := \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) (\mathbf{x} - \mathbf{h}^i) \otimes (\mathbf{x} - \mathbf{h}^i) \, d\mathbf{x}. \quad (2.29)$$

When  $N = 2$ , the scalar moment of inertia reads:

$$I^i(\mathbf{q}_d^i) := \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i, \mathbf{x}) |\mathbf{x} - \mathbf{h}_0^i|^2 \, d\mathbf{x}, \quad (2.30)$$

and equality (2.28) becomes, with notation of Definition 2:

$$M_{r,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) = \delta \Omega_{j_1}^i(\mathbf{q}_r^i) \delta \Omega_{j_2}^i(\mathbf{q}_r^i) I^i(\mathbf{q}_d^i) + m^i \delta \mathbf{h}_{j_1}^i(\mathbf{q}_r^i) \cdot \delta \mathbf{h}_{j_2}^i(\mathbf{q}_r^i). \quad (2.31)$$

Let us now go back to the general case (i.e.  $N = 2, 3$ ). We claim that, for all  $i = 1, \dots, n$ , all  $j = 1, \dots, \mathcal{N}_r^i$  and all  $k = 1, \dots, \mathcal{N}_d^i$ :

$$\int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x}) \cdot \delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \, d\mathbf{x} = 0.$$

This result is deduced straightforwardly from the following Lemma:

**Lemma 2.2** For all  $i = 1, \dots, n$ , all  $k = 1, \dots, \mathcal{N}_d^i$  and all  $i$ . v. r. d.  $\delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i, \mathbf{x}) \in \mathcal{D}_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ :

$$\int_{S^i(\mathbf{q}_r, \mathbf{q}_d)} \rho_S(\mathbf{q}_r, \mathbf{q}_d, \mathbf{x}) \delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \cdot \delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i, \mathbf{x}) \, d\mathbf{x} = 0. \quad (2.32)$$

**Proof :** Shape-changes of the swimmers are assumed to be produced by inner forces only. Then, consider for the  $i$ -th body, being at rest at the time  $t = 0$ , a deformation involving the parameter  $q_{d,j}^i$  only ( $j = 1, \dots, \mathcal{N}_d^i$  being fixed). In absence of fluid (and hence of hydrodynamic forces) and of gravity, no exterior force acts on it. Newton's law of conservation of linear momentum ensures that:

$$\frac{d}{dt} \left( \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i, \mathbf{x}) \delta \widehat{\mathbf{x}}_{d,j}^i(\mathbf{q}_d^i, \mathbf{x}) \, d\mathbf{x} \dot{q}_{d,j}^i \right) = \mathbf{0}, \quad \forall t \geq 0.$$

The swimmer being quiescent at the time  $t = 0$ , we deduce that  $\dot{q}_{d,j}^i(0) = 0$  and hence that:

$$\int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i, \mathbf{x}) \delta \tilde{\mathbf{x}}_{d,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) d\mathbf{x} = \mathbf{0}, \quad \forall t \geq 0.$$

The change of variables  $\mathbf{x}' := \mathbf{X}_r^i(\mathbf{q}_r^i, \mathbf{x})$  together with identities (2.26) yield:

$$\int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}') \delta \mathbf{x}_{d,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}') d\mathbf{x}' = \mathbf{0}, \quad \forall t \geq 0. \quad (2.33)$$

The location of the center of mass must remain unchanged while the bodies undergo shape-changes. Thus, the coordinates of the center of mass of  $S_d^i(\mathbf{q}_d^i)$  remain equal to  $\mathbf{h}_0^i$  for all  $t \geq 0$ . Newton's law of conservation of angular momentum leads to:

$$\begin{aligned} \frac{d}{dt} \left( \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i, \mathbf{x}) \delta \tilde{\mathbf{x}}_{d,j}^i(\mathbf{q}_d^i, \mathbf{x}) \times (\mathbf{x} - \mathbf{h}_0^i) d\mathbf{x} \dot{q}_{d,j}^i \right) &= \mathbf{0}, \quad \forall t \geq 0, \quad (N = 3), \\ \frac{d}{dt} \left( \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i, \mathbf{x}) \delta \tilde{\mathbf{x}}_{d,j}^i(\mathbf{q}_d^i, \mathbf{x}) \cdot (\mathbf{x} - \mathbf{h}_0^i)^\perp d\mathbf{x} \dot{q}_{d,j}^i \right) &= 0, \quad \forall t \geq 0, \quad (N = 2). \end{aligned}$$

Assuming again that the swimmer is at rest at the time  $t = 0$ , we deduce that, when  $N = 3$ :

$$\int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i, \mathbf{x}) \delta \tilde{\mathbf{x}}_{d,j}^i(\mathbf{q}_d^i, \mathbf{x}) \times (\mathbf{x} - \mathbf{h}_0^i) d\mathbf{x} = \mathbf{0}, \quad \forall t \geq 0.$$

The same change of variables as in (2.33) yields:

$$\int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) [R^i(\mathbf{q}_r^i)]^T \delta \mathbf{x}_{d,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \times [R^i(\mathbf{q}_r^i)]^T (\mathbf{x} - \mathbf{h}^i) d\mathbf{x} = \mathbf{0},$$

and then:

$$\int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \delta \mathbf{x}_{d,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \times (\mathbf{x} - \mathbf{h}^i) d\mathbf{x} = \mathbf{0}. \quad (2.34a)$$

When  $N = 2$ , this equality turns into:

$$\int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \delta \mathbf{x}_{d,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \cdot (\mathbf{x} - \mathbf{h}^i)^\perp d\mathbf{x} = 0. \quad (2.34b)$$

Identities (2.33) and (2.34) leads to (2.32) taking into account Definition 1.  $\blacksquare$

We can now figure out the expression of the Lagrangian function of the swimmers. Bodies' kinetic energy related to rigid motions and to motions of deformation are respectively, according to Definition 4:

$$K_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i) := \frac{1}{2} \dot{\mathbf{q}}_r^i \cdot [M_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i)] \dot{\mathbf{q}}_r^i \quad (2.35a)$$

$$= \frac{1}{2} \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) |\mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i, \mathbf{x})|^2 d\mathbf{x}, \quad (2.35b)$$

and

$$K_d^i(\mathbf{q}_d^i, \dot{\mathbf{q}}_d^i) := \frac{1}{2} \dot{\mathbf{q}}_d^i \cdot [M_d^i(\mathbf{q}_d^i)] \dot{\mathbf{q}}_d^i \quad (2.35c)$$

$$= \frac{1}{2} \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) |\mathbf{v}_d^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_d^i, \mathbf{x})|^2 d\mathbf{x}. \quad (2.35d)$$

Notice that, upon a change of variables, one easily checks that  $K_d^i(\mathbf{q}_d^i, \dot{\mathbf{q}}_d^i)$  does actually not depend on  $\mathbf{q}_r^i$  neither on  $\dot{\mathbf{q}}_r^i$ . Setting the potential function  $G(\mathbf{x}) := -gx_N$  ( $g$  stands for the gravitational constant) and by virtue of Archimedes' principle, the buoyant force on the  $i$ -th body is:

$$\mathbf{F}_b^i(\mathbf{q}_r^i, \mathbf{q}_d^i) := \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) - \rho_F) \nabla G(\mathbf{x}) \, d\mathbf{x},$$

where  $\rho_F > 0$  is the constant density of the fluid. The associated potential energy reads:

$$\mathcal{P}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) := - \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) - \rho_F) G(\mathbf{x}) \, d\mathbf{x}. \quad (2.36)$$

We deduce from Lemma 2.2 that the total kinetics energy of the  $i$ -th body reduces to the sum of  $K_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)$  and  $K_d^i(\mathbf{q}_d^i, \dot{\mathbf{q}}_d^i)$ . Thus, the Lagrangian function of the  $i$ -th body is classically defined by:

$$\mathcal{L}_b^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) := K_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) + K_d^i(\mathbf{q}_d^i, \dot{\mathbf{q}}_d^i) - \mathcal{P}^i(\mathbf{q}_r^i, \mathbf{q}_d^i). \quad (2.37)$$

## 2.5 Dynamics of the system and coupling conditions

For all  $t \geq 0$  and all  $\mathbf{x} \in \Omega(\mathbf{q}_r, \mathbf{q}_d)$ , we denote by  $\mathbf{u}(t, \mathbf{x})$  the Eulerian velocity of the fluid at the point  $\mathbf{x}$  and  $p(t, \mathbf{x})$  stands for the pressure. The fluid flow is governed by Euler's equations:

$$\rho_F \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p = \rho_F \nabla G \text{ in } \Omega(\mathbf{q}_r, \mathbf{q}_d), \, t > 0, \quad (2.38a)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega(\mathbf{q}_r, \mathbf{q}_d), \, t > 0. \quad (2.38b)$$

The kinematics coupling conditions between the bodies, the fluid and the thrusters is partly given by the boundary conditions:

$$\begin{aligned} \mathbf{u}(t) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) &= (\mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) + \mathbf{v}_d^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_d^i) \\ &\quad + \mathbf{v}_t^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_t^i)) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) \text{ on } \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \, \forall i = 1, \dots, n. \end{aligned} \quad (2.39a)$$

This equality means that the normal component of the fluid's velocity matches the normal component of the velocity of the bodies's boundary, along  $\Gamma_2(\mathbf{q}_r, \mathbf{q}_d)$ . On  $\Gamma_1$ , the classical *slip condition* for non-viscous fluid holds:

$$\mathbf{u}(t) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1. \quad (2.39b)$$

Taking into account expressions (2.11), (2.15) and (2.17) of the Eulerian velocities  $\mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)$ ,  $\mathbf{v}_d^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_d^i)$  and  $\mathbf{v}_t^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_t^i)$ , we can rewrite the boundary conditions for the fluid velocity:

$$\begin{aligned} \mathbf{u}(t) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) &= \sum_{i=1}^n \left( \sum_{j=1}^{\mathcal{N}_r^i} \dot{q}_{r,j}^i b_{r,j}^i(\mathbf{q}_r^i) + \sum_{k=1}^{\mathcal{N}_d^i} \dot{q}_{d,k}^i b_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \right. \\ &\quad \left. + \sum_{l=1}^{\mathcal{N}_t^i} \dot{q}_{t,l}^i b_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \right) \quad \text{on } \Gamma_2(\mathbf{q}_r, \mathbf{q}_d), \end{aligned} \quad (2.40a)$$

or in short form, using the relabelling process (A.4) and notation therein:

$$\mathbf{u}(t) \cdot \mathbf{n}(\mathbf{q}_r, \mathbf{q}_d) = \mathbf{b}(\mathbf{q}_r, \mathbf{q}_d) \cdot \dot{\mathbf{q}}, \quad \text{on } \Gamma_2(\mathbf{q}_r, \mathbf{q}_d). \quad (2.40b)$$

The stress tensor inside the fluid for an inviscid flow reads merely  $\sigma = -(p - \rho_F G)$  [Id]. For any swimmer  $S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ , Newton's law for linear momentum yields, for all  $i = 1, \dots, n$  and all  $t > 0$ :

$$\begin{aligned} \frac{d}{dt} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) d\mathbf{x} \right) = \\ \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} p(t) \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) d\Gamma + \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) - \rho_F) \nabla G d\mathbf{x}, \end{aligned} \quad (2.41a)$$

while Newton's law for angular momentum leads to, when  $N = 2$ :

$$\begin{aligned} \frac{d}{dt} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot (\mathbf{x} - \mathbf{h}^i)^\perp d\mathbf{x} \right) = \\ \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} p(t) \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) \cdot (\mathbf{x} - \mathbf{h}^i)^\perp d\Gamma. \end{aligned} \quad (2.41b)$$

When  $N = 3$ , this equality becomes:

$$\begin{aligned} \frac{d}{dt} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \times (\mathbf{x} - \mathbf{h}^i) d\mathbf{x} \right) = \\ \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} p(t) \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) \times (\mathbf{x} - \mathbf{h}^i) d\Gamma. \end{aligned} \quad (2.41c)$$

Thus, latter term on the right of (2.41a) is the buoyant force. Equations (2.38), (2.39) and (2.41) provide the expected system of governing equations of motion for the coupled system fluid-bodies. Keep in mind that our main purpose is to compare these equations with those we will obtain later on with Lagrangian formalism. The first step in carrying out this task consists in reformulating equations (2.41) in an equivalent but more convenient form.

### 3 Statement of the governing equations of motion

In this section, we will display properly both systems of equations of motion, the first one deriving from Newton's laws and the second being deduced from D'Alembert's variational principle. Beforehand, we focus on the dynamics of the bodies.

#### 3.1 First equivalence results

The proof of the equivalence between Newtonian and Lagrangian approaches, as long as only monogenic systems are concerned, can be found in many books of Mechanics (see for instance [16, §2.3 pages 44-45] or [4, §13 pages 64-67]). In [33], the authors extend this result to multibody systems with constraint forces (like friction forces). In a first time, we will prove that the same conclusion can be drawn for the set of deformable bodies we are dealing with.

**Lemma 3.1** *Newton's laws (2.41) are equivalent to anyone of the following assertions:*

1. For any  $i$ . v. r. d.  $\delta\widehat{\mathbf{x}}_r(\mathbf{q}_r^i) \in \mathcal{D}_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ , the identity below holds:

$$\begin{aligned} & \frac{d}{dt} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \delta\widehat{\mathbf{x}}_r(\mathbf{q}_r^i) \, d\mathbf{x} \right) \\ & - \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \frac{d}{dt} (\delta\widehat{\mathbf{x}}_r(\mathbf{q}_r^i)) \, d\mathbf{x} = \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} p(t) \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) \cdot \delta\widehat{\mathbf{x}}_r(\mathbf{q}_r^i) \, d\Gamma \\ & \quad + \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) - \rho_F) \nabla G \cdot \delta\widehat{\mathbf{x}}_r(\mathbf{q}_r^i) \, d\mathbf{x}. \quad (3.1) \end{aligned}$$

2. For any  $i$ . v. r. d.  $\delta\widehat{\mathbf{x}}_r(\mathbf{q}_r^i) \in \{\delta\mathbf{x}_{r,j}^i(\mathbf{q}_r^i), j = 1, \dots, \mathcal{N}_r^i\}$ , identity (3.1) holds.

The proof is given in Appendix B. We can then state the first equivalence result:

**Proposition 3.1** *Newton's laws (2.41) are equivalent to the following system of Euler-Lagrange's equations:*

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}_b^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) - \frac{\partial \mathcal{L}_b^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) &= \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} p(t) b_{r,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \, d\Gamma, \\ i = 1, \dots, n, \quad j = 1, \dots, \mathcal{N}_r^i. \quad (3.2) \end{aligned}$$

**Proof :** Actually, we will prove that equalities (3.2) are equivalent to the second assertion of Lemma 3.1. The potential  $\mathcal{P}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  can be expressed, upon a change of variables, as:

$$\mathcal{P}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) = - \int_{S_d^i(\mathbf{q}_d^i)} (\rho_S^d(\mathbf{q}_d^i, \mathbf{x}) - \rho_F) G(\mathbf{X}_r^i(\mathbf{q}_r^i, \mathbf{x})) \, d\mathbf{x}.$$

Under this form, it is easy to prove that:

$$\frac{d}{dt} \frac{\partial \mathcal{P}^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) - \frac{\partial \mathcal{P}^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) = \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) - \rho_F) \nabla G \cdot \delta\mathbf{x}_{r,j}^i(\mathbf{q}_r^i) \, d\mathbf{x}. \quad (3.3)$$

Concerning now the kinetic energy  $K^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) := K_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i) + K_d^i(\mathbf{q}_d^i, \dot{\mathbf{q}}_d^i)$  we have, on the one hand:

$$\frac{d}{dt} \left( \frac{\partial K^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) \right) = \frac{d}{dt} \left( \frac{\partial K_r^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) \right),$$

because  $K_d^i(\mathbf{q}_d^i, \dot{\mathbf{q}}_d^i)$  does not depend on  $\dot{\mathbf{q}}_r^i$ . It comes, taking into account definitions (2.35) and since  $\delta\mathbf{x}_{r,j}^i(\mathbf{q}_r^i) = \partial \mathbf{v}_r^i(\mathbf{q}_r^i) / \partial \dot{q}_{r,j}^i$ :

$$\frac{d}{dt} \left( \frac{\partial K^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) \right) = \frac{d}{dt} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \delta\mathbf{x}_{r,j}^i(\mathbf{q}_r^i) \, d\mathbf{x} \right). \quad (3.4)$$

On the other hand, the same arguments lead to:

$$\begin{aligned} \frac{\partial K^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) &= \frac{1}{2} \frac{\partial}{\partial q_{r,j}^i} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) |\mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)|^2 \, d\mathbf{x} \right) \\ &= \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i) \widetilde{\mathbf{v}}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \frac{\partial \widetilde{\mathbf{v}}_r^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \, d\mathbf{x}. \end{aligned}$$



Making use of formula (2.19) of Lemma 2.1, we obtain:

$$\frac{\partial K^i}{\partial \dot{\mathbf{q}}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) = \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i) \tilde{\mathbf{v}}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \frac{d}{dt}(\delta \tilde{\mathbf{x}}_r^i(\mathbf{q}_r^i)) \, d\mathbf{x}.$$

Latter formula (2.20) of Lemma 2.1 yields eventually:

$$\frac{\partial K^i}{\partial \dot{\mathbf{q}}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) = \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \frac{d}{dt}(\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i)) \, d\mathbf{x}. \quad (3.5)$$

Plugging together (3.3), (3.4) and (3.5) we obtain (3.2) by virtue of (3.1).  $\blacksquare$

In (3.2), the left hand side term can be explicitly computed.

**Proposition 3.2** *For all  $i = 1, \dots, n$  and all  $j_0 = 1, \dots, \mathcal{N}_r^i$ :*

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}_b^i}{\partial \dot{\mathbf{q}}_{r,j_0}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) - \frac{\partial \mathcal{L}_b^i}{\partial \mathbf{q}_{r,j_0}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) &= \left( \sum_{j=1}^{\mathcal{N}_r^i} M_{r,j_0,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \ddot{\mathbf{q}}_{r,j}^i \right) \\ &+ \left( \sum_{j_1=1}^{\mathcal{N}_r^i} \sum_{j_2=1}^{\mathcal{N}_r^i} A_{j_0,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \dot{\mathbf{q}}_{r,j_1}^i \dot{\mathbf{q}}_{r,j_2}^i \right) + \left( \sum_{j=1}^{\mathcal{N}_r^i} \sum_{k=1}^{\mathcal{N}_d^i} B_{j_0,j,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \dot{\mathbf{q}}_{r,j}^i \dot{\mathbf{q}}_{d,k}^i \right) \\ &+ \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) - \rho_F) \nabla G \cdot \delta \mathbf{x}_{r,j_0}^i(\mathbf{q}_r^i) \, d\mathbf{x}. \quad (3.6) \end{aligned}$$

In this identity,  $M_{r,j_0,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  is given in Definition 4 (or equivalently by (2.28) and (2.31)) and the matrices  $[I^i(\mathbf{q}_r^i, \mathbf{q}_d^i)]$  and  $[J^i(\mathbf{q}_r^i, \mathbf{q}_d^i)]$  in (2.27) and (2.29). Furthermore, we have with notation of Definition 2 and when  $N = 3$ :

$$\begin{aligned} A_{j_0,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) &:= \delta \Omega_{j_1}^i(\mathbf{q}_r^i) \cdot [J^i(\mathbf{q}_r^i, \mathbf{q}_d^i)] (\delta \Omega_{j_2}^i(\mathbf{q}_r^i) \times \delta \Omega_{j_0}^i(\mathbf{q}_r^i)) \\ &+ \frac{\partial \delta \Omega_{j_1}^i}{\partial \mathbf{q}_{j_2}^i}(\mathbf{q}_r^i) \cdot [I^i(\mathbf{q}_r^i, \mathbf{q}_d^i)] \delta \Omega_{j_0}^i(\mathbf{q}_r^i) + m^i \frac{\partial \delta \mathbf{h}_{j_1}^i}{\partial \mathbf{q}_{j_2}^i}(\mathbf{q}_r^i) \cdot \delta \mathbf{h}_{j_0}^i(\mathbf{q}_r^i), \quad (3.7) \end{aligned}$$

and

$$B_{j_0,j,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) := \delta \Omega_{j_0}^i(\mathbf{q}_r^i) \cdot \frac{\partial [I^i(\mathbf{q}_r^i, \mathbf{q}_d^i)]}{\partial \mathbf{q}_{d,k}^i} \delta \Omega_j^i(\mathbf{q}_r^i). \quad (3.8)$$

When  $N = 2$ , expressions (3.7) and (3.8) simplify as follows (keeping notation of Definition 2):

$$A_{j_0,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) := \frac{\partial \delta \Omega_{j_1}^i}{\partial \mathbf{q}_{j_2}^i}(\mathbf{q}_r^i) I^i(\mathbf{q}_d^i) \delta \Omega_{j_0}^i(\mathbf{q}_r^i) + m^i \frac{\partial \delta \mathbf{h}_{j_1}^i}{\partial \mathbf{q}_{j_2}^i}(\mathbf{q}_r^i) \cdot \delta \mathbf{h}_{j_0}^i(\mathbf{q}_r^i), \quad (3.9)$$

and

$$B_{j_0,j,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) := \delta \Omega_{j_0}^i(\mathbf{q}_r^i) \frac{\partial I^i(\mathbf{q}_d^i)}{\partial \mathbf{q}_{d,k}^i} \delta \Omega_j^i(\mathbf{q}_r^i), \quad (3.10)$$

where  $I^i(\mathbf{q}_d^i)$  is defined by (2.30).

The proof of this proposition is performed in Appendix B. About motion of solid, one is more familiar with the classical so-called *Euler equations* written down for example in [16, §5.5 pages 198-200] and which are expressed in the moving frame attached to the body. However, under this form, the trajectories are only obtained upon a *reconstruction process*. The form taken by the equations in the proposition above is not new. A similar Lagrangian approach is used in [11] and more recently in [28].

### 3.2 Statement of the governing equations of motion derived from Newton's Laws

Summing up (2.38), (2.40) and (3.2) and adding initial data (namely initial positions  $q_{r,j,0}^i \in \mathbb{R}$  and initial velocities  $\dot{q}_{r,j,0}^i \in \mathbb{R}$  for the bodies and also initial velocity  $\mathbf{u}_0$  for the fluid), we get the full coupled system of equations governing the system fluid-bodies, deduced from Newton's laws. Let us redisplay it properly:

$$\rho_F \left( \frac{\partial \mathbf{u}}{\partial t}(t) + (\mathbf{u}(t) \cdot \nabla) \mathbf{u}(t) \right) + \nabla p(t) = \rho_F \nabla G \quad \text{in } \Omega(\mathbf{q}_r, \mathbf{q}_d), \quad t > 0, \quad (3.11a)$$

$$\nabla \cdot \mathbf{u}(t) = 0 \quad \text{in } \Omega(\mathbf{q}_r, \mathbf{q}_d), \quad t > 0, \quad (3.11b)$$

$$\mathbf{u}(t) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1, \quad t > 0, \quad (3.11c)$$

$$\mathbf{u}(t) \cdot \mathbf{n}(\mathbf{q}_r, \mathbf{q}_d) - \mathbf{b}(\mathbf{q}_r, \mathbf{q}_d) \cdot \dot{\mathbf{q}} = 0 \quad \text{on } \Gamma_2(\mathbf{q}_r, \mathbf{q}_d), \quad t > 0, \quad (3.11d)$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}_b^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) - \frac{\partial \mathcal{L}_b^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) \\ = - \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} p(t) b_{r,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) d\Gamma, \quad t > 0, \quad (3.11e) \end{aligned}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{q}_{r,j}^i(0) = \dot{q}_{r,j,0}^i, \quad q_{r,j}^i(0) = q_{r,j,0}^i, \quad 1 \leq j \leq \mathcal{N}_r^i, \quad 1 \leq i \leq n. \quad (3.11f)$$

It is worth recalling that in this system,  $\mathbf{u}$ ,  $p$  and  $\mathbf{q}_r$  are the unknowns whereas  $\mathbf{q}_d$  and  $\mathbf{q}_t$  are assigned as functions of  $t$  and used to control the motion of the bodies. The coupling between the initial and boundary value problem (3.11a-3.11b) driving the fluid flow and the ODE's (3.11e) governing the rigid motions of the swimmers arises not only through the boundary condition (3.11d) but also through the moving domain  $\Omega(\mathbf{q}_r, \mathbf{q}_d)$ . Indeed, since  $\Omega(\mathbf{q}_r, \mathbf{q}_d)$  depends on the position of the swimmers, it is somehow also an unknown of the problem.

Up to now we have only assumed the fluid to be perfect. Indeed, System (3.11) is valid for nonzero vorticity flow. We seek now the fluid's velocity  $\mathbf{u}$  as the gradient of a potential function  $\psi(t)$ :

$$\mathbf{u}(t) = \nabla \psi(t) \quad \text{in } \Omega(\mathbf{q}_r, \mathbf{q}_d), \quad t > 0. \quad (3.12)$$

We can rewrite equations (3.11a-3.11d) of System (3.11) with Bernoulli's formula ( $p_0(t) > 0$  stands for a real valued function of time only):

$$\rho_F \left( \frac{\partial \psi}{\partial t}(t) + \frac{|\nabla \psi|^2}{2}(t) \right) + p(t) - p_0(t) = \rho_F G \quad \text{in } \Omega(\mathbf{q}_r, \mathbf{q}_d), \quad t > 0, \quad (3.13a)$$

$$-\Delta \psi(\mathbf{q}, \dot{\mathbf{q}}) = 0 \quad \text{in } \Omega(\mathbf{q}_r, \mathbf{q}_d), \quad t > 0, \quad (3.13b)$$

$$\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{q}, \dot{\mathbf{q}}) = 0 \quad \text{on } \Gamma_1, \quad t > 0, \quad (3.13c)$$

$$\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{b}(\mathbf{q}_r, \mathbf{q}_d) \cdot \dot{\mathbf{q}} = 0 \quad \text{on } \Gamma_2(\mathbf{q}_r, \mathbf{q}_d), \quad t > 0, \quad (3.13d)$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}_b^i}{\partial \dot{q}_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) - \frac{\partial \mathcal{L}_b^i}{\partial q_{r,j}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) \\ = - \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} p(t) b_{r,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) d\Gamma, \quad t > 0, \quad (3.13e) \end{aligned}$$

$$\dot{q}_{r,j}^i(0) = \dot{q}_{r,j,0}^i, \quad q_{r,j}^i(0) = q_{r,j,0}^i, \quad 1 \leq j \leq \mathcal{N}_r^i, \quad 1 \leq i \leq n. \quad (3.13f)$$

Conditions (2.14), (2.16) and (2.18) ensure that the compatibility condition required for the Neumann boundary value problem to admit a solution and that is:

$$\int_{\Gamma(\mathbf{q}_r, \mathbf{q}_d)} \frac{\partial \psi}{\partial \mathbf{n}}(t) d\Gamma = 0, \quad \forall t \geq 0,$$

is satisfied. Since, for all  $i = 1, \dots, n$ ,  $\Gamma^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  is of class  $C^2$ , the unitary normal  $\mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i)$  is continuously differentiable and the functions  $b_{r,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ ,  $b_{r,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  and  $b_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  defined by (A.1) are, at least, also continuously differentiable. We deduce, referring to [7, Section IX.5], that when  $\Omega$  is bounded,  $\psi(t)$  is well defined for all  $t \geq 0$  and belongs to the classical Sobolev space  $H^2(\Omega(\mathbf{q}_r, \mathbf{q}_d))$ . When  $\Omega$  is not bounded, results in [2, 3, 5] allow us to draw the same conclusion in suitable *weighted* Sobolev spaces. The important point is that  $\psi(t)$  is always regular *enough* nearby the boundaries of the bodies. The potential function is defined only up to an additive constant (because  $\Omega(\mathbf{q}_r, \mathbf{q}_d)$  is a connected set by hypothesis). To remove this indetermination, we add for instance the condition:

$$\int_{\Gamma_2(\mathbf{q}_r, \mathbf{q}_d)} \psi(t, \mathbf{x}) \, d\Gamma = 0, \quad \forall t \geq 0. \quad (3.14)$$

Observe that within this model, the state of the fluid is completely determined by the potential function  $\psi(t)$  through identity (3.12). In addition,  $\psi(t)$  itself is uniquely determined by the generalized coordinates  $\mathbf{q}, \dot{\mathbf{q}}$ , for it is the solution of a Neumann boundary value problem. We shall denote it  $\psi(\mathbf{q}, \dot{\mathbf{q}})$  from now on. System (3.11) above can formally be reduced to a system of ODE's of unknowns  $\mathbf{q}_r^i$ ,  $i = 1, \dots, n$  and its time derivatives.

### 3.3 Euler-Lagrange's equations of motion

Let us now tackle the same problem of modelling but with a different point of view. We consider the system fluid-bodies in its entirety and we compute its Lagrangian function. It is obtained by adding the kinetic energy of the fluid to the Lagrangian function of the bodies, defined earlier by (2.37). It comes:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^n \mathcal{L}_b^i(\mathbf{q}, \dot{\mathbf{q}}) + \frac{1}{2} \int_{\Omega(\mathbf{q}_r, \mathbf{q}_d)} \rho_F |\nabla \psi(\mathbf{q}, \dot{\mathbf{q}})|^2 \, d\mathbf{x}. \quad (3.15)$$

The full system of coupled equations is next obtained by replacing (3.11e) and (3.13a) by Euler-Lagrange's equations for the system bodies-fluid:

$$-\Delta \psi(\mathbf{q}, \dot{\mathbf{q}}) = 0 \quad \text{in } \Omega(\mathbf{q}_r, \mathbf{q}_d), \quad t > 0, \quad (3.16a)$$

$$\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{q}, \dot{\mathbf{q}}) = 0 \quad \text{on } \Gamma_1, \quad t > 0, \quad (3.16b)$$

$$\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{b}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \cdot \dot{\mathbf{q}}^i \quad \text{on } \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \quad t > 0, \quad (3.16c)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_{r,j}^i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{\partial \mathcal{L}}{\partial q_{r,j}^i}(\mathbf{q}, \dot{\mathbf{q}}) = F_j^i(\mathbf{q}, \dot{\mathbf{q}}), \quad \begin{array}{l} 1 \leq i \leq n, \\ 1 \leq j \leq \mathcal{N}_r^i \end{array} \quad t > 0, \quad (3.16d)$$

$$\dot{q}_{r,j}^i(0) = \dot{q}_{r,j,0}^i, \quad q_{r,j}^i(0) = q_{r,j,0}^i, \quad \begin{array}{l} 1 \leq j \leq \mathcal{N}_r^i, \\ 1 \leq i \leq n. \end{array} \quad (3.16e)$$

In (3.16d),  $\mathbf{F}^i(\mathbf{q}^i, \dot{\mathbf{q}}^i) := (F_1^i(\mathbf{q}^i, \dot{\mathbf{q}}^i), \dots, F_{\mathcal{N}_r^i}^i(\mathbf{q}^i, \dot{\mathbf{q}}^i))^T$  stand for generalized forces that remain to be determined. They are due to the presence of nonholonomic constraints resulting from the way we have modelled the action of the thrusters. Indeed, the velocity of the fluid blown out by the thrusters is prescribed.

As for System (3.13), within this latter model,  $\psi(\mathbf{q}, \dot{\mathbf{q}})$  can be expressed in terms of  $\mathbf{q}$  and  $\dot{\mathbf{q}}$  (for being uniquely determined by the data  $\mathbf{q}$  and  $\dot{\mathbf{q}}$ ) and the resulting system can be reduced to a system of ODE's in the variables  $\mathbf{q}_r^i$ ,  $i = 1, \dots, n$  and its time derivatives.

The natural question arising now is: does the reduced system of ODE's obtained with Euler-Lagrange's equations fits with the one obtained with Newton's laws ?

### 3.4 Main equivalence result

The first one of the main results brought in in this paper is:

**Theorem 3.1** *Let us assume that  $\Gamma_0$  (the boundary of the fluid at the time  $t = 0$ ) is twice continuously differentiable as well as the functions of deformation (2.2) and that the thrusters remain inactive (what means that  $\dot{\mathbf{q}}_t^i = \mathbf{0}$ ) for all  $t > 0$ . Then,  $F_j^i(\mathbf{q}^i, \dot{\mathbf{q}}^i) = 0$  in (3.16d) for all  $i = 1, \dots, n$ , all  $j = 1, \dots, \mathcal{N}_r^i$  and all  $t > 0$ . In this case, System (3.13) is equivalent (i.e. yield the same system of ODE's) than System (3.16).*

When thrusters work,  $\mathbf{F}^i(\mathbf{q}^i, \dot{\mathbf{q}}^i) \neq \mathbf{0}$  and we are not able to determine their expressions with Lagrangian approach only. We argue that under the smoothness assumption of Theorem 3.1, both systems of governing equations (3.13) and (3.16) must yield the same systems of ODE's, what allow us to deduce the expression of these generalized forces.

The smoothness assumption on  $\Gamma_0$  is certainly not optimal. For instance, the proof of Theorem 3.1 above can be carried out assuming that  $\Gamma_0$  is of class  $C^{1,1}$  (continuously differentiable with derivative lipschitz continuous) only. The point is that when  $\Gamma_0$  is not smooth *enough* (with singular points like cuspid points for instance), both models do not fit any longer as proved in Section 5 by exhibiting a counterexample.

In this paper, we are not concerned with studying whether Systems (3.13) or (3.16) admit solutions. Nevertheless, existence and regularity results for similarly systems are given in [28].

The proof of Theorem 3.1 deduces straightforwardly from the explicit computations of the governing ODE's of motion that will be done in the following subsection.

### 3.5 Fully explicit statement of the equations of motion

According to Kirchhoff's principle, we can decompose the potential  $\psi(\mathbf{q}, \dot{\mathbf{q}})$  into the form:

$$\psi(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^n \left( \sum_{j=1}^{\mathcal{N}_r^i} \dot{q}_{r,j}^i \varphi_{r,j}^i(\mathbf{q}_r, \mathbf{q}_d) + \sum_{k=1}^{\mathcal{N}_d^i} \dot{q}_{d,k}^i \varphi_{d,k}^i(\mathbf{q}_r, \mathbf{q}_d) + \sum_{l=1}^{\mathcal{N}_t^i} \dot{q}_{t,l}^i \varphi_{t,l}^i(\mathbf{q}_r, \mathbf{q}_d) \right), \quad (3.17)$$

where the functions  $\varphi_{r,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ ,  $\varphi_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  and  $\varphi_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  are harmonic in  $\Omega(\mathbf{q}_r, \mathbf{q}_d)$  and satisfy the boundary conditions for all  $i = 1, \dots, n$ :

$$\begin{aligned} \frac{\partial \varphi_{r,j}^i}{\partial \mathbf{n}}(\mathbf{q}_r, \mathbf{q}_d) &= b_{r,j}^i(\mathbf{q}_r) && \text{on } \Gamma(\mathbf{q}_r, \mathbf{q}_d), \quad j = 1, \dots, \mathcal{N}_r^i, \\ \frac{\partial \varphi_{d,k}^i}{\partial \mathbf{n}}(\mathbf{q}_r, \mathbf{q}_d) &= b_{d,k}^i(\mathbf{q}_r, \mathbf{q}_d) && \text{on } \Gamma(\mathbf{q}_r, \mathbf{q}_d), \quad k = 1, \dots, \mathcal{N}_d^i, \\ \frac{\partial \varphi_{t,l}^i}{\partial \mathbf{n}}(\mathbf{q}_r, \mathbf{q}_d) &= b_{t,l}^i(\mathbf{q}_r, \mathbf{q}_d) && \text{on } \Gamma(\mathbf{q}_r, \mathbf{q}_d), \quad l = 1, \dots, \mathcal{N}_t^i. \end{aligned}$$

Arguing as for the function  $\psi(t)$ , at the end of Subsection 3.2, we claim that the functions  $\varphi_{r,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ ,  $\varphi_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  and  $\varphi_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  are well defined and belong to suitable Sobolev spaces. In particular, they are smooth *enough* in a neighborhood of  $\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ .

We can now give the expression of the generalized forces due to the action of the thrusters. The following proposition will be restated (see Corollary 4.1) and proved later on.

**Proposition 3.3** *Under the same smoothness assumptions as in Theorem 3.1. For all  $i_0 = 1, \dots, n$  and all  $j_0 = 1, \dots, \mathcal{N}_r^{i_0}$ , the expressions of the generalized forces in (3.16d), are:*

$$F_{j_0}^{i_0}(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^n \sum_{j=1}^{\mathcal{N}_r^i} \sum_{l=1}^{\mathcal{N}_t^{i_0}} \mathfrak{U}_{j_0, j, l}^{i_0, i}(\mathbf{q}) \dot{q}_{r, j}^i \dot{q}_{t, l}^{i_0} + \sum_{i=1}^n \sum_{k=1}^{\mathcal{N}_d^i} \sum_{l=1}^{\mathcal{N}_t^{i_0}} \mathfrak{V}_{j_0, k, l}^{i_0, i}(\mathbf{q}) \dot{q}_{d, k}^i \dot{q}_{t, l}^{i_0} \\ + \sum_{i=1}^n \sum_{l_1=1}^{\mathcal{N}_t^{i_0}} \sum_{l_2=1}^{\mathcal{N}_t^i} \mathfrak{W}_{j_0, l_1, l_2}^{i_0, i}(\mathbf{q}) \dot{q}_{t, l_1}^{i_0} \dot{q}_{t, l_2}^i,$$

where:

$$\mathfrak{U}_{j_0, j, l}^{i_0, i}(\mathbf{q}) := - \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\mathcal{H} b_{r, j_0}^{i_0} b_{t, l}^{i_0} + \nabla_{\Gamma} b_{t, l}^{i_0} \cdot \delta \mathbf{x}_{r, j_0}^{i_0}) \varphi_{r, j}^i d\Gamma \\ + \delta_{i, i_0} \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r, j_0}^{i_0} b_{r, j}^i b_{t, l}^{i_0} d\Gamma,$$

and

$$\mathfrak{V}_{j_0, k, l}^{i_0, i}(\mathbf{q}) := - \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\mathcal{H} b_{r, j_0}^{i_0} b_{t, l}^{i_0} + \nabla_{\Gamma} b_{t, l}^{i_0} \cdot \delta \mathbf{x}_{r, j_0}^{i_0}) \varphi_{d, k}^i d\Gamma \\ + \delta_{i, i_0} \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r, j_0}^{i_0} b_{d, k}^i b_{t, l}^{i_0} d\Gamma,$$

and

$$\mathfrak{W}_{j_0, l_1, l_2}^{i_0, i}(\mathbf{q}) := - \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\mathcal{H} b_{r, j_0}^{i_0} b_{t, l_1}^{i_0} + \nabla_{\Gamma} b_{t, l_1}^{i_0} \cdot \delta \mathbf{x}_{r, j_0}^{i_0}) \varphi_{t, l_2}^i d\Gamma \\ + \delta_{i, i_0} \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r, j_0}^{i_0} b_{t, l_1}^{i_0} b_{t, l_2}^i d\Gamma.$$

In these identities,  $\delta_{i_0, i_1}$  is the Kronecker's symbol that takes the value 1 when  $i_0 = i_1$  and 0 otherwise. We recall that  $\mathcal{H}$  is defined in Subsection 2.1 and stands for the mean curvature of the surface  $\Gamma_2^{i_0}(\mathbf{q})$  ( $N = 3$ ) or the curvature of the curve  $\Gamma_2^{i_0}(\mathbf{q})$  ( $N = 2$ ) and  $\nabla_{\Gamma}$  is the tangential gradient ( $N = 3$ ) or merely the tangential derivative when  $N = 2$ .

Next, we give the second main result of this article:

**Theorem 3.2** *The  $nN(N-1)/2$  ODE's in  $q_{r, j}^i$  driving the motion of the bodies under the action of the prescribed controls  $\mathbf{q}_d$  and  $\mathbf{q}_t$  are, for all  $i_0 = 1, \dots, n$  and all  $j_0 = 1, \dots, \mathcal{N}_r^{i_0}$ :*

$$\sum_{i=1}^n \sum_{j=1}^{\mathcal{N}_r^i} \mathfrak{M}_{j_0, j}^{i_0, i}(\mathbf{q}) \ddot{q}_{r, j}^i + \frac{1}{2} \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{j_1=1}^{\mathcal{N}_r^{i_1}} \sum_{j_2=1}^{\mathcal{N}_r^{i_2}} \mathfrak{A}_{j_0, j_1, j_2}^{i_0, i_1, i_2}(\mathbf{q}) \dot{q}_{r, j_1}^{i_1} \dot{q}_{r, j_2}^{i_2} \\ + \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{j=1}^{\mathcal{N}_r^{i_1}} \sum_{k=1}^{\mathcal{N}_d^{i_2}} \mathfrak{B}_{j_0, j, k}^{i_0, i_1, i_2}(\mathbf{q}) \dot{q}_{r, j}^{i_1} \dot{q}_{d, k}^{i_2} + \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{j=1}^{\mathcal{N}_r^{i_1}} \sum_{l=1}^{\mathcal{N}_t^{i_2}} \mathfrak{C}_{j_0, j, l}^{i_0, i_1, i_2}(\mathbf{q}) \dot{q}_{r, j}^{i_1} \dot{q}_{t, l}^{i_2} = \\ \frac{1}{2} \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{k_1=1}^{\mathcal{N}_d^{i_1}} \sum_{k_2=1}^{\mathcal{N}_d^{i_2}} \mathfrak{D}_{j_0, k_1, k_2}^{i_0, i_1, i_2}(\mathbf{q}) \dot{q}_{d, k_1}^{i_1} \dot{q}_{d, k_2}^{i_2} + \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{k=1}^{\mathcal{N}_d^{i_1}} \sum_{l=1}^{\mathcal{N}_t^{i_2}} \mathfrak{E}_{j_0, k, l}^{i_0, i_1, i_2}(\mathbf{q}) \dot{q}_{d, k}^{i_1} \dot{q}_{t, l}^{i_2} \\ + \frac{1}{2} \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{l_1=1}^{\mathcal{N}_t^{i_1}} \sum_{l_2=1}^{\mathcal{N}_t^{i_2}} \mathfrak{F}_{j_0, l_1, l_2}^{i_0, i_1, i_2}(\mathbf{q}) \dot{q}_{t, l_1}^{i_1} \dot{q}_{t, l_2}^{i_2} + \sum_{i=1}^n \sum_{k=1}^{\mathcal{N}_d^i} \mathfrak{G}_{j_0, k}^{i_0, i}(\mathbf{q}) \ddot{q}_{d, k}^i \\ + \sum_{i=1}^n \sum_{l=1}^{\mathcal{N}_t^i} \mathfrak{H}_{j_0, l}^{i_0, i}(\mathbf{q}) \ddot{q}_{t, l}^i + \mathfrak{J}_{j_0}^{i_0}(\mathbf{q}).$$

All of the amounts involved in these equations can be expressed only in terms of the values of the functions  $\varphi_{r,j}^i(\mathbf{q}_r, \mathbf{q}_d)$ ,  $\varphi_{d,k}^i(\mathbf{q}_r, \mathbf{q}_d)$  and  $\varphi_{t,l}^i(\mathbf{q}_r, \mathbf{q}_d)$  on the boundaries of the bodies and of geometric data (namely the second fundamental form) related to these boundaries. We get the following expressions (we drop  $\mathbf{q}_r$  and  $\mathbf{q}_d$  most of time to shorten notation and use notation of Proposition 3.3):

$$\mathfrak{M}_{j_0,j}^{i_0,i}(\mathbf{q}) := \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F \varphi_{r,j}^i b_{r,j_0}^{i_0} d\Gamma + \delta_{i_0,i} M_{r,j_0,j}^{i_0}, \quad (3.18a)$$

are the entries of the mass matrix of the system bodies-fluid,  $M_{r,j_0,j}^{i_0}$  being given in Definition 4. Furthermore, we have:

$$\begin{aligned} \mathfrak{A}_{j_0,j_1,j_2}^{i_0,i_1,i_2}(\mathbf{q}) := & - \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\nabla_{\Gamma} \varphi_{r,j_0}^{i_0} \cdot \nabla_{\Gamma} \varphi_{r,j_2}^{i_2}) b_{r,j_1}^{i_1} d\Gamma \\ & - \int_{\Gamma_2^{i_2}(\mathbf{q})} \rho_F (\nabla_{\Gamma} \varphi_{r,j_0}^{i_0} \cdot \nabla_{\Gamma} \varphi_{r,j_1}^{i_1}) b_{r,j_2}^{i_2} d\Gamma + \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\nabla_{\Gamma} \varphi_{r,j_1}^{i_1} \cdot \nabla_{\Gamma} \varphi_{r,j_2}^{i_2}) b_{r,j_0}^{i_0} d\Gamma \\ & - 2\delta_{i_1,i_2} \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\mathcal{H} b_{r,j_1}^{i_1} b_{r,j_2}^{i_2} + \langle \Pi_2, \delta_{\Gamma} \mathbf{x}_{r,j_1}^{i_1}, \delta_{\Gamma} \mathbf{x}_{r,j_2}^{i_2} \rangle - \beta_{rr,j_1,j_2}^{i_1}) \\ & + \nabla_{\Gamma} b_{r,j_1}^{i_1} \cdot \delta_{\Gamma} \mathbf{x}_{r,j_2}^{i_2} + \nabla_{\Gamma} b_{r,j_2}^{i_2} \cdot \delta_{\Gamma} \mathbf{x}_{r,j_1}^{i_1}) \varphi_{r,j_0}^{i_0} d\Gamma \\ & + \delta_{i_1,i_0} \delta_{i_2,i_0} \left( A_{j_0,j_1,j_2}^{i_0}(\mathbf{q}) + A_{j_0,j_2,j_1}^{i_0}(\mathbf{q}) + \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r,j_0}^{i_0} b_{r,j_1}^{i_1} b_{r,j_2}^{i_2} d\Gamma \right), \end{aligned} \quad (3.18b)$$

where  $A_{j_0,j_1,j_2}^{i_0}(\mathbf{q})$  is given in (3.7) and  $\beta_{rr,j_1,j_2}^i$ ,  $\beta_{rd,j,k}^i$ ,  $\beta_{dd,k_1,k_2}^i$  and in (A.3). We have, next:

$$\begin{aligned} \mathfrak{B}_{j_0,j,k}^{i_0,i_1,i_2}(\mathbf{q}) := & - \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\nabla_{\Gamma} \varphi_{r,j_0}^{i_0} \cdot \nabla_{\Gamma} \varphi_{d,k}^{i_2}) b_{r,j}^{i_1} d\Gamma \\ & - \int_{\Gamma_2^{i_2}(\mathbf{q})} \rho_F (\nabla_{\Gamma} \varphi_{r,j_0}^{i_0} \cdot \nabla_{\Gamma} \varphi_{r,j}^{i_1}) b_{d,k}^{i_2} d\Gamma + \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\nabla_{\Gamma} \varphi_{r,j}^{i_1} \cdot \nabla_{\Gamma} \varphi_{d,k}^{i_2}) b_{r,j_0}^{i_0} d\Gamma \\ & - 2\delta_{i_1,i_2} \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\mathcal{H} b_{r,j}^{i_1} b_{d,k}^{i_2} + \langle \Pi_2, \delta_{\Gamma} \mathbf{x}_{r,j}^{i_1}, \delta_{\Gamma} \mathbf{x}_{d,k}^{i_2} \rangle - \beta_{rd,j,k}^{i_1}) \\ & + \nabla_{\Gamma} b_{r,j}^{i_1} \cdot \delta_{\Gamma} \mathbf{x}_{d,k}^{i_2} + \nabla_{\Gamma} b_{d,k}^{i_2} \cdot \delta_{\Gamma} \mathbf{x}_{r,j}^{i_1}) \varphi_{r,j_0}^{i_0} d\Gamma \\ & + \delta_{i_1,i_0} \delta_{i_2,i_0} \left( B_{j_0,j,k}^{i_0}(\mathbf{q}) + \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r,j_0}^{i_0} b_{r,j}^{i_1} b_{d,k}^{i_2} d\Gamma \right), \end{aligned} \quad (3.18c)$$

where  $B_{j_0,j,k}^{i_0}(\mathbf{q})$  is given in (3.8). Other terms read:

$$\begin{aligned} \mathfrak{C}_{j_0,j,l}^{i_0,i_1,i_2}(\mathbf{q}) := & - \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\nabla_{\Gamma} \varphi_{r,j_0}^{i_0} \cdot \nabla_{\Gamma} \varphi_{t,l}^{i_2}) b_{r,j}^{i_1} d\Gamma \\ & + \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\nabla_{\Gamma} \varphi_{r,j}^{i_1} \cdot \nabla_{\Gamma} \varphi_{t,l}^{i_2}) b_{r,j_0}^{i_0} d\Gamma \\ & - \delta_{i_1,i_2} \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\mathcal{H} b_{r,j}^{i_1} b_{t,l}^{i_2} + \nabla_{\Gamma} b_{t,l}^{i_2} \cdot \delta_{\Gamma} \mathbf{x}_{r,j}^{i_1}) \varphi_{r,j_0}^{i_0} d\Gamma \\ & + \delta_{i_1,i_0} \delta_{i_2,i_0} \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r,j_0}^{i_0} b_{r,j}^{i_1} b_{t,l}^{i_2} d\Gamma, \end{aligned} \quad (3.18d)$$

and

$$\begin{aligned}
\mathfrak{D}_{j_0, k_1, k_2}^{i_0, i_1, i_2}(\mathbf{q}) &:= \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\nabla_\Gamma \varphi_{r, j_0}^{i_0} \cdot \nabla_\Gamma \varphi_{d, k_2}^{i_2}) b_{d, k_1}^{i_1} \, d\Gamma \\
&+ \int_{\Gamma_2^{i_2}(\mathbf{q})} \rho_F (\nabla_\Gamma \varphi_{r, j_0}^{i_0} \cdot \nabla_\Gamma \varphi_{d, k_1}^{i_1}) b_{d, k_2}^{i_2} \, d\Gamma - \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\nabla_\Gamma \varphi_{d, k_1}^{i_1} \cdot \nabla_\Gamma \varphi_{d, k_2}^{i_2}) b_{r, j_0}^{i_0} \, d\Gamma \\
&+ 2\delta_{i_1, i_2} \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\mathcal{H} b_{d, k_1}^{i_1} b_{d, k_2}^{i_2} + \langle \Pi_2, \delta_\Gamma \mathbf{x}_{d, k_1}^{i_1}, \delta_\Gamma \mathbf{x}_{d, k_2}^{i_2} \rangle - \beta_{dd, k_1, k_2}^{i_1} \\
&\quad + \nabla_\Gamma b_{d, k_1}^{i_1} \cdot \delta_\Gamma \mathbf{x}_{d, k_2}^{i_2} + \nabla_\Gamma b_{d, k_2}^{i_2} \cdot \delta_\Gamma \mathbf{x}_{d, k_1}^{i_1}) \varphi_{r, j_0}^{i_0} \, d\Gamma \\
&\quad - \delta_{i_1, i_0} \delta_{i_2, i_0} \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r, j_0}^{i_0} b_{d, k_1}^{i_1} b_{d, k_2}^{i_2} \, d\Gamma, \quad (3.18e)
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{E}_{j_0, k, l}^{i_0, i_1, i_2}(\mathbf{q}) &:= \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\nabla_\Gamma \varphi_{r, j_0}^{i_0} \cdot \nabla_\Gamma \varphi_{t, l}^{i_2}) b_{d, k}^{i_1} \, d\Gamma \\
&\quad - \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\nabla_\Gamma \varphi_{d, k}^{i_1} \cdot \nabla_\Gamma \varphi_{t, l}^{i_2}) b_{r, j_0}^{i_0} \, d\Gamma \\
&\quad + \delta_{i_1, i_2} \int_{\Gamma_2^{i_1}(\mathbf{q})} \rho_F (\mathcal{H} b_{d, k}^{i_1} b_{t, l}^{i_2} + \nabla_\Gamma b_{t, l}^{i_2} \cdot \delta_\Gamma \mathbf{x}_{d, k}^{i_1}) \varphi_{r, j_0}^{i_0} \, d\Gamma \\
&\quad - \delta_{i_1, i_0} \delta_{i_2, i_0} \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r, j_0}^{i_0} b_{d, k}^{i_1} b_{t, l}^{i_2} \, d\Gamma, \quad (3.18f)
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{F}_{j_0, l_1, l_2}^{i_0, i_1, i_2}(\mathbf{q}) &:= - \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F (\nabla_\Gamma \varphi_{t, l_1}^{i_1} \cdot \nabla_\Gamma \varphi_{t, l_2}^{i_2}) b_{r, j_0}^{i_0} \, d\Gamma \\
&\quad - \delta_{i_1, i_0} \delta_{i_2, i_0} \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F b_{r, j_0}^{i_0} b_{t, l_1}^{i_1} b_{t, l_2}^{i_2} \, d\Gamma, \quad (3.18g)
\end{aligned}$$

and, at last:

$$\mathfrak{G}_{j_0, k}^{i_0, i}(\mathbf{q}) := - \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F \varphi_{d, k}^i b_{r, j_0}^{i_0} \, d\Gamma, \quad (3.18h)$$

$$\mathfrak{H}_{j_0, l}^{i_0, i}(\mathbf{q}) := - \int_{\Gamma_2^{i_0}(\mathbf{q})} \rho_F \varphi_{t, l}^i b_{r, j_0}^{i_0} \, d\Gamma, \quad (3.18i)$$

$$\mathfrak{J}_{j_0}^{i_0}(\mathbf{q}) := \int_{S^{i_0}(\mathbf{q})} (\rho_S - \rho_F) \nabla G \cdot \delta \mathbf{x}_{r, j_0}^{i_0} \, d\mathbf{x}. \quad (3.18j)$$

These equations fit with those given in [28] in a more abstract form and for deformable bodies without thrusters.

Observe that the equations are deeply coupled, what means in particular that the presence of any additional swimmer impacts the motion of the others. When the converse is assumed, one commonly says that the dynamics of the bodies are *hydrodynamically decoupled*. This assumption yields massive simplifications. Although pertinent when the bodies are far one from the others, it is not any longer true for closed bodies as shown in [18]. Indeed, in this article, we show that the velocity damping of a ball getting close to a wall in a perfect fluid, is roughly thirty percents of its initial velocity and hence that D'Alembert's paradox does not hold in this case.

Let us now explain how expressions (3.18) are obtained.

## 4 Comparison between both formalisms

In this section we perform the proof of Theorem 3.2.

### 4.1 Governing equations of motion derived from System (3.13)

The functions  $\varphi_{r,j}^i(\mathbf{q}_r, \mathbf{q}_d)$ ,  $\varphi_{d,k}^i(\mathbf{q}_r, \mathbf{q}_d)$  and  $\varphi_{t,l}^i(\mathbf{q}_r, \mathbf{q}_d)$  will play, most of time, symmetric roles in the subsequent computations, hence we apply the relabelling process described at the end of Appendix A by (A.4). It leads to the definition of the functions  $\psi_j(\mathbf{q})$  for all  $j = 1, \dots, \mathcal{N}$  and identity (3.17) can hence be rewritten in the short form:

$$\psi(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^{\mathcal{N}} \dot{q}_i \psi_i(\mathbf{q}).$$

The same relabelling process is applied to define  $\delta \mathbf{x}_j(\mathbf{q})$  and  $\beta_{j,k}(\mathbf{q})$ ,  $j, k = 1, \dots, \mathcal{N}$  from the quantities defined in (A.3). We introduce the Lagrangian function for the set of bodies:  $\mathcal{L}_b(\mathbf{q}, \dot{\mathbf{q}}) := \sum_{i=1}^n \mathcal{L}_b^i(\mathbf{q}, \dot{\mathbf{q}})$  and since, for all  $i_1, i_2 = 1, \dots, n$  such that  $i_1 \neq i_2$  we have:

$$\frac{\partial \mathcal{L}_b^{i_1}}{\partial q_{r,j_2}^{i_2}}(\mathbf{q}, \dot{\mathbf{q}}) = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}_b^{i_1}}{\partial \dot{q}_{j_2}^{i_2}}(\mathbf{q}, \dot{\mathbf{q}}) = 0,$$

for all  $j_1 = 1, \dots, \mathcal{N}_r^{i_1}$  and all  $j_2 = 1, \dots, \mathcal{N}_r^{i_2}$ , we can rewrite the  $\mathcal{N}_r$  ODE's (3.11e) under the form, for all  $i = 1, \dots, \mathcal{N}_r$ :

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}_b}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{\partial \mathcal{L}_b}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) + \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_j = \\ - \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \left( \frac{d}{dt} (\psi_j(\mathbf{q})) + \frac{1}{2} \sum_{k=1}^{\mathcal{N}} \nabla \psi_j(\mathbf{q}) \cdot \nabla \psi_k(\mathbf{q}) \dot{q}_k \right) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_j, \end{aligned}$$

where we have use Bernoulli's formula (3.13a). But, remarking again that each function  $\psi_j(\mathbf{q})$  depends on  $t$  only through the variable  $\mathbf{q}$ , we get, applying the chain rule:

$$\frac{d}{dt} (\psi_j(\mathbf{q})) = \sum_{k=1}^{\mathcal{N}} \frac{\partial \psi_j}{\partial q_k}(\mathbf{q}) \dot{q}_k, \quad \forall j = 1, \dots, \mathcal{N}, \forall t > 0,$$

and hence, we obtain that, for all  $i = 1, \dots, \mathcal{N}_r$ :

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}_b}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{\partial \mathcal{L}_b}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) + \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_j \\ = - \sum_{k=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \left( \frac{\partial \psi_j}{\partial q_k}(\mathbf{q}) + \frac{1}{2} \nabla \psi_j(\mathbf{q}) \cdot \nabla \psi_k(\mathbf{q}) \right) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_j \dot{q}_k. \end{aligned}$$

The quadratic form on the right can be symmetrized in order to obtain the equivalent formulation:

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}_b}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{\partial \mathcal{L}_b}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) + \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_j \\ = - \frac{1}{2} \sum_{k=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \left( \frac{\partial \psi_j}{\partial q_k}(\mathbf{q}) + \frac{\partial \psi_k}{\partial q_j}(\mathbf{q}) \right. \right. \\ \left. \left. + \nabla \psi_j(\mathbf{q}) \cdot \nabla \psi_k(\mathbf{q}) \right) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_j \dot{q}_k. \quad (4.1) \end{aligned}$$



## 4.2 Governing equations of motion derived from System (3.16)

Plugging the expression (3.17) of the potential function into (3.15) and applying the relabelling process described in Appendix A, we get, by virtue of Green's formula:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{L}_b(\mathbf{q}, \dot{\mathbf{q}}) + \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_k(\mathbf{q}) d\Gamma \right) \dot{q}_j \dot{q}_k.$$

Differentiating with respect to  $\dot{q}_i$  ( $i = 1, \dots, \mathcal{N}_r$ ) and since (invoking again Green's formula)  $\int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_i d\Gamma = \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_i b_j d\Gamma$ , one obtains:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial \mathcal{L}_b}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) + \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_j. \quad (4.2)$$

Differentiating again with respect to  $t$  this equality, it comes:

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{d}{dt} \frac{\partial \mathcal{L}_b}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) + \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \ddot{q}_j \\ &\quad + \sum_{j=1}^{\mathcal{N}} \frac{d}{dt} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_j. \end{aligned}$$

But according to a still used argument, we get on the one hand:

$$\frac{d}{dt} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) = \sum_{k=1}^{\mathcal{N}} \frac{\partial}{\partial q_k} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \dot{q}_k, \quad (4.3)$$

while on the other hand we have, for all  $i = 1, \dots, \mathcal{N}_r$ :

$$\frac{\partial \mathcal{L}}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial \mathcal{L}_b}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) + \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \frac{\partial}{\partial q_i} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_k(\mathbf{q}) d\Gamma \right) \dot{q}_j \dot{q}_k. \quad (4.4)$$

Eventually, summarizing (4.3) and (4.4), we get, for all  $i = 1, \dots, \mathcal{N}_r$ :

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{\partial \mathcal{L}}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{d}{dt} \frac{\partial \mathcal{L}_b}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{\partial \mathcal{L}_b}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) \\ &\quad + \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \ddot{q}_j + \sum_{j=1}^{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \left[ \frac{\partial}{\partial q_k} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \right. \\ &\quad \left. - \frac{1}{2} \frac{\partial}{\partial q_i} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_k(\mathbf{q}) d\Gamma \right) \right] \dot{q}_j \dot{q}_k. \end{aligned}$$

We symmetrize the quadratic form on the right, and substitute the identity above into (3.16d) to obtain:

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}_b}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{\partial \mathcal{L}_b}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) &+ \sum_{j=1}^{\mathcal{N}} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \ddot{q}_j = F_i(\mathbf{q}, \dot{\mathbf{q}}) \\ &- \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \left[ \frac{\partial}{\partial q_k} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) + \frac{\partial}{\partial q_j} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_k(\mathbf{q}) b_i(\mathbf{q}) d\Gamma \right) \right. \\ &\quad \left. - \frac{\partial}{\partial q_i} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j(\mathbf{q}) b_k(\mathbf{q}) d\Gamma \right) \right] \dot{q}_j \dot{q}_k, \quad (4.5) \end{aligned}$$

where the expressions of  $F_i(\mathbf{q}, \dot{\mathbf{q}})$  for all  $i = 1, \dots, \mathcal{N}_r^i$  are obtained from those of the generalized forces  $F_k^i(\mathbf{q}, \dot{\mathbf{q}})$  via the relabelling process. The game consists now in comparing both ODE's (4.1) and (4.5). It is clear that we will need to compute the derivatives  $\partial\psi_i(\mathbf{q})/\partial q_k$ , that are the derivatives of a Neumann boundary value problem with respect to variables relating to the shape of the PDE's domain. This task will be achieved by using methods of shape optimization theory.

### 4.3 Explicit computation of both quadratic forms

In this section, due to the length of the formula we will handle, we will drop  $\mathbf{q}$  most of time to shorten notation.

**Lemma 4.1** *Assume that  $\Gamma_0$  is of class  $C^2$  as well as the functions of deformation (2.2). Then, for any  $i = 1, \dots, \mathcal{N}$  and any  $k = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$ , the function  $\partial\psi_i/\partial q_k$  is well defined. It is the variational solution, in the Sobolev space  $H^1(\Omega(\mathbf{q}))$  when  $\Omega$  is bounded or in suitable weighted Sobolev spaces when  $\Omega$  is not, of the Neumann boundary value problem:*

$$-\Delta \left( \frac{\partial\psi_i}{\partial q_k} \right) = 0 \text{ in } \Omega(\mathbf{q}), \quad \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial\psi_i}{\partial q_k} \right) = \begin{cases} g_{i,k} & \text{on } \Gamma_2(\mathbf{q}), \\ 0 & \text{on } \Gamma_1. \end{cases} \quad (4.6a)$$

In this system,  $g_{i,k} \in H^{-1/2}(\Gamma_2(\mathbf{q}))$  and reads, when  $i = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$ :

$$g_{i,k} := \Delta_\Gamma \psi_i b_k - \mathcal{H} b_i b_k - \langle \Pi_2, \delta_\Gamma \mathbf{x}_k, \delta_\Gamma \mathbf{x}_i \rangle + \nabla_\Gamma \psi_i \cdot \nabla_\Gamma b_k + \beta_{i,k} - \nabla_\Gamma b_k \cdot \delta_\Gamma \mathbf{x}_i - \nabla_\Gamma b_i \cdot \delta_\Gamma \mathbf{x}_k, \quad (4.6b)$$

where  $\Delta_\Gamma$  is the Laplace-Beltrami operator on  $\Gamma_2(\mathbf{q})$  defined for instance in [17, Definition 5.4.11 page 196]. When  $i = \mathcal{N}_r + \mathcal{N}_d + 1, \dots, \mathcal{N}$ , some terms vanish and expression (4.6b) turns into:

$$g_{i,k} := \Delta_\Gamma \psi_i b_k - \mathcal{H} b_i b_k + \nabla_\Gamma \psi_i \cdot \nabla_\Gamma b_k - \nabla_\Gamma b_i \cdot \delta_\Gamma \mathbf{x}_k. \quad (4.6c)$$

Moreover we have, for all  $i, j = 1, \dots, \mathcal{N}$  and  $k = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$ :

$$\begin{aligned} \frac{\partial}{\partial q_k} \left( \int_{\Omega(\mathbf{q})} \rho_F (\nabla \psi_i \cdot \nabla \psi_j) \, d\mathbf{x} \right) &= \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla \psi_i \cdot \nabla \psi_j) b_k \, d\Gamma \\ &+ \int_{\Omega(\mathbf{q})} \rho_F \nabla \left( \frac{\partial\psi_i}{\partial q_k} \right) \cdot \nabla \psi_j \, d\mathbf{x} + \int_{\Omega(\mathbf{q})} \rho_F \nabla \left( \frac{\partial\psi_j}{\partial q_k} \right) \cdot \nabla \psi_i \, d\mathbf{x}. \end{aligned} \quad (4.7)$$

Notice that when  $k \geq \mathcal{N}_r + \mathcal{N}_d + 1$ , then  $\partial\psi/\partial q_k \equiv 0$  since none of the terms depends on  $q_k$ . The proof of this lemma is given in Appendix B.

We can now realize our aim, that is to compare both quadratic forms in (4.1) and in (4.5). Theorem 3.2 is a straight consequence of the following lemma:

**Lemma 4.2** *Under the assumptions of Lemma 4.1 and for all  $i = 1, \dots, \mathcal{N}_r$ , all  $j = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$  and all  $k = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$ , we have the equalities:*

$$\begin{aligned} \int_{\Gamma_2(\mathbf{q})} \rho_F \left( \frac{\partial\psi_j}{\partial q_k} + \frac{\partial\psi_k}{\partial q_j} + \nabla \psi_j \cdot \nabla \psi_k \right) b_i \, d\Gamma = \\ \frac{\partial}{\partial q_k} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_i \, d\Gamma \right) + \frac{\partial}{\partial q_j} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_k b_i \, d\Gamma \right) - \frac{\partial}{\partial q_i} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_k \, d\Gamma \right). \end{aligned} \quad (4.8)$$

Both, left and right hand side terms, are equal to:

$$\begin{aligned}
& - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_i \cdot \nabla_\Gamma \psi_j) b_k \, d\Gamma - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_i \cdot \nabla_\Gamma \psi_k) b_j \, d\Gamma \\
& + \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_j \cdot \nabla_\Gamma \psi_k) b_i \, d\Gamma - 2 \int_{\Gamma_2(\mathbf{q})} \rho_F \left( \mathcal{H} b_j b_k + \langle \Pi_2, \delta_\Gamma \mathbf{x}_j, \delta_\Gamma \mathbf{x}_k \rangle \right. \\
& \quad \left. - \beta_{j,k} + \nabla_\Gamma b_k \cdot \delta_\Gamma \mathbf{x}_j + \nabla_\Gamma b_j \cdot \delta_\Gamma \mathbf{x}_k \right) \psi_i \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F b_i b_j b_k \, d\Gamma. \quad (4.9)
\end{aligned}$$

For all  $i = 1, \dots, \mathcal{N}_r$  and all  $j = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$  and all  $k = \mathcal{N}_r + \mathcal{N}_d + 1, \dots, \mathcal{N}$ , equality (4.8) does not hold any longer. Expression above turns into:

$$\begin{aligned}
& \int_{\Gamma_2(\mathbf{q})} \rho_F \left( \frac{\partial \psi_j}{\partial q_k} + \frac{\partial \psi_k}{\partial q_j} + \nabla \psi_j \cdot \nabla \psi_k \right) b_i \, d\Gamma = \\
& \quad - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_i \cdot \nabla_\Gamma \psi_k) b_j \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_j \cdot \nabla_\Gamma \psi_k) b_i \, d\Gamma \\
& \quad - \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H} b_j b_k + \nabla_\Gamma b_k \cdot \delta_\Gamma \mathbf{x}_j) \psi_i \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F b_i b_j b_k \, d\Gamma,
\end{aligned}$$

whereas:

$$\begin{aligned}
& \frac{\partial}{\partial q_k} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_i \, d\Gamma \right) + \frac{\partial}{\partial q_j} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_k b_i \, d\Gamma \right) - \frac{\partial}{\partial q_i} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_k \, d\Gamma \right) = \\
& \quad - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_i \cdot \nabla_\Gamma \psi_k) b_j \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_j \cdot \nabla_\Gamma \psi_k) b_i \, d\Gamma \\
& \quad - \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H} b_j b_k + \nabla_\Gamma b_k \cdot \delta_\Gamma \mathbf{x}_j) \psi_i \, d\Gamma \\
& \quad + \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H} b_i b_k + \nabla_\Gamma b_k \cdot \delta_\Gamma \mathbf{x}_i) \psi_j \, d\Gamma.
\end{aligned}$$

At last, for all  $i = 1, \dots, \mathcal{N}_r$ , all  $j = \mathcal{N}_r + \mathcal{N}_d + 1, \dots, \mathcal{N}$  and all  $k = \mathcal{N}_r + \mathcal{N}_d + 1, \dots, \mathcal{N}$ :

$$\begin{aligned}
& \int_{\Gamma_2(\mathbf{q})} \rho_F \left( \frac{\partial \psi_j}{\partial q_k} + \frac{\partial \psi_k}{\partial q_j} + \nabla \psi_j \cdot \nabla \psi_k \right) b_i \, d\Gamma = \\
& \quad \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_j \cdot \nabla_\Gamma \psi_k) b_i \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F b_i b_j b_k \, d\Gamma,
\end{aligned}$$

while

$$\begin{aligned}
& \frac{\partial}{\partial q_k} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_i \, d\Gamma \right) + \frac{\partial}{\partial q_j} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_k b_i \, d\Gamma \right) - \frac{\partial}{\partial q_i} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_k \, d\Gamma \right) = \\
& \quad \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_\Gamma \psi_j \cdot \nabla_\Gamma \psi_k) b_i \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H} b_i b_j + \nabla_\Gamma b_j \cdot \delta_\Gamma \mathbf{x}_i) \psi_k \, d\Gamma \\
& \quad + \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H} b_i b_k + \nabla_\Gamma b_k \cdot \delta_\Gamma \mathbf{x}_i) \psi_j \, d\Gamma - \int_{\Gamma_2(\mathbf{q})} \rho_F b_i b_j b_k \, d\Gamma.
\end{aligned}$$

**Proof :** Taking into account (4.6a) and applying twice Green's formula, we can rewrite the left hand side of (4.8) as follows:

$$\begin{aligned} \int_{\Gamma_2(\mathbf{q})} \rho_F \left( \frac{\partial \psi_j}{\partial q_k} + \frac{\partial \psi_k}{\partial q_j} + \nabla \psi_j \cdot \nabla \psi_k \right) b_i \, d\Gamma = \\ \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_j}{\partial q_k} \right) \psi_i \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_k}{\partial q_j} \right) \psi_i \, d\Gamma \\ + \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_{\Gamma} \psi_j \cdot \nabla_{\Gamma} \psi_k) b_i \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F b_i b_j b_k \, d\Gamma. \end{aligned} \quad (4.10)$$

Concerning the right hand side terms in (4.8), we use formula (4.7) combined with Green's formula to obtain, after simplifying:

$$\begin{aligned} \frac{\partial}{\partial q_k} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_i \, d\Gamma \right) + \frac{\partial}{\partial q_j} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_k b_i \, d\Gamma \right) - \frac{\partial}{\partial q_i} \left( \int_{\Gamma_2(\mathbf{q})} \rho_F \psi_j b_k \, d\Gamma \right) = \\ \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_j}{\partial q_k} \right) \psi_i \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_i}{\partial q_k} \right) \psi_j \, d\Gamma \\ + \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_{\Gamma} \psi_j \cdot \nabla_{\Gamma} \psi_i) b_k \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_k}{\partial q_j} \right) \psi_i \, d\Gamma \\ + \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_i}{\partial q_j} \right) \psi_k \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_{\Gamma} \psi_k \cdot \nabla_{\Gamma} \psi_i) b_j \, d\Gamma \\ - \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_j}{\partial q_i} \right) \psi_k \, d\Gamma - \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_k}{\partial q_i} \right) \psi_j \, d\Gamma \\ - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_{\Gamma} \psi_j \cdot \nabla_{\Gamma} \psi_k) b_i \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F b_i b_j b_k \, d\Gamma. \end{aligned} \quad (4.11)$$

Green-like formula [17, Theorem 5.4.13, formula (5.59) pages 196-197] yields:

$$\int_{\Gamma_2(\mathbf{q})} \rho_F \Delta_{\Gamma} \psi_i b_k \psi_j \, d\Gamma = - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_{\Gamma} \psi_i \cdot \nabla_{\Gamma} \psi_j) b_k \, d\Gamma - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_{\Gamma} \psi_i \cdot \nabla_{\Gamma} b_k) \psi_j \, d\Gamma,$$

and then, taking into account (4.6a) and (4.6b), we obtain, for all  $i, k = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$ :

$$\begin{aligned} \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_i}{\partial q_k} \right) \psi_j \, d\Gamma = - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_{\Gamma} \psi_i \cdot \nabla_{\Gamma} \psi_j) b_k \, d\Gamma \\ - \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H} b_i b_k + \langle \Pi_2, \delta_{\Gamma} \mathbf{x}_k, \delta_{\Gamma} \mathbf{x}_i \rangle - \beta_{i,k} \\ + \nabla_{\Gamma} b_k \cdot \delta_{\Gamma} \mathbf{x}_i + \nabla_{\Gamma} b_i \cdot \delta_{\Gamma} \mathbf{x}_k) \psi_j \, d\Gamma. \end{aligned} \quad (4.12a)$$

When  $i = \mathcal{N}_r + \mathcal{N}_d + 1, \dots, \mathcal{N}$ , this identity becomes, according to (4.6c):

$$\begin{aligned} \int_{\Gamma_2(\mathbf{q})} \rho_F \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_i}{\partial q_k} \right) \psi_j \, d\Gamma = - \int_{\Gamma_2(\mathbf{q})} \rho_F (\nabla_{\Gamma} \psi_i \cdot \nabla_{\Gamma} \psi_j) b_k \, d\Gamma \\ - \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H} b_i b_k + \nabla_{\Gamma} b_i \cdot \delta_{\Gamma} \mathbf{x}_k) \psi_j \, d\Gamma. \end{aligned} \quad (4.12b)$$

Of course,  $\partial\psi_i/\partial q_k \equiv 0$  for all  $k = \mathcal{N}_r + \mathcal{N}_d + 1, \dots, \mathcal{N}$ . Next, plugging (4.12) into (4.10) and (4.11), we obtain the equalities of the lemma and the proof is then complete. ■

Since, obviously, there is only one set of governing equations of motion valid for our problem, we deduce from the previous lemma the expressions of the generalized forces given in Proposition 3.3, by equalizing (4.1) and (4.5) and taking into account the results of Lemma 4.2. We display them with the notation used in this section:

**Corollary 4.1** *The expression of the generalized force  $F_i$  is, for any  $i = 1, \dots, \mathcal{N}_r$ :*

$$F_i(\mathbf{q}) := \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} f_{i,j,k}(\mathbf{q}) \dot{q}_j \dot{q}_k,$$

where  $f_{i,j,k}(\mathbf{q}) = f_{i,k,j}(\mathbf{q})$  for all  $j, k = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$ . Moreover, for all  $j, k = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$ :

$$f_{i,j,k}(\mathbf{q}) := 0. \quad (4.13a)$$

For all  $j = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$  and all  $k = \mathcal{N}_r + \mathcal{N}_d + 1, \dots, \mathcal{N}$ , the expression of  $f_{i,j,k}(\mathbf{q})$  becomes:

$$f_{i,j,k}(\mathbf{q}) := - \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H}b_i b_k + \nabla_{\Gamma} b_k \cdot \delta_{\Gamma} \mathbf{x}_i) \psi_j \, d\Gamma + \int_{\Gamma_2(\mathbf{q})} \rho_F b_i b_j b_k \, d\Gamma. \quad (4.13b)$$

At last, for all  $j, k = \mathcal{N}_r + \mathcal{N}_d + 1, \dots, \mathcal{N}$ , we have:

$$\begin{aligned} f_{i,j,k}(\mathbf{q}) := & - \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H}b_i b_j + \nabla_{\Gamma} b_j \cdot \delta_{\Gamma} \mathbf{x}_i) \psi_k \, d\Gamma \\ & - \int_{\Gamma_2(\mathbf{q})} \rho_F (\mathcal{H}b_i b_k + \nabla_{\Gamma} b_k \cdot \delta_{\Gamma} \mathbf{x}_i) \psi_j \, d\Gamma + 2 \int_{\Gamma_2(\mathbf{q})} \rho_F b_i b_j b_k \, d\Gamma. \end{aligned} \quad (4.13c)$$

## 5 On the vertical motion of a nonsmooth shaped solid

In this section, we will provide a counterexample to Theorem 3.1 by exhibiting a two dimensional nonsmooth solid moving vertically in a fluid and for which Newtonian and Lagrangian approaches lead to distinct results. The solid is the one pictured on the right of Figure 2, and we assume that the system fluid-solid fills the whole two dimensional space. We will use complex analysis to compute explicitly the potential function of the fluid flow, so we identify the space with  $\mathbb{C}$  (the set of the complex numbers). However, when there is no confusion possible, we will mix the complex notation  $\xi = \xi_1 + i\xi_2$  ( $i^2 = -1$ ) with the real one  $\xi = (\xi_1, \xi_2)^T$  and even with the polar coordinates  $(r, \theta)$ ,  $r = |\xi|$  and  $\theta = \text{Arg}(\xi)$  (i.e.  $\xi = r e^{i\theta}$ ). The solid has an axis of symmetry and we assume that at the time  $t = 0$  this axis coincides with the imaginary axis and that the center of mass of the rigid body lies at the origin of the frame. If we assume in addition that the density of the rigid body is constant, then, for symmetry reasons, its center of mass will remain on the imaginary axis for all  $t > 0$  whenever the initial velocity of the solid is vertical. Therefore, within this configuration, the system has only one degree of freedom. We denote by  $ih$  the position of the center of mass of the solid and  $i\dot{h}$  stands for its velocity. The domain occupied by the solid is  $S(h)$  while  $\Gamma(h)$  stands for its boundary and  $\Omega(h) := \mathbb{C} \setminus S(h)$  is the domain of the surrounded fluid. We denote merely  $S := S(0)$ ,  $\Gamma := \Gamma(0)$  and  $\Omega := \Omega(0)$ . The boundary  $\Gamma$  is the image of the unitary circle by the mapping:

$$f(\xi) := \frac{3\xi^2 - 1}{2\xi} + \frac{2\xi}{3\xi^2 - 1}, \quad \xi := \xi_1 + i\xi_2 \in \mathbb{C}, \quad (5.1)$$

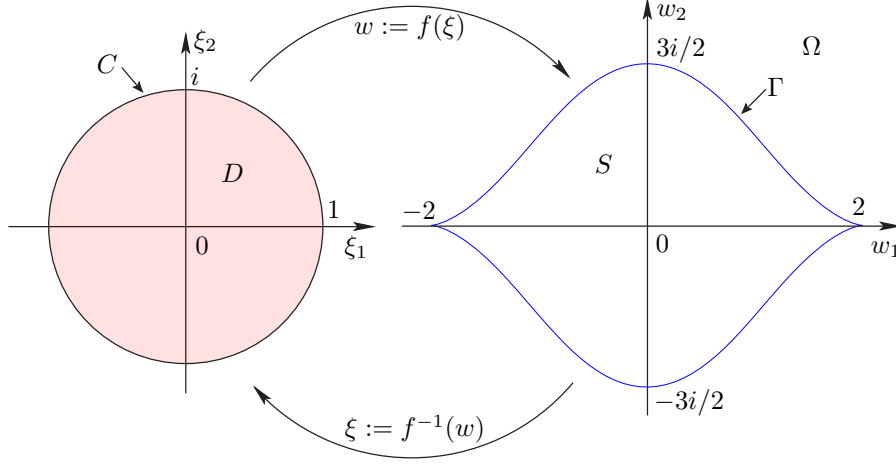


Figure 2: Function  $f$  is a conformal mapping from  $D^c := \mathbb{C} \setminus D$  onto  $\Omega$ .

which is a conformal mapping from  $D^c$ , the exterior of the unitary disk  $D$ , onto  $\Omega$ . We denote  $w := f(\xi)$  for all  $\xi \in D^c$ . Following [14, Chap XIII], we introduce, for all  $w \in \Omega(h)$ ,  $\psi(h, \dot{h}, w) := \psi_1(h, \dot{h}, w) + i\psi_2(h, \dot{h}, w)$  the complex holomorphic potential function relating to the fluid flow. Commonly,  $\psi_1(h, \dot{h}, w)$  is the actual real valued potential function while  $\psi_2(h, \dot{h}, w)$  is the stream function. The potential can be rewritten under the form:

$$\psi(h, \dot{h}, w) = \dot{h}\varphi(w - ih), \quad (5.2)$$

where  $\varphi(w) := \varphi_1(w) + i\varphi_2(w)$ . The functions  $\varphi_1$  and  $\varphi_2$  are both defined and harmonic in the fixed domain  $\Omega$ , and do not depend on  $h$ . In order to compute  $\varphi$ , we introduce the function:

$$\phi(\xi) := \varphi(f(\xi)), \quad \forall \xi \in D^c,$$

and we denote  $\phi_1 := \Re(\phi)$  and  $\phi_2 := \Im(\phi)$ . According to [14, formula (5) page 222],  $\phi_2(h)$  solves a Dirichlet boundary value problem:

$$\begin{aligned} -\Delta\phi_2(\xi) &= 0 \text{ in } D^c, & \phi_2(\xi) &= -\Re(f(\xi)) \text{ on } C, \\ |\nabla\phi_2(\xi)| &\rightarrow 0 \text{ as } |\xi| \rightarrow +\infty. \end{aligned}$$

The last assertion tells us that the fluid is quiescent at infinity. We can extend  $f(\xi)$  and  $f(1/\xi)$  as Laurent's series at  $\xi = 0$ . We get:

$$\begin{aligned} f(\xi) &= \frac{3}{2}\xi + \frac{1}{6}\xi^{-1} + \sum_{k \geq 1} \frac{2}{3^{k+1}}\xi^{-2k-1} \quad \forall \xi \in \mathbb{C}, |\xi| > 1/3, \\ f(1/\xi) &= \frac{3}{2}\xi^{-1} + \frac{1}{6}\xi + \sum_{k \geq 1} \frac{2}{3^{k+1}}\xi^{2k+1} \quad \forall \xi \in \mathbb{C}, |\xi| < 3. \end{aligned}$$

Writing then that  $\phi_2(e^{i\theta}) = -(f(e^{i\theta}) + f(e^{-i\theta}))/2$  we obtain the expression of  $\phi_2(e^{i\theta})$  under the form of a Fourier's series:

$$\phi_2(e^{i\theta}) = -\frac{5}{6}\cos(\theta) - \sum_{k \geq 1} 3^{-k-1}\cos((2k+1)\theta), \quad \forall \theta \in [-\pi, \pi[,$$

and we deduce that, (using here the polar coordinates  $(r, \theta)$ , due to the geometry of the domain):

$$\phi_2(r, \theta) = -\frac{5}{6}\frac{\cos(\theta)}{r} - \sum_{k \geq 1} 3^{-k-1}\frac{\cos((2k+1)\theta)}{r^{2k+1}}, \quad \forall \theta \in [-\pi, \pi[, \forall r \geq 1. \quad (5.3)$$

Observe that  $\phi_2$  can actually be extended as an harmonic function outside the disk of center  $\mathbf{0}$  and radius  $1/\sqrt{3}$ . Let us compute now the Lagrangian function  $\mathcal{L}(h, \dot{h})$  of the system fluid-solid. We assume that the density of the solid and the density of the fluid,  $\rho_F > 0$  are equal. Hence the solid is neutrally buoyant and the Lagrangian function reduces to the kinematic energy of the system, namely:

$$\mathcal{L}(h, \dot{h}) = \frac{1}{2} \left( m |\dot{h}|^2 + \int_{\Omega(h)} \rho_F |\nabla \psi_2(h, \dot{h}, w)|^2 dw \right),$$

where  $m > 0$  stands for the mass of the solid. Notice that  $|\nabla \psi_2(h, \dot{h}, w)| = |\nabla \psi_1(h, \dot{h}, w)|$  for  $\psi(h, \dot{h})$  is an holomorphic function. This expression can be turned into, upon a change of variables and applying Green's formula:

$$\mathcal{L}(h, \dot{h}) = \frac{1}{2} |\dot{h}|^2 \left( m + \rho_F \int_C \phi_2 \frac{\partial \phi_2}{\partial \mathbf{n}} d\Gamma \right),$$

and we deduce that the Lagrangian function does actually not depend on  $h$ . Euler-Lagrange's equation of motion is merely:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{h}}(\dot{h}) = \left( m + \rho_F \int_C \phi_2 \frac{\partial \phi_2}{\partial \mathbf{n}} d\Gamma \right) \ddot{h} = 0, \quad (5.4)$$

what, after integrating, leads to so-called D'Alembert's paradox:  $\dot{h}(t) = \dot{h}_0$  for all  $t > 0$ . The solid has a straight uniform velocity and hence the hydrodynamic resultant force is null.

Let us now invoke Newton's laws with the same configuration. Making use of Bernoulli's formula, we get:

$$m \ddot{h} = - \int_{\Gamma(h)} \rho_F \left( \frac{\partial \psi_1}{\partial t}(h, \dot{h}) + \frac{1}{2} |\nabla \psi_1(h, \dot{h})|^2 \right) n_2 d\Gamma, \quad (5.5)$$

where  $\mathbf{n} = (n_1, n_2)^T$  is the unitary normal to  $C$  directed toward the interior of  $D$ . According to (5.2), we obtain that:

$$\frac{\partial \psi_1}{\partial t}(h, \dot{h}, w) = \ddot{h} \varphi_1(w - ih) - |\dot{h}|^2 \frac{\partial \varphi_1}{\partial w_2}(w - ih), \quad \forall w \in \Omega(h).$$

Plugging this expression into (5.5), and upon a change of variables, we get:

$$\left( m + \rho_F \int_C \phi_2 \frac{\partial \phi_2}{\partial \mathbf{n}} d\Gamma \right) \ddot{h} = - |\dot{h}|^2 \rho_F \int_{\Gamma} \left( \frac{1}{2} |\nabla \varphi_2|^2 - \frac{\partial \varphi_1}{\partial w_2} \right) n_2 d\Gamma, \quad (5.6)$$

where we are use again that  $|\nabla \phi_1| = |\nabla \phi_2|$  and  $|\nabla \varphi_1| = |\nabla \varphi_2|$  for  $\phi$  and  $\varphi$  are both holomorphic functions. This equation fits with (5.4) if and only if the right hand side term vanishes. We are going to show that it is actually undetermined, in the sense that the amount inside the integral does not belong to  $L^1(\Gamma)$ . We have, on the one hand:

$$\int_{\Gamma} \left| \frac{\partial \varphi_1}{\partial w_2} n_2 \right| d\Gamma \leq \int_{\Gamma} |\nabla \varphi_1| d\Gamma,$$

and arguing again that  $|\nabla \varphi_1| = |\nabla \varphi_2|$ , it comes, upon a change of variables:

$$\begin{aligned} \int_{\Gamma} \left| \frac{\partial \varphi_1}{\partial w_2} n_2 \right| d\Gamma &\leq (\text{mes}(\Gamma))^{1/2} \left( \int_{\Gamma} |\nabla \varphi_2|^2 d\Gamma \right)^{1/2} \\ &\leq (\text{mes}(\Gamma))^{1/2} \left( \int_C |\nabla \phi_2|^2 d\Gamma \right)^{1/2} < +\infty. \end{aligned}$$

On the other hand, going back to (5.6), we have, applying the same change of variables:

$$\int_{\Gamma} |\nabla \varphi_2|^2 n_2 \, d\Gamma = \int_{-\pi}^{\pi} |\nabla \phi_2(e^{i\theta})|^2 |f'(e^{i\theta})|^{-2} \Im(-e^{i\theta} f'(e^{i\theta})) \, d\theta. \quad (5.7)$$

Let us then compute the behavior of  $|\nabla \phi_2(e^{i\theta})|^2 |f'(e^{i\theta})|^{-2} \Im(-e^{i\theta} f'(e^{i\theta}))$  nearby  $\theta = 0$ . By virtue of (5.3), we get:

$$\frac{\partial \phi_2}{\partial \theta}(1, 0) = 0 \quad \text{and} \quad \frac{\partial \phi_2}{\partial r}(1, 0) = \gamma, \quad (5.8)$$

where

$$\gamma := \frac{5}{6} + \sum_{k \geq 1} \frac{(2k+1)}{3^{k+1}} > 0.$$

In addition, one readily obtains that:

$$|f'(e^{i\theta})|^{-2} \Im(-e^{i\theta} f'(e^{i\theta})) = -\frac{1}{8} \theta^{-1} + \mathcal{O}(\theta). \quad (5.9)$$

Summarizing (5.8), (5.9) we get

$$|\nabla \phi_2(e^{i\theta})|^2 |f'(e^{i\theta})|^{-2} \Im(-e^{i\theta} f'(e^{i\theta})) = -\gamma^2 \theta^{-1} + \mathcal{O}(\theta),$$

what proves that (5.7) is undetermined.

To sum up, Newton's approach fails in providing the equation of motion for the solid. The problem is that the velocity of the fluid is infinite nearby the singular points of  $\Gamma(h)$  as well as the hydrodynamic forces although the energy of the system be always finite.

This phenomena is well known in airfoils theory (see [14, Chap. XII, §c. pages 211-212]). This problem can be overcome by introducing, for any time  $t \geq 0$ , circulation in the fluid flow in such a way that the velocity be finite at the singular point. In other words, circulation is used as a new degree of freedom of the fluid flow and its value is set according to the so-called Kutta-Joukowski's law. However, notice that this method does not apply with our problem since the solid we are handled has two singular points. Furthermore, to our knowledge, it has not been proved so far that the solution agreeing with Kutta-Joukowski's law solves System 3.11.

## 6 Concluding comments

Lagrangian formalism consists in computing first the energy of the fluid flow: it is obtained by integrating the gradient of the potential (or stream) function over the fluid domain. This is a smoothing process for the resulting kinetic energy is more regular with respect to the generalized coordinates than the potential function. Then, according to the least action principle, Euler-Lagrange's equations are obtained partly by differentiating the fluid's kinetic energy. Notice that this process is carried out without difficulties in the example treated in Section 5.

Conversely, with Newtonian formalism, forces are computed first, what requires to differentiate the potential (or the stream) function and yields, in the nonsmooth case of Section 5 an infinite and nonintegrable quantity. Next these forces are integrated (when it is allowable) along the boundary of the bodies in order to obtain the hydrodynamic resultant force and next to apply Newton's laws.

We have proved that both approaches coincide for smooth bodies. Although Lagrangian formalism is more simple to handle, it does not allow to obtain the expected governing equations of motion when the bodies are endowed with thrusters.



At last, we guess that the underlying question relating to the phenomena encountered in the example of Section 5 is: does it actually exist a solution to System (3.11) that remains free of vortex for all time, even in the presence of a nonsmooth shaped solid inside the fluid? Indeed, it is known that vortices are only carried by the fluid flow along the stream lines and hence, assuming that the fluid is quiescent at the initial time, they can only appear on the fluid boundary. What could explain the paradoxical results of Section 5 is that vortices actually do appear at singular points and therefore that there does not exist any potential function solving Euler's equations in this case.

## A Additional technical definitions

The vectors  $\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i)$ ,  $\delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  and  $\delta \mathbf{x}_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  defined in Subsection 2.3 can be decomposed into their tangential and normal components on  $\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ . They are next extended by 0 on  $\Gamma(\mathbf{q}) \setminus \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ .

**Definition 5** We introduce, for all  $i = 1, \dots, n$  and all  $j = 1, \dots, \mathcal{N}_r^i$ :

$$b_{r,j}^i(\mathbf{q}_r^i) := \begin{cases} \delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) & \text{on } \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \\ 0 & \text{on } \Gamma(\mathbf{q}_r, \mathbf{q}_d) \setminus \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \end{cases} \quad (\text{A.1a})$$

and for all  $k = 1, \dots, \mathcal{N}_d^i$ :

$$b_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) := \begin{cases} \delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) & \text{on } \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \\ 0 & \text{on } \Gamma(\mathbf{q}_r, \mathbf{q}_d) \setminus \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \end{cases} \quad (\text{A.1b})$$

and for all  $l = 1, \dots, \mathcal{N}_t^i$ :

$$b_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) := \begin{cases} \delta \mathbf{x}_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) & \text{on } \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \\ 0 & \text{on } \Gamma(\mathbf{q}_r, \mathbf{q}_d) \setminus \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \end{cases} \quad (\text{A.1c})$$

while the tangential components are defined by:

$$\begin{aligned} \delta_{\Gamma} \mathbf{x}_{r,j}^i(\mathbf{q}_r^i) &:= \delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i) - b_{r,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) && \text{on } \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \\ \delta_{\Gamma} \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) &:= \delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) - b_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) && \text{on } \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \\ \delta_{\Gamma} \mathbf{x}_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) &:= \delta \mathbf{x}_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) - b_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i) && \text{on } \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \end{aligned}$$

all of them being extended by 0 on  $\Gamma(\mathbf{q}_r, \mathbf{q}_d) \setminus \Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ . Statement of the governing equations of motion will also involve the quantities:

$$\delta^2 \mathbf{x}_{rr,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{x}) := \frac{\partial^2 \mathbf{X}^i}{\partial q_{r,j_1}^i \partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})), \quad (\text{A.2a})$$

$$\delta^2 \mathbf{x}_{rd,j,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) := \frac{\partial^2 \mathbf{X}^i}{\partial q_{r,j}^i \partial q_{d,k}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})), \quad (\text{A.2b})$$

$$\delta^2 \mathbf{x}_{dd,k_1,k_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) := \frac{\partial^2 \mathbf{X}^i}{\partial q_{d,k_1}^i \partial q_{d,k_2}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{Y}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})), \quad (\text{A.2c})$$

for all  $i = 1, \dots, n$ , all  $j, j_1, j_2 = 1, \dots, \mathcal{N}_r^i$ , all  $k, k_1, k_2 = 1, \dots, \mathcal{N}_d^i$  and all  $\mathbf{x} \in S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ . Their normal components are denoted by:

$$\beta_{rr,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{x}) := \delta^2 \mathbf{x}_{rr,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{x}) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}), \quad (\text{A.3a})$$

$$\beta_{rd,j,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) := \delta^2 \mathbf{x}_{rd,j,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}), \quad (\text{A.3b})$$

$$\beta_{dd,k_1,k_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) := \delta^2 \mathbf{x}_{dd,k_1,k_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x}), \quad (\text{A.3c})$$

also extended by 0 to the whole boundary  $\Gamma(\mathbf{q}^r, \mathbf{q}_d)$ .

All the quantities defined above can be explicitly computed once the generalized coordinates have been set up. The expressions of  $\delta^2 \mathbf{x}_{rr,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{x})$  and  $\delta^2 \mathbf{x}_{rd,j,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})$  are connected to those of  $\delta \mathbf{x}_{r,j}^i(\mathbf{q}_r^i, \mathbf{x})$  and  $\delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \mathbf{x})$ . Such formula are given in Lemma B.1 in Appendix B.

## Relabelling process

In the subsequent computations, some quantities defined in this Section will play symmetric roles. It would be then convenient when necessary to relabel them. We proceed as follows, for instance with  $b_{r,j}^i(\mathbf{q}_r^i)$ ,  $b_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  and  $b_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ :

$$1 \leq i \leq n, \quad 1 \leq j \leq \mathcal{N}_r^i, \quad b_{\mathcal{N}_r^i(i-1)+j}(\mathbf{q}) := b_{r,j}^i(\mathbf{q}_r^i), \quad (\text{A.4a})$$

$$1 \leq i \leq n, \quad 1 \leq k \leq \mathcal{N}_d^i, \quad b_{\mathcal{N}_r^i+\mathcal{N}_d^i+\dots+\mathcal{N}_d^{i-1}+k}(\mathbf{q}) := b_{d,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i), \quad (\text{A.4b})$$

$$1 \leq i \leq n, \quad 1 \leq l \leq \mathcal{N}_t^i, \quad b_{\mathcal{N}_r^i+\mathcal{N}_d^i+\mathcal{N}_t^i+\dots+\mathcal{N}_t^{i-1}+l}(\mathbf{q}) := b_{t,l}^i(\mathbf{q}_r^i, \mathbf{q}_d^i). \quad (\text{A.4c})$$

We define then  $\mathbf{b}(\mathbf{q}) := (b_1(\mathbf{q}), \dots, b_{\mathcal{N}}(\mathbf{q}))^T$ . For quantities carrying double subscripts, like  $\beta_{rr,j_1,j_2}^i(\mathbf{q}_r^i)$ ,  $\beta_{rd,j,k}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$  and  $\beta_{dd,k_1,k_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ , the relabelling process yields without confusion possible, the expressions of  $\beta_{j,k}(\mathbf{q})$ ,  $j, k = 1, \dots, \mathcal{N}$ .

## B Technical results and proofs

**Lemma B.1** *When  $N = 3$  and for all  $i = 1, \dots, n$  and all  $j_1, j_2 = 1, \dots, \mathcal{N}_r^i$ ,  $\delta^2 \mathbf{x}_{rr,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{x})$  is equal to anyone of the expressions on the right of the following equality:*

$$\delta^2 \mathbf{x}_{rr,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{x}) = \begin{cases} \delta \Omega_{j_1}^i(\mathbf{q}_r^i) \times (\delta \Omega_{j_2}^i(\mathbf{q}_r^i) \times (\mathbf{x} - \mathbf{h}^i)) + \frac{\partial \delta \Omega_{j_1}^i(\mathbf{q}_r^i)}{\partial q_{r,j_2}^i} \times (\mathbf{x} - \mathbf{h}^i) + \frac{\partial \delta \mathbf{h}_{j_1}^i(\mathbf{q}_r^i)}{\partial q_{r,j_2}^i}, \\ \delta \Omega_{j_2}^i(\mathbf{q}_r^i) \times (\delta \Omega_{j_1}^i(\mathbf{q}_r^i) \times (\mathbf{x} - \mathbf{h}^i)) + \frac{\partial \delta \Omega_{j_2}^i(\mathbf{q}_r^i)}{\partial q_{r,j_1}^i} \times (\mathbf{x} - \mathbf{h}^i) + \frac{\partial \delta \mathbf{h}_{j_2}^i(\mathbf{q}_r^i)}{\partial q_{r,j_1}^i}, \end{cases}$$

whereas, for all  $j = 1, \dots, \mathcal{N}_r^i$  and all  $k = 1, \dots, \mathcal{N}_d^i$ , we have merely:

$$\delta^2 \mathbf{x}_{rd,j,k}^i(\mathbf{q}_r^i, \mathbf{x}) = \delta \Omega_j^i(\mathbf{q}_r^i) \times \delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{x}). \quad (\text{B.1})$$

When  $N = 2$  and for all  $j_1, j_2 = 1, \dots, \mathcal{N}_r^i$ , the equalities above turn into:

$$\delta^2 \mathbf{x}_{rr,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{x}) = \begin{cases} -\delta \Omega_{j_1}^i(\mathbf{q}_r^i) \delta \Omega_{j_2}^i(\mathbf{q}_r^i) (\mathbf{x} - \mathbf{h}^i) + \frac{\partial \delta \Omega_{j_1}^i(\mathbf{q}_r^i)}{\partial q_{r,j_2}^i} (\mathbf{x} - \mathbf{h}^i)^\perp + \frac{\partial \delta \mathbf{h}_{j_1}^i(\mathbf{q}_r^i)}{\partial q_{r,j_2}^i}, \\ -\delta \Omega_{j_2}^i(\mathbf{q}_r^i) \delta \Omega_{j_1}^i(\mathbf{q}_r^i) (\mathbf{x} - \mathbf{h}^i) + \frac{\partial \delta \Omega_{j_2}^i(\mathbf{q}_r^i)}{\partial q_{r,j_1}^i} (\mathbf{x} - \mathbf{h}^i)^\perp + \frac{\partial \delta \mathbf{h}_{j_2}^i(\mathbf{q}_r^i)}{\partial q_{r,j_1}^i}, \end{cases} \quad (\text{B.2})$$

and for all  $j = 1, \dots, \mathcal{N}_r^i$  and all  $k = 1, \dots, \mathcal{N}_d^i$ :

$$\delta^2 \mathbf{x}_{rd,j,k}^i(\mathbf{q}_r^i, \mathbf{x}) = \delta \Omega_j^i(\mathbf{q}_r^i) \delta \mathbf{x}_{d,k}^i(\mathbf{q}_r^i, \mathbf{x})^\perp. \quad (\text{B.3})$$

Moreover, we have, for all  $j_1, j_2 = 1, \dots, \mathcal{N}_r^i$ :

$$\frac{\partial \delta \mathbf{h}_{j_1}^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i) = \frac{\partial \delta \mathbf{h}_{j_2}^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i), \quad (\text{B.4})$$

$$\frac{\partial \delta \Omega_{j_1}^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i) - \frac{\partial \delta \Omega_{j_2}^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i) = \delta \Omega_{j_2}^i(\mathbf{q}_r^i) \times \delta \Omega_{j_1}^i(\mathbf{q}_r^i), \quad (N = 3), \quad (\text{B.5})$$

$$\frac{\partial \Omega_{j_1}^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i) = \frac{\partial \Omega_{j_2}^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i), \quad (N = 2). \quad (\text{B.6})$$

**Proof :** Differentiating with respect to  $q_{r,j_1}^i$  equality (2.10), we get, for all  $i = 1, \dots, n$  and all  $j_1 = 1, \dots, \mathcal{N}_r^i$ :

$$\frac{\partial \mathbf{X}_r^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i, \mathbf{x}) = \frac{\partial [R^i(\mathbf{q}_r^i)]}{\partial q_{r,j_1}^i}(\mathbf{x} - \mathbf{h}_0^i) + \frac{\partial \mathbf{h}^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i). \quad (\text{B.7})$$

Next, differentiating likewise the identity  $\mathbf{X}_r^i(\mathbf{q}_r^i, \mathbf{Y}_r^i(\mathbf{q}_r^i, \mathbf{x})) = \mathbf{x}$  with respect to  $q_{r,j_2}^i$ , it comes:

$$\frac{\partial \mathbf{X}_r^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{Y}_r^i(\mathbf{q}_r^i, \mathbf{x})) + [D_{\mathbf{x}} \mathbf{X}_r^i(\mathbf{q}_r^i, \mathbf{x})] \frac{\partial \mathbf{Y}_r^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{x}) = \mathbf{0}, \quad (\text{B.8})$$

whence we deduce, invoking again (2.10), that:

$$\frac{\partial \mathbf{Y}_r^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{x}) = -[R^i(\mathbf{q}_r^i)]^T \delta \mathbf{x}_{r,j_2}^i(\mathbf{q}_r^i, \mathbf{x}). \quad (\text{B.9})$$

Eventually, we compute that, for all  $j_2 = 1, \dots, \mathcal{N}_r^i$ :

$$\begin{aligned} \frac{\partial \delta \mathbf{x}_{r,j_1}^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{x}) &= \frac{\partial^2 \mathbf{X}_r^i}{\partial q_{r,j_1}^i \partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{Y}_r^i(\mathbf{q}_r^i, \mathbf{x})) \\ &\quad + \left[ D_{\mathbf{x}} \left( \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j_1}^i} \right) (\mathbf{q}_r^i, \mathbf{Y}_r^i(\mathbf{q}_r^i, \mathbf{x})) \right] \frac{\partial \mathbf{Y}_r^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{x}). \end{aligned} \quad (\text{B.10})$$

Summarizing (B.7), (B.9) and (B.10) and taking into account Definition 2, we obtain the expressions of  $\delta^2 \mathbf{x}_{rr,j_1,j_2}^i(\mathbf{q}_r^i)$  given in the lemma. We proceed likewise to obtain the other expressions. ■

**Proof of Proposition 3.2:** Last term on the right of identity (3.6) has been obtained in (3.3), so we focus on the kinematic energy  $K^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) := K_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i) + K_d^i(\mathbf{q}_d^i, \dot{\mathbf{q}}_d^i)$ , which reads:

$$\begin{aligned} K^i(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) &:= \frac{1}{2} \sum_{j_1=1}^{\mathcal{N}_r^i} \sum_{j_2=1}^{\mathcal{N}_r^i} M_{r,j_1,j_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \dot{q}_{r,j_1}^i \dot{q}_{r,j_2}^i \\ &\quad + \frac{1}{2} \sum_{k_1=1}^{\mathcal{N}_d^i} \sum_{k_2=1}^{\mathcal{N}_d^i} M_{d,k_1,k_2}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \dot{q}_{d,k_1}^i \dot{q}_{d,k_2}^i. \end{aligned} \quad (\text{B.11})$$

After some algebra, we get:

$$\begin{aligned} \frac{d}{dt} \frac{\partial K^i}{\partial \dot{q}_{r,j_0}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) - \frac{\partial K^i}{\partial q_{r,j_0}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i, \dot{\mathbf{q}}_r^i, \dot{\mathbf{q}}_d^i) = \\ \sum_{j=1}^{\mathcal{N}_r^i} M_{r,j_0,j}^i(\mathbf{q}_r^i, \mathbf{q}_d^i) \ddot{q}_{r,j}^i + \frac{1}{2} \sum_{j_1=1}^{\mathcal{N}_r^i} \sum_{j_2=1}^{\mathcal{N}_r^i} \left( \frac{\partial M_{r,j_0,j_1}^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) + \frac{\partial M_{r,j_0,j_2}^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \right. \\ \left. - \frac{\partial M_{r,j_1,j_2}^i}{\partial q_{r,j_0}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \right) \dot{q}_{r,j_1}^i \dot{q}_{r,j_2}^i + \sum_{j=1}^{\mathcal{N}_r^i} \sum_{k=1}^{\mathcal{N}_d^i} \frac{\partial M_{r,j_0,j}^i}{\partial q_{d,k}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \dot{q}_{r,j}^i \dot{q}_{d,k}^i. \quad (\text{B.12}) \end{aligned}$$

Computation of the derivatives arising in the right hand side can be carried out as follows: going back to Definition 4, we obtain:

$$\begin{aligned} \frac{\partial M_{r,j_0,j_1}^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) = \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i) \frac{\partial^2 \mathbf{X}_r^i}{\partial q_{r,j_0}^i \partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \cdot \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \, d\mathbf{x} \\ + \int_{S_d^i(\mathbf{q}_d^i)} \rho_S^d(\mathbf{q}_d^i) \frac{\partial \mathbf{X}_r^i}{\partial q_{r,j_0}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \cdot \frac{\partial^2 \mathbf{X}_r^i}{\partial q_{r,j_1}^i \partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \, d\mathbf{x}. \quad (\text{B.13}) \end{aligned}$$

Therefore, with notation of Definition 2 and (A.2), we get:

$$\begin{aligned} \frac{1}{2} \sum_{j_1=1}^{\mathcal{N}_r^i} \sum_{j_2=1}^{\mathcal{N}_r^i} \left( \frac{\partial M_{r,j_0,j_1}^i}{\partial q_{r,j_2}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) + \frac{\partial M_{r,j_0,j_2}^i}{\partial q_{r,j_1}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \right. \\ \left. - \frac{\partial M_{r,j_1,j_2}^i}{\partial q_{r,j_0}^i}(\mathbf{q}_r^i, \mathbf{q}_d^i) \right) \dot{q}_{r,j_1}^i \dot{q}_{r,j_2}^i = \\ \sum_{j_1=1}^{\mathcal{N}_r^i} \sum_{j_2=1}^{\mathcal{N}_r^i} \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \delta \mathbf{x}_{r,j_0}^i(\mathbf{q}_r^i) \cdot \delta^2 \mathbf{x}_{r,r,j_1,j_2}^i(\mathbf{q}_r^i) \, d\mathbf{x}. \quad (\text{B.14}) \end{aligned}$$

Expressions given in Lemma B.1 leads to (3.7) and (3.9). Concerning the identities (3.8) and (3.10), they derive straightforwardly from the last term on the right of (B.13), together with (2.28) and (2.31). ■

**Proof of Lemma 3.1:** Introducing  $\mathbf{e}_1 := (1, 0)^T$ ,  $\mathbf{e}_2 := (0, 1)^T$  when  $N = 2$  and  $\mathbf{e}_1 := (1, 0, 0)^T$ ,  $\mathbf{e}_2 := (0, 1, 0)^T$  and  $\mathbf{e}_3 := (0, 0, 1)^T$  when  $N = 3$ , we can specify either  $\delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) = \mathbf{e}_j$  or  $\delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) = \mathbf{e}_j \times (\mathbf{x} - \mathbf{h}^i(\mathbf{q}_r^i))$  in (3.1) and we obtain (2.41).

Conversely, assume that the velocities  $\mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)$  satisfies (2.41). Invoking formula [15, formula (85) page 46], we get, on the one hand:

$$\begin{aligned} \frac{d}{dt} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \, d\mathbf{x} \right) = \\ \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) (\mathbf{v}^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i)) \, d\Gamma \\ + \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \frac{d}{dt} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)) \, d\mathbf{x}, \quad (\text{B.15a}) \end{aligned}$$

and also, when  $N = 3$ :

$$\begin{aligned} \frac{d}{dt} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \times (\mathbf{x} - \mathbf{h}^i(\mathbf{q}_r^i)) d\mathbf{x} \right) = \\ \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \times (\mathbf{x} - \mathbf{h}^i) (\mathbf{v}^i(\mathbf{q}^i, \dot{\mathbf{q}}^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i)) d\Gamma \\ + \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \frac{d}{dt} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)) \times (\mathbf{x} - \mathbf{h}^i(\mathbf{q}_r^i)) d\mathbf{x}, \quad (\text{B.15b}) \end{aligned}$$

since, by definition of the center of mass:

$$\begin{aligned} \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \times \frac{d}{dt} (\mathbf{x} - \mathbf{h}^i(\mathbf{q}_r^i)) d\mathbf{x} = \\ - \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) d\mathbf{x} \right) \times \dot{\mathbf{h}}^i(\mathbf{q}_r^i) = \mathbf{0}. \quad (\text{B.16}) \end{aligned}$$

On the other hand, the same formula [15, formula (85) page 46] yields also, for any  $\delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) \in \mathcal{D}_r^i(\mathbf{q}_r^i, \mathbf{q}_d^i)$ :

$$\begin{aligned} \frac{d}{dt} \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) d\mathbf{x} \right) \\ - \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \frac{d}{dt} (\delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i)) d\mathbf{x} = \\ \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) (\mathbf{v}^i(\mathbf{q}^i, \dot{\mathbf{q}}^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i)) d\Gamma \\ + \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \frac{d}{dt} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)) \cdot \delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) d\mathbf{x}. \quad (\text{B.17}) \end{aligned}$$

But according to Definition 1 and when  $N = 3$ , we have:

$$\delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) = \delta \widehat{\mathbf{h}}(\mathbf{q}_r^i) + \delta \widehat{\boldsymbol{\Omega}}(\mathbf{q}_r^i) \times (\mathbf{x} - \mathbf{h}^i(\mathbf{q}_r^i)),$$

for some vectors  $\delta \widehat{\mathbf{h}}(\mathbf{q}_r^i)$  and  $\delta \widehat{\boldsymbol{\Omega}}(\mathbf{q}_r^i)$ . Therefore, we can write that:

$$\begin{aligned} \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \cdot \delta \widehat{\mathbf{x}}_r(\mathbf{q}_r^i) (\mathbf{v}^i(\mathbf{q}^i, \dot{\mathbf{q}}^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i)) d\Gamma = \\ \delta \widehat{\mathbf{h}}(\mathbf{q}_r^i) \cdot \left( \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) (\mathbf{v}^i(\mathbf{q}^i, \dot{\mathbf{q}}^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i)) d\Gamma \right) \\ - \delta \widehat{\boldsymbol{\Omega}}(\mathbf{q}_r^i) \cdot \left( \int_{\Gamma_2^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i) \times (\mathbf{x} - \mathbf{h}^i(\mathbf{q}_r^i)) (\mathbf{v}^i(\mathbf{q}^i, \dot{\mathbf{q}}^i) \cdot \mathbf{n}(\mathbf{q}_r^i, \mathbf{q}_d^i)) d\Gamma \right), \quad (\text{B.18}) \end{aligned}$$

and

$$\begin{aligned} & \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \frac{d}{dt} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)) \cdot \delta \tilde{\mathbf{x}}_r(\mathbf{q}_r^i) \, d\mathbf{x} = \\ & \delta \hat{\mathbf{h}}(\mathbf{q}_r^i) \cdot \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \frac{d}{dt} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)) \, d\mathbf{x} \right) \\ & - \delta \hat{\boldsymbol{\Omega}}(\mathbf{q}_r^i) \cdot \left( \int_{S^i(\mathbf{q}_r^i, \mathbf{q}_d^i)} \frac{d}{dt} (\rho_S(\mathbf{q}_r^i, \mathbf{q}_d^i) \mathbf{v}_r^i(\mathbf{q}_r^i, \dot{\mathbf{q}}_r^i)) \times (\mathbf{x} - \mathbf{h}^i(\mathbf{q}_r^i)) \, d\mathbf{x} \right). \quad (\text{B.19}) \end{aligned}$$

We have thus proved the second point of the lemma when  $N = 3$  by substituting (B.18) and (B.19) into (B.17) and taking into account (B.15). We proceed similarly when  $N = 2$ .

At last, second and third assertions of the lemma are equivalent by virtue of Definition 3. ■

**Proof of Lemma 4.1:** One wishes to differentiate, with respect to  $q_k$ , at a given point  $\mathbf{q}_0$ , the boundary value problem:

$$-\Delta \psi_i = 0 \text{ in } \Omega(\mathbf{q}), \quad \frac{\partial \psi_i}{\partial \mathbf{n}} = b_i(\mathbf{q}) \text{ on } \Gamma(\mathbf{q}). \quad (\text{B.20})$$

This problem is the one treated in [17, Subsection 5.5 pages 202-206], in [18, Section 4] and in [28, Section 4]. In these references, it is shown that  $\partial \psi_i / \partial q_k$  is well defined and belongs to the Sobolev space  $H^1(\Omega(\mathbf{q}))$  when  $\Omega$  is bounded or corresponding weighted Sobolev spaces when  $\Omega$  is not bounded. For the sake of completeness, we give here the formal approach leading to formula (4.6). As regards the first equation of System (B.20), we obtain merely, by differentiating with respect to  $q_k$ :

$$-\Delta \frac{\partial \psi_i}{\partial q_k} = 0 \text{ in } \Omega(\mathbf{q}).$$

The derivation of the boundary conditions for  $\partial \psi_i / \partial q_k$  are more involved. First, we introduce the mapping  $\tilde{\mathbf{X}}(\mathbf{q}, \cdot) := \mathbf{X}(\mathbf{q}_0 + \mathbf{q}, \mathbf{Y}(\mathbf{q}_0, \cdot))$  and the functions  $B_i(\mathbf{q}, \cdot) := b_i(\mathbf{q}_0 + \mathbf{q}, \tilde{\mathbf{X}}(\mathbf{q}, \cdot))$  and  $\mathbf{N}(\mathbf{q}, \cdot) := \mathbf{n}(\mathbf{q}_0 + \mathbf{q}, \tilde{\mathbf{X}}(\mathbf{q}, \cdot))$ . Differentiating with respect to  $q_k$  the identity:

$$\nabla \psi_i(\mathbf{q}, \tilde{\mathbf{X}}(\mathbf{q}, \mathbf{x})) \cdot \mathbf{N}(\mathbf{q}, \mathbf{x}) = B_i(\mathbf{q}, \mathbf{x}), \quad \forall \mathbf{x} \in \Gamma(\mathbf{q}_0),$$

when  $\mathbf{q} = \mathbf{0}$ , we obtain:

$$\begin{aligned} & \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_i}{\partial q_k} \right) (\mathbf{q}_0, \cdot) + \langle [D_{\mathbf{x}}^2 \psi_i(\mathbf{q}_0, \cdot)], \delta \mathbf{x}_k(\mathbf{q}_0, \cdot), \mathbf{n}(\mathbf{q}_0, \cdot) \rangle \\ & + \nabla \psi_i(\mathbf{q}_0, \cdot) \cdot \frac{\partial \mathbf{N}}{\partial q_k}(\mathbf{0}, \cdot) = \frac{\partial B_i}{\partial q_k}(\mathbf{0}, \cdot) \text{ on } \Gamma(\mathbf{q}_0), \end{aligned}$$

where  $[D_{\mathbf{x}}^2 \psi_i(\mathbf{q}_0)]$  is the Hessian matrix of  $\psi(\mathbf{q}_0)$ . We can decompose  $\delta \mathbf{x}_k(\mathbf{q}_0, \cdot)$  into its normal and tangential components (we drop  $\mathbf{q}_0$  subsequently to shorten notation):  $\delta \mathbf{x}_k = b_k \mathbf{n} + \delta_{\Gamma} \mathbf{x}_k$  and we obtain that:

$$\frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial \psi_i}{\partial q_k} \right) + \frac{\partial^2 \psi_i}{\partial \mathbf{n}^2} b_k + \langle [D_{\mathbf{x}}^2 \psi_i], \delta_{\Gamma} \mathbf{x}_k, \mathbf{n} \rangle + \nabla \psi_i \cdot \frac{\partial \mathbf{N}}{\partial q_k} = \frac{\partial B_i}{\partial q_k} \text{ on } \Gamma(\mathbf{q}_0). \quad (\text{B.21})$$

We use on the one hand, the identity of [17, Proposition 5.4.12 page 196], valid for any function  $\psi$  regular enough in a neighborhood of  $\Gamma(\mathbf{q}_0)$ :

$$\Delta\psi = \Delta_\Gamma\psi - \mathcal{H}\frac{\partial\psi}{\partial\mathbf{n}} + \frac{\partial^2\psi}{\partial\mathbf{n}^2}, \quad \text{on } \Gamma(\mathbf{q}_0),$$

where  $\Delta_\Gamma$  is the Laplace-Beltrami operator on  $\Gamma(\mathbf{q}_0)$  and  $\mathcal{H}$  the mean curvature of  $\Gamma(\mathbf{q}_0)$ . On the other hand, computing the tangential derivative in the direction  $\delta_\Gamma\mathbf{x}_k$  of the boundary condition of System (B.20), we get:

$$\langle [D_{\mathbf{x}}^2\psi_i], \delta_\Gamma\mathbf{x}_k, \mathbf{n} \rangle - \langle \Pi_2, \nabla_\Gamma\psi_i, \delta_\Gamma\mathbf{x}_k \rangle = \nabla_\Gamma b_i \cdot \delta_\Gamma\mathbf{x}_k, \quad \text{on } \Gamma(\mathbf{q}_0). \quad (\text{B.22})$$

Proposition 4 in Section 5 of [28] yields:

$$\nabla\psi_i \cdot \frac{\partial\mathbf{N}}{\partial q_k} = -\nabla_\Gamma\psi_i \cdot \nabla_\Gamma b_k - \langle \Pi_2, \nabla_\Gamma\psi_i, \delta_\Gamma\mathbf{x}_k \rangle \quad \text{on } \Gamma(\mathbf{q}_0). \quad (\text{B.23})$$

We get hence, plugging (B.22) and (B.23) into (B.21):

$$\frac{\partial}{\partial\mathbf{n}} \left( \frac{\partial\psi_i}{\partial q_k} \right) = \Delta_\Gamma\psi_i b_k - \mathcal{H}b_i b_k + \nabla_\Gamma\psi_i \cdot \nabla_\Gamma b_k + \frac{\partial B_i}{\partial q_k} - \nabla_\Gamma b_i \cdot \delta_\Gamma\mathbf{x}_k.$$

It remains to compute the derivative of  $B_i(\mathbf{q})$  with respect to  $q_k$ . Taking into account its definition, we have, for all  $i = 1, \dots, \mathcal{N}_r + \mathcal{N}_d$ :

$$\frac{\partial B_i}{\partial q_k}(\mathbf{q}, \cdot) = \frac{\partial}{\partial q_k} \left( \frac{\partial\mathbf{X}}{\partial q_i}(\mathbf{q}_0 + \mathbf{q}, \mathbf{Y}(\mathbf{q}_0, \cdot)) \cdot \mathbf{N}(\mathbf{q}, \cdot) \right) \quad \text{on } \Gamma(\mathbf{q}). \quad (\text{B.24})$$

Introducing the amount:

$$\beta_{i,k}(\mathbf{q}) := \frac{\partial^2\mathbf{X}}{\partial q_i \partial q_k}(\mathbf{q}, \mathbf{Y}(\mathbf{q})) \cdot \mathbf{n}(\mathbf{q}),$$

and making use again of formula (B.23), we obtain, when  $\mathbf{q} = \mathbf{0}$ :

$$\frac{\partial B_i}{\partial q_k} = \beta_{i,k} - \langle \Pi_2, \delta_\Gamma\mathbf{x}_k, \delta_\Gamma\mathbf{x}_i \rangle - \nabla_\Gamma b_k \cdot \delta_\Gamma\mathbf{x}_i \quad \text{on } \Gamma(\mathbf{q}_0). \quad (\text{B.25})$$

The last identity (4.7), classical for smooth functions (see for instance [15, formula (85) page 46]), can be found, in the context of Sobolev spaces, in [17, Theorem 5.2.2 page 172]. More precisely, we apply here [17, Corollary 5.2.2 page 172]. The proof is then complete. ■

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