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LARGE DEVIATIONS FOR RANDOM SPECTRAL MEASURES AND SUM RULES

F. GAMBOA AND A. ROUAULT

ABSTRACT. We prove a Large Deviation Principle for the random spectral measure associated to the pair (H_N, e) where H_N is sampled in the $GUE(N)$ and e is a fixed unit vector (and more generally in the β extension of this model). The rate function consists of two parts. The contribution of the absolutely continuous part of the measure is the reversed Kullback information with respect to the semicircle distribution and the contribution of the singular part is connected to the rate function of the extreme eigenvalue in the GUE. This method is also applied to the Laguerre and Jacobi ensembles, but in those cases the expression of the rate function is not so explicit.

1. INTRODUCTION

Let U be a unitary operator in a Hilbert space \mathcal{H} , and e be a unit cyclic vector (the span generated by the iterates $(U^n x)$ is \mathcal{H}). The spectral measure associated with the pair (U, e) plays an important role and will be one of the object studied here. This measure is the unique probability measure (p.m.) μ on the unit circle \mathbb{T} such that

$$\langle e, U^n e \rangle = \int_{\mathbb{T}} z^n d\mu(z) \quad (n \geq 1).$$

Assume further that $\dim \mathcal{H} = N$ and that e_1 the first vector of the canonical basis is cyclic for U . Let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of U (all lying on \mathbb{T}), and let ψ_1, \dots, ψ_N be a system of unit eigenvectors. The spectral measure is then

$$\mu_w^{(N)} = \sum_{k=1}^N \pi_k \delta_{\lambda_k} \quad (1)$$

with $\pi_k := |\langle \psi_k, e_1 \rangle|^2$ $k = 1, \dots, N$. Notice that given λ_k , the vector ψ_k is determined up to a phase, but the number π_k is completely determined. To

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avoid confusion, we put an index \mathbf{w} (for weight) to distinguish this measure from the classical empirical spectral distribution (ESD) defined by

$$\mu_{\mathbf{u}}^{(N)} = \frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k}. \quad (2)$$

When U is uniformly sampled from $\mathbb{U}(N)$ (the unitary group of order N) with the Haar distribution, it is well known that the joint distribution of $(\lambda_1, \dots, \lambda_N)$ has a density proportional to

$$|\Delta(\lambda_1, \dots, \lambda_N)|^2$$

where Δ is the Vandermonde determinant (see for example [Meh04]). Furthermore, e_1 is almost surely (a.s.) cyclic and (π_1, \dots, π_N) is independent of $(\lambda_1, \dots, \lambda_N)$. Moreover, (π_1, \dots, π_N) is uniformly distributed on the simplex $\mathcal{S}_n = \{(\pi_1, \dots, \pi_N) : \pi_k > 0, (k = 1, \dots, N), \pi_1 + \dots + \pi_N = 1\}$. As N tends to infinity, both sequences of random measures $(\mu_{\mathbf{w}}^{(N)})$ and $(\mu_{\mathbf{u}}^{(N)})$ converge weakly to the equilibrium measure, i.e. the uniform distribution on \mathbb{T} . In a previous work ([GR08]), we have established a strong connection between the asymptotic properties of $(\mu_{\mathbf{w}}^{(N)})$ and the random moment problem studied in [GLC04] and [CKS93]. In particular we proved that the sequence $(\mu_{\mathbf{w}}^{(N)})$ satisfies a Large Deviation Principle (denoted hereafter LDP), with speed N and good rate function given by reversed Kullback entropy with respect to the equilibrium measure (see the Appendix where we recall the definition of this quantity). For the sake of completeness, we also recall in the appendix the definition of a LDP and some useful results on large deviations. Notice that there is a quite important difference in the large deviation behaviour of $(\mu_{\mathbf{w}}^{(N)})$ and $(\mu_{\mathbf{u}}^{(N)})$. Indeed, this last sequence of probability measures (p.ms.) satisfies a LDP with speed N^2 and with a rate function connected to the Voiculescu entropy (see for example [HP00]). To show a LDP for $(\mu_{\mathbf{w}}^{(N)})$ one may think of two kinds of proof. The first one, which could be called the direct way, uses the representation (1) [GR08]. Besides, it is possible to code a measure μ on \mathbb{T} by the system of its Verblunsky (or Schur) coefficients, via the Favard theorem [Sim05b]; they are also the canonical moments of μ (see [DS97] for the definition). The second method uses this coding. It turns out that, under the Haar distribution, the canonical moments $(c_1^{(N)}, \dots, c_N^{(N)})$ of $\mu_{\mathbf{w}}^{(N)}$ are independent random variables (r.vs.) with explicit distribution depending on N . It is then possible in a first step to check the LDP on these variables and in a second step to lift the LDP and the rate function on the space of measures [LC05].

The precise form of the rate function can be explained, in the first method by the Dirichlet weighting of the random measure, and in the second method

by the Szegő formula, which enters in the class of the so-called sum rules. The same thing can be done for the Jacobi ensemble with the arcsine distribution (on $[0, 1]$ or on $[-2, 2]$) playing the role of the uniform distribution on \mathbb{T} (see [GR08]).

In this paper we will focus on models of self-adjoint matrices and their extensions. If H is a self-adjoint bounded operator in a Hilbert space \mathcal{H} and e a cyclic vector, the spectral measure is the unique p.m. μ on \mathbb{R} such that

$$\langle e, H^n e \rangle = \int_{\mathbb{R}} x^n d\mu(x) \quad (n \geq 1).$$

If $\dim \mathcal{H} = N$ and e_1 is cyclic for H , the spectral measure is

$$\mu_w^{(N)} = \sum_{k=1}^N \pi_k \delta_{\lambda_k} \quad (3)$$

with the same notation as above, except that, now, the eigenvalues are real.

The first two models studied here leads to an eigenvalue distribution that is not almost surely (a.s.) supported by a fixed compact set. We will first study the β -Hermite ensemble. It is a family extending the Gaussian ensembles (GOE, GUE, GSE). The second model considered is the β -Laguerre ensemble that generalizes Wishart matrices. In both cases, we could expect that the sequence $(\mu_w^{(N)})$ satisfies a LDP with speed N and with a rate function given by the reversed Kullback entropy with respect to the limit distribution (respectively semicircle and Marchenko-Pastur distributions). Actually the difference with the unitary case comes from the problem of support. We prove results of the same flavour that those we previously obtained in the unitary case, but with an extra contribution in the rate function due to the singular part of measures. The third model studied is the β -Jacobi ensemble in which the eigenvalues are confined in a compact set.

The paper is organized as follows. The next section is devoted to the introduction of notation and models : topology on space of moments and real matrix models that we will study later. In Section 3, we discuss some relationships between the random spectral measures and coefficients appearing in the construction of the associated random orthogonal polynomials. The LDP for real matrix models are studied in last two sections. The case of the β -Hermite ensemble is completely tackled in Section 4. Surprisingly, we manage to compute explicitly the rate function, with the help of a convenient sum rule [Kil07]. The β -Laguerre and β -Jacobi ensembles are studied in Section 5 and 6. Here, the rate functions are not so explicit. All useful distributions we work with are defined in Section 7.2. Furthermore, basic facts on LDPs are recalled in Section 7.1.

2. NOTATION AND MODELS

2.1. Topology on moments spaces. Let \mathcal{M}^1 be the set of all p.ms. on \mathbb{R} and let \mathcal{M}_m^1 be the subset consisting of p.ms. on \mathbb{R} having all their moments finite. For $\mu \in \mathcal{M}_m^1$ we set

$$m_k(\mu) = \int_{\mathbb{R}} x^k d\mu(x), \quad k \geq 1,$$

and $\mathbf{m}(\mu) = (m_k(\mu))_{k \geq 1}$. As it is classical in moment problems, we consider the set \mathcal{M}_m^1 as a subset of $\mathbb{R}[[X]]$, (the set of formal series with real coefficients), or equivalently as a subset of the set of linear forms on the space $\mathbb{R}[X]$ of polynomials with real coefficients, or eventually as a subset of $\mathbb{R}^{\mathbb{N}}$. We may identify μ either with

- The formal series $\sum_{n=0}^{\infty} m_n(\mu) X^n$,
- The linear form on $\mathbb{R}[X] : P(X) \mapsto \int_{\mathbb{R}} P(x) d\mu(x)$
- The sequence $\mathbf{m}(\mu)$.

We endow \mathcal{M}_m^1 with the distance of convergence of moments:

$$d(\mu, \nu) = \sum_{k=1}^{\infty} 2^{-k} \frac{|m_k(\mu) - m_k(\nu)|}{1 + |m_k(\mu) - m_k(\nu)|}. \quad (4)$$

If $\mathcal{M}_{m,d}^1$ denotes the subset of \mathcal{M}_m^1 consisting in all p.ms. determined by their moments, then the mapping \mathbf{m} is injective and continuous from $\mathcal{M}_{m,d}^1$ to $\mathbb{R}^{\mathbb{N}}$.

2.2. β -Hermite ensemble. Let us recall some classical Gaussian matrix models and their extensions.

- GOE(N) The diagonal entries are independent and $\mathcal{N}(0; 2/N)$ distributed and the non diagonal entries are independent up to symmetry and $\mathcal{N}(0; 1/N)$ distributed. The joint density on \mathbb{R}^N of the eigenvalues is proportional to

$$\Delta(\lambda_1, \dots, \lambda_N) \exp -\frac{N}{4} \sum_j \lambda_j^2.$$

The matrix of eigenvectors is orthogonal, so its first line is uniformly distributed on the N -dimensional sphere, i.e. the vector (π_1, \dots, π_N) has the distribution $\text{Dir}_N(1/2)$.

- GUE(N) The diagonal entries are independent and $\mathcal{N}(0; 1/N)$ distributed and the non diagonal entries are independent up to symmetry and distributed as $\mathcal{N}(0; 1/2N) + \sqrt{-1} \mathcal{N}(0; 1/2N)$ where both normal

variables are independent. The joint density of the eigenvalues is proportional to

$$\Delta(\lambda_1, \dots, \lambda_N)^2 \exp -\frac{N}{2} \sum_j \lambda_j^2.$$

The matrix of eigenvectors is unitary, so the first line is uniformly distributed on the N -dimensional (complex) sphere, i.e. the vector (π_1, \dots, π_N) has the distribution $\text{Dir}_N(1)$.

If M is sampled from the GOE(N) or GUE(N), e_1 is a.s. cyclic, the eigenvalues are a.s. distinct and then we will consider the (random) spectral measure $\mu_w^{(N)}$ given by (3).

We do not recall the definition of the symplectic ensemble GSE(N). Nevertheless, some of the previous objects may also be defined in this context.

- More generally, it is now classical to consider a parameter $\beta = 2\beta' > 0$, and a density in \mathbb{R}^N proportional to

$$|\Delta(\lambda_1, \dots, \lambda_N)|^\beta \exp -\frac{N\beta}{2} \sum_j \lambda_j^2.$$

This expression extends the above formulas so that $\beta = 1$ for the GOE, $\beta = 2$ for the GUE and $\beta = 4$ for the GSE. It is often called a Coulomb gas model and $(\lambda_1, \dots, \lambda_N)$ are called charges.

Dumitriu and Edelman ([DE02] Theorem 2.12) found a matrix model for this distribution, i.e. a random real symmetric matrix whose eigenvalues follows the above distribution. Moreover they proved that the corresponding vector (π_1, \dots, π_N) is independent of the eigenvalues and $\text{Dir}_N(\beta')$ distributed. A specific description of the matrix will be given in the next section.

When $N \rightarrow \infty$, it is known that $(\mu_u^{(N)})$ converges weakly to the semicircle distribution, and satisfies a LDP with speed N^2 and with a rate function connected to the Voiculescu entropy.

2.3. β -Laguerre and β -Jacobi ensembles.

- The classical Wishart real ensemble is formed by $W = G^t G$ with G a $m \times N$ matrix with independent $\mathcal{N}(0, 2/N)$ entries. The joint density of eigenvalues is proportional to

$$|\Delta(\lambda)| \prod_{j=1}^m \lambda_j^{\frac{1}{2}(N-m+1)-1} \exp -\frac{N}{4} \sum_{j=1}^m \lambda_j$$

and the distribution of weights (π_1, \dots, π_m) is $\text{Dir}_m(1/2)$.

This eigenvalues distribution is classically extended to the β -Laguerre distribution of charges, with density proportional to :

$$|\Delta(\lambda)|^\beta \prod_{j=1}^m \lambda_j^{\beta'(N-m+1)-1} \exp -\frac{N\beta}{4} \sum_{j=1}^m \lambda_j.$$

For this case, Dumitriu and Edelman ([DE02] Theorem 3.4) also gave a (real symmetric) matrix model and proved that the vector of weights (π_1, \dots, π_m) are also independent of the eigenvalues and is $\text{Dir}_N(\beta')$ distributed.

- The $J\beta E(N; \mathbf{a}, \mathbf{b})$ ensemble (with $\mathbf{a} > -1, \mathbf{b} > -1$) has been defined to extend the MANOVA ensemble known in statistics for $\beta = 1$ and $\beta = 2$. It is defined by a density of N charges on $[-2, 2]$

$$|\Delta(x_1, \dots, x_N)|^\beta \prod_{j=1}^N (2 - x_j)^{\mathbf{a}} (2 + x_j)^{\mathbf{b}}.$$

Killip and Nenciu ([KN04]) found a matrix (real symmetric) model and proved that the corresponding vector of weights is again independent of the eigenvalues and $\text{Dir}_N(\beta')$ distributed. A variant is the $\widehat{J\beta E}(N, \mathbf{a}, \mathbf{b})$ ensemble where the charges are distributed on $[0, 1]$ according to a density proportional to

$$|\Delta(x_1, \dots, x_N)|^\beta \prod_{j=1}^N x_j^{\mathbf{b}} (1 - x_j)^{\mathbf{a}},$$

In the matrix model, the weights have the same properties as above.

3. TRIDIAGONAL REPRESENTATIONS

3.1. Spectral map. In this section, we will describe the Jacobi mapping between tridiagonal matrices and spectral measures. This mapping will be one of the key tools for our large deviations results. We consider finite size matrices corresponding to measures supported by a finite number of points and semi-infinite matrices corresponding to measures with bounded infinite support. The material of this section is largely borrowed from [Sim05a], [Sim07] [Sim98].

If μ is a probability measure with a finite support consisting of N points the orthonormal polynomials (with positive leading coefficients) obtained by Gram-Schmidt procedure from the sequence $1, x, x^2, \dots, x^{N-1}$ satisfy the recurrence relation

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x)$$

for $n \leq N - 1$, where $a_n > 0$ for those n . In the basis $\{p_0, p_1, \dots, p_{n-1}\}$, the linear transformation $x \mapsto xf(x)$ in $L^2(d\mu)$ is represented by the matrix

$$J_\mu = \begin{pmatrix} b_0 & a_0 & 0 & \dots & 0 \\ a_0 & b_1 & a_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & \dots & 0 & a_{N-2} & b_{N-1} \end{pmatrix} \quad (5)$$

So, measures supported by N points lead to Jacobi matrices, i.e. $N \times N$ symmetric tridiagonal matrices with subdiagonal positive terms ; in fact, there is a one-to-one correspondence between them. Let be given such a Jacobi matrix J and let e_1 be a cyclic vector. If μ is the spectral measure associated to the pair (J, e_1) , then J is nothing more than the representation of the multiplication by x in the basis of orthonormal polynomials associated to μ (and $J = J_\mu$). More generally, if μ is a p.m. on \mathbb{R} , with bounded infinite support, we may apply the same Gram-Schmidt process and consider the associated semi-infinite Jacobi matrix:

$$J_\mu = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & \dots \\ 0 & a_1 & b_2 & a_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (6)$$

Notice that again we have $a_k > 0$ for every k . The mapping $\mu \mapsto J_\mu$ (which we call Jacobi mapping) is a one to one correspondence between p.ms on \mathbb{R} having compact infinite support and this kind of tridiagonal matrices with $\sup_n (|a_n| + |b_n|) < \infty$. This result is sometimes called Favard's theorem (see [Sim05a] p.432).

Furthermore, a compactly supported p.m. μ is completely determined through the knowledge of all its moments $m_k(\mu)$ for $k \geq 1$. So, an inversion formula for the Jacobi mapping may be performed by using J_μ to compute the moments of μ (see for example [Sim98]). Actually, there is a recursive procedure connecting successive moments with successive sections of the matrix. For a general Jacobi semi-infinite (resp. $N \times N$) matrix A , let $A^{[j]}$ for $j \geq 1$ (resp. for $j \leq N$) the left top submatrix of A . It is known from [Sim98] formula (5.37), that if A is semi-infinite, we have the identity

$$\langle e_1, A^k e_1 \rangle = \langle e_1, (A^{[j]})^k e_1 \rangle, \quad k = 1, \dots, 2j - 1. \quad (7)$$

It is straightforward that this formula holds true when A is a Jacobi $N \times N$ matrix, as soon as $j \leq N$ and $k \leq 2j - 2$. When $A = J_\mu$, the Jacobi matrix

associated to a p.m. μ , we get, in terms of the moments :

$$m_k(\mu) = \langle e_1, \left(J_\mu^{[j]} \right)^k e_1 \rangle, \quad k = 1, \dots, 2j - 1. \quad (8)$$

for every j if μ as an infinite support, and for $j \leq N$ if μ is supported by N points. Notice that this kind of formula leads to Gauss-Jacobi quadratures. It means that, there exists a sequence of polynomials f_r of $2[N/2] + 1$ variables, such that

$$m_r(\mu) = f_r(b_0, \dots, b_{[r/2]}; a_0, \dots, a_{[r/2]-1}), \quad (9)$$

for any r if μ as an infinite support, and for $r \leq 2N - 1$ if μ is supported by N points.

Notice that the inverse relations are quite intricated (see for instance Simon [Sim98] Theorem A2). Actually, a_n depends on m_1, \dots, m_{2n+2} and b_n depends on m_1, \dots, m_{2n+1} .

3.2. Tridiagonal representations of β -ensembles. We now consider the Jacobi mapping for our random matrix models. The case of the β -ensembles is directly obtained by the representation proposed by Dumitriu and Edelman ([DE02]).

- For the normalized $G\beta E$ this representation is

$$H_\beta^{(N)} = \begin{pmatrix} b_0^{(N)} & a_0^{(N)} & 0 & \dots & 0 \\ a_0^{(N)} & b_1^{(N)} & a_1^{(N)} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & a_{N-3}^{(N)} & b_{N-2}^{(N)} & a_{N-2}^{(N)} \\ 0 & \dots & 0 & a_{N-2}^{(N)} & b_{N-1}^{(N)} \end{pmatrix}$$

where the variables $a_0^{(N)}, \dots, a_{N-2}^{(N)}, b_0^{(N)}, \dots, b_{N-1}^{(N)}$ are independent and

$$\begin{aligned} b_j^{(N)} &\stackrel{(d)}{=} \mathcal{N}(0; (\beta' N)^{-1}), \\ a_j^{(N)} &\stackrel{(d)}{=} \sqrt{\gamma \left(\beta'(N-1-j), (\beta' N)^{-1} \right)}. \end{aligned} \quad (10)$$

It means that $H_\beta^{(N)}$ has the same joint distribution of eigenvalues as for the $G\beta E(N)$. Moreover the weights are independent of the eigenvalues and have the required distribution.

- For the $L\beta E(N, m(N))$ the representation is $L_\beta^{(N)} = B_\beta^{(N)} (B_\beta^{(N)})^T$

$$B_\beta^{(N)} = \begin{pmatrix} d_1^{(N)} & 0 & 0 & \dots & 0 \\ s_1^{(N)} & d_2^{(N)} & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & s_{m(N)-2}^{(N)} & d_{m(N)-1}^{(N)} & 0 \\ 0 & \dots & 0 & s_{m(N)-1}^{(N)} & d_{m(N)}^{(N)} \end{pmatrix}$$

where the variables $d_1^{(N)}, \dots, d_{m(N)}^{(N)}, s_1^{(N)}, \dots, s_{m(N)-1}^{(N)}$ are independent and

$$\begin{aligned} s_j^{(N)} &\stackrel{(d)}{=} \sqrt{\gamma(\beta'(m(N) - j), (\beta'N)^{-1})}, \\ d_j^{(N)} &\stackrel{(d)}{=} \sqrt{\gamma(\beta'(N + 1 - j), (\beta'N)^{-1})}. \end{aligned} \quad (11)$$

- The representation of the $J\beta E(N; a, b)$ has been obtained by Killip and Nenciu ([KN04]). Actually, they consider a measure μ on $[-2, 2]$ with finite support as the projection of a symmetric measure $\tilde{\mu}$ on the unit circle $\mathbb{T} = \{z : |z| = 1\}$ by the mapping $z \mapsto z + z^{-1}$. The Jacobi parameters $(a_0, \dots; b_0, \dots)$ of μ are in bijection with the Verblunsky coefficients (α_0, \dots) of $\tilde{\mu}$ by the Geronimus relations (this is also true for measures with infinite support, see [Sim05a] section 11)). Notice that choosing a probability distribution to sample Verblunsky coefficients leads to a probability distribution on Jacobi matrices.

Theorem 3.1 (Killip-Nenciu, Theorem 2). *Given $\beta > 0$, let $\alpha_k^{(N)}, 0 \leq k \leq 2N - 2$ be independent and distributed as follows:*

$$\begin{aligned} \alpha_{2p}^{(N)} &\stackrel{(d)}{=} \beta_s \left((N - p - 1)\beta' + \mathbf{a} + 1, (N - p - 1)\beta' + \mathbf{b} + 1 \right), \\ \alpha_{2p-1}^{(N)} &\stackrel{(d)}{=} \beta_s \left((N - p - 1)\beta' + \mathbf{a} + \mathbf{b} + 2, (N - p)\beta' \right), \end{aligned} \quad (12)$$

for $p = 0, \dots, N - 1$. Let $\alpha_{2N-1}^{(N)} = \alpha_{-1}^{(N)} = -1$ and define¹

$$\begin{aligned} b_k^{(N)} &= (1 - \alpha_{2k-1}^{(N)})\alpha_{2k}^{(N)} - (1 + \alpha_{2k-1}^{(N)})\alpha_{2k-2}^{(N)} \\ a_k^{(N)} &= \sqrt{(1 - \alpha_{2k-1}^{(N)})(1 - (\alpha_{2k}^{(N)})^2)(1 + \alpha_{2k+1}^{(N)})} \end{aligned} \quad (13)$$

¹these are the Geronimus relations

Then the eigenvalues of the tridiagonal matrix $A^{(N)}$ built with these coefficients $a_k^{(N)}$ and $b_k^{(N)}$ have a joint density proportional to

$$|\Delta(x_1, \dots, x_N)|^\beta \prod_{j=1}^N (2 - x_j)^a (2 + x_j)^b$$

and the vector of weights is $\text{Dir}(\beta')$ distributed.

We call $J\beta E(N, \mathbf{a}, \mathbf{b})$ ensemble the above distribution on tridiagonal $N \times N$ matrices.

Since it is often convenient to work on $[0, 1]$ instead of $[-2, 2]$, let us introduce the affine mappings :

$$x \in [0, 1] \xrightarrow{r} 4x - 2 \quad (14)$$

$$y \in [-2, 2] \xrightarrow{s} \frac{y + 2}{4} \quad (15)$$

We call $\widehat{J\beta E}(N, \mathbf{a}, \mathbf{b})$ the image of $J\beta E(N, \mathbf{a}, \mathbf{b})$ by s . The preceding result may be rephrased in the following way:

Corollary 3.2. *If $A^{(N)}$ is sampled in the $\widehat{J\beta E}(N, \mathbf{a}, \mathbf{b})$ ensemble, its eigenvalues have a joint density proportional to*

$$|\Delta(x_1, \dots, x_N)|^\beta \prod_{j=1}^N x_j^b (1 - x_j)^a,$$

and the vector of weights is $\text{Dir}(\beta')$ distributed.

4. LARGE DEVIATIONS IN THE β -HERMITE ENSEMBLE

4.1. Introduction. Recall that the sequence of ESD

$$\mu_{\mathbf{u}}^{(N)} = \frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k}$$

satisfies the LDP with speed $\beta' N^2$ and good rate function

$$I^{\mathbf{u}}(\mu) = -\Sigma(\mu) + \int_{\mathbb{R}} \frac{x^2}{2} d\mu(x) + K_H,$$

where K_H is a constant (see [BAG97]) and

$$\Sigma(\mu) = \iint_{\mathbb{R}^2} \log|x - y| d\mu(x) d\mu(y).$$

The equilibrium measure, unique minimizer of $I^{\mathbf{u}}$, is the semicircle distribution (denoted hereafter SC, see Section 7.2). In particular, the sequence $(\mu_{\mathbf{u}}^{(N)})$ converges weakly in probability to SC.

To manage the large deviations of $(\mu_{\vec{w}}^{(N)})$, we will first tackle the large deviations of $(a_k^{(N)}, b_k^{(N)}, k \geq 0)$. It is important to notice already that, in view of (10), as $N \rightarrow \infty$, we have for fixed $k \geq 0$, $a_k^{(N)} \rightarrow 1$ and $b_k^{(N)} \rightarrow 0$ (in probability). The corresponding infinite Jacobi matrix which satisfies

$$b_k = 0, \quad a_k = 1, \quad k \geq 0,$$

(often called the free Jacobi matrix, see Simon ([Sim05b] p.13) is J_μ with $\mu = SC$.

In the large deviations properties of $(\mu_{\vec{w}}^{(N)})$, the extremes eigenvalues will play an important role. As a matter of fact, the following function will appear in our rate function. Let, for $x \geq 2$

$$\begin{aligned} \mathcal{F}_G(x) &= \int_2^x \sqrt{t^2 - 4} dt = 4 \int_1^{\frac{x}{2}} \sqrt{t^2 - 1} dt \\ &= \frac{x}{2} \sqrt{x^2 - 4} - 2 \log \left(\frac{x + \sqrt{x^2 - 4}}{2} \right). \end{aligned}$$

Further, for $x < -2$ set $\mathcal{F}_G(x) = \mathcal{F}_G(-x)$. The following theorem gives the large deviations properties for the largest eigenvalues in the $G\beta E(N)$ model frame.

Theorem 4.1. *For the $G\beta E(N)$ model the sequence $(\lambda_{max}^{(N)})$ satisfies for $x \geq 2$*

$$\lim_N \frac{1}{\beta' N} \log \mathbb{P}(\lambda_{max}^{(N)} \geq x) = -\mathcal{F}_G(x). \quad (16)$$

The statement and proof for the GOE are due to [BADG01] Theorem 6.1, the case GUE is in [Mai07] Prop. 3.1 and the general case, although not treated completely is quoted in Féral [Fér08]. Actually, the proof depends only of the explicit form of the joint distribution of charges and is easily extended.

To prepare the statement of our main result, we need another definition.

Definition 4.2 (Simon). *We say that a p.m. μ on \mathbb{R} satisfies the Blumenthal-Weyl condition (B.W.c) if*

- i) $Supp(\mu) = [-2, 2] \cup \{E_j^-\}_{j=1}^{N^-} \cup \{E_j^+\}_{j=1}^{N^+}$ where N^+ (resp. N^-) is either 0, finite or infinite,

$$E_1^- < E_2^- < \dots < -2 \quad \text{and} \quad E_1^+ > E_2^+ > \dots > 2$$

are isolated points of the support.

- ii) *If $N^+ = \infty$ (resp. $N^- = \infty$) then E_j^+ converges towards 2 (resp. E_j^- converges towards -2).*

4.2. Main result. Here is our main result. Notice that, of course, SC is the unique minimizer of the rate function, in accordance with the remark at the beginning of this section.

Theorem 4.3. *The sequence $(\mu_w^{(N)})$ satisfies the LDP in $\mathcal{M}_{m,d}^1$ with speed $\beta'N$ and good rate function*

$$I^w(\nu) = \begin{cases} \mathcal{K}(SC|\nu) + \sum_{n=1}^{N^+} \mathcal{F}_G(E_n^+) + \sum_{n=1}^{N^-} \mathcal{F}_G(E_n^-) & \text{if } \nu \text{ satisfies B.W.c,} \\ +\infty & \text{otherwise.} \end{cases} \quad (17)$$

Hence, the rate function $I(\nu)$ is finite if and only if

$$\nu(dx) = f_a(x)SC(dx) + \nu_s(dx) + \sum_{n=1}^{N^+} \kappa_n \delta_{E_n^+}(dx) + \sum_{n=1}^{N^-} \kappa_n \delta_{E_n^-}(dx)$$

where ν_s is singular (with respect to the Lebesgue measure) and is supported by a subset of $[-2, +2]$ and

$$-\int_{-2}^2 \log f_a(x)SC(dx) < \infty, \quad \sum_{n=1}^{N^+} \mathcal{F}_G(E_n^+) + \sum_{n=1}^{N^-} \mathcal{F}_G(E_n^-) < \infty.$$

In this case

$$I(\nu) = \int_{-2}^2 \log \left(\frac{\sqrt{4-x^2}}{2\pi f_a(x)} \right) \frac{\sqrt{4-x^2}}{2\pi} dx + \sum_{n=1}^{N^+} \mathcal{F}_G(E_n^+) + \sum_{n=1}^{N^-} \mathcal{F}_G(E_n^-). \quad (18)$$

Proof : For $k > 0$, the subset $M(k)$ of \mathcal{M}_m^1 of all p.ms supported by $[-k, +k]$ is compact for our topology. Indeed, for p.ms in $M(k)$ the moment maps are continuous function ($M(k)$ is tight for the convergence in law). From Theorem 4.1 we know that

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}(\lambda_{max}^{(N)} > k) = -\infty.$$

By symmetry, we have also

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}(\lambda_{min}^{(N)} < -k) = -\infty.$$

This implies

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}(\mu_w^{(N)} \notin M(k)) = -\infty,$$

hence the sequence $(\mu_w^{(N)})_N$ is exponentially tight.

From the inverse contraction principle (see [DZ98] Theorem 4.2.4 and Remark a)) it is a consequence of the two following theorems: the first one is

a LDP for the sequence of moments and the second one is a *magic* formula which allows a powerful identification of the rate function. ■

We now give one of the main ingredients of our LDP proof for $G\beta E(N)$ ensembles. First define the functions $g(x) := x - 1 - \log x$ if $x > 0$ and $g(x) = \infty$ otherwise and let

$$G(x) := \begin{cases} g(x^2) & \text{if } x > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 4.4. *The sequence $(\mathbf{m}(\mu_w^{(N)}))$ satisfies in \mathbb{R}^N the LDP with speed $\beta'N$ and good rate function I defined as follows. This function is finite if and only if there exist $(b_0, \dots; a_0, \dots) \in \mathbb{R}^N \times (0, \infty)^N$ satisfying*

$$\sum_{j=0}^{\infty} b_j^2 < \infty, \quad \sum_{j=0}^{\infty} (a_j - 1)^2 < \infty \quad (19)$$

such that $m_r = \langle e_1, A^r e_1 \rangle$ for every $r \geq 1$ with A infinite tridiagonal matrix built with $(b_0, \dots; a_0, \dots)$. In that case

$$I(m_1, \dots) = \frac{1}{2} \sum_{j=0}^{\infty} b_j^2 + \sum_{j=0}^{\infty} G(a_j) < \infty.$$

Theorem 4.5 (Killip-Simon [KS03], [Sim05c] Theorem 13.8.6). *Let J be a Jacobi matrix built with $(a_0, \dots; b_0, \dots) \in (0, \infty)^N \times \mathbb{R}^N$ satisfying $\sup[a_n + \sup|b_n|] < \infty$. Let μ be the associated measure obtained by Favard's theorem. Then*

$$\sum_k [b_k^2 + (a_k - 1)^2] < \infty \quad (20)$$

if, and only if, the p.m. μ satisfies B.W.c. and the two following conditions:

$$\sum_{j=1}^{N_+} (E_j^+ - 2)^{3/2} + \sum_{j=1}^{N_-} (-2 - E_j^-)^{3/2} < \infty \quad (21)$$

$$\int_{-2}^2 \log(f_a(x)) \sqrt{4 - x^2} dx > -\infty. \quad (22)$$

In that case

$$I^w(\mu) = \sum_n \left[\frac{1}{2} b_n^2 + G(a_n) \right] \quad (23)$$

where both sides may be (simultaneously) infinite.

The proof of Theorem 4.4 will use the following result.

Lemma 4.6. For fixed k , $(b_0^{(N)}, \dots, b_k^{(N)}; a_0^{(N)}, \dots, a_{k-1}^{(N)})_{N \geq k}$ satisfies in \mathbb{R}^{2k-1} a LDP with speed $\beta'N$ and good rate function

$$I_k(b_0, \dots, b_k; a_0, \dots, a_{k-1}) = \frac{1}{2} \sum_{j=0}^k b_j^2 + \sum_{j=0}^{k-1} G(a_j). \quad (24)$$

Proof It is an immediate consequence of independence and the LDP for Gaussian and Gamma r.vs. recalled in the following lemma. \blacksquare

Lemma 4.7.

- (1) The sequence of distributions $\mathcal{N}(0; n^{-1})$ satisfies the LDP with speed n and good rate function $x \mapsto x^2/2$.
- (2) For $\alpha > 0$ and c fixed, the sequence of distributions $\gamma((n-c), (\alpha n)^{-1})$ satisfies the LDP with speed n and good rate function $x \mapsto g(\alpha x)$.
- (3) For $u, v > 0$ and δ, δ' fixed, the sequence of distributions $\beta_s(un + \delta, vn + \delta')$ satisfies the LDP with speed n and good rate function:

$$h(q) = \begin{cases} q(u-v) - u \log(1+q) - v \log(1-q) & ; \quad q \in (-1, 1) \\ \infty & ; \quad \text{otherwise.} \end{cases} \quad (25)$$

Proof The points 1 and 2 are well known. For point 3, we use the representation

$$\beta_s(un + \delta, vn + \delta') \stackrel{(d)}{=} \frac{\gamma(un + \delta) - \gamma(vn + \delta')}{\gamma(un + \delta) + \gamma(vn + \delta')}$$

hence by contraction the rate function is

$$h(q) = \inf \left\{ ug(x/u) + vg(y/v); \frac{x-y}{x+y} = q \right\},$$

which yields easily (25). \blacksquare

Proof of Theorem 4.4 : Fix $\ell > 1$. By Lemma 4.6 and the contraction principle, the sequence $(m_1(\mu_w^{(N)}), \dots, m_{2\ell-1}(\mu_w^{(N)}))$ satisfies the LDP in $\mathbb{R}^{2\ell-1}$ with speed $\beta'N$ and rate function $\tilde{I}_{2\ell-1}$ defined as follows. Notice that there is at most only one tridiagonal matrix A_ℓ built from $(b_0, \dots, b_{\ell-1}; a_0, \dots, a_{\ell-2})$ as in (5) such that

$$m_r = \langle e_1, A_\ell^r e_1 \rangle, \quad r = 1, \dots, 2\ell - 1. \quad (26)$$

Hence, if $(m_1, \dots, m_{2\ell-1})$ satisfies (26), then

$$\tilde{I}_{2\ell-1}(m_1, \dots, m_{2\ell-1}) = I_{\ell-1}(b_0, \dots, b_{\ell-1}; a_0, \dots, a_{\ell-2}) \quad (27)$$

Otherwise, $\tilde{I}_{2\ell-1}(m_1, \dots, m_{2\ell-1})$ is infinite. We do not consider the even case since there is no injectivity in that case.

We now apply the Dawson-Gärtner theorem. Let us endow $\mathbb{R}[[X]]$ with the topology of pointwise convergence of coefficients. It can be viewed as the projective limit

$$\mathbb{R}[X] = \varprojlim \mathbb{R}_k[X]$$

where $\mathbb{R}_k[X]$ is the set of polynomials of degree equal or less than k . The rate function is

$$I(m_1, \dots) = \sup\{\tilde{I}_{2k+1}(m_1, \dots, m_{2k+1}) : k \geq 0\}. \quad (28)$$

It is clear that

$$\begin{aligned} \sup\{\tilde{I}_{2k+1}(m_1, \dots, m_{2k+1}) : k \geq 0\} &= \sup_k \left\{ \frac{1}{2} \sum_{j=0}^k b_j^2 + \sum_{j=0}^{k-1} G(a_j) \right\} \\ &= \frac{1}{2} \sum_{j=0}^{\infty} b_j^2 + \sum_{j=0}^{\infty} G(a_j) \leq \infty. \end{aligned}$$

4.3. Failure of the direct method. Mimicking the unitary case ([GR08]), it is tempting to define the random measure

$$\tilde{\mu}_{\mathbf{w}}^{(N)} = \sum_{k=1}^N Y_k \delta_{\lambda_k}$$

with the Y_k independent and $\gamma(\beta')$ distributed so that

$$\mu_{\mathbf{w}}^{(N)} = \frac{\tilde{\mu}_{\mathbf{w}}^{(N)}}{\tilde{\mu}_{\mathbf{w}}^{(N)}(1)}$$

The problem is that the method of Najim [Naj02] cannot be applied as the important assumption on the range of the eigenvalues is violated. Indeed, not all the eigenvalues lie in the support of the semicircle law. So that, the conclusion given by this approach is wrong. The rate function candidate only contains the Kullback part of the LDP but loses the *outer* part.

4.4. Optimization of the rate function. In this subsection we discuss the problem of minimizing the rate function $I^{\mathbf{w}}$ under moment constraints. Surprisingly, using basic tools of large deviations, we show that this optimum value may be expressed in terms of the coefficients (a_k) and (b_k) . This result has the same spirit as the relation between the optimum of Burg entropy under trigonometric moment restrictions and the Verblunsky coefficients (see [GS84])

and [Bur75]). To begin with, for $l > 1$, we define the moment sets

$$\begin{aligned} \mathbb{M}_{2l-1} &:= \left\{ \left(\int x^j \mu(dx) \right)_{j=1, \dots, 2l-1} : \mu \in \mathcal{M}^1 \right\}, \\ \mathbb{M}_{2l-1}^{[-2, 2]} &:= \left\{ \left(\int x^j \mu(dx) \right)_{j=1, \dots, 2l-1} : \mu \in \mathcal{M}^1, \text{Supp} \mu \subset [-2, 2] \right\}. \end{aligned}$$

For $c \in \mathbb{M}_{2l-1}$ (resp. $c \in \mathbb{M}_{2l-1}^{[-2, 2]}$), let $\mathcal{S}_{2l-1}(c)$ (resp. $\mathcal{S}_{2l-1}^{[-2, 2]}(c)$) denotes the set of all p.ms on \mathbb{R} (resp. on $[-2, 2]$) having c as $2l - 1$ first moments.

Corollary 4.8.

1) Let $c \in \text{Int } \mathbb{M}_{2l-1}$, then

$$\inf_{\mu \in \mathcal{S}_{2l-1}(c)} I^w(\mu) = \frac{1}{2} \sum_{j=0}^{l-1} b_j^2 + \sum_{j=0}^{l-2} G(a_j).$$

Where as before, the real $(a_j)_{j=1, \dots, l-2}$ and $(b_j)_{j=1, \dots, l-1}$ are the coefficients of the tridiagonal matrix (5).

2) Let $c \in \text{Int } (\mathbb{M}_{2l-1}^{[-2, 2]})$ then

$$\begin{aligned} \inf_{\mu \in \mathcal{S}_{2l-1}(c)} I^w(\mu) &= \inf_{\mu \in \mathcal{S}_{2l-1}^{[-2, 2]}(c)} \mathcal{K}(SC|\mu) = \frac{1}{2} \sum_{j=0}^{l-1} b_j^2 + \sum_{j=0}^{l-2} G(a_j) \\ &= \sup_{(v_0, v) \in \mathbb{R}^{2l}} \left(v_0 + \sum_{j=1}^{2l-1} v_j c_j + \int_{-2}^2 \log \left(1 - v_0 - \sum_{j=1}^{2l-1} v_j x^j \right) SC(dx) \right), \end{aligned}$$

(here $\log(u) = -\infty$ whenever $u < 0$).

Proof The proof of the first part directly follows from the contraction principle (see [DZ98]). The first equality of the second part is obvious as \mathcal{F}_G is non negative. The last equality is a direct application of general optimization procedures developed in [GG97] or [BL93]. \blacksquare

5. LARGE DEVIATIONS IN THE β -LAGUERRE ENSEMBLE

In the Laguerre case, in the usual asymptotics $N \rightarrow \infty$, $m(N)/N \rightarrow \tau < 1$, we observe similar phenomena. Recall that the sequence of ESD

$$\mu_{\mathbf{u}}^{(N)} = \frac{1}{m(N)} \sum_{k=1}^{m(N)} \delta_{\lambda_k}$$

satisfies the LDP with speed $\beta' N^2$ and good rate function

$$I^u(\mu) = -\Sigma(\mu) + \int_0^\infty \left(\frac{x}{2} - (1 - \tau) \log x \right) d\mu(x) + K_L,$$

where K_L is a constant ([HP98]). The equilibrium measure, unique minimizer of I^u is the Marchenko-Pastur distribution of parameter τ (denoted hereafter by MP, see Appendix). In particular, the sequence $(\mu_u^{(N)})$ converges weakly in probability to MP.

To manage the large deviations of $(\mu_w^{(N)})$, we will first tackle the large deviations of $(s_k^{N,m}, d_k^{N,m}, k \geq 0)$. Recall that the elements of the tridiagonal matrix $L_\beta^{(N)}$ are

$$\begin{aligned} b_0^{(N)} &= (d_1^{(N)})^2, \quad b_k^{(N)} = (s_k^{(N)})^2 + (d_{k+1}^{(N)})^2 \quad (1 \leq k \leq N-1) \\ a_k^{(N)} &= s_{k+1}^{(N)} d_{k+1}^{(N)} \quad (0 \leq k \leq N-2). \end{aligned} \quad (29)$$

We can see already that, in view of (11), we have for fixed $k \geq 1$ and $N \rightarrow \infty$, $\lim d_k^{(N)} = 1$ and $\lim s_k^{(N)} = \sqrt{\tau}$ (in probability). From (29), this yields $\lim b_0^{(N)} = 1$ and for fixed $k \geq 1$, $\lim b_k^{(N)} = 1 + \tau$, $\lim a_{k-1}^{(N)} = \sqrt{\tau}$ (in probability). The corresponding infinite Jacobi matrix which satisfies

$$b_0 = 1, \quad b_k = 1 + \tau, \quad (k \geq 1) \quad ; \quad a_k = \sqrt{\tau}; \quad (k \geq 0).$$

is J_μ with $\mu = MP$.

Let \mathcal{F}_L defined by

$$\mathcal{F}_L(x) = \begin{cases} \int_x^x \frac{\sqrt{(t-a(\tau))(t-b(\tau))}}{t} dt & x \geq b(\tau), \\ \int_x^{a(\tau)} \frac{\sqrt{(a(\tau)-t)(b(\tau)-t)}}{t} dt & 0 < x \leq a(\tau). \end{cases}$$

This is actually the rate function of the LDP for the extreme eigenvalue, as quoted in Féral [Fér08].

Proposition 5.1. *For the $L\beta E(N, \tau N)$ model,*

(1) *the sequence $(\lambda_{max}^{(N)})$ satisfies for $x \geq b(\tau)$*

$$\lim_N \frac{1}{\beta' N} \log \mathbb{P}(\lambda_{max}^{(N)} \geq x) = -\mathcal{F}_L(x). \quad (30)$$

(2) *the sequence $(\lambda_{min}^{(N)})$ satisfies for $0 < x \leq a(\tau)$*

$$\lim_N \frac{1}{\beta' N} \log \mathbb{P}(\lambda_{min}^{(N)} \leq x) = -\mathcal{F}_L(x). \quad (31)$$

For a general double sequence of positive numbers $(d_k)_{k \geq 1}$ and $(s_k)_{k \geq 1}$ we set $d \circ s = (d_1, \dots; s_1, \dots)$. We deduce the elements

$$\begin{aligned} b_0 &= d_1^2, \quad b_k = s_k^2 + d_{k+1}^2 \quad (k \geq 1) \\ a_k &= s_{k+1}d_{k+1} \quad (k \geq 0). \end{aligned} \tag{32}$$

Conversely, if $(a_0, \dots; b_0, \dots)$ is given in $(0, \infty)^{\mathbb{N} \times \mathbb{N}}$ such that the tridiagonal matrix is positive, there exists a unique $d \circ s$ satisfying (32). Here is a direct consequence of Lemma 4.7.

Theorem 5.2. *Under the $L\beta E(N, \tau N)$ model, the sequence $(\mu_{\mathfrak{w}}^{(N)})$ satisfies in $\mathcal{M}_{m,d}^1((0, \infty))$ a LDP with speed $\beta' N$ and good rate function $I^{\mathfrak{w}}$ defined as follows. This function is finite at ν if and only if there exists $d \circ s \in [0, \infty)^{\mathbb{N}} \times [0, \infty)^{\mathbb{N}}$ (necessarily unique) satisfying*

$$\sum_k G(d_k) + \tau \sum_k G(s_k/\sqrt{\tau}) < \infty,$$

such that $m_r(\nu) = \langle e_1, A^r e_1 \rangle$ for every $r \geq 1$ with A infinite tridiagonal matrix built with $(b_0, \dots; a_0, \dots)$ satisfying (32). In that case

$$I^{\mathfrak{w}}(\nu) = \sum_k G(d_k) + \tau \sum_k G(s_k/\sqrt{\tau}). \tag{33}$$

Remark 5.3.

- It is clear from (33) that the unique minimizer of $I^{\mathfrak{w}}$ corresponds to $d_k \equiv 1$ and $s_k \equiv \sqrt{\tau}$ which corresponds to MP.
- When $\tau = 1$, we can write:

$$\begin{aligned} I^{\mathfrak{w}}(\nu) &= \sum_{k \geq 1} [d_k^2 - 1 - \log d_k^2 + s_k^2 - 1 - \log s_k^2] \\ &= d_1^2 - 1 + \sum_{k \geq 1} [d_{k+1}^2 + s_k^2 - 2] - 2 \sum_{k \geq 1} \log(d_k s_k) \\ &= b_0 - 1 + \sum_{k \geq 1} (b_k - 2) - 2 \sum_{k \geq 0} \log a_k. \end{aligned}$$

This expression of $I^{\mathfrak{w}}$ in terms of the Jacobi coefficients makes plausible the existence of a convenient sum rule and we propose the following conjecture :

Conjecture 5.4. *The rate function is*

$$I^{\mathfrak{w}}(\nu) = \mathcal{K}(MP | \nu) + \sum_j \mathcal{F}_L(E_j^{\pm}).$$

Proof of Theorem 5.2 For k fixed, $(d_k^{(N)})$ (resp. $(s_k^{(N)})$) satisfies a LDP with good rate function $G(d_k)$ (resp. $\tau G(s_k/\sqrt{\tau})$) hence, by independence, the rate function is the sum (33). ■

6. LARGE DEVIATIONS IN THE β -JACOBI ENSEMBLE

Let us consider the $\widehat{J\beta E}(N, \mathbf{a}(N), \mathbf{b}(N))$ ensemble. The usual asymptotics is $N \rightarrow \infty$, $\mathbf{b}(N)/N \rightarrow \beta' \kappa_1$, $\mathbf{a}(N)/N \rightarrow \beta' \kappa_2$. The sequence of ESD

$$\mu_N = \frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k}$$

satisfies the LDP with speed $\beta' N$ and good rate function :

$$I^u(\mu) = -\Sigma(\mu) - \int_0^1 (\kappa_1 \log x + \kappa_2 \log(1-x)) d\mu(x) + K_J, \quad (34)$$

where K_J is a constant (see [HP05]). The equilibrium measure (the unique minimizer of I^u) is the Kesten-MacKay distribution (denoted hereafter KMK) of parameter (u_-, u_+) , where

$$u_-, u_+ = u_{\pm} \left(\frac{1 + \kappa_1}{2 + \kappa_1 + \kappa_2}, \frac{1 + \kappa_1 + \kappa_2}{2 + \kappa_1 + \kappa_2} \right)$$

(see Section 7.2). In particular, the sequence $(\mu_u^{(N)})$ converges weakly in probability to KMK.

To manage the large deviations of $(\mu_w^{(N)})$, we will first tackle the large deviations of $(\alpha_k^{(N)}, k \geq 0)$. It is important to notice already that, in view of (12), we have for fixed $p \geq 0$,

$$\lim_N \alpha_{2p}^{(N)} = \frac{\kappa_2 - \kappa_1}{2 + \kappa_1 + \kappa_2}, \quad \lim_N \alpha_{2p+1}^{(N)} = -\frac{\kappa_1 + \kappa_2}{2 + \kappa_1 + \kappa_2}.$$

The symmetric measure admitting these limiting Verblunsky coefficients is well understood by its Cauchy-Stieltjes transform since the work of Geronimus ([Ger44], see also the books of Simon). We do not give details here to shorten the paper. After projection, we obtain the KMK distribution.

Let \mathcal{F}_J defined by

$$\mathcal{F}_J(x) = \begin{cases} \int_{u_+}^x \frac{\sqrt{(t-u_+)(t-u_-)}}{t(1-t)} dt & u_+ \leq x < 1, \\ \int_x^{u_-} \frac{\sqrt{(u_- - t)(u_+ - t)}}{t(1-t)} dt & 0 < x \leq u_-. \end{cases}$$

This is actually the rate function of the LDP for the extreme eigenvalue. Indeed as in Féral [Fér08], we can prove

Lemma 6.1. *For the $\widehat{J\beta E}(N, \mathbf{a}(N), \mathbf{b}(N))$ model with the above notations, if $\kappa_1, \kappa_2 > 0$,*

(1) *the sequence $(\lambda_{max}^{(N)})$ satisfies for $x \in (u_+, 1)$*

$$\lim_N \frac{1}{\beta' N} \log \mathbb{P}(\lambda_{max}^{(N)} \geq x) = -\mathcal{F}_J(x). \quad (35)$$

(2) *the sequence $(\lambda_{min}^{(N)})$ satisfies for $x \in (0, u_-)$*

$$\lim_N \frac{1}{\beta' N} \log \mathbb{P}(\lambda_{min}^{(N)} \leq x) = -\mathcal{F}_J(x). \quad (36)$$

Proof We follow the scheme of proof of [Fér08], Section 5.2. The potential is

$$V(x) = -\kappa_1 \log x - \kappa_2 \log(1 - x).$$

It is convex and \mathcal{C}^∞ on the compact support of the equilibrium measure. It fulfills assumption (iv) of Theorem 4.2 therein. Besides from the formula (46) giving the Cauchy-Stieltjes transform, we see that we are exactly in the same situation as in Féral. \blacksquare

Theorem 6.2.

(1) *(Gamboa-Rouault [GR08]) Under the $\widehat{J\beta E}(N, \mathbf{a}, \mathbf{b})$ model, the sequence $(\mu_{\mathbf{w}}^{(N)})$ satisfies in $\mathcal{M}^1([0, 1])$ endowed with the weak topology the LDP with speed N and good rate function*

$$I(\nu) = \mathcal{K}(\text{ARCSINE} \mid \nu).$$

(2) *Under the $\widehat{J\beta E}(N, \kappa_1 N, \kappa_2 N)$ model, the sequence $(\mu_{\mathbf{w}}^{(N)})$ satisfies in $\mathcal{M}^1([0, 1])$ endowed with the weak topology the LDP with speed N and with a good rate function $I^{\mathbf{w}}$ defined as follows. This function is finite at ν if and only if there exists $\vec{\alpha} \in (-1, 1)^{\mathbb{N}}$ (necessarily unique) such that*

$$\begin{aligned} \mathcal{I}(\vec{\alpha}) &:= (\kappa_1 - \kappa_2) \sum_0^\infty \alpha_{2k} + (\kappa_1 + \kappa_2) \sum_0^\infty \alpha_{2k+1} \\ &\quad - (1 + \kappa_1) \sum_0^\infty \log(1 + \alpha_{2k}) - (1 + \kappa_2) \sum_0^\infty \log(1 - \alpha_{2k}) \\ &\quad - (1 + \kappa_1 + \kappa_2) \sum_0^\infty \log(1 + \alpha_{2k+1}) - \sum_0^\infty \log(1 - \alpha_{2k+1}) \end{aligned}$$

is finite. In that case

$$I^{\mathbf{w}}(\nu) = \mathcal{I}(\vec{\alpha}).$$

Proof We apply Lemma 4.7 (3), with $n = \beta' N$, and for an even index we have $u = 1 + \kappa_1, v = 1 + \kappa_2$ and with odd index $u = 1 + \kappa_1 + \kappa_2, v = 1$

$$\begin{aligned} I_{\alpha_{2k}}(x) &= x(\kappa_1 - \kappa_2) - (1 + \kappa_1) \log(1 + x) - (1 + \kappa_2) \log(1 - x) \\ I_{\alpha_{2k+1}}(x) &= x(\kappa_1 + \kappa_2) - (1 + \kappa_1 + \kappa_2) \log(1 + x) - \log(1 - x) \end{aligned}$$

Then it is enough to add up.

In the particular case of \mathbf{a} and \mathbf{b} fixed, we have $\kappa_1 = \kappa_2 = 0$ and

$$I(\vec{\alpha}) = - \sum_0^{\infty} \log(1 - \alpha_k^2).$$

But the Szegő formula ([Sim05b]) says that it is exactly the reversed Kullback with respect to the ARCSINE distribution. \blacksquare

In the general case, there is up to our knowledge, no known sum rule. Besides it is very intricate to express the above sums in terms of the tridiagonal coefficients. Nevertheless it is tempting to propose the conjecture.

Conjecture 6.3. *Under the $\widehat{J\beta E}(N, \kappa_1 N, \kappa_2 N)$ model, the rate function is given by*

$$I(\nu) = \mathcal{K}(KMK | \nu) + \sum_j \mathcal{F}_J(E_j^{\pm})$$

7. APPENDIX

7.1. Large deviations. We present here large deviations principles (LDPs) for sequences of random measures defined in the previous Section. For the sake of completeness we briefly recall the LDP definition. Let (u_n) be a decreasing positive sequence of real numbers with $\lim_{n \rightarrow \infty} u_n = 0$.

Definition 7.1. *We say that a sequence (R_N) of p.ms. on a measurable Hausdorff space $(G, \mathcal{B}(G))$ satisfies a LDP with speed u_N^{-1} and rate function I if:*

- i) I is lower semicontinuous (lsc), with values in $\mathbb{R}^+ \cup \{+\infty\}$.
- ii) For any measurable set A of G :

$$-I(\text{int } A) \leq \liminf_{N \rightarrow \infty} u_N \log R_N(A) \leq \limsup_{N \rightarrow \infty} u_N \log R_N(A) \leq -I(\text{clo } A),$$

where $I(A) = \inf_{\xi \in A} I(\xi)$ and $\text{int } A$ (resp. $\text{clo } A$) is the interior (resp. the closure) of A .

We say that the rate function I is good if its level set $\{x \in G : I(x) \leq a\}$ is compact for any $a \geq 0$. More generally, a sequence of G -valued r.v.s. is said to satisfy a LDP if the sequence of their distributions satisfies a LDP.

The rate functions obtained in this paper are built with be the so-called Kullback information or cross entropy. Let P and Q be p.ms. on G , the Kullback information between P and Q is defined by

$$\begin{cases} \mathcal{K}(P|Q) &= \int_G \log \frac{dP}{dQ} dP & \text{if } P \ll Q \text{ and } \log \frac{dP}{dQ} \in L^1(P), \\ &= +\infty & \text{otherwise.} \end{cases} \quad (37)$$

7.2. The distributions.

7.2.1. *Gamma distribution.* For $a, b > 0$, the $\gamma(a, b)$ distribution is supported by $[0, \infty)$ with density

$$\frac{e^{-x/b} x^{a-1}}{b^a \Gamma(a)}$$

Its mean is ab .

7.2.2. *Beta distribution.* For $a, b > 0$, the beta symmetric distribution of parameter (a, b) , denoted by $\beta_s(a, b)$, is supported by $(-1, 1]$ and has density

$$2^{1-a-b} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (1-x)^{a-1} (1+x)^{b-1}$$

Its mean is $\frac{b-a}{b+a}$.

7.2.3. *Dirichlet distribution.* For $k \geq 1$, we set

$$\begin{aligned} \mathcal{S}_k &:= \{(x_1, \dots, x_k) : x_i > 0, (i = 1, \dots, k), x_1 + \dots + x_k = 1\} \\ \mathcal{S}_k^< &:= \{(x_1, \dots, x_k) : x_i > 0, (i = 1, \dots, k), x_1 + \dots + x_k < 1\}. \end{aligned}$$

Obviously, the mapping $(x_1, \dots, x_{k+1}) \mapsto (x_1, \dots, x_k)$ is a bijection from the simplex \mathcal{S}_{k+1} onto $\mathcal{S}_k^<$.

For $a_j > 0$, $j = 1, \dots, k+1$, the Dirichlet distribution $\text{Dir}(a_1, \dots, a_{k+1})$ on \mathcal{S}_{k+1} has the density

$$\frac{\Gamma(a_1 + \dots + a_{k+1})}{\Gamma(a_1) \cdots \Gamma(a_{k+1})} x_1^{a_1-1} \cdots x_{k+1}^{a_{k+1}-1} \quad (38)$$

with respect to the Lebesgue measure on \mathcal{S}_{k+1} . When $a_1 = \dots = a_{k+1} = a > 0$, we will denote the Dirichlet distribution by $\text{Dir}_k(a)$. If $a = 1$ we recover the uniform distribution on $\mathcal{S}_k^<$.

7.2.4. *Semicircle distribution.* The semicircle distribution denoted by SC is supported by $[-2, 2]$ with density

$$\frac{\sqrt{4-x^2}}{2\pi}.$$

Its Cauchy-Stieltjes transform² is

$$m(z) = \int \frac{d\mu(x)}{x-z} = \frac{-z + \sqrt{z^2 - 4}}{2} \quad (39)$$

When $0 < \tau \leq 1$, the Marchenko-Pastur distribution, denoted by MP is supported by $(a(\tau), b(\tau))$ where $a(\tau) = (1 - \sqrt{\tau})^2$, $b(\tau) = (1 + \sqrt{\tau})^2$ with density

$$\frac{\sqrt{(x-a(\tau))(b(\tau)-x)}}{2\pi\tau x}. \quad (40)$$

Its Cauchy-Stieltjes transform is

$$m(z) = \frac{-z + (1-\tau) + \sqrt{(z-1-\tau)^2 - 4\tau}}{2\tau z}. \quad (41)$$

7.2.5. *Kesten-McKay distribution.* The Kesten-McKay distribution is supported by (u_-, u_+) with $0 \leq u_- < u_+ \leq 1$ and its density is

$$C_{u_-, u_+} \frac{\sqrt{(x-u_-)(u_+-x)}}{2\pi x(1-x)} \quad (42)$$

where

$$C_{u_-, u_+}^{-1} := \frac{1}{2} \left[1 - \sqrt{u_- u_+} - \sqrt{(1-u_-)(1-u_+)} \right].$$

To express its Cauchy-Stieltjes transform, let us give some notation. For $(b, c) \in (0, 1) \times (0, 1)$ we put

$$\sigma_{\pm}(b, c) = \frac{1}{2} \left[1 + \sqrt{bc} \pm \sqrt{(1-b)(1-c)} \right], \quad (43)$$

and for $(x, y) \in (0, 1) \times (0, 1)$

$$\begin{aligned} u_{\pm}(x, y) &= (1-x-y+2xy) \pm 2\sqrt{x(1-x)y(1-y)} \\ &= \left(\sqrt{(1-x)(1-y)} \pm \sqrt{xy} \right)^2. \end{aligned} \quad (44)$$

The mappings σ_{\pm} and u_{\pm} are inverse in the following sense :

$$\{(b, c) : 0 < b < c < 1\} \xrightarrow[(u_-, u_+)]{(\sigma_-, \sigma_+)} \{(x, y) : 0 < x < y < 1 \text{ and } x + y > 1\} \quad (45)$$

²Throughout, all branches of the square roots are taken in accordance to the definition of Cauchy transform

The Cauchy-Stieltjes is then (see for instance [Dem06] p.129, or [CC04] p.425)

$$m(z) = \frac{(1 - \sigma_+ - \sigma_-)}{2(1 - \sigma_+)z} + \frac{(\sigma_+ - \sigma_-)}{2(1 - \sigma_+)(1 - z)} + \frac{\sqrt{(z - a_-)(z - a_+)}}{2z(1 - z)}. \quad (46)$$

ARCSINE corresponds to $u_- = 0$ and $u_+ = 1$.

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