

The Connectivity Order of Links

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ABSTRACT — We associate at each link a connectivity space which describes its splittability properties. Then, the notion of order for finite connectivity spaces results in the definition of a new numerical invariant for links, their connectivity order. A section of this short paper presents a theorem which asserts that every finite connectivity structure can be realized by a link : the Brunn-Debrunner-Kanenobu Theorem.

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1 Order of Finite Connectivity Spaces

Let us recall the definition of a connectivity space [1, 4].

Definition 1 (Connectivity spaces) A connectivity space is a couple $\mathbf{X} = (X, \mathcal{K})$ where X is a set and \mathcal{K} is a set of nonempty subsets of X such that

$$\forall \mathcal{I} \in \mathcal{P}(\mathcal{K}), \bigcap_{K \in \mathcal{I}} K \neq \emptyset \implies \bigcup_{K \in \mathcal{I}} K \in \mathcal{K}.$$

The set X is called the support of the space \mathbf{X} , the set \mathcal{K} is its connectivity structure. The elements of \mathcal{K} are called the connected subsets of \mathbf{X} . The morphisms between two connectivity spaces are the functions which transform connected subsets into connected subsets. A connectivity space is called integral if every singleton subset is connected. A connectivity space is called finite if the number of its points is finite.

Definition 2 (Irreducible connected subsets) Let $\mathbf{X} = (X, \mathcal{K})$ be a connectivity space. A connected subset $K \in \mathcal{K}$ is called reducible if there are two connected subsets $A \subsetneq K$ and $B \subsetneq K$ such that

$$K = A \cup B \text{ and } A \cap B \neq \emptyset.$$

A connected subset is said to be irreducible if it is not reducible.

In the sequel, all connectivity spaces will be supposed to be integral and finite.

Definition 3 (Order of irreducible connected subsets) Let $\mathbf{X} = (X, \mathcal{K})$ be a (finite and integral) connectivity space. We define by induction the order of each irreducible subset. Singleton subsets are said to be of order 0. The order $\omega(L)$ of an irreducible subset L which has more than one point is defined by

$$\omega(L) = 1 + \max_{K \in \mathcal{S}(L)} \omega(K)$$

where $\mathcal{S}(L)$ is the set of irreducible connected subsets which are strictly included in L .

Definition 4 (Order of a connectivity space) The order $\omega(\mathbf{X})$ of a (finite and integral) nonempty connectivity space $\mathbf{X} = (X, \mathcal{K})$ is the maximum of the order of its irreducible connected subsets.

Instead of $\omega(\mathbf{X})$, we generally write just $\omega(X)$.

Remark. It is useful to represent the structure of a (finite and integral) connectivity space \mathbf{X} by a graph G whose vertices are the irreducible connected subsets of \mathbf{X} and edges express the inclusion relations : if a and b are irreducible connected subsets of \mathbf{X} , (a, b) is an edge of G iff $a \subseteq b$ and there is no irreducible connected subset c such that $a \subsetneq c \subsetneq b$. In [5], I called this graph the *generic graph* of the connectivity space \mathbf{X} ; the vertices of this graph, *i.e.* the irreducible connected subsets of \mathbf{X} , are the *generic points* of \mathbf{X} . The order of a space is then the maximal length of paths in the generic graph.

Proposition 1 The order $\omega(X)$ of a nonempty finite integral connectivity space is always less than the number $\sharp X$ of its points : $\omega(X) \leq \sharp X - 1$. The order $\omega(X)$ is zero iff the space is totally disconnected, that is the only connected subsets are the singletons.

Example. Let $n \geq 1$ an integer, and (A_n, \mathcal{A}_n) the connectivity space defined by $A_n = \{1, \dots, n\}$ and

$$\mathcal{A}_n = \{\{1\}, \{2\}, \dots, \{n\}\} \cup \{\{1, 2\}, \{1, 2, 3\}, \dots, \{1, 2, \dots, n\}\}.$$

Then $\omega(A_n) = n - 1$. Indeed, in this space each connected subset is irreducible, and its order is, by induction, equal to its cardinal minus one. It is, up to isomorphism, the only connectivity space with n points which is of order $n - 1$.

2 The Brunn-Debrunner-Kanenobu Theorem

At each (tame) link L , we can associate a connectivity space \mathbf{X}_L taking the components of the link L as points of \mathbf{X}_L and nonsplittable sublinks of L (one considers knots, *i.e.* sublinks with only one component, as nonsplittable links) defining the connected nonempty subsets of \mathbf{X}_L .

Definition 5 The connectivity structure of the connectivity space \mathbf{X}_L associated to a link L is called the splittability structure of L .

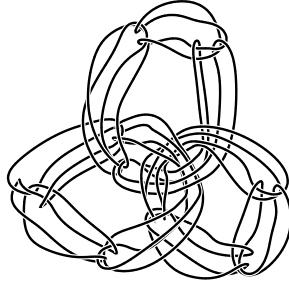


Figure 1: A Borromean ring of borromean rings.

Example. The splittability structure of the Borromean ring is (isomorphic to) $\{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$.

In [4, 5], I asked whether every finite connectivity space can be represented by a link, that is whether exists (at least) one link whose connectivity structure is (isomorphic to) the one given. It appears that in 1892, Brunn [2] first asked this question, without formalizing the concept of a connectivity space. His answer was positive, and he gave the idea of a proof based on a construction using some of the links now called “brunnian”. In 1964, Debrunner [3], rejecting the Brunnian “proof”, gave another construction, proving it but only for n -dimensional links with $n \geq 2$. In 1985, Kanenobu [6, 7] seems to be the first to give a proof of the possibility to represent every finite connectivity structure by a classical link, a result which has not been, up to now, very wellknown. The key idea of those different constructions is already in the Brunn’s article : for him, what we call “Brunnian links” are not so interesting in themselves, but for the constructions they allow to make, that is the representation of *all* finite connectivity structures by links.

Theorem 2 (Brunn-Debrunner-Kanenobu) *Every finite connectivity structure is the splittability structure of a link in \mathbf{R}^3 .*

3 The Connectivity Order of Links

At each link L , we associate its connectivity order $\omega(L)$, *i.e.* the order of the connectivity space associated to L .

Examples. The link pictured on the figure 1 has a connectivity order 2. The one of the figure 2 has a connectivity order 8, which is the maximal order for a link with 9 components. The way this very asymmetrical link splits when a component is erased or cut highly depends on the position of this component in the link, as shows its connectivity structure, which is the one we called \mathcal{A}_9 .

Remark. The connectivity order is not a Vassiliev finite type invariant for links. For example, it is easy to check that the connectivity order of the singular link

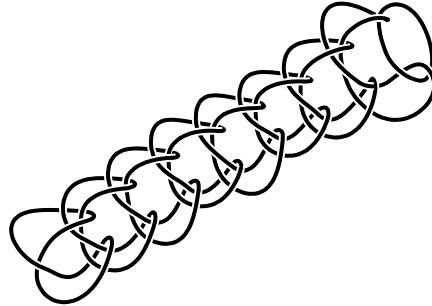


Figure 2: A link with a connectivity order 8.

with two components, a circle and another component crossing this circle at $2n$ double-points, is greater than 2^n .

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