

# On-line Bayesian Cramér-Rao Bound for Oversampled Dynamical Phase Offset Estimation

Jordi Vilà Valls, Jean-Marc Brossier and Laurent Ros

GIPSA-lab, Departement Image Signal

BP 46 - 38402 Saint Martin d'Hères - FRANCE

E-mail: jordi.vilavalls@gipsa-lab.inpg.fr, jean-marc.brossier@gipsa-lab.inpg.fr, laurent.ros@gipsa-lab.inpg.fr

**Abstract**—In this paper, we present a closed-form expression of a Bayesian Cramér-Rao lower bound for the estimation of a dynamical carrier phase offset. This bound is derived in an on-line oversampled scenario. Our numerical results show that using the oversampled signal for estimating the phase offset we can obtain better performances than using a classical synchronizer.

**Index Terms**—Phase estimation, bayesian Cramér-Rao lower bound, oversampling, carrier synchronization.

## I. INTRODUCTION

Synchronization is a fundamental part in modern digital receivers. A synchronizer has to estimate some parameters, such as carrier frequency, carrier phase and timing epoch, to correct the distortions introduced by the channel. This knowledge makes possible to recover the signal of interest correctly. In this paper we focus our attention in the phase estimation problem.

Many methods for estimating the phase introduced by an unknown channel have been proposed over the past decades, from Phase Locked Loops (PLL) to the most sophisticated signal processing techniques. When having an estimation process we need to have lower bounds on the estimation performance to be used as a benchmark. The family of Cramér-Rao Bounds (CRBs) has been shown to give accurate estimation lower bounds in many scenarios [1].

Several Cramér-Rao lower bounds have appeared in the literature. Rife *et al.* [10] have derived CRB closed-form expressions for constant phase-offset estimation in the so-called Data-Aided (DA) scenario, and Cowley [11] did so in the Non-Data-Aided (NDA) scenario. In many cases, these lower bounds require a big computational complexity. A modified CRB (MCRB) has been introduced in [8], [9], to avoid this problem. The MCRB is much simpler but in general looser. CRBs for time-varying parameter estimation has also been derived. Tichavský [7] derived a general *on-line* recursive Bayesian CRB (BCRB) analytical expression. Bay *et al.* [5] [6] presented an Asymptotic BCRB(ABCRB) and an analytical expression of the *off-line* CRB and BCRB. Most of the lower bounds mentioned before, consider signal models with white observation noise.

In this contribution we investigate the BCRB related to the estimation of the carrier phase in a DA scenario. We consider an *oversampled* signal model after receiver matched filter, this implies having a *coloured reception noise*. In Section II, we set the signal model. In Section III, we recall general BCRB

expressions and we derive a closed-form expression of the on-line Cramér-Rao Bound for oversampled dynamical phase estimation. Finally in Section IV, the different numerical results for the BCRB resulting from BPSK transmission are presented and interpreted. Appendix gives also some complementary results.

*Notations*: italic indicates a scalar quantity, as in  $a$ ; bold-face indicates a vector quantity, as in  $\mathbf{a}$  and capital boldface indicates a matrix quantity as in  $\mathbf{A}$ . The  $(k, l)^{th}$  entry of a matrix  $\mathbf{A}$  is denoted  $[\mathbf{A}]_{k,l}$ . The matrix transpose and self-adjoint operators are indicated by the superscripts  $T$  and  $H$  respectively as in  $\mathbf{A}^T$  and  $\mathbf{A}^H$ .  $\Re(\cdot)$  and  $(\cdot)^*$  are the real part and conjugate of a complex number or matrix, respectively.  $E_x$  denotes the expectation over  $x$ .  $\text{mod}(\cdot)$  refers to the modulo operator,  $[\cdot]$  means integer part and  $|a|$  is the absolute value of  $a$ .  $\nabla_{\boldsymbol{\theta}}$  and  $\Delta_{\boldsymbol{\psi}}^{\boldsymbol{\theta}}$  represent the first and second-order partial derivatives operator *i.e.*,  $\nabla_{\boldsymbol{\theta}} = \left[ \frac{\partial}{\partial \theta_1} \cdots \frac{\partial}{\partial \theta_K} \right]^T$  and  $\Delta_{\boldsymbol{\psi}}^{\boldsymbol{\theta}} = \nabla_{\boldsymbol{\psi}} \nabla_{\boldsymbol{\theta}}^T$ .  $\mathbf{1}$  stands for the all-ones matrix.

## II. SIGNAL MODEL

Let us consider a detailed signal model. We propose the signal model for the transmission of a known sequence  $\{a_m\}_{m \in \mathbb{Z}}$  over an Additive White Gaussian Noise (AWGN) channel affected by a carrier phase offset  $\theta(t)$ .

### A. Oversampled Signal Model

The received complex baseband signal after matched filtering is

$$y(t) = \left[ \left\{ T \sum_m a_m \Pi(t - mT) + n(t) \right\} e^{i\theta(t)} \right] * \Pi^*(-t) \quad (1)$$

where  $T$ ,  $\Pi(t)$  and  $n(t)$  stands for the symbol period, shaping pulse and circular gaussian noise with a known bilateral power spectral density (psd)  $N_0$ .

We define

$$b(t) = \left[ n(t) e^{i\theta(t)} \right] * \Pi^*(-t) \quad (2)$$

we note that  $n(t)$  and  $n(t)e^{i\theta(t)}$  have the same statistical properties as  $n(t)$  is supposed to be circular. We define  $\sigma_n^2 = N_0 D$  as the variance of the AWGN  $n(t)$  measured in the

band  $[-\frac{D}{2}, \frac{D}{2}]$  where  $D$  is the shaping pulse noise equivalent bandwidth.

We also define  $\tilde{g}_m(t)$  as

$$\tilde{g}_m(t) = T \int_{-\infty}^{+\infty} \Pi^*(-\alpha) e^{i\theta(t-\alpha+mT)} \Pi(t-\alpha) d\alpha \quad (3)$$

Then the received signal can be written as

$$y(t) = \sum_m a_m \tilde{g}_m(t - mT) + b(t) \quad (4)$$

Hereafter we suppose a shaping pulse with support in  $[0, T]$  and a slow varying phase evolution during a period  $T$ . In this case we can approximate  $\tilde{g}_m(t)$  by

$$\tilde{g}_m(t) \approx g(t) e^{i\theta(t+(m+\frac{1}{2})T)} \quad (5)$$

where

$$g(t) = T \int_{-T}^0 \Pi^*(-\alpha) \Pi(t-\alpha) d\alpha \quad (6)$$

If the received signal is fractionally-spaced at  $t_k = k\frac{T}{s} + \tau$ , where  $s$  is an integer oversampling factor and  $\tau$  a known offset from the optimum sampling instants, we have that

$$y\left(k\frac{T}{s} + \tau\right) = \sum_m a_m \tilde{g}_m\left(k\frac{T}{s} + \tau - mT\right) + b\left(k\frac{T}{s} + \tau\right) \quad (7)$$

and from (eq.5) we have that

$$y\left(k\frac{T}{s} + \tau\right) = e^{i\theta(k\frac{T}{s} + \tau + \frac{T}{2})} A_k + b\left(k\frac{T}{s} + \tau\right) \quad (8)$$

where

$$A_k = \sum_m a_m g\left(k\frac{T}{s} + \tau - mT\right) \quad (9)$$

and we can finally write the received oversampled signal as

$$y_k = A_k e^{i\theta_k} + b_k \quad (10)$$

where  $k$  refers to  $t_k$  instants.

The samples  $b_k$  are a non-white noise with variance  $\sigma_n^2$ , and covariance matrix  $\mathbf{\Gamma}$  which depends on the oversampling factor  $s$ . This is a  $N \times N$  matrix where  $N$  is the size of the oversampled vector  $\mathbf{y}$ . For example with  $s = 2$ , we have

$$\mathbf{\Gamma} = \frac{N_0}{T} \begin{pmatrix} g(0) & g(-\frac{T}{2}) & 0 & 0 & \cdots & 0 \\ g(\frac{T}{2}) & g(0) & g(-\frac{T}{2}) & 0 & \cdots & 0 \\ 0 & g(\frac{T}{2}) & g(0) & g(-\frac{T}{2}) & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & g(0) & g(-\frac{T}{2}) \\ 0 & 0 & \cdots & 0 & g(\frac{T}{2}) & g(0) \end{pmatrix} \quad (11)$$

When considering the offset  $\tau$  in the interval  $[0, \frac{T}{s}]$ . The coefficients  $A_k$  can be written as

$$A_k = a_m g(\tau + pT) + a_{m+1} g(\tau - (1-p)T) \quad (12)$$

where  $p = \frac{k}{s} \bmod(s)$  and  $m = \lfloor \frac{k}{s} \rfloor$ .

Note that  $\{A_k\}_{k \in \mathbb{Z}}$  is a non-stationary sequence for  $s > 1$ , even if  $\{a_m\}_{m \in \mathbb{Z}}$  is a stationary symbols sequence.

### B. Phase-offset evolution model

We consider the case of a Wiener phase-offset evolution

$$\theta_k = \theta_{k-1} + w_k \quad k \geq 2 \quad (13)$$

where  $w_k$  is an i.i.d. zero-mean Gaussian noise with known variance  $\frac{\sigma_w^2}{s}$ . Here  $\sigma_w^2$  stands for the variance growth of the phase noise in one symbol period. If the initial condition  $\theta_1$  has a zero-mean gaussian distribution with variance  $\sigma_{\theta_1}^2$ , we have that the covariance matrix of the phase-offset evolution is

$$\Sigma = \frac{\sigma_w^2}{s} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & N \end{pmatrix} + \sigma_{\theta_1}^2 \mathbf{1} \quad (14)$$

We note that  $\boldsymbol{\theta} = [\theta_1 \cdots \theta_N]^T$ .

## III. CRAMÉR-RAO BOUNDS

In this section we present the family of Cramér-Rao Bounds. When dealing with an estimation problem we aim to know the ultimate accuracy that can be achieved by the estimator. The Cramér-Rao Bounds provide a lower bound on the Mean Square Error (MSE) achievable by any unbiased estimator. In the sequel we give the general expression of one of the bounds of this family, the Bayesian Cramér-Rao Bound (BCRB). Also the closed-form expression for an on-line BCRB for an oversampled dynamical phase estimation is presented. In the on-line synchronization mode, at time  $k$  the receiver updates the observation vector  $\mathbf{y} = [y_1 \cdots y_{k-1}]^T$  including the new observation  $y_k$  to obtain the updated vector  $\mathbf{y} = [y_1 \cdots y_k]^T$  in order to estimate  $\theta_k$ : only the past and the current observations are available.

### A. Bayesian Cramér-Rao Bound (BCRB)

Also called Posterior Cramér-Rao Bound (PCRB), this bound is particularly suited for problems where an *a priori* information is available.

We have a set of measurements  $\mathbf{y}$  and we want to estimate a  $N$ -dimensional real random parameter  $\boldsymbol{\theta}$ . The joint probability density of the pair  $(\mathbf{y}, \boldsymbol{\theta})$  is  $p_{\mathbf{y}, \boldsymbol{\theta}}(\mathbf{y}, \boldsymbol{\theta})$ . If  $\hat{g}(\mathbf{y})$  is our estimate of  $\boldsymbol{\theta}$ , a Bayesian CRB has been proposed in [1] such that

$$E_{\mathbf{y}, \boldsymbol{\theta}} \{ [g(\mathbf{y}) - \boldsymbol{\theta}] [g(\mathbf{y}) - \boldsymbol{\theta}]^T \} \geq \text{BCRB} \quad (15)$$

The BCRB is the inverse of the Bayesian Information Matrix (BIM), which can be written as

$$\mathbf{B} = E_{\boldsymbol{\theta}} [\mathbf{F}(\boldsymbol{\theta})] + E_{\boldsymbol{\theta}} \left[ -\Delta_{\boldsymbol{\theta}}^{\theta} \ln p(\boldsymbol{\theta}) \right] \quad (16)$$

where  $p(\boldsymbol{\theta})$  is the unknown parameter distribution and  $\mathbf{F}(\boldsymbol{\theta})$  is the Fisher Information Matrix (FIM) defined as

$$\mathbf{F}(\boldsymbol{\theta}) = E_{\mathbf{y}|\boldsymbol{\theta}} \left[ -\Delta_{\boldsymbol{\theta}}^2 \ln p(\mathbf{y} | \boldsymbol{\theta}) \right] \quad (17)$$

Equivalently we can say that the  $(i, j)^{th}$  element of the BIM is

$$[\mathbf{B}]_{i,j} = E_{\mathbf{y},\boldsymbol{\theta}} \left\{ -\frac{\partial^2 \ln p_{\mathbf{y},\boldsymbol{\theta}}(\mathbf{y}, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right\} \quad (18)$$

and since  $p_{\mathbf{y},\boldsymbol{\theta}}(\mathbf{y}, \boldsymbol{\theta}) = p_{\mathbf{y}|\boldsymbol{\theta}}(\mathbf{y} | \boldsymbol{\theta})p_{\boldsymbol{\theta}}(\boldsymbol{\theta})$  we can also write that

$$[\mathbf{B}]_{i,j} = [\mathbf{B}^D]_{i,j} + [\mathbf{B}^P]_{i,j} \quad (19)$$

where

$$[\mathbf{B}^D]_{i,j} = E_{\mathbf{y},\boldsymbol{\theta}} \left\{ -\frac{\partial^2 \ln p_{\mathbf{y}|\boldsymbol{\theta}}(\mathbf{y} | \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right\} \quad (20)$$

$$[\mathbf{B}^P]_{i,j} = E_{\mathbf{y},\boldsymbol{\theta}} \left\{ -\frac{\partial^2 \ln p_{\boldsymbol{\theta}}(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right\} \quad (21)$$

We can see that  $\mathbf{B}_{ij}^D$  represents the average information about  $\boldsymbol{\theta}$  brought by the observations  $\mathbf{y}$  and  $\mathbf{B}_{ij}^P$  represents the information available from the prior knowledge on  $\boldsymbol{\theta}$ , i.e.,  $p(\boldsymbol{\theta})$ .

### B. The On-Line BCRB for Dynamical Phase Estimation

In this paragraph, an analytical expression of the BCRB for an on-line fractionally-spaced phase-offset estimation problem is presented.

We use the model presented in section II (eq.10, eq.13)

$$\begin{aligned} y_k &= A_k e^{i\theta_k} + b_k \\ \theta_k &= \theta_{k-1} + w_k \end{aligned} \quad (22)$$

where, as stated before,  $b_k$  is a non-white noise with covariance matrix  $\boldsymbol{\Gamma}$ . The index  $k$  refers to  $t_k$  instants and  $A_k$  are the coefficients specified in (eq.9).

The first step to obtain (eq.16) is to find the joint distribution  $p(\mathbf{y}, \boldsymbol{\theta})$ , so the likelihood function  $p(\mathbf{y} | \boldsymbol{\theta})$ , to compute the FIM (see eq.17). From the state-space model (eq.22) we have that

$$p(\mathbf{y} | \boldsymbol{\theta}) = \frac{1}{\pi^N |\det(\boldsymbol{\Gamma})|} e^{-[\mathbf{y}-\mathbf{m}]^H \boldsymbol{\Gamma}^{-1} [\mathbf{y}-\mathbf{m}]} \quad (23)$$

where  $\mathbf{y}$  is the  $N$ -dimensional received signal array and  $\mathbf{m}$  is the mean vector of  $\mathbf{y}$ , where the  $k^{th}$  component is  $[\mathbf{m}]_k = A_k e^{i\theta_k}$ .

As the elements of the FIM (see eq.17) are

$$[\mathbf{F}]_{k,l} = E_{\mathbf{y}|\boldsymbol{\theta}} \left\{ -\frac{\partial^2 \ln p(\mathbf{y} | \boldsymbol{\theta})}{\partial \theta_k \partial \theta_l} \right\} \text{ for } k, l = 1, \dots, N \quad (24)$$

we first compute  $\Lambda(\boldsymbol{\theta}) = \ln p(\mathbf{y} | \boldsymbol{\theta})$

$$\Lambda(\boldsymbol{\theta}) = -\ln(\pi^N |\det(\boldsymbol{\Gamma})|) - [\mathbf{y} - \mathbf{m}]^H \boldsymbol{\Gamma}^{-1} [\mathbf{y} - \mathbf{m}] \quad (25)$$

Then we find the first derivative of  $\Lambda(\boldsymbol{\theta})$

$$\begin{aligned} \frac{\partial \Lambda(\boldsymbol{\theta})}{\partial \theta_l} &= \frac{\partial}{\partial \theta_l} \left\{ -[\mathbf{y} - \mathbf{m}]^H \boldsymbol{\Gamma}^{-1} [\mathbf{y} - \mathbf{m}] \right\} \\ &= \left\{ \frac{\partial \mathbf{m}^H}{\partial \theta_l} \boldsymbol{\Gamma}^{-1} [\mathbf{y} - \mathbf{m}] + [\mathbf{y} - \mathbf{m}]^H \boldsymbol{\Gamma}^{-1} \frac{\partial \mathbf{m}}{\partial \theta_l} \right\} \\ &= 2\Re \left\{ \frac{\partial \mathbf{m}^H}{\partial \theta_l} \boldsymbol{\Gamma}^{-1} [\mathbf{y} - \mathbf{m}] \right\} \end{aligned} \quad (26)$$

and then the second one

$$\frac{\partial^2 \Lambda(\boldsymbol{\theta})}{\partial \theta_k \partial \theta_l} = 2\Re \left\{ \frac{\partial^2 \mathbf{m}^H}{\partial \theta_k \partial \theta_l} \boldsymbol{\Gamma}^{-1} [\mathbf{y} - \mathbf{m}] - \frac{\partial \mathbf{m}^H}{\partial \theta_l} \boldsymbol{\Gamma}^{-1} \frac{\partial \mathbf{m}}{\partial \theta_k} \right\} \quad (27)$$

If we compute the expectation over  $\mathbf{y} | \boldsymbol{\theta}$  we obtain directly the FIM's coefficients,

$$[\mathbf{F}]_{k,l} = E_{\mathbf{y}|\boldsymbol{\theta}} \left\{ -\frac{\partial^2 \Lambda(\boldsymbol{\theta})}{\partial \theta_k \partial \theta_l} \right\} = 2\Re \left\{ \frac{\partial \mathbf{m}^H}{\partial \theta_l} \boldsymbol{\Gamma}^{-1} \frac{\partial \mathbf{m}}{\partial \theta_k} \right\} \quad (28)$$

We note that

$$\frac{\partial \mathbf{m}^H}{\partial \theta_l} = [0, \dots, 0, -iA_l^* e^{-i\theta_l}, 0, \dots, 0] \quad (29)$$

$$\frac{\partial \mathbf{m}}{\partial \theta_k} = [0, \dots, 0, iA_k e^{i\theta_k}, 0, \dots, 0]^T \quad (30)$$

So the coefficients can be written as

$$[\mathbf{F}]_{k,l} = 2\Re \left\{ A_l^* A_k \cdot [\boldsymbol{\Gamma}^{-1}]_{k,l} e^{j(\theta_k - \theta_l)} \right\} \quad (31)$$

From this result we can compute the coefficients  $[\mathbf{B}^D]_{k,l} = E_{\boldsymbol{\theta}} \left\{ [\mathbf{F}]_{k,l} \right\}$  as

$$\begin{aligned} [\mathbf{B}^D]_{k,l} &= E_{\boldsymbol{\theta}} \left\{ 2\Re \left\{ A_l^* A_k [\boldsymbol{\Gamma}^{-1}]_{k,l} e^{j(\theta_k - \theta_l)} \right\} \right\} \\ &= 2\Re \left\{ A_l^* A_k [\boldsymbol{\Gamma}^{-1}]_{k,l} E_{\boldsymbol{\theta}} \left\{ e^{j(\theta_k - \theta_l)} \right\} \right\} \end{aligned} \quad (32)$$

We can write that

$$\begin{aligned} E_{\boldsymbol{\theta}} \left\{ e^{j(\theta_k - \theta_l)} \right\} &= E_{\boldsymbol{\theta}} \left\{ e^{i(\mathbf{u}_{kl}^T \boldsymbol{\theta})} \right\} \\ &= \phi(\mathbf{u}_{kl}) \end{aligned} \quad (33)$$

where  $\mathbf{u}_{kl}^T = [0, \dots, 0, (+1), 0, \dots, 0, (-1), 0, \dots, 0]$ ,  $+1$  in the  $k^{th}$  position and  $-1$  in the  $l^{th}$  position of the array,  $\phi(\cdot)$  is the characteristic function of a Gaussian random variable  $\boldsymbol{\theta}$ :

$$\begin{aligned} \phi(\mathbf{u}_{kl}) &= \exp \left\{ -\frac{1}{2} \mathbf{u}_{kl}^T \boldsymbol{\Sigma}^{-1} \mathbf{u}_{kl} \right\} \\ &= \exp \left\{ -\frac{1}{2} \left( [\boldsymbol{\Sigma}^{-1}]_{k,k} + [\boldsymbol{\Sigma}^{-1}]_{l,l} - 2[\boldsymbol{\Sigma}^{-1}]_{k,l} \right) \right\} \end{aligned} \quad (34)$$

with  $\boldsymbol{\Sigma}$  the covariance matrix of the phase evolution  $\boldsymbol{\theta}$  (eq.14), finally

$$[\mathbf{B}^D]_{k,l} = 2\Re \left\{ A_l^* A_k [\boldsymbol{\Gamma}^{-1}]_{k,l} e^{\Psi} \right\} \quad (35)$$

where

$$\Psi = \left\{ -\frac{1}{2} \left( [\Sigma^{-1}]_{k,k} + [\Sigma^{-1}]_{l,l} - 2 [\Sigma^{-1}]_{k,l} \right) \right\} \quad (36)$$

But to compute the BCRB we need not only the coefficients  $[\mathbf{B}^D]_{k,l}$  but also the *a priori* coefficients  $[\mathbf{B}^P]_{k,l}$ . We aim to compute the *a priori* matrix

$$\mathbf{B}^P = E_{\theta} \left\{ -\Delta_{\theta} \ln p(\theta) \right\} \quad (37)$$

As we have a Wiener phase-offset evolution model (see eq.13), the distribution can be computed as

$$p(\theta) = p(\theta_1) \cdot \prod_{k=2}^N p(\theta_k | \theta_{k-1}) \quad (38)$$

and due to this expansion we can rewrite the expression as

$$\Delta_{\theta} \ln p(\theta) = \Delta_{\theta} \ln p(\theta_1) + \sum_{k=2}^N \Delta_{\theta} \ln p(\theta_k | \theta_{k-1}) \quad (39)$$

We have to compute the following

$$E_{\theta} \left\{ -\Delta_{\theta} \ln p(\theta) \right\} \quad (40)$$

The first term is a matrix with only one non-zero element, namely, the entry (1,1) which is equal to

$$\left[ \Delta_{\theta} \ln p(\theta_1) \right]_{1,1} = \frac{\partial^2 \ln p(\theta_1)}{\partial \theta_1^2} \quad (41)$$

The other terms in (eq.39) are matrices with only four non-zero elements, namely, the entries (k-1,k-1), (k-1,k), (k,k-1) and (k,k). Due to the Gaussian nature of the noise, one finds

$$\begin{aligned} \left[ \Delta_{\theta} \ln p(\theta_k | \theta_{k-1}) \right]_{k,k} &= \left[ \Delta_{\theta} \ln p(\theta_k | \theta_{k-1}) \right]_{k-1,k-1} \\ &= \frac{-s}{\sigma_w^2} \end{aligned} \quad (42)$$

$$\begin{aligned} \left[ \Delta_{\theta} \ln p(\theta_k | \theta_{k-1}) \right]_{k,k-1} &= \left[ \Delta_{\theta} \ln p(\theta_k | \theta_{k-1}) \right]_{k-1,k} \\ &= \frac{s}{\sigma_w^2} \end{aligned} \quad (43)$$

So we obtain

$$\begin{aligned} \mathbf{B}^P &= E_{\theta} \left\{ -\Delta_{\theta} \ln p(\theta) \right\} \\ &= \frac{1}{\sigma_w^2/s} \begin{pmatrix} 1 - E_{\theta_1} \left[ \Delta_{\theta} \ln p(\theta_1) \right] & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \end{aligned} \quad (44)$$

We can consider  $E_{\theta_1} \left[ \Delta_{\theta} \ln p(\theta_1) \right] = 0$ . This corresponds to the case of non-informative prior about  $\theta_1$  (see [2]).

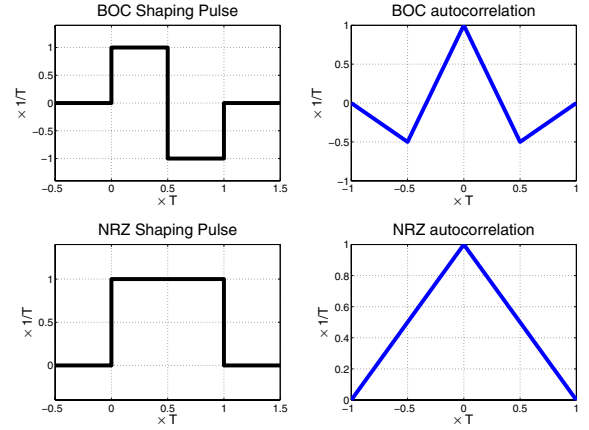


Fig. 1. BOC and NRZ shaping functions  $\Pi(t)$  and their autocorrelation  $g(t)$

Finally we obtain the BCRB matrix as

$$\mathbf{BCRB} = \{\mathbf{B}\}^{-1} = \{\mathbf{B}^D + \mathbf{B}^P\}^{-1} \quad (45)$$

The on-line BCRB associated to observation vector  $\mathbf{y} = [y_1 \cdots y_N]$  is clearly equal to entry  $(N, N)$  of the inverse of the BIM,  $[\mathbf{BCRB}]_{N,N}$ .

#### IV. DISCUSSION

In this section we show the behaviour of the previous bound by considering different scenarios. We assume the transmission of BPSK symbols over an AWGN channel. We consider three oversampling factors ( $s = 1, 2$  and  $4$ ) and two shaping pulses, BOC and NRZ (see figure 1). The NRZ shaping is used in GPS systems and the BOC shaping is used in Galileo.

Equation (45) gives us the bound for a known sequence  $\mathbf{a}$ , so the bound depends on the sequence,  $\mathbf{BCRB}(\mathbf{a})$ . As we want to obtain a lower bound on the estimation error independent from the sequence we take the minimal bound over the set of sequences.

$$\mathbf{BCRB} = \min_{\mathbf{a}} \left\{ [\mathbf{BCRB}(\mathbf{a})]_{N,N} \right\} \quad (46)$$

We obtain the bound by Monte Carlo trials.

In the BCRB presented we plot the Root Mean Square Error (RMSE) versus the Signal to Noise Ratio (SNR). The SNR corresponds to the Carrier to Noise Ratio ( $\frac{C}{N}$ ) in entrance of the receiver. In our case, as shaping pulses and symbols  $a_k$  are normalised (i.e.  $\sigma_a^2 = 1$ ;  $g(0) = 1$ ) this ratio is simply  $\frac{C}{N} = \frac{1}{\sigma_n^2}$ .

Figure 2 and 3 superimpose versus the SNR, the on-line BCRB (see eq.(45)) for both BOC and NRZ shaping pulses and oversampling factors  $s = 1, 2$  and  $4$ . We consider a phase noise with variance  $\sigma_w^2 = 0.01 \text{ rad}^2$  and  $\sigma_w^2 = 0.1 \text{ rad}^2$  respectively, and a null offset  $\tau = 0$ .

First of all we note that the performances are the same for both shaping pulses except for  $s = 4$  in figure 3. One can see that performance increase with the oversampling factor  $s$  and the interest of oversampling becomes clear at low SNR. The gain due to oversampling goes to zero as the SNR goes to  $\infty$ .

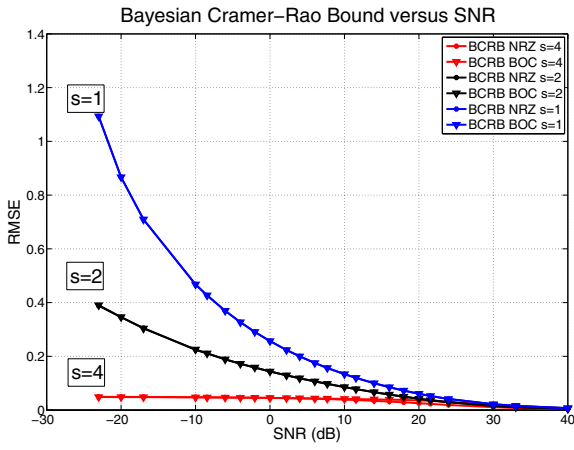


Fig. 2. Bayesian CRBs for three different oversampling factors  $s = 1, 2$  and  $4$ , with a phase-noise variance  $\sigma_w^2 = 0.01 \text{ rad}^2$ . BOC and NRZ shaping

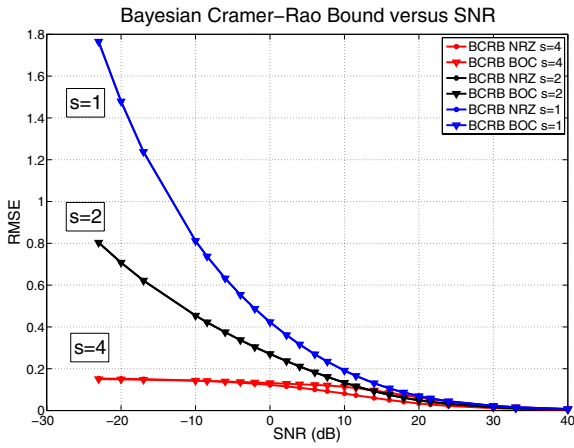


Fig. 3. Bayesian CRBs for three different oversampling factors  $s = 1, 2$  and  $4$ , with a phase-noise variance  $\sigma_w^2 = 0.1 \text{ rad}^2$ . BOC and NRZ shaping

Note that in the  $s = 4$  case, for a fixed SNR and  $\tau = 0$ , the SNR computed after matched filter and sampling, called discrete-time SNR (SNRd), is higher for the NRZ pulse than for the BOC pulse (see Appendix I).

In figure 4 we analyse the bound behaviour for a fixed SNR versus phase-noise variance. We present a scenario with a low SNR value,  $SNR = 0\text{dB}$ . Here we can still measure the gain given by the oversampling.

We have analysed the Bayesian CRB which uses the *a priori* information. But what do we gain using this *a priori* information? We show in figure 5 the difference between a Bayesian CRB and a Standard CRB (SCRb) for a BOC shaping, two different oversampling factors  $s = 1$  and  $2$  and a variance  $\sigma_w^2 = 0.1 \text{ rad}^2$ . We can easily see that at low SNR the BCRB is more accurate than the SCRb.

The SCRb is equal to the inverse of the Fisher Information Matrix (see eq.17).

In figure 6 we show the performance of the SCRb and the BCRB for a variance  $\sigma_w^2 = 0.01 \text{ rad}^2$ . We can see that using

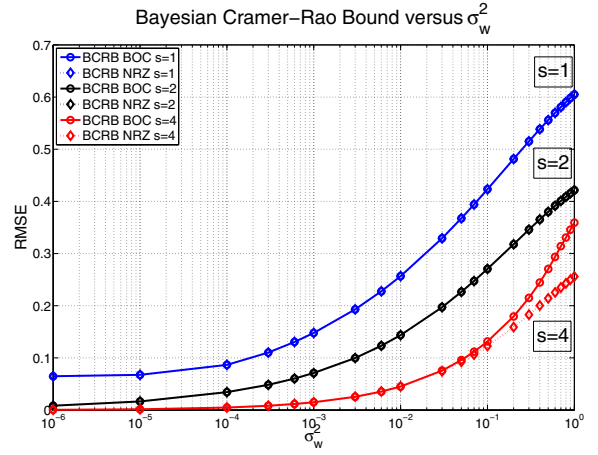


Fig. 4. Bayesian CRBs for three different oversampling factors  $s = 1, 2$  and  $4$ ,  $SNR = 0\text{dB}$ .

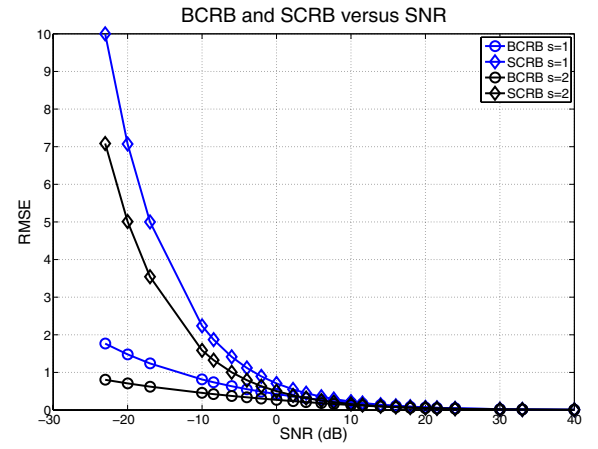


Fig. 5. BCRB and SCRb for  $s = 1$  and  $2$ , BOC shaping and  $\sigma_w^2 = 0.1 \text{ rad}^2$

the *a priori* is more important when having weak phase-noise variances.

The performance of the bound for a known non-null offset  $\tau$  are presented in Appendix II. We can see in this scenario (figure 9) that we have an important loose of performance when  $s = 1$  and  $2$ , otherwise when having  $s = 4$ , the performance are the same as obtained with the optimal offset,  $\tau = 0$ .

## V. CONCLUSION

In this contribution, we have derived an analytical expression of a Bayesian Cramér-Rao Bound for a realistic dynamical carrier phase estimation in an oversampled scenario. This bound is useful when having phase-tracking estimators in an on-line oversampled scenario and looking for a benchmark on the estimator performance. We have presented numerical results with time limited pulses as used in satellite positioning systems. In such scenario, where the Shannon sampling theorem is not respected, we have shown the interest of using a fractionally-spaced method for phase estimation.



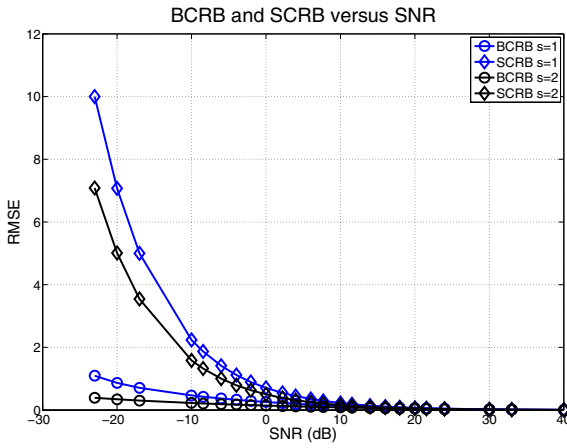


Fig. 6. BCRB and SCRB for  $s = 1$  and  $2$ , BOC shaping and  $\sigma_w^2 = 0.01$  rad<sup>2</sup>

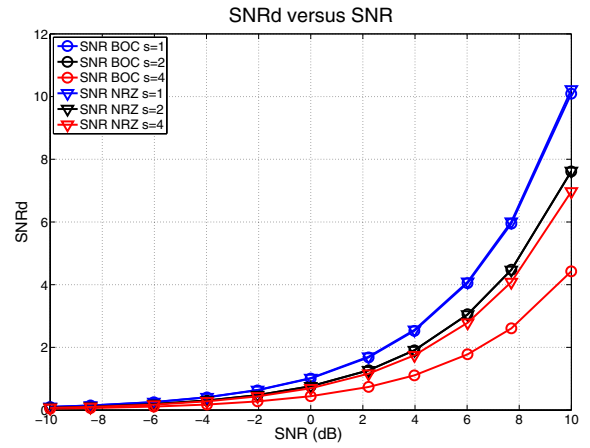


Fig. 7. Discrete-time SNR for different oversampling factors  $s = 1, 2$  and  $4$  and  $\tau = 0$ . BOC and NRZ shaping.

#### ACKNOWLEDGEMENT

This work was supported in part by the French ANR (Agence Nationale de la Recherche), LURGA project.

#### REFERENCES

- [1] H. L. Van Trees, *Detection, Estimation and Modulation Theory*. New York: Wiley, 1968, vol.1.
- [2] S. M. Kay, *Fundamentals of statistical signal processing: estimation theory*. Upper Saddle River, NJ, USA: Prentice Hall, Inc., 1993.
- [3] B. D. O Anderson and J. B. Moore, *Optimal Filtering*. Englewood Cliffs, NJ, USA: Prentice Hall, 1979.
- [4] P. O. Amblard, J. M. Brossier and E. Moisan, "Phase tracking: what do we gain from optimality? Particle filtering versus phase-locked loops", *Elsevier Signal Processing*, vol. 83, pp. 151-167, Oct. 2003.
- [5] S. Bay, C. Herzet, J. M. Brossier, J. P. Barbot, and B. Geller, "Analytic and Asymptotic Analysis of Bayesian Cramér-Rao Bound for Dynamical Phase Offset Estimation", to appear in *IEEE Trans. Signal Processing*.
- [6] S. Bay, C. Herzet, J. M. Brossier, J. P. Barbot, A. Renaux and B. Geller, "Bayesian Cramér-Rao bound for dynamical phase offset estimation", *IEEE 8th Workshop on Signal Processing Advances in Wireless Communications 2007*, Helsinki, 17-20 June 2007.
- [7] P. Tichavský, C. H. Muravchik, A. Nehorai, "Posterior Cramér-Rao bounds for discrete-time nonlinear filtering", *IEEE Trans. Signal Processing*, vol. 46, pp. 1386-1396, May 1998.
- [8] A. N. D'Andrea, U. Mengali and R. Reggiannini, "The modified Cramér-Rao bounds and its applications to synchronization problems", *IEEE Trans. Commun.*, vol. 42, pp. 1391-1399, Feb.-Apr. 1994.
- [9] F. Gini, R. Reggiannini and U. Mengali, "The modified Cramér-Rao bound in vector parameter estimation", *IEEE Trans. Commun.*, vol. 46, pp. 52-60, Jan. 1998.
- [10] B. M. F. Rice, B. Cowley and M. Rice, "Cramér-Rao Lower Bounds for QAM Phase and Frequency Estimation", *IEEE Trans. Commun.*, vol. 49, pp. 1582-1591, Sept. 2001.
- [11] W. G. Cowley, "Phase and frequency estimation for PSK packets: bounds and algorithms", *IEEE Trans. Commun.*, vol. 44, pp. 26-28, Jan. 1996.

#### APPENDIX I: DISCRETE-TIME SNR AFTER MATCHED FILTER AND SAMPLING

The average signal to noise ratio on the discrete-time sequence  $A_k$ , obtained after matched filter and sampling, for a known offset  $\tau$  and oversampling factor  $s$  is

$$SNRd(\tau, s) = \frac{E\{|A_k|^2\}}{\sigma_n^2} \quad (47)$$

$$= \frac{\sigma_a^2}{s\sigma_n^2} \sum_{p=0}^{s-1} [g^2(\tau + pT) + g^2(\tau - (1-p)T)]$$

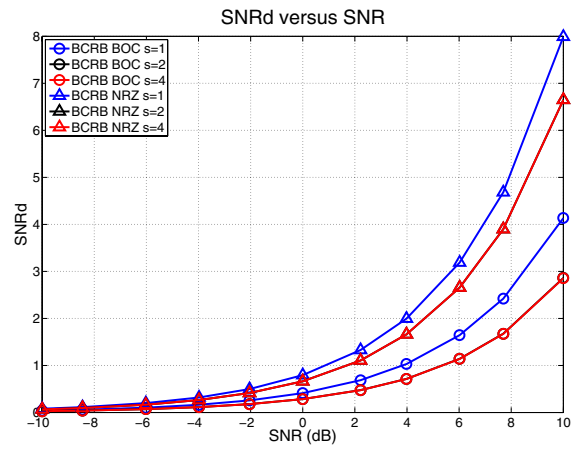


Fig. 8. Discrete-time SNR for different oversampling factors  $s = 1, 2$  and  $4$  and  $\tau = \frac{T}{8}$ . BOC and NRZ shaping.

where  $p$  is defined as in (eq.12). Note that, with  $\tau = 0$ , the mean variance of the cyclostationary samples  $A_k$ ,  $\sigma_A^2$ , is the same for NRZ and BOC pulses if  $s = 1$  and  $s = 2$  but is in the advantage of NRZ if  $s = 4$ . For  $\tau \neq 0$  the mean variance  $\sigma_A^2$  is different for both oversampling factors and shaping pulses, for a given entrance SNR. This influence the different performances obtained when using different shaping pulses.

Figures 7 and 8 present the discrete-time SNR after matched filter versus the SNR for  $\tau = 0$  and  $\tau = \frac{T}{8}$  respectively.

We note that for  $\tau = 0$  and  $s = 1$ , the SNRd is equal to the SNR and that for  $s > 1$ , SNRd < SNR because we use the intermediate samples to compute it.

#### APPENDIX II: PERFORMANCES FOR NON-NULL OVERSAMPLING OFFSET

In figure 9 we superimpose the BCRB for  $\tau = 0$  and  $\tau = \frac{T}{8}$  for three different oversampling factors,  $s = 1, 2$  and  $4$ , and

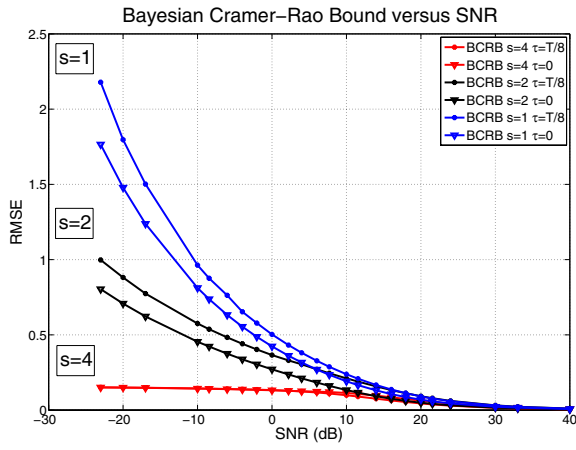


Fig. 9. Bayesian CRBs for three different oversampling factors  $s = 1, 2$  and  $4$ , with oversampling offset  $\tau = 0$  and  $\tau = \frac{T}{8}$ .  $\sigma_w^2 = 0.1 \text{ rad}^2$  and BOC shaping

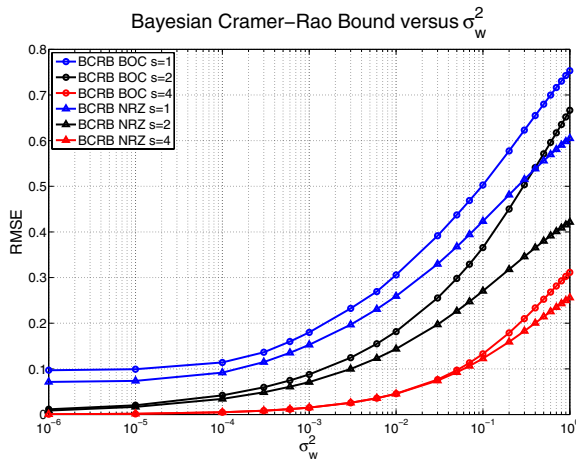


Fig. 10. Bayesian CRBs for three different oversampling factors  $s = 1, 2$  and  $4$ , with oversampling offset  $\tau = \frac{T}{8}$  and SNR = 0dB

BOC shaping. We can measure a loose of performance up to 2dB when having a positive known offset  $\tau$  for the cases  $s = 1$  and  $s = 2$ . We note that for the case  $s = 4$  the offset only affects slightly the bound.

Figure 10 shows the bound for both shaping pulses, a fixed SNR and a non-null offset  $\tau = \frac{T}{8}$  versus phase-noise variance. Comparing figure 10 to figure 4 we can see that for a non-null offset  $\tau$  we obtain different performances for both shaping pulses even with  $s = 1$  and  $2$ , due to the difference between the  $A_k$  values, necessary to compute the bound. We can see in Appendix I that the SNRd obtained are different.