# Inverse conductivity problem on Riemann surfaces 

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#### Abstract

An electrical potential $U$ on a bordered Riemann surface $X$ with conductivity function $\sigma>0$ satisfies equation $d\left(\sigma d^{c} U\right)=0$. The problem of effective reconstruction of $\sigma$ from electrical currents measurements (Dirichlet-to-Neumann mapping) on the boundary: $\left.\left.U\right|_{b X} \mapsto \sigma d^{c} U\right|_{b X}$ is studied. We extend to the case of Riemann surfaces the reconstruction scheme given, firstly, by R.Novikov [N1] for simply connected $X$. We apply for this new kernels for $\bar{\partial}$ on the affine algebraic Riemann surfaces constructed in [H2].


## 0. Introduction

### 0.1. Inverse conductivity problem.

Let $X$ be bordered oriented real two-dimensional manifold in $\mathbb{R}^{3}$ equiped with a smooth symmetric and positive tensor $\hat{\sigma}: T^{*} X \rightarrow T^{*} X$ on cotangent bundle $T^{*} X$, called anisotropic conductivity tensor on $X, \hat{\sigma}$ is called symmetric and positive if $\hat{\sigma} a \wedge b=\hat{\sigma} b \wedge a$ and $\hat{\sigma} a \wedge a>0$ for any $a, b \in T^{*} X$.

Let $u$ (correspondingly $U$ ) be a smooth function on $b X$ (correspondingly on $X$ ) such that $\left.U\right|_{b X}=u$, called electric potential on $b X$ and correspondingly on $X$. 1-form $\hat{\sigma} d U$ on $X$ is called electrical current on $X$. By Maxwell equation

$$
d(\hat{\sigma} d U)=0 \quad \text { on } X
$$

Inverse conductivity problem consists in this case in the following: what kind of information about $X$ and $\hat{\sigma}$ can be efficiently extracted from the knowledge of Dirichlet-to-Neumann mapping

$$
\left.\left.u\right|_{b X} \mapsto \hat{\sigma} d U\right|_{b X} \forall u \in C^{(1)}(b X)
$$

This important problem in the mathematical setting goes back to the inverse boundary values problems posed by I.M.Gelfand [G] and by A.P.Calderon [C].

This real problem is deeply related with complex analysis on Riemann surfaces. The first indication to this relation gives the following statement, obtaind at first by J.Sylvester $[\mathrm{S}]$ for simply connected $X$.

Under conditions above $\forall$ couple $(X, \hat{\sigma})$ there exist a unique complex structure $c$ on $X$ and smooth scalar valued, positive conductivity function $\sigma$ such that the equation $d(\hat{\sigma} d U)=0$ takes the form $d\left(\sigma d^{c} U\right)=0$, where $d^{c}=i(\bar{\partial}-\partial), \sigma^{2}(x)=\operatorname{det} \hat{\sigma}(x), x \in X$.

This statement permits to reduce the inverse conductivity problem to the questions about reconstruction from Dirichlet-to-Neumann mapping of the genus of $X$, of the complex structure of $X$ and of the scalar conductivity function $\sigma$ on $X$.

These questions are well answered for the important case when $X$ is a domain in $\mathbb{R}^{2}$, due to the sequence of works: [F1], [F2], [BC1], [SU], [N3], [N1], [GN], [BC2], [N2], [Na].

The exact reconstruction scheme for this case was discovered by R.Novikov [N1].
Formulated questions are well answered also for the case when conductivity function $\sigma$ is known to be constant on $X$, i.e. when only Riemann surface $X$ must be reconstructed from Dirichlet-to-Neumann data [LU], [Be], [HM].

In this paper we study another important case of this problem, when bordered twodimensional manifold $X$ and complex structure on $X$ are known, but conductivity function $\sigma$ on $X$ must be reconstructed from Dirichlet-to-Neumann data.
0.2. Main results.

We extend here the R.Novikov's reconstruction scheme for the case of bordered Riemann surfaces. Our method (announced firstly in [H1]) is based on the appropriate new kernels for $\bar{\partial}$ on the affine algebraic Riemann surfaces constructed in [H2].

By this reason we use some special embedding of $X$ into $\mathbb{C}^{2}$.
Let $\hat{X}$ be compactification of $X$ such that $\hat{X}=\overline{X \cup X_{0}}$ be compact Riemann surface of genus $g$. Let $A=\left\{A_{1}, \ldots, A_{d}\right\}$ be divisor, generic, effective with support in $X_{0}$, consisting of $d=g+2$ points.

By Riemann-Roch formula there exist three independent functions $f_{0}, f_{1}, f_{2} \in \mathcal{M}(\hat{X}) \cap$ $\mathcal{O}(\hat{X} \backslash A)$ having at most simple poles in the points of $A$. Without restriction of generality one can put $f_{0}=$ const. Let $V$ be algebraic curve in $\mathbb{C}^{2}$ of the form

$$
V=\left\{\left(z_{1}, z_{2}\right) \in \mathbb{C}^{2}: z_{1}=f_{1}(x), z_{2}=f_{2}(x), x \in \hat{X} \backslash A\right\} .
$$

Let $\tilde{V}$ be compactification of $V$ in $\mathbb{C} P^{2}$ of the form $\tilde{V}=\left\{w \in \mathbb{C} P^{2}: \tilde{P}(w)=0\right\}$, where $\tilde{P}$ is homogeneous holomorphic polynomial of homogeneous coordinates $w=\left(w_{0}: w_{1}: w_{2}\right)$. Without loss of generality one can suppose: functions $f_{1}, f_{2}$ are such that
i) $\tilde{V}$ intersects $\mathbb{C} P_{\infty}^{1}=\left\{w \in \mathbb{C} P^{2}: w_{0}=0\right\}$ transversally $\tilde{V} \cap \mathbb{C} P_{\infty}^{1}=\left\{a_{1}, \ldots, a_{d}\right\}$, where points $a_{j}=\left(0,1, \lim _{x \rightarrow A_{j}} \frac{f_{2}(x)}{f_{1}(x)}\right), j=1,2, \ldots, d$ are different points of $\mathbb{C} P_{\infty}^{1}$.
ii) $V=\tilde{V} \backslash \underset{\sim}{\mathbb{P}} P_{\infty}^{1}$ is connected curve in $\mathbb{C}^{2}$ with equation $V=\left\{z \in \mathbb{C}^{2}: P(z)=0\right\}$, where $P(z)=\tilde{P}\left(1, z_{1}, z_{2}\right)$ such that $\left|\frac{\partial P}{\partial z_{1}}\right| \leq \operatorname{const}(V)\left|\frac{\partial P}{\partial z_{2}}\right|$, if $\left|z_{1}\right| \geq r_{0}=\operatorname{const}(V)$.
iii) For any $z^{*} \in V$, where $\frac{\partial P}{\partial z_{2}}\left(z^{*}\right)=0$ we have $\frac{\partial^{2} P}{\partial z_{2}^{2}}\left(z^{*}\right) \neq 0$.

With certain restriction of generality we suppose, in addition, that
iv) curve $V$ is a regular curve, i.e. $\operatorname{grad} P(z) \neq 0 \forall z \in V$. This restriction must be eliminated in other publication.
Let us equip $V$ by euclidean volume form $\left.d d^{c}|z|^{2}\right|_{V}$.
Let $\varphi \mapsto f=\hat{R} \varphi$ be operator for solution of $\bar{\partial} f=\varphi$ on $V$
from [H2], Proposition 2, $f \mapsto u=R_{\lambda} f$ be operator for solution of $\left(\partial+\lambda d z_{1}\right) u=f-\mathcal{H} f$ on $V$, where $\mathcal{H} f$ is projection of $f$ on subspace of holomorphic (1,0)-forms on $\tilde{V}$ from [H2], Proposition 3, $\varphi \in L_{1,1}^{\infty} \cap L_{1,1}^{1}(V), f \in W_{1,0}^{1, \tilde{p}}(V), u \in W^{2, \tilde{p}}(V), \tilde{p}>2$.

Let $g_{\lambda}(z, \xi), z, \xi \in V, \lambda \in \mathbb{C}$ be kernel of operator $R_{\lambda} \circ \hat{R}$ from [H2].
Let $V_{X}=\left\{\left(z_{1}, z_{2}\right) \in V: z_{1}=f_{1}(x), z_{2}=f_{2}(x), x \in X\right\}$.
Let $\sigma$ be conductivity function on $V$ with conditions $\sigma \in C^{(3)}(V), \sigma>0$ on $V$, $\sigma(z(x))=\sigma(x), x \in X, \sigma=$ const on $V \backslash V_{X}$.

Function $\psi(z, \lambda), z \in V, \lambda \in \mathbb{C}$ will be called Faddeev type function on $V \times \mathbb{C}$ if $d d^{c} \psi=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}} \psi$ on $V$ and $\forall \lambda \in \mathbb{C} e^{-\lambda z} \psi(z, \lambda) \stackrel{\text { def }}{=} \mu(z, \lambda) \rightarrow 1, z \rightarrow \infty,|\bar{\partial} \mu|=O\left(\frac{1}{|z|+1}\right)$, $z \in V$.

Theorem Under formulated conditions
I. There exists unique Faddeev type function $\psi(z, \lambda), z \in V, \lambda \in \mathbb{C}$.
II. Function $\psi$ and as a conseqence conductivity function $\sigma$ can be reconstructed through Dirichlet-to-Neumann data by the following procedure.
$I I_{a}$. From Dirichlet-to-Neumann data on $b X$ by Proposition 3.1 (section 3) one can find restriction $\left.\psi\right|_{b V_{X}}$ of the Faddeev type function $\psi(z, \lambda), z \in V, \lambda \in \mathbb{C}$ as a unique solution of the Fredholm integral equation

$$
\left.\psi(z, \lambda)\right|_{b V_{X}}=e^{\lambda z_{1}}+\int_{\xi \in b V_{X}} e^{\lambda\left(z_{1}-\xi_{1}\right)} g_{\lambda}(z, \xi) \cdot\left(\hat{\Phi} \psi(\xi)-\hat{\Phi}_{0} \psi(\xi)\right)
$$

where $\hat{\Phi} \psi=\left.\bar{\partial} \psi\right|_{b V_{X}}, \hat{\Phi}_{0} \psi=\left.\bar{\partial} \psi_{0}\right|_{b V_{X}},\left.d d^{c} \psi_{0}\right|_{V_{X}}=0,\left.\psi_{0}\right|_{b V_{X}}=\left.\psi\right|_{b V_{X}}$.
$I I_{b}$. Using values of $\psi(z, \lambda)$ in arbitrary point $z^{*} \in b V_{X}$ by Proposition 2.2 (section 2) one can find " $\bar{\partial}$ scattering data":

$$
b(\lambda) \stackrel{\text { def }}{=} \lim _{\substack{z \rightarrow \infty \\ z \in V}} \frac{\bar{z}_{1}}{\bar{\lambda}} e^{-\bar{\lambda} \bar{z}_{1}} \frac{\partial \psi}{\partial \bar{z}_{1}}(z, \lambda)=\left(\bar{\psi}\left(z^{*}, \lambda\right)\right)^{-1} \frac{\partial \psi}{\partial \bar{\lambda}}\left(z^{*}, \lambda\right), \quad z^{*} \in b V_{X}
$$

with estimate (2.12).
$I I_{c}$. Using $b(\lambda), \lambda \in \mathbb{C}$ by Proposition 2.3 (section 2) one can find values of

$$
\left.\mu(z, \lambda)\right|_{V_{X}}=\left.\psi(z, \lambda) e^{-\lambda z_{1}}\right|_{V_{X}}, \quad \lambda \in \mathbb{C}
$$

as a unique solution of Fredholm integral equation

$$
\mu(z, \lambda)=1-\frac{1}{2 \pi i} \int_{\xi \in \mathbb{C}} b(\xi) e^{\bar{\xi} \bar{z}_{1}-\xi z_{1}} \overline{\mu(z, \xi)} \frac{d \xi \wedge d \bar{\xi}}{\xi-\lambda}
$$

From equality $d d^{c} \psi=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}} \psi$ on $X$ we find finally $\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}$ on $X$.
Remarks
For the case $V=\mathbb{C}$ the reconstraction scheme I-II for potential $q$ in the Schrödinger equation $-\Delta U+q U=E U$ on $X \subset V$ through the Dirichlet-to-Neumann data on $b X$ was given for the first time by R.Novikov [N1]. However, in [N1] this scheme was rigorously justified only for the case when estimates of the type (2.12) are available, for example, if for $E \neq 0\|q\| \leq \operatorname{const}(E)$. By the additional result of A.Nachman [Na] the estimates of the type (2.12) are valid also if $E=0$ and $q=\frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}}, \sigma>0, \sigma \in C^{(2)}(X)$.

Part $I I_{a}$ in the present paper is completely similar to the related result of [N1] for $V=\mathbb{C}$.

Part $I I_{b}$ of this scheme for $V=\mathbb{C}$ is a consequence of works R.Beals, R.Coifman [BC1], P.Grinevich, S.Novikov [GN] and R.Novikov [N2].

Part $I I_{c}$ of this scheme follows from part $I I_{b}$ and the classical result of I.Vekua [V].

## $\S 1$. Faddeev type function on affine algebraic Riemann surface. Uniqueness and existence

Let $V$ be smooth algebraic curve in $\mathbb{C}^{2}$ defined in introduction, equiped by euclidean volume form $\left.d d^{c}|z|^{2}\right|_{V}$.

Let $V_{0}=\left\{z \in V:\left|z_{1}\right| \leq r_{0}\right\}$, where $r_{0}$ satisfies condition ii) of introduction.

## Definition

Let $q$ be (1,1)-form in $C_{1,1}(\tilde{V})$ with support of $q$ in $V_{0}$. For $\lambda \in \mathbb{C}$ function $z \mapsto \psi(z, \lambda)$, $z \in V$ will be called here the Faddeev type function associated with form (potential) $q$ on $V$ (and zero level of energy $E$ ) if

$$
\begin{equation*}
-d d^{c} \psi+q \psi=0, \quad z \in V \tag{1.1}
\end{equation*}
$$

and function $\mu=e^{-\lambda z_{1}} \psi$ satisfies the properties:

$$
\mu \in C(\tilde{V}), \quad \lim _{\substack{z \rightarrow \infty \\ z \in V}} \mu(z, \lambda)=1 \text { and }|\bar{\partial} \mu|=O\left(\frac{1}{1+|z|}\right), z \in V .
$$

From [F1], [F2], [N2] it follows that in the case $V=C=\left\{z \in \mathbb{C}^{2}: z_{2}=0\right\}$, for almost all $\lambda \in \mathbb{C}$ the Faddeev type function $\psi=e^{\lambda z_{1}} \mu$ exists, unique and satisfies the Faddeev type integral equation

$$
\begin{aligned}
& \mu(z, \lambda)=1+\frac{i}{2} \int_{\xi \in V} g(z-\xi, \lambda) \mu(\xi, \lambda) q(\xi) \text { where } \\
& g(z, \lambda)=\frac{i}{2(2 \pi)^{2}} \int_{w \in \mathbb{C}} \frac{e^{i(w \bar{z}+\bar{w} z)} d w \wedge d \bar{w}}{w(\bar{w}-i \lambda)}
\end{aligned}
$$

is so called the Faddeev-Green function for operator $\mu \mapsto \bar{\partial}\left(\partial+\lambda d z_{1}\right) \mu$ on $V=\mathbb{C}$.
Faddeev type functions $z \mapsto \psi(z, \lambda)$ are especially useful for solutions of inverse scattering or inverse boundary problems for equation (1.1) when such functions exist and unique for any $\lambda \in \mathbb{C}$. It was remarked by P.Grinevich and S.Novikov [GN] (see also [T]) that for some continuous $q$ with compact support in $\mathbb{C}$ even with arbitrary small norm there exists the subset of exceptional $\lambda^{*}$ for which the Faddeev type integral equation is not uniquely solvable.

From R.Novikov's works [N1], [N2] it follows that the Faddeev type functions associated with potential $q$ on $V=\mathbb{C}$ and non-zero level of energy $E$ exist $\forall \lambda \in \mathbb{C}$ if $\|q\| \leq \operatorname{const}(|E|)$.

From R.Beals, R.Coifman works [BC2] develloped by A.Nachman [Na] and R.Brown, G.Uhlmann $[\mathrm{BU}]$ it follows that for any potential $q$ of the form $q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}$, where $\sigma \in$ $C^{2}(\mathbb{C}), \sigma(z) \equiv$ const if $|z| \geq$ const, Faddeev type function $z \mapsto \psi(z, \lambda)$ exists and unique for any $\lambda \in \mathbb{C}$.

Proposition 1.1 below gives the uniqueness of the Faddeev type function on affine algebraic Riemann surface $V$ for potential $q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}$, where $\sigma \in C^{2}(V), \sigma=$ const on $V \backslash V_{X} \subset V \backslash V_{0}$. The proof will be based on the approach going back to [BC2] in the case $V=\mathbb{C}$.

## Proposition 1.1 (Uniqueness)

Let $\sigma$ be positive function belonging to $C^{(2)}(V)$ such that $\sigma \equiv$ const $>0$ on

$$
V \backslash V_{X} \subset V \backslash V_{0}=\cup_{j=1}^{d} V_{j}
$$

where $\left\{V_{j}\right\}$ are connected components of $V \backslash V_{0}$.
Let $\mu \in L^{\infty}(V)$ such that $\frac{\partial \mu}{\partial z_{1}} \in L^{\tilde{p}}(V)$ for some $\tilde{p}>2$ and $\mu$ satisfies equation

$$
\begin{equation*}
\bar{\partial}\left(\partial+\lambda d z_{1}\right) \mu=\frac{i}{2} q \mu \text { where } q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}} \tag{1.2}
\end{equation*}
$$

and for some $j \in\{1,2, \ldots, d\} \mu(z) \rightarrow 0, z \rightarrow \infty, z \in V_{j}$.
Then $\mu \equiv 0$.
Remark
Proposition 1 is still valid if to replace the condition $\frac{\partial \mu}{\partial z_{1}} \in L^{\tilde{p}}(V), \tilde{p}>2$ by the weaker condition $\partial \mu \in L^{\tilde{p}}(V)$.

Lemma 1.1
Put $f=e^{\lambda z_{1}} \mu, f_{1}=\sqrt{\sigma} \frac{\partial f}{\partial z_{1}}, f_{2}=\sqrt{\sigma} \frac{\partial f}{\partial z_{1}}$, where $\mu$ satisfies conditions of Proposition 1.1. Then

$$
\begin{align*}
& d\left(\sigma d^{c} f\right)=0 \quad \text { on } V \quad \text { and }  \tag{1.3}\\
& \frac{\partial f_{1}}{\partial \bar{z}_{1}}=q_{1} f_{2}, \quad \frac{\partial f_{2}}{\partial z_{1}}=\bar{q}_{1} f_{1}, \tag{1.4}
\end{align*}
$$

where $q_{1}=-\frac{\partial \log \sqrt{\sigma}}{\partial z_{1}}$. Besides, $q_{1} \in L^{p}\left(V_{0}\right) \forall p<2, q_{1}=0$ on $V \backslash V_{0}$.
Proof
The property $\left.q_{1}\right|_{V \backslash V_{0}}=0$ follows from the property $\sigma \equiv$ const on $V \backslash V_{0}$. Put $V_{0}^{ \pm}=\left\{z \in V_{0}: \pm\left|\frac{\partial P}{\partial z_{2}}\right| \geq \pm\left|\frac{\partial P}{\partial z_{1}}\right|\right\}$. Put $\tilde{q}_{1}=\frac{\partial \log \sqrt{\sigma}}{\partial z_{2}}$. Then $\left.q_{1}\right|_{V_{0}^{+}} \in C^{(1)}\left(V_{0}^{+}\right)$and $\left.\tilde{q}_{1}\right|_{V_{0}^{-}} \in C^{(1)}\left(V_{0}^{-}\right)$. The identity $\left.q_{1}\right|_{V_{0}^{-}}=\left(\frac{\partial z_{1}}{\partial z_{2}}\right)^{-1} \tilde{q}_{1}$ and property iii) imply that $q_{1} \in L^{p}\left(V_{0}\right) \forall p<2$. Equation (1.3) for $f$ is equivalent to the equation (1.2) for $\mu=e^{-\lambda z_{1}} f$. Equation (1.3) for $f$ means that

$$
\begin{equation*}
\bar{\partial} F_{1}+(\partial \ln \sqrt{\sigma}) \wedge F_{2}=0 \text { and } \partial F_{2}+(\partial \ln \sqrt{\sigma}) \wedge F_{1}=0 \tag{1.5}
\end{equation*}
$$

where $F_{1}=\sqrt{\sigma} \cdot \partial f$ and $F_{2}=\sqrt{\sigma} \cdot \bar{\partial} f$. Using $z_{1}$ as a local coordinate on $V$ we obtain from (1.5) the system (1.4), where $\left.f_{1}=d z_{1}\right\rfloor F_{1}$ and $\left.f_{2}=d \bar{z}_{1}\right\rfloor F_{2}$.

## Lemma 1.2

Put $m_{1}=e^{-\lambda z_{1}} f_{1}$ and $m_{2}=e^{-\lambda z_{1}} f_{2}$, where $f_{1}, f_{2}$ are defined in Lemma 4.1. Then system (1.3) is equivalent to system

$$
\begin{equation*}
\frac{\partial m_{1}}{\partial \bar{z}_{1}}=q_{1} m_{2}, \quad \frac{\partial m_{2}}{\partial z_{1}}+\lambda m_{2}=\bar{q}_{1} m_{1} . \tag{1.6}
\end{equation*}
$$

Besides,

$$
\begin{equation*}
m_{1}=\sqrt{\sigma}\left(\lambda \mu+\frac{\partial \mu}{\partial z_{1}}\right) \text { and } m_{2}=\sqrt{\sigma} \frac{\partial \mu}{\partial \bar{z}_{1}} . \tag{1.7}
\end{equation*}
$$

Proof
Putting in (1.4) $f_{1}=e^{\lambda z_{1}} m_{1}$ and $f_{2}=e^{\lambda z_{1}} m_{2}$, we obtain

$$
\begin{aligned}
& \frac{\partial e^{\lambda z_{1}} m_{1}}{\partial \bar{z}_{1}}=q_{1} e^{\lambda z_{1}} m_{2} \Longleftrightarrow \frac{\partial m_{1}}{\partial \bar{z}_{1}}=q_{1} m_{2} \text { and } \\
& \frac{\partial e^{\lambda z_{1}} m_{2}}{\partial z_{1}}=\bar{q}_{1} e^{\lambda z_{1}} m_{1} \Longleftrightarrow \frac{\partial m_{2}}{\partial z_{1}}+\lambda z_{2}=\bar{q}_{1} m_{1} .
\end{aligned}
$$

Besides,

$$
f_{1}=\sqrt{\sigma} \frac{\partial f}{\partial z_{1}}=\sqrt{\sigma} e^{\lambda z_{1}}\left(\lambda \mu+\frac{\partial \mu}{\partial z_{1}}\right)=e^{\lambda z_{1}} m_{1}
$$

where $m_{1}=\sqrt{\sigma}\left(\lambda \mu+\frac{\partial \mu}{\partial z_{1}}\right)$ and

$$
f_{2}=\sqrt{\sigma} \frac{\partial f}{\partial \bar{z}_{1}}=\sqrt{\sigma} e^{\lambda z_{1}}\left(\frac{\partial \mu}{\partial \bar{z}_{1}}\right)=e^{\lambda z_{1}} m_{2}
$$

where $m_{2}=\sqrt{\sigma} \frac{\partial \mu}{\partial \bar{z}_{1}}$.
Lemma 1.3
Put $u_{ \pm}=m_{1} \pm e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}} \bar{m}_{2}$. Then system (1.4) is equivalent to the system

$$
\frac{\partial u_{ \pm}}{\partial \bar{z}_{1}}= \pm q_{1} e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}} \bar{u}_{ \pm}
$$

Proof
From definition of $u_{ \pm}$, using Lemma 1.2, we obtain

$$
\begin{aligned}
& \frac{\partial u_{ \pm}}{\partial \bar{z}_{1}}=\frac{\partial m_{1}}{\partial \bar{z}_{1}} \pm e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}}\left(\bar{\lambda} \bar{m}_{2}-\bar{\lambda} \bar{m}_{2}+q_{1} \bar{m}_{1}\right)= \\
& q_{1} m_{2} \pm q_{1} e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}} \bar{m}_{1}= \pm q_{1} e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}}\left(\bar{m}_{1} \pm e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}} m_{2}\right)= \\
& \pm e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}} q_{1} \bar{u}_{ \pm} .
\end{aligned}
$$

Proof of Proposition 1.1
Let $u_{ \pm}=m_{1} \pm e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}} \bar{m}_{2}$. We will prove that under conditions of Proposition 1.1 $\forall \lambda \neq 0 u_{ \pm} \equiv 0$ together with $m_{1}$ and $m_{2}$.

From equality $q_{1}=-\frac{\partial \log \sqrt{\sigma}}{\partial z_{1}}$ and Lemma 1.3 we obtain that

$$
\begin{equation*}
\bar{\partial} u_{ \pm}= \pm q_{1} e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}} \bar{u}_{ \pm} d \bar{z}_{1} \in L_{0,1}^{\tilde{p}}(V) \cap L_{0,1}^{1}(V), \quad \tilde{p}>2 \tag{1.8}
\end{equation*}
$$

From (1.6), (1.7) we obtain that $m_{1} \in L^{\tilde{p}}(V) \oplus L^{\infty}(V)$ and $\frac{\partial m_{1}}{\partial \tilde{z}_{1}} \in L^{p}(V), \forall p \geq 1$.

From (1.6), properties $\mu \in L^{\infty}(V), \frac{\partial \mu}{\partial z_{1}} \in L^{\tilde{p}}(V), \mu(z) \rightarrow 0, z \in V_{j}, z \rightarrow \infty$ and from [H2] Corollary 1.1 we deduce that there exists

$$
\begin{equation*}
\lim _{\substack{z \rightarrow \infty \\ z \in V}} m_{1}(z)=\lim _{\substack{z \rightarrow \infty \\ z \in V}} \sqrt{\sigma}\left(\lambda \mu+\frac{\partial \mu}{\partial z_{1}}\right)=\lim _{\substack{z \rightarrow \infty \\ z \in V}} \sqrt{\sigma} \lambda \mu(z)=0 \tag{1.9}
\end{equation*}
$$

From (1.8), (1.9) and generalized Liouville theorem from Rodin $[\mathrm{R}]$, Theorem 7.1, it follows that $u_{ \pm}=0$.

It means, in particular, that $\frac{\partial \mu}{\partial z_{1}}=0$, which together with (1.9) imply $\mu=0$. Proposition 1.1 is proved.

Proposition 1.2 (Existence)
Let conductivity $\sigma$ on $V$ satisfies conditions of Proposition 1.1. Then $\forall \lambda \in \mathbb{C}$ there exists the Faddeev type function $\psi=\mu e^{\lambda z}$ on $V$ associated with potential $q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}$, i.e.

$$
\begin{aligned}
& \text { a) } \bar{\partial}\left(\partial+\lambda d z_{1}\right) \mu=\frac{i}{2} q \mu, \quad \text { where } q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}} \\
& \text { b) } \mu-1 \in W^{1, \tilde{p}}(V) \cap C(\tilde{V}) \quad \forall \tilde{p}>2, \\
& \mu-1 \in C^{(\infty)}\left(V \backslash V_{0}\right) \text { and } \mu-1=O\left(\frac{1}{\left|z_{1}\right|}\right), \quad z \in V \backslash V_{0} .
\end{aligned}
$$

Moreover, $\forall \tilde{p}>2$

$$
\begin{aligned}
& \text { c) }\|\mu-1\|_{L^{\tilde{p}}(V)} \leq \operatorname{const}(V, \sigma, \tilde{p})\left\{\min \left(\frac{1}{\sqrt{|\lambda|}}, \frac{1}{|\lambda|}\right)\right\} \\
& \left\|\frac{\partial \mu}{\partial z_{1}}\right\|_{L^{\tilde{p}}(V)} \leq \operatorname{const}(V, \sigma, \tilde{p})\{\min (\sqrt{|\lambda|}, 1)\}, \quad \forall \tilde{p}>2 \\
& \text { d) } \forall \lambda \exists c \in \mathbb{C} \text { such that }\left(\frac{\partial \mu}{\partial \bar{\lambda}}-c\right) \in W^{1, \tilde{p}}(V) \\
& \text { e) under additional assumption } \sigma \in C^{(3)}(V) \\
& \left\|\frac{\partial^{2} \mu}{\partial z_{1}^{2}}\right\|_{L^{\tilde{p}}(V)} \leq \operatorname{const}(V, \sigma) \cdot \lambda^{1 /(1+1 / \tilde{p})}
\end{aligned}
$$

## Remark

Proposition 1.2 will be proved by approach going back to L.Faddeev [F1], [F2] and for the case $V=\mathbb{C}$ develloped in $[\mathrm{N} 1],[\mathrm{N} 2],[\mathrm{Na}]$.

Proof
Put $G_{\lambda}=R_{\lambda} \circ \hat{R}$, where $\hat{R}$ is operator, defined by formula (2.4) from Proposition 2 of [H2], and $R_{\lambda}$ operator, defined by formula (3.1) from Proposition 3 of [H2]. Let function $\mu(z, \lambda)$ be such that $\forall \lambda, \mu \in W^{1, \tilde{p}}(V) \oplus$ const with respect to $z \in V$. Then $\forall \lambda \neq 0$ we have properties $q \mu \in C(V), q \mu=0$ on $V \backslash V_{0}$, which imply, in particular, that mapping $\mu \mapsto q \mu$ is compact operator from $W^{1, \tilde{p}}(V) \oplus$ const to $L^{\tilde{p}}(V) \cap C(V)$. By Lemmas 2.1, 2.2 and Proposition 2 from [H2] we have $\hat{R} q \mu \in W^{1, \tilde{p}}(V)$. Proposition 3 ii) implies that
$R_{\lambda} \circ \hat{R}(q \mu) \in W^{1, \tilde{p}}(V)$. Hence, mapping $\mu \mapsto \mu-R_{\lambda} \circ \hat{R} \frac{i}{2} q \mu$ is a Fredholm operator on $W^{1, \tilde{p}}(V) \oplus$ const. By Proposition 1.1 the operator $I-R_{\lambda} \circ \hat{R}\left(\frac{i}{2} q \cdot\right)$ is invertible. Then there is unique function $\mu$ such that $(\mu-1) \in W^{1, \tilde{p}}(V), \forall \tilde{p}>2$ and

$$
\begin{equation*}
\mu=1+R_{\lambda} \circ \hat{R}\left(\frac{i}{2} q \mu\right) . \tag{1.10}
\end{equation*}
$$

Let us check now statement a) of Proposition 1.2. From (1.10) and Proposition 3i) from [H2] we obtain

$$
\begin{align*}
& \left(\partial+\lambda d z_{1}\right) \mu=\lambda d z_{1}+\left(\partial+\lambda d z_{1}\right) R_{\lambda} \circ \hat{R}\left(\frac{i}{2} q \mu\right)=  \tag{1.11}\\
& \lambda d z_{1}+\mathcal{H}\left(\hat{R}\left(\frac{i}{2} q \mu\right)\right)+\hat{R}\left(\frac{i}{2} q \mu\right)
\end{align*}
$$

From (1.11) and Proposition 2 of [H2] we obtain

$$
\bar{\partial}\left(\partial+\lambda d z_{1}\right) \mu=\frac{i}{2} q \mu+\bar{\partial}\left(\lambda d z_{1}+\mathcal{H}\left(\hat{R}\left(\frac{i}{2} q \mu\right)\right)=\frac{i}{2} q \mu\right.
$$

where we have used that $\mathcal{H}\left(\hat{R}\left(\frac{i}{2} q \mu\right)\right) \in H_{1,0}(\tilde{V})$.
Property a) is proved.
For proving of properties b), c) it is sufficient to remark that by Proposition 3ii) and 3iii) from [H2] we have property c) and

$$
\left\|\left(1+\left|z_{1}\right|\right)(\mu-1)\right\|_{L^{\infty}(V)} \leq \operatorname{const}(V, \sigma) \frac{1}{\sqrt{|\lambda|}}
$$

To prove property d) let us differentiate (1.10) with respect to $\bar{\lambda}$. We obtain

$$
\begin{aligned}
& \frac{\partial \mu}{\partial \bar{\lambda}}-R_{\lambda} \circ\left(\hat{R}\left(\frac{i}{2} q \frac{\partial \mu}{\partial \bar{\lambda}}\right)\right)=\bar{z}_{1}\left(R_{\lambda} \circ\left(\hat{R}\left(\frac{i}{2} q \mu\right)\right)\right)-R_{\lambda}\left(\bar{\xi}_{1} \hat{R}\left(\frac{i}{2} q \mu\right)\right)= \\
& \bar{z}_{1}(\mu-1)-R_{\lambda}\left(\bar{\xi}_{1} \hat{R}\left(\frac{i}{2} q \mu\right)\right)
\end{aligned}
$$

From Proposition 2 of [H2] we deduce that

$$
\bar{\xi}_{1} \hat{R}\left(\frac{i}{2} q \mu\right) \in W_{1,0}^{1, \tilde{p}}(V) \oplus(\text { const }) d z_{1} .
$$

From Proposition 3ii) of [H2] and Remark 1 to it we obtain

$$
R_{\lambda}\left(\bar{\xi}_{1} \hat{R}\left(\frac{i}{2} q \mu\right)\right) \in W_{1,0}^{1, \tilde{p}}(V) \oplus \text { const. }
$$

From Proposition 1.2 b) we deduce also

$$
\bar{z}_{1}(\mu-1) \in W^{1, \tilde{p}}(V) \oplus \text { const. }
$$

Hence,

$$
\frac{\partial \mu}{\partial \bar{\lambda}}=\left(I-R_{\lambda} \circ \hat{R}\left(\frac{i}{2} q \cdot\right)\right)^{-1}\left(\bar{z}_{1}(\mu-1)-R_{\lambda}\left(\bar{\xi}_{1} \hat{R}\left(\frac{i}{2} q \mu\right)\right) \in W^{1, \tilde{p}}(V) \oplus(\text { const }) .\right.
$$

Property e) follows (under condition $\sigma \in C^{(3)}(V)$ ) from Proposition 3iv) of [H2].
§2. Equation $\frac{\partial \mu(z, \lambda)}{\partial \lambda}=b(\lambda) e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{\mu}(z, \lambda), \lambda \in \mathbb{C}$
For further results it is important to obtain asymptotic development of $\frac{\partial \mu}{\partial \bar{z}_{1}}(z, \lambda)$ for $z_{1} \rightarrow \infty$.

## Proposition 2.1

Let conductivity function $\sigma$ on $V$ satisfy the conditions of Proposition 1.1. Let function $\psi(z, \lambda)=\mu(z, \lambda) e^{\lambda z}$ be the Faddeev type function on $V$ associated with potential $q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}$. Then $\forall \lambda \in \mathbb{C}$ function $\frac{\partial \mu(z, \lambda)}{\partial \bar{z}_{1}}$ has the following asymptotic for $z_{1} \rightarrow \infty$ :

$$
\begin{equation*}
\left.e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}} \frac{\partial \mu}{\partial \bar{z}_{1}}\right|_{V_{j}}=\frac{B_{1, j}(\lambda)}{\bar{z}_{1}}+\sum_{k=2}^{\infty} \frac{B_{k, j}(\lambda)}{\bar{z}_{1}^{k}}, \tag{2.1}
\end{equation*}
$$

where $\left|B_{1, j}(\lambda)\right|=O(\min (1, \sqrt{|\lambda|})), j=1,2, \ldots, d$ and
under additional condition $\sigma \in C^{(3)}(V)$ we have

$$
\begin{equation*}
\left.e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}} \frac{\partial^{2} \mu}{\partial \bar{z}_{1}^{2}}\right|_{V_{j}}=\bar{\lambda} \frac{B_{1, j}}{\bar{z}_{1}}+\sum_{k=2}^{\infty} \frac{\bar{\lambda} B_{k, j}(\lambda)-(k-1) B_{k-1, j}(\lambda)}{\bar{z}_{1}^{k}}, \tag{2.2}
\end{equation*}
$$

where $\left|B_{1, j}(\lambda)\right|=O\left(\min \left(\sqrt{|\lambda|},|\lambda|^{-1 /(1+\tilde{p})}\right)\right), j=1,2, \ldots, d \forall \tilde{p}>2$.
For proving of Proposition 2.1 we need the following decomposition statement for the Faddeev type function $\mu=\psi e^{-\lambda z}$ on $V \backslash V_{0}$.

## Lemma 2.1

i) Let $\mu$ be function on $V \backslash V_{0}$, which satisfies equation

$$
\begin{equation*}
\bar{\partial}\left(\partial+\lambda d z_{1}\right) \mu=0 \text { on } V \backslash V_{0} \tag{2.3}
\end{equation*}
$$

and the property

$$
\left.(\mu-1)\right|_{V \backslash V_{0}} \in W^{1, \tilde{p}}\left(V \backslash V_{0}\right), \forall \tilde{p}>2 .
$$

Then

$$
\begin{align*}
& A \stackrel{\text { def }}{=} \frac{\partial \mu}{\partial z_{1}}+\lambda \mu \in \mathcal{O}\left(\tilde{V} \backslash V_{0}\right) ;\left.\quad A\right|_{V_{j}}=\lambda+\sum_{k=1}^{\infty} A_{k, j} \frac{1}{z_{1}^{k}} \\
& B \stackrel{\text { def }}{=} e^{-\lambda z_{1}+\bar{\lambda} \bar{z}_{1}} \frac{\overline{\partial \mu}}{\partial \bar{z}_{1}}=\mathcal{O}\left(\tilde{V} \backslash V_{0}\right) ;\left.\quad \bar{B}\right|_{V_{j}}=\sum_{k=1}^{\infty} B_{k, j} \frac{1}{\bar{z}_{1}^{k}} \tag{2.4}
\end{align*}
$$

$\left|z_{1}\right|>r_{0}, j=1,2, \ldots, d$.
ii) Let

$$
\left.M\right|_{V_{j}}=1+\sum_{k=1}^{\infty} \frac{a_{k, j}}{z_{1}^{k}} \text { and }\left.\bar{N}\right|_{V_{j}}=1+\sum_{k=1}^{\infty} \frac{b_{k, j}}{\bar{z}_{1}^{k}}
$$

be formal series with coefficients $a_{k, j}$ and $b_{k, j}$ determined by relations

$$
\begin{aligned}
& \lambda a_{k, j}-(k-1) a_{k-1, j}=A_{k, j} \\
& \bar{\lambda} b_{k, j}-(k-1) b_{k-1, j}=B_{k, j} ; j=1,2 \ldots, d ; k=1,2, \ldots
\end{aligned}
$$

Then function $\mu$ has asymptotic decomposition

$$
\begin{align*}
& \left.\mu\right|_{V_{j}}=\left.M\right|_{V_{j}}+\left.e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{N}\right|_{V_{j}}, \quad z_{1} \rightarrow \infty, \text { i.e. }  \tag{2.5}\\
& \left.\mu\right|_{V_{j}}=\left.M_{\nu}\right|_{V_{j}}+\left.e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{N}_{\nu}\right|_{V_{j}}+O\left(\frac{1}{\left|z_{1}\right|^{\nu+1}}\right), \text { where } \\
& \left.M_{\nu=1}\right|_{V_{j}}=1+\sum_{k=1}^{\nu} \frac{a_{k, j}}{z_{1}^{k}},\left.\quad \bar{N}_{\nu}\right|_{V_{j}}=\sum_{k=1}^{\nu} \frac{b_{k, j}}{\bar{z}_{1}^{k}} .
\end{align*}
$$

iii) Moreover, for any $A \in \mathcal{O}\left(\tilde{V} \backslash V_{0}\right), A(z) \rightarrow \lambda, z \rightarrow \infty$, there exist $B \in \mathcal{O}\left(\tilde{V} \backslash V_{0}\right)$, $B(z) \rightarrow 0, z \rightarrow \infty$, and $\mu$ such that $\mu$ satisfies (2.3), (2.4) and $(\mu-1) \in W^{1, \tilde{p}}\left(V \backslash V_{0}\right)$.

## Proof

i) From equation (2.3) it follows that

$$
\left.\partial \bar{\partial}\left(e^{\lambda z_{1}} \mu(z, \lambda)\right)\right|_{V \backslash V_{0}}=0
$$

It means that $\bar{\partial}\left(e^{\lambda z_{1}} \mu(z, \lambda)\right)=e^{\lambda z_{1}} \bar{\partial} \mu$ is antiholomorphic form on $V \backslash V_{0}$ and $\left(\partial \mu+\lambda \mu d z_{1}\right)$ is holomorphic form on $V \backslash V_{0}$. From this, condition $\bar{\partial} \mu \in L_{0,1}^{\tilde{p}}\left(V \backslash V_{0}\right)$ and the Cauchy theorem it follows that

$$
\begin{aligned}
& \left.e^{\lambda z_{1}} \bar{\partial} \mu\right|_{V_{j}}=\left.e^{\bar{\lambda} \bar{z}_{1}} \bar{B} d \bar{z}_{1}\right|_{V_{j}}=\left.e^{\bar{\lambda} \bar{z}_{1}} \sum_{k=1}^{\infty} \frac{B_{k, j}}{\bar{z}_{1}^{k}} d \bar{z}_{1}\right|_{V_{j}} \\
& \text { and }\left.\quad\left(\partial \mu+\lambda \mu d z_{1}\right)\right|_{V_{j}}=\left.A d z_{1}\right|_{V_{j}}=\left.\left(\lambda+\sum_{k=1}^{\infty} \frac{A_{k, j}}{z_{1}^{k}}\right) d z_{1}\right|_{V_{j}} .
\end{aligned}
$$

ii) From (2.4), (2.5) we obtain, at first, that

$$
\bar{\partial} \mu=e^{-\lambda z_{1}} \bar{\partial}\left(e^{\overline{z_{1}}} \bar{N}_{\nu}\right)+O\left(\frac{1}{\left|\bar{z}_{1}\right|^{\nu+1}}\right)
$$

and then

$$
\begin{equation*}
\mu=M_{\nu}+e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{N}_{\nu}+O\left(\frac{1}{\left|z_{1}\right|^{\nu+1}}\right) . \tag{2.6}
\end{equation*}
$$

iii) Let

$$
\left.A\right|_{V_{j}}=\lambda+\sum_{k=1}^{\infty} \frac{A_{k, j}}{z_{1}^{k}} \in L^{\tilde{p}}\left(V \backslash V_{0}\right)
$$

Proposition 3 iii) from [H2], applied to ( 1,0 ) forms on the complex plan with coordinate $z_{1}$, gives that $\forall \lambda \neq 0$ there exists $\mu \in W^{1, \tilde{p}}\left(V \backslash V_{0}\right) \oplus 1$ such that $A=\frac{\partial \mu}{\partial z_{1}}+\lambda \mu$. Put $\bar{B}=e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}} \frac{\partial \mu}{\partial \bar{z}_{1}}$. Then $\bar{B} \in L^{\tilde{p}}\left(V \backslash V_{0}\right)$ and

$$
\frac{\partial \bar{B}}{\partial z_{1}}=e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}}\left(\frac{\partial}{\partial \bar{z}_{1}}\left(\lambda \mu+\frac{\partial \mu}{\partial z_{1}}\right)\right)=e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}}\left(\frac{\partial A}{\partial \bar{z}_{1}}\right)=0
$$

i.e. $B \in \mathcal{O}\left(\tilde{V} \backslash V_{0}\right)$. By construction $\mu$ satisfies (2.3), (2.4) and $(\mu-1) \in W^{1, \tilde{p}}$.

## Lemma 2.2

Functions $M_{\nu}$ and $N_{\nu}$ from decomposition (2.6) have the following properties:

$$
\begin{aligned}
& \forall z \in \tilde{V} \backslash V_{0} \quad \exists \lim _{\nu \rightarrow \infty}\left(\frac{\partial M_{\nu}}{\partial z_{1}}+\lambda M_{\nu}\right) \stackrel{\text { def }}{=} \frac{\partial M}{\partial z_{1}}+\lambda M \text { and } \\
& \exists \lim _{\nu \rightarrow \infty}\left(\frac{\partial N_{\nu}}{\partial z_{1}}+\lambda N_{\nu}\right) \stackrel{\text { def }}{=} \frac{\partial N}{\partial z_{1}}+\lambda N .
\end{aligned}
$$

Functions $\frac{\partial M}{\partial z_{1}}+\lambda M$ and $\frac{\partial N}{\partial z_{1}}+\lambda N$ belongs to to $\mathcal{O}\left(\tilde{V} \backslash V_{0}\right)$ and

$$
\begin{align*}
& \frac{\partial \mu}{\partial \bar{z}_{1}}=e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}}\left(\frac{\partial \bar{N}}{\partial \bar{z}_{1}}+\bar{\lambda} \bar{N}\right),  \tag{2.7}\\
& \frac{\partial \mu}{\partial z_{1}}+\lambda \mu=\frac{\partial M}{\partial z_{1}}+\lambda M  \tag{2.8}\\
& \frac{\partial N}{\partial z_{1}}+\lambda N \rightarrow 0, \text { if } \quad z_{1} \rightarrow \infty
\end{align*}
$$

## Proof

Let us show that (2.4) implies (2.7) and (2.8). Indeed,

$$
\begin{aligned}
& \frac{\partial \mu}{\partial \bar{z}_{1}}=\lim _{\nu \rightarrow \infty}\left(\bar{\lambda} e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{N}_{\nu}+e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \frac{\partial \bar{N}_{\nu}}{\partial \bar{z}_{1}}\right)=e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \lim _{\nu \rightarrow \infty}\left(\frac{\partial \bar{N}_{\nu}}{\partial \bar{z}_{1}}+\bar{\lambda} \bar{N}_{\nu}\right)= \\
& e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}}\left(\frac{\partial \bar{N}}{\partial \bar{z}_{1}}+\bar{\lambda} \bar{N}\right), \\
& \frac{\partial \mu}{\partial z_{1}}+\lambda \mu=\lim _{\nu \rightarrow \infty}\left(\frac{\partial M_{\nu}}{\partial z_{1}}-\lambda e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{N}_{\nu}+\lambda M_{\nu}+\lambda e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{N}_{\nu}\right)= \\
& \lim _{\nu \rightarrow \infty}\left(\frac{\partial \bar{M}_{\nu}}{\partial z_{1}}+\lambda M_{\nu}\right)=\frac{\partial M}{\partial z_{1}}+\lambda M .
\end{aligned}
$$

Properties (2.6), (2.7), (2.8), $\bar{\partial} \mu \in L_{0,1}^{\tilde{p}}(V)$ and Riemann extension theorem imply that $\frac{\partial M}{\partial z_{1}}+\lambda M$ and $\frac{\partial N}{\partial z_{1}}+\lambda N$ belongs to $\mathcal{O}\left(\tilde{V} \backslash V_{0}\right)$ and $\frac{\partial N}{\partial z_{1}}+\lambda N \rightarrow 0$, if $z_{1} \rightarrow \infty$.

## Corollary

In conditions of Lemmas 2.1, 2.2 we have convergence $M_{\nu} \rightarrow M$ and $N_{\nu} \rightarrow N, \nu \rightarrow \infty$, in general, only in the space of formal series, in spite of that convergence
$\frac{\partial M_{\nu}}{\partial z_{1}}+\lambda M_{\nu} \rightarrow \frac{\partial M}{\partial z_{1}}+\lambda M$ and $\frac{\partial N_{\nu}}{\partial z_{1}}+\lambda N_{\nu} \rightarrow \frac{\partial N}{\partial z_{1}}+\lambda N$ take place in the space $\mathcal{O}\left(\tilde{V} \backslash V_{0}\right)$.
Simple example
Put $V_{0}=\left\{z \in V:\left|z_{1}\right|<1\right\}, \lambda=1, A=\lambda+\sum_{k=1}^{\infty} \frac{A_{k}}{z_{1}^{k}}$ with $A_{k}=1, k=1,2, \ldots$
By Lemma 2.1 there exists $\mu=M+e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{N}$, which satisfies (2.3), (2.4), where $M=1+\sum_{k=1}^{\infty} \frac{a_{k}}{z_{1}^{k}}$ with $a_{k}$ determined by relations $a_{k}-(k-1) a_{k-1}=A_{k}=1 ; k=1,2, \ldots$. It gives $a_{k}=(k-1)!\left(1+\frac{1}{2!}+\ldots+\frac{1}{(k-1)!}\right)$. We have $\left|a_{k}\right|^{1 / k} \rightarrow \infty, k \rightarrow \infty$, i.e. radius of convergence of serie for $M$ is equal to zero, in spite of that $\left|A_{k}\right|^{1 / k}=1, k=1,2, \ldots$.

Proof of Proposition 2.1
Estimate for $\frac{\partial \mu}{\partial z_{1}}$ from Proposition 1.2 c) and the Cauchy theorem applied to antiholomorphic function $e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}} \frac{\partial \mu}{\partial \bar{z}_{1}}$ implies development (2.1) and estimate

$$
\begin{equation*}
\left|B_{1, j}(\lambda)\right|=O(\min (1, \sqrt{|\lambda|})) . \tag{2.9}
\end{equation*}
$$

Estimate for $\frac{\partial^{2} \mu}{\partial \bar{z}_{1}^{2}}$ from Proposition 1.2 e) and the Cauchy theorem for antiholomorphic in $z_{1}$ function $\left.e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}} \frac{\partial^{2} \mu}{\partial \bar{z}_{1}^{2}}\right|_{V_{j}}$ imply development (2.2) and estimate

$$
\begin{equation*}
\left|B_{1, j}(\lambda)\right|=O\left(\min \left(\sqrt{|\lambda|},|\lambda|^{-1 /(1+\tilde{p})}\right)\right) . \tag{2.10}
\end{equation*}
$$

Proposition 2.1 is proved.
The next proposition gives $\bar{\partial}$-equation on Faddeev function $\mu(z, \lambda)$ with respect to parameter $\lambda \in \mathbb{C}$. For the case $V=\mathbb{C}$ this proposition goes back to Beals, Coifmann [BC1], Grinevich, S.Novikov [GN] and R.Novikov [N2].

## Proposition 2.2

Let conductivity function $\sigma$ on $V$ satisfy the conditions of Proposition 1.1. Let function $\psi(z, \lambda)=\mu(z, \lambda) e^{\lambda z}$ be the Faddeev type function on $V$ constructed in Proposition 1.2 and associated with potential $q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}$. Then $\forall z \in V$ the following $\bar{\partial}$-equation with respect to $\lambda \in \mathbb{C}$ takes place

$$
\begin{equation*}
\frac{\partial \mu}{\partial \bar{\lambda}}=b(\lambda) e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \bar{\mu}, \tag{2.11}
\end{equation*}
$$

where

$$
\begin{align*}
& b(\lambda) \stackrel{\text { def }}{=} \lim _{z \rightarrow \infty} \frac{\bar{z}_{1}}{\bar{\lambda}} e^{\lambda z_{1}-\bar{\lambda} \bar{z}_{1}} \frac{\partial \mu}{\partial \bar{z}_{1}}, \\
& |b(\lambda)| \leq \operatorname{const}(V, \sigma)\left\{\min \left(\frac{1}{\sqrt{|\lambda|}}, \frac{1}{|\lambda|}\right)\right\} \text { if } \sigma \in C^{(2)}(V) \text { and }  \tag{2.12}\\
& |b(\lambda)| \leq \operatorname{const}(V, \sigma, \tilde{p})\left\{\min \left(\frac{1}{\sqrt{|\lambda|}},\left(\frac{1}{|\lambda|}\right)^{1+1 / \tilde{p}}\right)\right\} \text { if } \sigma \in C^{(3)}(V), \forall \tilde{p}>2 .
\end{align*}
$$

## Remark

The proof below is a generalization of the R.Novikov proof [N2] of the corresponding statement for the case $V=\mathbb{C}$.

Proof
Equation $d d^{c} \psi=q \psi$ for the Faddeev type function $\psi=\mu \cdot e^{\lambda z}$ is equivalent to the equation $\bar{\partial}\left(\partial+\lambda d z_{1}\right) \mu=\frac{i}{2} q \mu$. Put $\psi_{\lambda}=\partial \psi / \partial \bar{\lambda}$ and $\mu_{\lambda}=\partial \mu / \partial \bar{\lambda}$. By Proposition 1.2 d$)$ $\forall \lambda \in \mathbb{C}$ function $\mu_{\lambda}$ belongs to $W^{1, \tilde{p}}(V) \oplus \operatorname{const}(\lambda)$. Besides, from equation $d d^{c} \psi=q \psi$ it follows the equation

$$
d d^{c} \psi_{\lambda}=q \psi_{\lambda} \quad \text { on } \quad V .
$$

From Lemmas 2.1, 2.2, Proposition 2.1 and Proposition 1.2 b),c) we obtain

$$
\begin{align*}
& \left.\frac{\partial \mu}{\partial \bar{z}_{1}}\right|_{V_{j}}=e^{\overline{\bar{z}} \bar{z}_{1}-\lambda z_{1}} \frac{B_{1, j}(\lambda)}{\bar{z}_{1}}+O\left(\frac{1}{\left|z_{1}\right|^{2}}\right) \text { and } \\
& \left.\left(\frac{\partial \mu}{\partial z_{1}}+\lambda \mu\right)\right|_{V_{j}}=\lambda+\frac{A_{1, j}(\lambda)}{z_{1}}+O\left(\frac{1}{\left|z_{1}\right|^{2}}\right) \tag{2.13}
\end{align*}
$$

From (2.4)-(2.6) and (2.13) we deduce that

$$
\begin{align*}
& \mu=1+\frac{a_{j}(\lambda)}{z_{1}}+e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}} \frac{b_{j}(\lambda)}{\bar{z}_{1}}+O\left(\frac{1}{\left|z_{1}\right|^{2}}\right),  \tag{2.14}\\
& \text { where } \bar{\lambda} b_{j}(\lambda)=B_{1, j}, \quad \lambda a_{j}(\lambda)=A_{1, j}, \quad j=1,2, \ldots, d \tag{2.15}
\end{align*}
$$

From (2.14) with help of Proposition 1.2 d) we obtain

$$
\begin{aligned}
& \psi=e^{\lambda z_{1}} \mu=e^{\lambda z_{1}}\left(1+\frac{a_{j}(\lambda)}{z_{1}}+e^{\overline{\bar{\lambda}} \bar{z}_{1}-\lambda z_{1}} \frac{b_{j}(\lambda)}{\bar{z}_{1}}+O\left(\frac{1}{\left|z_{1}\right|^{2}}\right)\right), \\
& \text { and } \psi_{\lambda}=\frac{\partial \psi}{\partial \bar{\lambda}}=e^{\overline{\bar{\lambda} \bar{z}_{1}}}\left(b_{j}(\lambda)+O\left(\frac{1}{\left|z_{1}\right|}\right)\right), \quad z \in V_{j} .
\end{aligned}
$$

Put $\mu_{\lambda}=e^{-\lambda z_{1}} \psi_{\lambda}$. We obtain $\bar{\partial}\left(\partial+\lambda d z_{1}\right) \mu_{\lambda}=q \mu_{\lambda}$ and

$$
\mu_{\lambda}=e^{\bar{\lambda} \bar{z}_{1}-\lambda z_{1}}\left(b_{j}+O\left(\frac{1}{\left|z_{1}\right|}\right)\right), \quad z \in V_{j} .
$$

Proposition 1.1 about uniqueness of the Faddeev type function implies equality (2.11), where

$$
\begin{equation*}
b(\lambda) \stackrel{\text { def }}{=} b_{1}(\lambda)=\ldots=b_{d}(\lambda) \tag{2.16}
\end{equation*}
$$

We have also equalities

$$
B_{1,1}(\lambda)=\ldots=B_{1, d}(\lambda) .
$$

Put

$$
B(\lambda) \stackrel{\text { def }}{=} B_{1,1}(\lambda)=\ldots=B_{1, d}(\lambda)
$$

Estimates (2.12) follow from equalities (2.15), (2.16) and inequalities (2.9), (2.10).

## Proposition 2.3

Let, under conditions of Proposition 2.2, conductivity function $\sigma \in C^{(3)}(V)$. Then $\forall z \in V$ function $\lambda \mapsto \mu(z, \lambda), \lambda \in \mathbb{C}$, is a unique solution of the following integral equation

$$
\begin{equation*}
\mu(z, \lambda)=1-\frac{1}{2 \pi i} \int_{\xi \in \mathbb{C}} b(\xi) e^{\bar{\xi} \bar{z}_{1}-\xi z_{1}} \overline{\mu(z, \xi)} \frac{d \xi \wedge d \bar{\xi}}{\xi-\lambda} \tag{2.17}
\end{equation*}
$$

where function $\lambda \mapsto b(\lambda)$ from (2.11) belongs to $L^{q}(\mathbb{C}) \forall q \in(4 / 3,4)$.

## Proof

Estimates (2.12) imply that $\forall q \in(4 / 3,4)$ function $b(\lambda)$ from Proposition 2.2 belongs to $L^{q}(\mathbb{C})$. By the classical Vekua result [Ve] with such functions $b(\lambda)$ the equation (2.17) is the uniquely solvable Fredholm integral equation in the space $C(\overline{\mathbb{C}})$.
§3. Reconstruction of Faddeev type function $\psi$ and of conductivity $\sigma$ on $X$ through Dirichlet to Neumann data on $b X$

Let $X$ be domain with smooth (de class $C^{(2)}$ ) boundary on $V$ such that $X \supseteq \bar{V}_{0}$. Let $\sigma \in C^{(3)}(V), \sigma>0$ on $V$ and $\sigma \equiv$ const on $V \backslash X$. Let $q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}$, then $q \in C_{1,1}^{(1)}(X)$ and supp $q \subseteq X$.

Definition (Dirichlet-Neumann operator)
Let $u \in C^{(1)}(b X)$ and $U \in W^{1, \tilde{p}}(X), \tilde{p}>2$ be solution of the Dirichlet problem $\left.d \sigma d^{c} U\right|_{X}=0,\left.U\right|_{b X}=u$, where $d^{c}=i(\bar{\partial}-\partial)$. Operator $\left.\left.u\right|_{b X} \rightarrow \sigma d^{c} U\right|_{b X}$ is called usually by Dirichlet to Neumann operator. Put $\tilde{\psi}=\sqrt{\sigma} U$ and $\psi=\sqrt{\sigma} u$. Then

$$
\begin{equation*}
d d^{c} \tilde{\psi}=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}} \tilde{\psi}=q \tilde{\psi} \text { on } X \tag{3.1}
\end{equation*}
$$

Operator $\left.\left.\psi\right|_{b X} \mapsto \bar{\partial} \tilde{\psi}\right|_{b X}$ will be called here by Dirichlet-Neumann operator. The data contained in operator $\left.\left.u\right|_{b X} \rightarrow \sigma d^{c} U\right|_{b X}$ and in $\left.\left.\psi\right|_{b X} \rightarrow \bar{\partial} \tilde{\psi}\right|_{b X}$ are equivalent, but for our further statements operator $\left.\left.\psi\right|_{b X} \rightarrow \bar{\partial} \tilde{\psi}\right|_{b X}$ is more convenient.

Let $\psi_{0}$ be solution of Dirichlet problem

$$
d d^{c} \psi_{0}=0,\left.\quad \psi_{0}\right|_{b X}=\left.\psi\right|_{b X}
$$

Put

$$
\begin{equation*}
\hat{\Phi} \psi=\left.\bar{\partial} \tilde{\psi}\right|_{b X} \text { and } \hat{\Phi}_{0} \psi=\left.\bar{\partial} \tilde{\psi}_{0}\right|_{b X} \tag{3.2}
\end{equation*}
$$

where operator $\left.\psi\right|_{b X} \rightarrow \hat{\Phi}_{0} \psi$ is Dirichlet-Neumann operator for equation (3.1) with potential $q \equiv 0$.

Proposition 3.1 (Reconstruction of $\left.\psi\right|_{b X}$ through Dirichlet-Neumann data)
Let $\psi=\mu(z, \lambda) \cdot e^{\lambda z_{1}}$ be the Faddeev function associated with potential $q=\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}$. Then $\forall \lambda \in \mathbb{C} \backslash\{0\}$ the restriction $\left.\psi\right|_{b X}$ of $\psi$ on $b X$ is a unique solution in $C(b X)$ of the Fredholm integral equation:

$$
\begin{equation*}
\left.\psi(z, \lambda)\right|_{b X}=e^{\lambda z_{1}}-\int_{\xi \in b X} e^{\lambda\left(z_{1}-\xi_{1}\right)} g_{\lambda}(z, \xi) \cdot\left(\hat{\Phi} \psi(\xi)-\hat{\Phi}_{0} \psi(\xi)\right) \tag{3.3}
\end{equation*}
$$

where $g_{\lambda}(z, \xi)$ - kernel of operator $R_{\lambda} \circ \hat{R}$,

$$
\hat{\Phi} \psi(\xi)-\hat{\Phi}_{0} \psi(\xi)=\int_{w \in b X}\left(\Phi(\xi, w)-\Phi_{0}(\xi, w)\right) \psi(w)
$$

$\Phi(\xi, w), \Phi_{0}(\xi, w)$ are kernels of operators $\hat{\Phi}$ and $\hat{\Phi}_{0}$.

## Remark

This proposition for the case $V=\mathbb{C}$ coincide with the second part of Theorem 1 from [N1].

Lemma 3.1 (Green-Riemann formula)
$\forall f, g \in C^{(1)}(X)$ we have equality

$$
\int_{X} g \wedge \partial \bar{\partial} f-\int_{X} f \wedge \partial \bar{\partial} g=\int_{b X} g \wedge \bar{\partial} f+\int_{b X} f \wedge \partial g
$$

Proof

$$
\begin{aligned}
& \int_{X} g \wedge \partial \bar{\partial} f-\int_{X} f \wedge \partial \bar{\partial} g=\int_{X} g \wedge \partial \bar{\partial} f+\int_{X} f \wedge \bar{\partial} \partial g= \\
& \int_{b X} g \wedge \bar{\partial} f-\int_{X} \partial g \wedge \bar{\partial} f+\int_{b X} f \wedge \partial g-\int_{X} \bar{\partial} f \wedge \partial g= \\
& \int_{b X} g \wedge \bar{\partial} f+\int_{b X} f \wedge \partial g
\end{aligned}
$$

## Lemma 3.2

Let $\psi=e^{\lambda z_{1}} \mu$ be the Faddeev function associated with $q \in C_{1,1}(V)$, supp $q \subseteq X$. Let $G_{\lambda}(z, \xi)=e^{\lambda\left(z_{1}-\xi_{1}\right)} g_{\lambda}(z, \lambda)$, where $g_{\lambda}(z, \xi)-$ kernel of operator $R_{\lambda} \circ \hat{R}$. Then $\forall z \in V \backslash X$ we have equality

$$
\begin{equation*}
\psi(z, \lambda)=e^{\lambda z_{1}}-\int_{\xi \in b X} G_{\lambda}(z, \xi) \bar{\partial} \psi(\xi)-\int_{\xi \in b X} \psi(\xi) \partial G_{\lambda}(z, \xi) \tag{3.4}
\end{equation*}
$$

Proof
From definition of the Faddeev function $\psi=e^{\lambda z_{1}} \mu$ we have

$$
\begin{equation*}
\psi(z, \lambda)=e^{\lambda z_{1}}+\hat{G}_{\lambda}\left(\frac{i}{2} q \psi\right) \quad \text { on } \quad V \tag{3.5}
\end{equation*}
$$

where $\hat{G}_{\lambda}$ - operator with kernel $G_{\lambda}(z, \xi)$ and $d d^{c} \psi=q \psi$. Besides,

$$
\int_{X} G_{\lambda}\left(\frac{i}{2} q \psi\right)=-\int_{X} G_{\lambda} \partial \bar{\partial} \psi
$$

Using the Green-Riemann formula from Lemma 3.1 we have

$$
\int_{X} G_{\lambda} \partial \bar{\partial} \psi=\int_{X} \psi \partial \bar{\partial} G_{\lambda}+\int_{b X} G_{\lambda} \bar{\partial} \psi+\int_{b X} \psi \partial G_{\lambda}
$$

For $z \in V \backslash X$ we have $\partial \bar{\partial} G_{\lambda}=0$. Hence,

$$
\begin{equation*}
-\int_{X} G_{\lambda}\left(\frac{i}{2} q \psi\right)=\int_{b X} G_{\lambda} \bar{\partial} \psi+\int_{b X} \psi \partial G_{\lambda} \tag{3.6}
\end{equation*}
$$

From (3.5) and (3.6) we obtain (3.4).
Proof of Proposition 3.1
Let $\psi_{0}: \bar{\partial} \partial \psi=0$ and $\left.\psi_{0}\right|_{b X}=\psi$. Then by Lemma $3.1 \forall z \in V \backslash X$ we have

$$
\begin{equation*}
\int_{b X} \psi_{0} \partial G_{\lambda}+\int_{b X} G_{\lambda} \bar{\partial} \psi_{0}=0 \tag{3.7}
\end{equation*}
$$

Combining this relation with (3.4) we obtain

$$
\begin{equation*}
\psi(z, \lambda)=e^{\lambda z_{1}}-\int_{b X} G_{\lambda}(z, \xi)\left(\bar{\partial} \psi(\xi)-\bar{\partial} \psi_{0}(\xi)\right) \tag{3.8}
\end{equation*}
$$

Equality (3.8) implies (3.3), where $\hat{\Phi} \psi, \hat{\Phi}_{0} \psi$ are defined by (3.2). Integral equation (3.3) is the Fredholm equation in $L^{\infty}(b X)$, because operator $\left(\hat{\Phi}-\hat{\Phi}_{0}\right)$ is a compact operator in $L^{\infty}(b X)$. For the case $V=\mathbb{C}$ Proposition 1 of [N1], contains explicit inequality:

$$
\begin{equation*}
\lim _{\substack{w \rightarrow \xi \\ w, \xi \in b X}} \frac{\left|\Phi(\xi, w)-\Phi_{0}(\xi, w)\right|}{|\ln | \xi-w| |} \leq(\text { const })\left|\frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}\right|(w) . \tag{3.9}
\end{equation*}
$$

Because of its local nature, equality (3.9) is valid for our more general case.
Existence of the Faddeev function $\forall \lambda \neq 0$, proved in Proposition 1.2, implies existence of solution of equation (3.3). The uniqueness of solution of equation (3.3) we deduce (as in proof of Proposition 2 in [N1]) from the following statement.

## Lemma 3.3

Let $q \in C_{1,1}(V)$, supp $q \subseteq X$. Then each solution $\psi=\psi(z, \lambda)$ of equation $d d^{c} \psi=q \psi$ on $V$ which coincides on $b X$ with solution of integral equation (3.3) in space $C(b X)$ is the Faddeev type function associated with $q$.

Proof
Let $\psi$ satisfy $d d^{c} \psi=q \psi$ on $V,\left.\psi\right|_{b X} \in C(b X)$ and $\left.\psi\right|_{b X}$ satisfy integral equation (3.3) $=(3.8)$. From (3.8) with help of (3.7) we obtain formula (3.4) for $\psi(z, \lambda), z \in V \backslash X$,
$\lambda \in \mathbb{C}$. From the Sohotsky-Plemelj jump formula we deduce (see [HM], Lemma 15) that $\forall z^{*} \in b X$

$$
\begin{equation*}
\psi\left(z^{*}\right)=\lim _{\substack{z \rightarrow z^{*} \\ z \in X}}\left(\int_{b X} G_{\lambda} \bar{\partial} \psi+\psi \partial G_{\lambda}\right)-\lim _{\substack{z \rightarrow z^{*} \\ z \in V \backslash X}}\left(\int_{b X} G_{\lambda} \bar{\partial} \psi+\psi \partial G_{\lambda}\right) \tag{3.10}
\end{equation*}
$$

From (3.4) and (3.10) we obtain equality

$$
\begin{equation*}
e^{\lambda z_{1}}=\int_{b X} G_{\lambda} \bar{\partial} \psi+\psi \partial G_{\lambda} \text { for } z \in X, \quad \lambda \in \mathbb{C} \tag{3.11}
\end{equation*}
$$

Using the Green-Riemann formula (Lemma 3.1) we obtain further

$$
\begin{align*}
& \int_{b X} G_{\lambda} \bar{\partial} \psi+\psi \partial G_{\lambda}=\int_{X} \psi \bar{\partial} \partial G_{\lambda}-\int_{X} G_{\lambda} \bar{\partial} \partial \psi= \\
& \left\{\begin{array}{cl}
\psi(z)-\int_{X} G_{\lambda} \bar{\partial} \partial \psi, & \text { if } z \in X, \lambda \in \mathbb{C} \\
-\int_{X} G_{\lambda} \bar{\partial} \partial \psi, & \text { if } z \in V \backslash X, \lambda \in \mathbb{C}
\end{array}\right. \tag{3.12}
\end{align*}
$$

Equalities (3.4), (3.11) and (3.12) imply

$$
\psi(z)=e^{\lambda z_{1}}+\int_{V} G_{\lambda} \bar{\partial} \partial \psi=e^{\lambda z_{1}}+\hat{G}_{\lambda}\left(\frac{i}{2} q \psi\right)
$$

i.e. $\psi$ satisfies (3.5). From equality (3.5) and Proposition 3 ii) from [H2] we obtain that $\mu-1=\psi e^{-\lambda z_{1}}-1 \in W^{1, \tilde{p}}(V)$ and $\psi$ is the Faddeev type function associated with $q$.

Lemma 3.3 is proved.
The uniqueness of solution of equation (3.3) in $C(b X)$ follows now from Lemma 3.3 and the uniqueness of the Faddeev type function associated with $q$ proved in Proposition 1.1.

Proposition 3.1 is proved.
Proposition 3.2 (Reconstruction of $\left.\psi\right|_{X}$ through $\left.\psi\right|_{b X}$ )
In conditions of Propositions 1.1, 1.2 the following properties for " $\bar{\partial}$-scattering data $b(\lambda) "$ permit to reconstruct $\left.\psi\right|_{X}$ through $\left.\psi\right|_{b X}$

$$
\forall z^{*} \in b X \exists \lim _{\substack{z \rightarrow \infty \\ z \in V}} \frac{\bar{z}_{1}}{\bar{\lambda}} e^{-\bar{\lambda} \bar{z}_{1}} \frac{\partial \psi}{\partial \bar{z}_{1}}(z, \lambda) \stackrel{\text { def }}{=} b(\lambda)=\bar{\psi}\left(z^{*}, \lambda\right)^{-1} \frac{\partial \psi}{\partial \bar{\lambda}}\left(z^{*}, \lambda\right)
$$

$\forall z \in X$ function $\lambda \mapsto \psi(z, \lambda), \lambda \in \mathbb{C}$, is a unique solution of the integral equation

$$
\psi(z, \lambda)=e^{\lambda z_{1}}-\frac{1}{2 \pi i} \int_{\xi \in \mathbb{C}} b(\xi) e^{(\lambda-\xi) z_{1}} \bar{\psi}(\xi, \lambda) \frac{d \xi \wedge d \bar{\xi}}{\xi-\lambda}
$$

This proposition is a consequence of Propositions 2.1-2.3.
Proposition 3.3 (Reconstruction formulas for $\sigma$ )
Conductivity function $\left.\sigma\right|_{X}$ can be reconstructed through Dirichlet-Neumann data by the R.Novikov's scheme:

$$
\text { DN data }\left.\rightarrow \psi\right|_{b X} \rightarrow \bar{\partial}-\text { scattering data }\left.\left.\rightarrow \psi\right|_{X} \rightarrow \frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}\right|_{X}
$$

The last step of this scheme one can realize by any of the following formulas:

$$
\begin{aligned}
& \text { A) } \frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}(z)=\left(d d^{c} \psi(z, \lambda)\right) \psi^{-1}(z, \lambda), \quad z \in X, \lambda \in \mathbb{C} \\
& \text { B) } \frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}(z)=2 i \lim _{\lambda \rightarrow \infty} \lambda e^{-\lambda z_{1}} d z_{1} \wedge \bar{\partial} \psi(z, \lambda), z \in X \\
& \text { C) } \frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}}(z)=-2 i \lim _{\lambda \rightarrow 0} \frac{\bar{\partial} \partial \mu(z, \lambda)}{\mu(z, \lambda)}, z \in X .
\end{aligned}
$$

## Proof

The first three steps of the scheme above were done in Propositions 3.1, 3.2. The formula A) is an immediate consequense of equation (3.1). The formula B) follows from equation $\bar{\partial}\left(\partial+\lambda d z_{1}\right) \mu=\frac{i}{2} \frac{d d^{c} \sqrt{\sigma}}{\sqrt{\sigma}} \mu$ (Proposition 1.2a) and estimates $\mu \rightarrow 1, \lambda \rightarrow \infty$ (Proposition 1.2b), $\partial \bar{\partial} \mu(z, \lambda) \rightarrow 0, \lambda \rightarrow \infty$ (Proposition 1.2e). The formula C) follows from the same equation and estimates $\left\|\frac{\partial \mu}{\partial z_{1}}\right\|=O(\sqrt{|\lambda|}), \lambda \rightarrow 0$, (Proposition 1.2c).

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