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# Asymptotic derivation of the section-averaged shallow water equations for natural river hydraulics

Luca Bonaventura\*, Astrid Decoene,†Fausto Saleri

*MOX – Modelling and Scientific Computing  
Dipartimento di Matematica “F. Brioschi”, Politecnico di Milano  
Via Bonardi 9, 20133 Milano, Italy*

## Abstract

The section-averaged shallow water model usually applied in river and open channel hydraulics is derived asymptotically up to second order in the vertical/longitudinal length ratio, starting from the three-dimensional Reynolds-averaged Navier-Stokes equations for incompressible free surface flows. The derivation is carried out under quite general assumptions on the geometry of the channel, thus allowing for the application of the resulting equations to natural rivers with arbitrarily shaped cross sections. As a result of the derivation, a generalized friction term is obtained, that does not rely on local uniformity assumptions and that can be computed directly from three-dimensional turbulence models, without need for local uniformity assumptions. The modified equations including the novel friction term are compared to the classical Saint Venant equations in the case of steady state open channel flows, where analytic solutions are available, showing that the solutions resulting from the modified equation set are much closer to the three-dimensional solutions than those of the classical equation set. Furthermore, it is shown that the proposed formulation yields results that are very similar to those obtained with empirical friction closures widely applied in computational hydraulics. The generalized friction term derived therefore justifies *a posteriori* these empirical closures, while allowing to avoid the assumptions on local flow uniformity on which these closures rely.

*Keywords: Computational hydraulics; shallow water equations; Saint Venant equations.*

## 1 Introduction

In environmental modelling of free surface flows, whenever the ratio between the vertical and longitudinal scales is small enough, the so-called *Shallow Water* approximation is usually introduced, in order to reduce the computational cost implied by the numerical solution of three-dimensional free surface flow equations. Models based on this approximation are extensively used to simulate various geophysical phenomena, such as rivers and coastal flows,[6, 8] oceans and even avalanches,[1] and they have been used in hydraulics for a very long time. When the viscosity is neglected and a rectangular channel section is assumed, the derivation of the one-dimensional Shallow Water system is classical, see e.g.[13]. However, this derivation is unsatisfactory, since viscosity effects are added *a posteriori* and the three-dimensional geometry is not arbitrary.

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\*luca.bonaventura@polimi.it

†astrid.decoene@mate.polimi.it

In Ref.[7], Gerbeau and Perthame derive rigorously, by asymptotic analysis, a one-dimensional viscous Saint-Venant system from the two-dimensional Navier-Stokes equations with molecular viscosity, linear friction boundary conditions and flat bathymetry. The effect of the viscosity is recovered in a one-dimensional friction term and in a one-dimensional diffusion term, both resulting from the derivation. The final system is a second order approximation – with respect to the ratio between the vertical and longitudinal scales – of the original two-dimensional model. Other systems have been derived in the same spirit. In Ref.[12], the asymptotic analysis is made through a variable change in a reference domain, independent of the ratio parameter and time. Marche proposes in Ref.[10] the derivation of a two-dimensional viscous shallow water system taking into account capillary effects, varying bathymetry and a molecular viscosity. However, in order to simulate realistic river flows, three-dimensional geometries and turbulence phenomena must be taken into account. Thus, the Reynolds-averaged Navier-Stokes equations (RANS) on an arbitrary three-dimensional domain are a more appropriate starting point for the derivation of simplified systems. In Ref.[5], Saleri *et al.* derived a two-dimensional viscous shallow water system from the three-dimensional RANS equations, taking into account a non-flat bathymetry, atmospheric pressure effects and considering a constant vertical eddy viscosity and linear friction boundary conditions.

In this paper, we have chosen to proceed as in Ref.[7], extending the analysis to the three-dimensional RANS equations with anisotropic Reynolds tensor for free surface flows in arbitrary geometry, with nonlinear friction boundary conditions entirely analogous to those actually used in practice in hydraulics applications. We present a rigorous derivation of the section-averaged system, including the effects of eddy viscosity and friction. This derivation is also aimed at providing an adequate framework for the rigorous derivation of coupling between three- and one-dimensional free surface models. The equation system obtained allows to compute the free surface level of the flow as well as a section-averaged velocity. If applied to channels with rectangular cross-section, this system is similar to the classical section-averaged shallow water equations [11], except for the friction term. Indeed, our derivation shows that, in order to take into account effects up to the second order in the asymptotic parameter, the classical friction term should be corrected by a term which depends on the turbulent vertical viscosity. This conclusion is in agreement with the one of Gerbeau *et al.* in Ref.[7] for two-dimensional flows with constant viscosity over a flat bathymetry. Indeed, if the vertical viscosity and friction coefficients are taken to be constant and the flow is homogeneous in the transversal direction, we retrieve the same friction correction as in Ref.[7]. However, our derivation provides the expression of the friction correction term in a more general case, which includes turbulent flows, nonlinear friction boundary conditions and three-dimensional arbitrary geometry.

In particular, we compute the correction term associated to a specific model for the vertical profile of turbulent velocity. Furthermore, for steady state open channel flows admitting analytic solutions of the three-dimensional as well as the simplified models, we show that the solutions computed including our correction term are much closer to those of the three dimensional model than those of the standard shallow water model. If empirical friction closures are introduced, as commonly done in computational hydraulics (see e.g.[2]), one obtains results very similar to those of our generalized friction term for steady state open channel solutions. Thus, the generalized friction term resulting from the present derivation justifies *a posteriori* these empirical closures, while allowing to avoid the assumptions on local flow uniformity on which these closures rely. The friction correction term can be easily included in section averaged models such as the one proposed by Deponti *et al.* in Ref.[3], [4]. Its use is also expected to ease the coupling of three- and one-dimensional free surface models in the framework of an integrated hydrological basin model.

In section 2 of this paper we review the three-dimensional RANS equations and their boundary conditions. Then, we derive the section-averaged shallow water model in section 3 and in section 4 we give the expression of the friction correction term in the laminar and turbulent cases. Finally, in section 5, we compare the analytical solutions of the three-dimensional and the section-averaged models in the particular case of steady state open channel flows with rectangular cross-section, in order to show the accuracy gain achieved by adding the friction correction.

## 2 The Reynolds-averaged Navier-Stokes equations

### 2.1 The three-dimensional equations with boundary conditions

We consider the motion of an incompressible fluid with constant density  $\rho > 0$ , in a three-dimensional domain  $\Omega_t = \Omega(t)$  which is normal with respect to the vertical direction  $z$ . We denote by  $\omega$  the projection of  $\Omega_t$  on the  $xy$ -plane, defined as follows:

$$\omega(t) = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq L, l_1(x, t) \leq y \leq l_2(x, t)\} ,$$

where  $l_1$  and  $l_2$  are the time and space dependent transversal limits of the flow, and  $L$  its length. We assume the bottom of the domain to be fixed and impervious. We call  $\eta$  and  $b$  the functions describing the free surface and the bottom, which is assumed to be fixed in time. The three-dimensional domain is defined by

$$\Omega_t = \{(x, y, z) \in \mathbb{R}^3 / (x, y) \in \omega(t), b(x, y) \leq z \leq \eta(x, y, t)\} ,$$

as illustrated in Figure 1. The boundary of the domain  $\Omega_t$  is denoted by  $\partial\Omega_t$  and can be decomposed into four separate parts: the free surface  $\Gamma_s(t)$ , the bottom surface  $\Gamma_b$ , the *inflow* boundary  $\Gamma_{in}(t)$  and the *outflow* boundary  $\Gamma_{out}(t)$ .

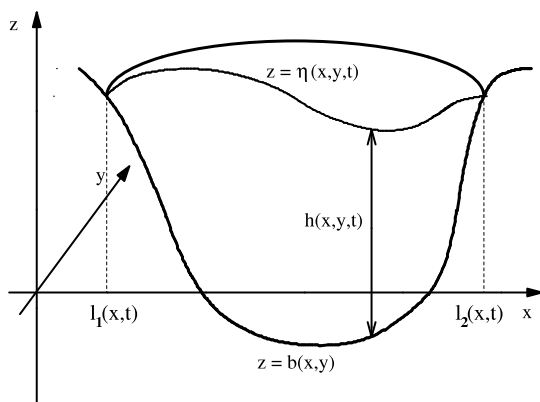


Figure 1: Three-dimensional domain

The governing equations for the motion of the fluid are the incompressible Reynolds-Averaged Navier-Stokes (RANS) equations in  $\Omega_t$ , valid for any  $t \in (0, T]$ , which can be written as follows:

$$\begin{cases} \frac{d\mathbf{U}}{dt} - \nabla \cdot \left( \frac{1}{\rho} \boldsymbol{\sigma}_T \right) = \mathbf{f} + \mathbf{g}, \\ \nabla \cdot \mathbf{U} = 0, \end{cases} \quad (1)$$

where  $\mathbf{U} = (u, v, w)^T$  is the velocity of the fluid,  $\boldsymbol{\sigma}_T$  is the stress tensor,  $\mathbf{f} = (f_x, f_y, f_z)^T$  the sum of the external forces applied on the fluid,  $\frac{d}{dt}$  denotes the total time derivative and  $\mathbf{g} = (0, 0, -g)^T$  the gravity acceleration. We only consider Newtonian fluids, for which the tensor  $\boldsymbol{\sigma}_T$  is written in the following way:

$$\boldsymbol{\sigma}_T = -p\mathbf{I} + \underline{\underline{\boldsymbol{\sigma}}}, \quad (2)$$

where  $p$  is the pressure and  $\underline{\underline{\boldsymbol{\sigma}}}$  the viscous stress tensor. Since we are considering flows in presence of gravity, that is aligned with the vertical direction, we consider a turbulence model given through an anisotropic relationship between the stress tensor  $\underline{\underline{\boldsymbol{\sigma}}}$  and the strain-rate tensor

$$\mathbb{D} = \nabla \mathbf{U} + (\nabla \mathbf{U})^T.$$

Following Levermore and Sammartino in Ref. [9], we take:

$$\underline{\underline{\boldsymbol{\sigma}}} = \begin{bmatrix} \sigma_{11} & \mu_h \mathbb{D}_{12} & \mu_v \mathbb{D}_{13} \\ \mu_h \mathbb{D}_{21} & \sigma_{22} & \mu_v \mathbb{D}_{23} \\ \mu_v \mathbb{D}_{31} & \mu_v \mathbb{D}_{32} & \mu_e \mathbb{D}_{33} \end{bmatrix}, \quad (3)$$

where

$$\sigma_{11} = \mu_h(\mathbb{D}_{11} - \frac{1}{2}(\mathbb{D}_{11} + \mathbb{D}_{22})) + \mu_e \frac{1}{2}(\mathbb{D}_{11} + \mathbb{D}_{22}),$$

and

$$\sigma_{22} = \mu_h(\mathbb{D}_{22} - \frac{1}{2}(\mathbb{D}_{11} + \mathbb{D}_{22})) + \mu_e \frac{1}{2}(\mathbb{D}_{11} + \mathbb{D}_{22}).$$

The positive coefficients  $\mu_h$ ,  $\mu_v$  and  $\mu_e$  are the eddy viscosities. They can be interpreted as the eddy viscosity relative to the horizontal shear motion, the eddy viscosity relative to the vertical shear motion, and the bulk viscosity relative to the expansion rate in the horizontal direction, respectively.

The system is closed by suitable initial and boundary conditions. We denote by  $\mathbf{n}_s$  the outward normal to the free surface, which depends on time:

$$\mathbf{n}_s = \frac{1}{\sqrt{1 + |\nabla \eta|^2}} \left( -\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1 \right)^T,$$

and by  $\mathbf{n}_b$  the outward normal to the bottom:

$$\mathbf{n}_b = \frac{1}{\sqrt{1 + |\nabla b|^2}} \left( \frac{\partial b}{\partial x}, \frac{\partial b}{\partial y}, -1 \right)^T,$$

while vectors  $\mathbf{t}_{b,1}$  and  $\mathbf{t}_{b,2}$  form a basis for the tangent plane to the bottom surface:

$$\mathbf{t}_{b,1} = \frac{1}{\sqrt{1 + \left| \frac{\partial b}{\partial x} \right|^2}} \left( 1, 0, \frac{\partial b}{\partial x} \right)^T,$$

and

$$\mathbf{t}_{b,2} = \frac{1}{\sqrt{1 + \left| \frac{\partial b}{\partial y} \right|^2}} \left( 0, 1, \frac{\partial b}{\partial y} \right)^T.$$

In a viscous flow, the velocity is zero on a solid wall, so that the so-called “no-slip” condition should be applied on the bottom:

$$\mathbf{U} = 0 \quad \text{on } \Gamma_b. \quad (4)$$

However, the boundary layer at the bottom is hardly ever resolved at typical resolutions of environmental models. Furthermore, it is necessary to describe in some approximate fashion the subgrid scale surface roughness. Thus, condition (4) is generally substituted by two boundary conditions assigned at a small distance  $\Delta z_r$  from the wall, which represents the typical length scale of the bottom boundary layer. In addition, the velocity is considered zero at a distance  $\Delta z_0$  of the wall, which represents the typical length scale of the bottom surface roughness, and should be much smaller than  $\Delta z_r$ :

$$\Delta z_0 \ll \Delta z_r. \quad (5)$$

The first boundary condition is a kinematic condition:

$$\mathbf{U} \cdot \mathbf{n}_b = 0 \quad \text{at } z = b(x, y) + \Delta z_r, \quad (6)$$

and the second one is a dynamic condition which accounts for friction effects:

$$\left( \frac{1}{\rho} \boldsymbol{\sigma}_T \cdot \mathbf{n}_b \right)^T \cdot \mathbf{t}_b = -\alpha \|\mathbf{U}\| \mathbf{U} \cdot \mathbf{t}_b \quad \text{at } z = b(x, y) + \Delta z_r, \quad (7)$$

where  $\alpha > 0$  is a dimensionless friction coefficient. Note that  $\mathbf{n}_b$  and  $\mathbf{t}_b$  are, respectively, the outward normal and a tangent vector to the bottom surface, so that condition (7) is indeed an assumption on the profile of the tangential velocity component along the direction normal to the bottom surface. A logarithmic wall law is usually assumed for tangential velocity near the bottom, see e.g. [11], so that a parabolic model is chosen for the vertical eddy viscosity, as well as a particular value of the friction coefficient  $\alpha$ , depending on the value of  $\Delta z_r$ , that will be described in greater detail later. In the following we will denote  $z_r(x, y) = b(x, y) + \Delta z_r$ .

At the free surface, the velocity of the fluid is equal to the velocity of the free surface itself. This is expressed by the following kinematic condition:

$$\frac{\partial \eta}{\partial t} - \mathbf{U} \cdot \mathbf{n}_s = 0 \quad \text{on } \Gamma_s(t). \quad (8)$$

The dynamical condition at the free surface takes into account the atmospheric stress,

$$\frac{1}{\rho} \boldsymbol{\sigma}_T \cdot \mathbf{n}_s = -\frac{1}{\rho} p_a \mathbf{n}_s \quad \text{on } \Gamma_s(t), \quad (9)$$

where  $p_a$  is the atmospheric pressure.

## 2.2 Adimensionalization of the system

Let us consider the following absolute scales:  $L$  for the total length,  $H$  for the depth and  $U$  for the  $x$ -component of the velocity. We denote by  $\epsilon$  the ratio between the vertical and the longitudinal scales:

$$\epsilon = \frac{H}{L}.$$

In addition we introduce the following dimensionless quantities:

$$\nu_h = \frac{\mu_h}{\rho U L}, \quad \nu_v = \frac{\mu_v}{\rho U L}, \quad \nu_e = \frac{\mu_e}{\rho U L}, \quad G = \frac{H}{U^2} g, \quad p_a = \frac{p_a}{\rho U^2}.$$

The scale for time is  $L/U$ , for the vertical velocity it is  $W = \epsilon U$ , and for the pressure  $P = \rho U^2$ . For the sake of simplicity we indicate again by  $u$ ,  $v$ ,  $w$ ,  $p$ ,  $\eta$  and  $b$ , respectively, velocity components, pressure, free surface and bottom elevation, after rescaling. Using these notations in (1) we obtain the following adimensionalized system, written as a function of the primitive unknowns  $u$ ,  $v$ ,  $w$  and  $p$ :

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \left( (\nu_h + \nu_e) \frac{\partial u}{\partial x} - (\nu_h - \nu_e) \frac{\partial v}{\partial y} \right) \\ \quad + \frac{\partial}{\partial y} \left( \nu_h \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{1}{\epsilon^2} \frac{\partial}{\partial z} \left( \nu_v \frac{\partial u}{\partial z} \right) + \frac{\partial}{\partial z} \left( \nu_v \frac{\partial w}{\partial x} \right), \\ \frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} + \frac{\partial p}{\partial y} = \frac{\partial}{\partial x} \left( \nu_h \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \\ \quad + \frac{\partial}{\partial y} \left( (\nu_h + \nu_e) \frac{\partial v}{\partial y} - (\nu_h - \nu_e) \frac{\partial u}{\partial x} \right) + \frac{1}{\epsilon^2} \frac{\partial}{\partial z} \left( \nu_v \frac{\partial v}{\partial z} \right) \\ \quad + \frac{\partial}{\partial z} \left( \nu_v \frac{\partial w}{\partial y} \right), \\ \epsilon^2 \left( \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} \right) + \frac{\partial p}{\partial z} = -G + \frac{\partial}{\partial x} \left( \nu_v \frac{\partial u}{\partial z} + \epsilon^2 \nu_v \frac{\partial w}{\partial x} \right) \\ \quad + \frac{\partial}{\partial y} \left( \nu_v \frac{\partial v}{\partial z} \right) + \epsilon^2 \frac{\partial}{\partial y} \left( \nu_v \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left( 2\nu_e \frac{\partial w}{\partial z} \right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \end{array} \right. \quad (10)$$

Coherently, the rescaled boundary conditions are, on the free surface  $\Gamma_s(t)$ ,

$$\left\{ \begin{array}{l} \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} = w, \\ \\ \frac{\partial \eta}{\partial x} \left( p - (\nu_h + \nu_e) \frac{\partial u}{\partial x} + (\nu_h - \nu_e) \frac{\partial v}{\partial y} \right) - \frac{\partial \eta}{\partial y} \left( \nu_h \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right. \\ \quad \left. + \nu_v \left( \frac{1}{\epsilon^2} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right) = p_a \frac{\partial \eta}{\partial x}, \\ \\ - \frac{\partial \eta}{\partial x} \left( \nu_h \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \frac{\partial \eta}{\partial y} \left( p - (\nu_h + \nu_e) \frac{\partial v}{\partial y} + (\nu_h - \nu_e) \frac{\partial u}{\partial x} \right) \\ \quad + \nu_v \left( \frac{1}{\epsilon^2} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = p_a \frac{\partial \eta}{\partial y}, \\ \\ - \frac{\partial \eta}{\partial x} \left( \nu_v \left( \frac{\partial u}{\partial z} + \epsilon^2 \frac{\partial w}{\partial x} \right) \right) - \frac{\partial \eta}{\partial y} \left( \nu_v \left( \frac{\partial v}{\partial z} + \epsilon^2 \frac{\partial w}{\partial y} \right) \right) - p \\ \quad + 2\nu_e \frac{\partial w}{\partial z} = -p_a, \end{array} \right. \quad (11)$$

and near the bottom at  $z = z_r$ ,

$$\left\{ \begin{array}{l} u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y} = w, \\ \\ \frac{\partial b}{\partial x} \left( (\nu_h + \nu_e) \frac{\partial u}{\partial x} - (\nu_h - \nu_e) \frac{\partial v}{\partial y} - 2\nu_e \frac{\partial w}{\partial z} \right) + \frac{\partial b}{\partial y} \left( \nu_h \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) \\ \quad + \left( \left( \frac{\partial b}{\partial x} \right)^2 - \frac{1}{\epsilon^2} \right) \left( \nu_v \left( \frac{\partial u}{\partial z} + \epsilon^2 \frac{\partial w}{\partial x} \right) \right) \\ \quad + \frac{\partial b}{\partial x} \frac{\partial b}{\partial y} \left( \nu_v \left( \frac{\partial v}{\partial z} + \epsilon^2 \frac{\partial w}{\partial y} \right) \right) \\ \quad = -\alpha \sqrt{u^2 + v^2 + \epsilon^2 w^2} \left( \frac{1}{\epsilon} u + \epsilon \frac{\partial b}{\partial x} w \right) N(b, \epsilon), \\ \\ \frac{\partial b}{\partial x} \left( \nu_h \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \frac{\partial b}{\partial y} \left( (\nu_h + \nu_e) \frac{\partial v}{\partial y} - (\nu_h - \nu_e) \frac{\partial u}{\partial x} - 2\nu_e \frac{\partial w}{\partial z} \right) \\ \quad + \left( \left( \frac{\partial b}{\partial y} \right)^2 - \frac{1}{\epsilon^2} \right) \left( \nu_v \left( \frac{\partial v}{\partial z} + \epsilon^2 \frac{\partial w}{\partial y} \right) \right) \\ \quad + \frac{\partial b}{\partial x} \frac{\partial b}{\partial y} \left( \nu_v \left( \frac{\partial u}{\partial z} + \epsilon^2 \frac{\partial w}{\partial x} \right) \right) \\ \quad = -\alpha \sqrt{u^2 + v^2 + \epsilon^2 w^2} \left( \frac{1}{\epsilon} v + \epsilon \frac{\partial b}{\partial y} w \right) N(b, \epsilon). \end{array} \right. \quad (12)$$

where

$$N(b, \epsilon) = \sqrt{1 + \epsilon^2 \left( \frac{\partial b}{\partial x} \right)^2 + \epsilon^2 \left( \frac{\partial b}{\partial y} \right)^2}.$$



### 3 Derivation of the section-averaged shallow water model

#### 3.1 Second order approximation in $\epsilon$

In order to derive our section-averaged shallow water model, a number of approximations have to be performed. Firstly, we assume that the vertical eddy viscosity is first order with respect to the ratio between the vertical and longitudinal scales, that is,

$$\nu_v = \epsilon \nu_{v,0}, \quad (13)$$

where  $\nu_{v,0}$  is a given positive quantity. This assumption can be justified by a simple dimensional analysis. Indeed, following the Prandtl hypothesis, the eddy viscosity is homogeneous to a length times a velocity, and more precisely

$$\frac{\mu}{\rho} \sim l_m^2 \|\mathbb{D}\|, \quad (14)$$

where  $l_m$  is the mixing length of the turbulent flow and  $\|\mathbb{D}\|$  is the norm of the strain-rate tensor. When considering the vertical eddy viscosity,  $l_m$  is homogeneous to a depth and the strain-rate tensor reduces to the vertical acceleration, therefore we conclude that

$$\frac{\mu}{\rho} \sim l_m^2 \left\| \frac{\partial \mathbf{U}}{\partial z} \right\|. \quad (15)$$

Note that Prandtl's mixing length model – see for instance Ref. [11] – is based on this assumption. Adimensionalizing this expression of  $\mu_v$  gives:

$$U L \hat{\mu}_v \sim H^2 \hat{l}_m^2 \sqrt{\frac{U^2}{H^2} \left( \left( \frac{\partial \hat{u}}{\partial z} \right)^2 + \left( \frac{\partial \hat{v}}{\partial z} \right)^2 + \left( \epsilon \frac{\partial \hat{w}}{\partial z} \right)^2 \right)} \sim U H \hat{l}_m^2 \left| \frac{\partial \hat{u}}{\partial z} \right|, \quad (16)$$

where the “hat” denotes here the adimensional variables. Thus

$$\nu_v = \frac{\hat{\mu}_v}{\rho U L} \sim \epsilon \frac{\hat{l}_m^2}{\rho} \left| \frac{\partial \hat{u}}{\partial z} \right| = O(\epsilon). \quad (17)$$

Moreover, the horizontal and bulk viscosities are of same order as the vertical eddy viscosity, and therefore we can write:

$$\nu_h = \epsilon \nu_{h,0}, \quad \nu_e = \epsilon \nu_{e,0}, \quad (18)$$

where  $\nu_{h,0}$  and  $\nu_{e,0}$  are two given positive quantities. Finally, we assume a slow varying bathymetry in the longitudinal direction, as it has been done often in these derivations – see for instance Ref. [5] –, and we consider a constant atmospheric pressure, that is

$$\frac{\partial b}{\partial x} = O(\epsilon) \quad \text{and} \quad \nabla p_a = 0. \quad (19)$$

Since our aim is to obtain a second order approximation with respect to  $\epsilon$  of the three-dimensional system, we neglect quantities of order  $O(\epsilon^2)$ . In this way, under the previous

assumptions, (10) becomes:

$$\left\{ \begin{aligned}
 & \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = \epsilon \frac{\partial}{\partial x} \left( (\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - (\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) \\
 & \quad + \epsilon \frac{\partial}{\partial y} \left( \nu_{h,0} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{1}{\epsilon} \frac{\partial}{\partial z} \left( \nu_{v,0} \frac{\partial u}{\partial z} \right) + \epsilon \frac{\partial}{\partial z} \left( \nu_{v,0} \frac{\partial w}{\partial x} \right), \\
 & \frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} + \frac{\partial p}{\partial y} = \epsilon \frac{\partial}{\partial x} \left( \nu_{h,0} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \\
 & \quad + \epsilon \frac{\partial}{\partial y} \left( (\nu_{h,0} + \nu_{e,0}) \frac{\partial v}{\partial y} - (\nu_{h,0} - \nu_{e,0}) \frac{\partial u}{\partial x} \right) + \frac{1}{\epsilon} \frac{\partial}{\partial z} \left( \nu_{v,0} \frac{\partial v}{\partial z} \right) \\
 & \quad + \epsilon \frac{\partial}{\partial z} \left( \nu_{v,0} \frac{\partial w}{\partial y} \right), \\
 & \frac{\partial p}{\partial z} = -G + \epsilon \frac{\partial}{\partial x} \left( \nu_{v,0} \frac{\partial u}{\partial z} \right) + \epsilon \frac{\partial}{\partial y} \left( \nu_{v,0} \frac{\partial v}{\partial z} \right) + \epsilon \frac{\partial}{\partial z} \left( 2\nu_{e,0} \frac{\partial w}{\partial z} \right), \\
 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
 \end{aligned} \right. \quad (20)$$

together with boundary conditions on the free surface  $\Gamma_s(t)$ ,

$$\left\{ \begin{aligned}
 & \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} = w, \\
 & \frac{\partial \eta}{\partial x} \left( p - \epsilon (\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} + \epsilon (\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) - \frac{\partial \eta}{\partial y} \left( \epsilon \nu_{h,0} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) \\
 & \quad + \frac{1}{\epsilon} \nu_{v,0} \frac{\partial u}{\partial z} + \epsilon \nu_{v,0} \frac{\partial w}{\partial x} = p_a \frac{\partial \eta}{\partial x}, \\
 & -\frac{\partial \eta}{\partial x} \left( \epsilon \nu_{h,0} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \frac{\partial \eta}{\partial y} \left( p - \epsilon (\nu_{h,0} + \nu_{e,0}) \frac{\partial v}{\partial y} + \epsilon (\nu_{h,0} - \nu_{e,0}) \frac{\partial u}{\partial x} \right) \\
 & \quad + \frac{1}{\epsilon} \nu_{v,0} \frac{\partial v}{\partial z} + \epsilon \nu_{v,0} \frac{\partial w}{\partial y} = p_a \frac{\partial \eta}{\partial y}, \\
 & -\frac{\partial \eta}{\partial x} \left( \epsilon \nu_{v,0} \frac{\partial u}{\partial z} \right) - \frac{\partial \eta}{\partial y} \left( \epsilon \nu_{v,0} \frac{\partial v}{\partial z} \right) - p + 2\epsilon \nu_{e,0} \frac{\partial w}{\partial z} = -p_a,
 \end{aligned} \right. \quad (21)$$

and near the bottom at  $z = z_r$ ,

$$\left\{ \begin{array}{l} u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y} = w, \\ \frac{\partial b}{\partial x} \left( \epsilon (\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - \epsilon (\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} - 2\epsilon \nu_{e,0} \frac{\partial w}{\partial z} \right) \\ \quad + \frac{\partial b}{\partial y} \left( \epsilon \nu_{h,0} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) - \frac{1}{\epsilon} \nu_{v,0} \frac{\partial u}{\partial z} + \epsilon \nu_{v,0} \frac{\partial w}{\partial x} \\ \qquad \qquad \qquad = -\frac{\alpha}{\epsilon} \|\mathbf{u}\| u, \\ \frac{\partial b}{\partial x} \left( \epsilon \nu_{h,0} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \frac{\partial b}{\partial y} \left( \epsilon (\nu_{h,0} + \nu_{e,0}) \frac{\partial v}{\partial y} - \epsilon (\nu_{h,0} - \nu_{e,0}) \frac{\partial u}{\partial x} \right. \\ \quad \left. - 2\epsilon \nu_{e,0} \frac{\partial w}{\partial z} \right) + \left( \left( \frac{\partial b}{\partial y} \right)^2 - \frac{1}{\epsilon^2} \right) \left( \epsilon \nu_{v,0} \frac{\partial v}{\partial z} \right) \\ \qquad \qquad \qquad = -\alpha \|\mathbf{u}\| \left( \frac{1}{\epsilon} v + \epsilon \frac{\partial b}{\partial y} w \right), \end{array} \right. \quad (22)$$

where  $\mathbf{u} = (u, v)$  is the horizontal velocity. Notice that, neglecting terms in  $O(\epsilon)$  in (22)<sub>2</sub> and (22)<sub>3</sub>, we obtain the classical boundary condition on the bottom (see e.g. Ref. [2]) :

$$\nu_{v,0} \frac{\partial u}{\partial z} = -\alpha \|\mathbf{u}\| u.$$

### 3.2 Vertical integration of the equations

We will now vertically-integrate system (20) between the free surface and the reference level  $z_r$  at which the bottom conditions are given. We will denote by  $h$  the corrected water depth, that is the real water depth corrected by the distance  $\Delta z_r$ :

$$\begin{aligned} h(x, y, t) &= \eta(x, y, t) - z_r(x, y) \\ &= \eta(x, y, t) - b(x, y) - \Delta z_r. \end{aligned} \quad (23)$$

Then, for any three-dimensional variable  $f$ , we denote with  $\bar{f}$  the average along the vertical direction,

$$\bar{f}(x, y, t) = \frac{1}{h(x, y, t)} \int_{z_r}^{\eta} f(x, y, z, t) dz.$$

Let us first vertically-integrate the momentum equation (20)<sub>1</sub> for  $u$  between the reference bottom level  $z_r$  and the free surface. Making use of the Leibnitz rule yields:

$$\begin{aligned}
 & \frac{\partial h\bar{u}}{\partial t} + \frac{\partial h\bar{u}^2}{\partial x} + \frac{\partial h\bar{u}\bar{v}}{\partial y} + \frac{\partial h\bar{p}}{\partial x} + uw(\eta) - uw(z_r) - \frac{\partial \eta}{\partial t}u(\eta) - \frac{\partial \eta}{\partial x}u^2(\eta) \\
 & + \frac{\partial b}{\partial x}u^2(z_r) - \frac{\partial \eta}{\partial y}uv(\eta) + \frac{\partial b}{\partial y}uv(z_r) - \frac{\partial \eta}{\partial x}p(\eta) + \frac{\partial b}{\partial x}p(z_r) \\
 & = \frac{\partial}{\partial x} \int_{z_r}^{\eta} \left( \epsilon(\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - \epsilon(\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) dz \\
 & + \frac{\partial}{\partial y} \int_{z_r}^{\eta} \left( \epsilon \nu_{h,0} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) dz \\
 & - \frac{\partial \eta}{\partial x} \left( \epsilon(\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - \epsilon(\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) \Big|_{z=\eta} \\
 & - \frac{\partial \eta}{\partial y} \left( \epsilon \nu_{h,0} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \Big|_{z=\eta} \\
 & + \frac{\partial b}{\partial x} \left( \epsilon(\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - \epsilon(\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) \Big|_{z=z_r} \\
 & + \frac{\partial b}{\partial y} \left( \epsilon \nu_{h,0} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \Big|_{z=z_r} \\
 & + \frac{1}{\epsilon} \nu_{v,0} \frac{\partial u}{\partial z} \Big|_{\eta-z_r} + \epsilon \nu_{v,0} \frac{\partial w}{\partial x} \Big|_{\eta-z_r}.
 \end{aligned}$$

Using the kinematic boundary conditions (21)<sub>1</sub> and (22)<sub>1</sub>, as well as (21)<sub>2</sub> and (22)<sub>2</sub>, the equation reduces to:

$$\begin{aligned}
 & \frac{\partial h\bar{u}}{\partial t} + \frac{\partial h\bar{u}^2}{\partial x} + \frac{\partial h\bar{u}\bar{v}}{\partial y} + \frac{\partial h\bar{p}}{\partial x} = p_a \frac{\partial \eta}{\partial x} - \frac{\alpha \|\mathbf{u}(z_r)\|}{\epsilon} u(z_r) - \frac{\partial b}{\partial x} p(z_r) \\
 & + 2\epsilon \frac{\partial b}{\partial x} \left( \nu_{e,0} \frac{\partial w}{\partial z} \right) \Big|_{z=z_r} + \epsilon \frac{\partial}{\partial x} \int_{z_r}^{\eta} \left( (\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - (\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) dz \quad (24) \\
 & + \epsilon \frac{\partial}{\partial y} \int_{z_r}^{\eta} \left( \nu_{h,0} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) dz.
 \end{aligned}$$

On the other hand, by vertically-integrating equation (20)<sub>3</sub> between  $z$  and the free surface, we obtain:

$$\begin{aligned}
p &= p(\eta) + G(\eta - z) - \epsilon \int_z^\eta \left( \frac{\partial}{\partial x} (\nu_{v,0} \frac{\partial u}{\partial \delta}) + \frac{\partial}{\partial y} (\nu_{v,0} \frac{\partial v}{\partial \delta}) \right) d\delta \\
&\quad - 2\epsilon (\nu_{e,0} \frac{\partial w}{\partial z}) \Big|_{z=\eta} + 2\epsilon \nu_{e,0} \frac{\partial w}{\partial z} \\
&= p(\eta) + G(\eta - z) - \epsilon \frac{\partial}{\partial x} \int_z^\eta \left( \nu_{v,0} \frac{\partial u}{\partial \delta} \right) d\delta + \epsilon \frac{\partial \eta}{\partial x} \left( \nu_{v,0} \frac{\partial u}{\partial z} \right) \Big|_{z=\eta} \\
&\quad - \epsilon \frac{\partial}{\partial y} \int_z^\eta \left( \nu_{v,0} \frac{\partial v}{\partial \delta} \right) d\delta + \epsilon \frac{\partial \eta}{\partial y} \left( \nu_{v,0} \frac{\partial v}{\partial z} \right) \Big|_{z=\eta} \\
&\quad - 2\epsilon (\nu_{e,0} \frac{\partial w}{\partial z}) \Big|_{z=\eta} + 2\epsilon \nu_{e,0} \frac{\partial w}{\partial z}.
\end{aligned}$$

Applying now the dynamic condition (21)<sub>4</sub> at the free surface, we deduce the following expression for the pressure:

$$\begin{aligned}
p &= p_a + G(\eta - z) + 2\epsilon \nu_{e,0} \frac{\partial w}{\partial z} - \epsilon \frac{\partial}{\partial x} \int_z^\eta \left( \nu_{v,0} \frac{\partial u}{\partial \delta} \right) d\delta \\
&\quad - \epsilon \frac{\partial}{\partial y} \int_z^\eta \left( \nu_{v,0} \frac{\partial v}{\partial \delta} \right) d\delta + O(\epsilon^2).
\end{aligned} \tag{25}$$

Note that the pressure near the bottom at  $z = z_r$  is given by

$$\begin{aligned}
p(z_r) &= p_a + Gh + 2\epsilon (\nu_{e,0} \frac{\partial w}{\partial z}) \Big|_{z=z_r} - \epsilon \frac{\partial}{\partial x} \int_{z_r}^\eta \left( \nu_{v,0} \frac{\partial u}{\partial \delta} \right) dz \\
&\quad - \epsilon \frac{\partial}{\partial y} \int_{z_r}^\eta \left( \nu_{v,0} \frac{\partial v}{\partial \delta} \right) d\delta + O(\epsilon^2),
\end{aligned} \tag{26}$$

and therefore, recalling that  $\frac{\partial b}{\partial x} = 0(\epsilon)$ , we can conclude

$$\frac{\partial b}{\partial x} p(z_r) = p_a \frac{\partial b}{\partial x} + Gh \frac{\partial b}{\partial x} + O(\epsilon^2). \tag{27}$$

Let us now vertically integrate this expression of the pressure from the bottom reference depth  $z = z_r$  to the surface:

$$\begin{aligned}
 h\bar{p} &= hp_a + G\frac{h^2}{2} + 2\epsilon \int_{z_r}^{\eta} (\nu_{e,0} \frac{\partial w}{\partial z}) dz \\
 &\quad - \epsilon \int_{z_r}^{\eta} \left( \frac{\partial}{\partial x} \int_z^{\eta} (\nu_{v,0} \frac{\partial u}{\partial \delta}) d\delta + \frac{\partial}{\partial y} \int_z^{\eta} (\nu_{v,0} \frac{\partial v}{\partial \delta}) d\delta \right) dz \\
 &= hp_a + G\frac{h^2}{2} + 2\epsilon \int_{z_r}^{\eta} (\nu_{e,0} \frac{\partial w}{\partial z}) dz \\
 &\quad - \epsilon \left( \frac{\partial}{\partial x} \int_{z_r}^{\eta} \int_z^{\eta} (\nu_{v,0} \frac{\partial u}{\partial \delta}) d\delta dz + \frac{\partial}{\partial y} \int_{z_r}^{\eta} \int_z^{\eta} (\nu_{v,0} \frac{\partial v}{\partial \delta}) d\delta dz \right) \\
 &\quad + \epsilon \left( \frac{\partial b}{\partial x} \int_{z_r}^{\eta} (\nu_{v,0} \frac{\partial u}{\partial z}) dz + \frac{\partial b}{\partial y} \int_{z_r}^{\eta} (\nu_{v,0} \frac{\partial v}{\partial z}) dz \right) \\
 &= hp_a + G\frac{h^2}{2} + 2\epsilon \int_{z_r}^{\eta} (\nu_{e,0} \frac{\partial w}{\partial z}) dz \\
 &\quad - \epsilon \left( \frac{\partial}{\partial x} \int_{z_r}^{\eta} \int_z^{\eta} (\nu_{v,0} \frac{\partial u}{\partial \delta}) d\delta dz + \frac{\partial}{\partial y} \int_{z_r}^{\eta} \int_z^{\eta} (\nu_{v,0} \frac{\partial v}{\partial \delta}) d\delta dz \right).
 \end{aligned} \tag{28}$$

The momentum equation (24) therefore writes:

$$\begin{aligned}
 \frac{\partial h\bar{u}}{\partial t} + \frac{\partial h\bar{u}^2}{\partial x} + \frac{\partial h\bar{u}\bar{v}}{\partial y} + G\frac{\partial h^2}{\partial x} &= - \frac{\alpha \|\mathbf{u}(z_r)\|}{\epsilon} u(z_r) - Gh\frac{\partial b}{\partial x} \\
 &\quad - 2\epsilon \frac{\partial}{\partial x} \left( \int_{z_r}^{\eta} (\nu_{e,0} \frac{\partial w}{\partial z}) dz \right) + \epsilon \frac{\partial}{\partial y} \int_{z_r}^{\eta} \left( \nu_{h,0} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) dz \\
 &\quad + \epsilon \frac{\partial}{\partial x} \int_{z_r}^{\eta} \left( (\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - (\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) dz \\
 &\quad + \epsilon \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \int_{z_r}^{\eta} \int_z^{\eta} (\nu_{v,0} \frac{\partial u}{\partial \delta}) d\delta dz + \frac{\partial}{\partial y} \int_{z_r}^{\eta} \int_z^{\eta} (\nu_{v,0} \frac{\partial v}{\partial \delta}) d\delta dz \right).
 \end{aligned} \tag{29}$$

An analogous equation can be obtained by vertically integrating the continuity equation (20)<sub>4</sub>:

$$\frac{\partial h}{\partial t} + \frac{\partial h\bar{u}}{\partial x} + \frac{\partial h\bar{v}}{\partial y} = 0. \tag{30}$$

### 3.3 Section averaged equations

Since the equations are now vertically-integrated, they are defined on  $\omega(x, t) \times I$ , where we recall that  $\omega(t) = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq L, l_1(x, t) \leq y \leq l_2(x, t)\}$ . We can therefore integrate

them along the  $y$ -axis between  $l_1(x, t)$  and  $l_2(x, t)$ . In addition we point out that, for any scalar quantity  $f$ ,

$$\left( \int_{z_r}^{\eta} f dz \right) \Big|_{y=l_1} = \left( \int_{z_r}^{\eta} f dz \right) \Big|_{y=l_2} = 0. \quad (31)$$

This assumption is justified in the case of a natural river, whose depth tends to zero as the banks are approached. Note however that we can retrieve the same section averaged model under the hypothesis that

$$\frac{\partial l_1}{\partial x} \equiv O(\epsilon^2) \quad \text{and} \quad \frac{\partial l_2}{\partial x} \equiv O(\epsilon^2), \quad (32)$$

as happens for instance in straight or mildly curved channels. For the sake of clarity we do only report the derivation in the first case, that is with hypothesis (31).

We denote

$$\begin{aligned} A(x, t) &= \int_{l_1}^{l_2} h(x, y, t) dy, \\ \bar{f}(x, y, t) &= \frac{1}{A(x, t)} \int_{l_1}^{l_2} \int_{z_r}^{\eta} f(x, y, z, t) dz dy, \\ Q(x, t) &= \int_{l_1}^{l_2} \int_{z_r}^{\eta} u(x, y, z, t) dz dy = A(x, t) \bar{u}(x, y, t). \end{aligned}$$

From now on we make the further assumption that the transversal velocity  $v$  is small with respect to the longitudinal velocity component  $u$ . This hypothesis is consistent with regimes that allow to use a section averaged model on  $u$ . We therefore assume that:

$$\frac{|v|}{|u|} = O(\epsilon). \quad (33)$$

We integrate the momentum equation (24) on  $u$ . Using the Leibnitz rule and (31), we obtain:

$$\begin{aligned} \frac{\partial(A\bar{u})}{\partial t} + \frac{\partial(A\bar{u}^2)}{\partial x} + G \int_{l_1}^{l_2} \frac{\partial}{\partial x} \left( \frac{h^2}{2} \right) dy &= - \frac{1}{\epsilon} \int_{l_1}^{l_2} \alpha ||\mathbf{u}(z_r)|| u(z_r) dy \\ &- G \int_{l_1}^{l_2} h \frac{\partial b}{\partial x} dy - 2\epsilon \frac{\partial}{\partial x} \left( \int_{l_1}^{l_2} \int_{z_r}^{\eta} (\nu_{e,0} \frac{\partial w}{\partial z}) dz dy \right) \\ &+ \epsilon \frac{\partial}{\partial x} \int_{l_1}^{l_2} \int_{z_r}^{\eta} \left( (\nu_{h,0} + \nu_{e,0}) \frac{\partial u}{\partial x} - (\nu_{h,0} - \nu_{e,0}) \frac{\partial v}{\partial y} \right) dz dy \\ &+ \epsilon \frac{\partial^2}{\partial x^2} \left( \int_{l_1}^{l_2} \int_{z_r}^{\eta} \int_z^{\eta} (\nu_{v,0} \frac{\partial u}{\partial \delta}) d\delta dz dy \right). \end{aligned} \quad (34)$$

Note that

$$G \int_{l_1}^{l_2} \frac{\partial}{\partial x} \left( \frac{h^2}{2} \right) dy = G \int_{l_1}^{l_2} \frac{\partial h}{\partial x} h dy = G \int_{l_1}^{l_2} \left( \frac{\partial \eta}{\partial x} h - \frac{\partial b}{\partial x} h \right) dy.$$

Note also that, from the continuity equation,

$$-2\epsilon \frac{\partial}{\partial x} \left( \int_{l_1}^{l_2} \int_{z_r}^{\eta} (\nu_{e,0} \frac{\partial w}{\partial z}) dz dy \right) = 2\epsilon \frac{\partial}{\partial x} \left( \int_{l_1}^{l_2} \int_{z_r}^{\eta} \nu_{e,0} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz dy \right),$$

and then, using this last expression in the right-hand side of (34), we obtain the following section averaged momentum equation:

$$\begin{aligned} \frac{\partial Q}{\partial t} + \frac{\partial \tilde{Q}^2}{\partial x} + G \int_{l_1}^{l_2} h \frac{\partial \eta}{\partial x} dy &= -\frac{1}{\epsilon} \int_{l_1}^{l_2} \alpha \|\mathbf{u}(z_r)\| |u(z_r)| dy \\ &+ \epsilon \frac{\partial}{\partial x} \int_{l_1}^{l_2} \int_{z_r}^{\eta} \left( \nu_{h,0} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \right) dz dy \\ &+ 3\epsilon \frac{\partial}{\partial x} \int_{l_1}^{l_2} \int_{z_r}^{\eta} \left( \nu_{e,0} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) dz dy \\ &+ \epsilon \frac{\partial^2}{\partial x^2} \left( \int_{l_1}^{l_2} \int_{z_r}^{\eta} \int_z^{\eta} (\nu_{v,0} \frac{\partial u}{\partial \delta}) d\delta dz dy \right), \end{aligned} \tag{35}$$

where

$$\tilde{Q} = \left( \int_{l_1}^{l_2} \int_{z_r}^{\eta} u^2(x, y, z, t) dz dy \right)^{1/2}.$$

Denoting by  $\beta$  the momentum correction (or Boussinesq) coefficient

$$\beta = \frac{1}{A} \int_{l_1}^{l_2} \int_{z_r}^{\eta} \frac{u^2}{\bar{u}^2} dz dy,$$

we have that

$$\frac{\partial \tilde{Q}^2}{\partial x} = \frac{\partial}{\partial x} \left( \beta \frac{Q^2}{A} \right).$$

The integration of the continuity equation (30)<sub>4</sub> gives

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \tag{36}$$

that is the classical continuity equation of the one-dimensional open channel equations.

### 3.4 Asymptotic analysis of the section-averaged equations

We now go back to the three-dimensional equations in order to model the friction term and show that we can neglect the last viscous term in the right-hand side of the momentum equation (35).

From the three-dimensional momentum equation (20)<sub>1</sub> we deduce that

$$\frac{\partial}{\partial z} \left( \nu_{v,0} \frac{\partial u}{\partial z} \right) = O(\epsilon). \tag{37}$$



In addition, boundary condition (21)<sub>2</sub> indicates that  $\nu_{v,0} \frac{\partial u}{\partial z} = O(\epsilon)$  at the free surface. Function  $z \rightarrow \nu_{v,0} \frac{\partial u}{\partial z}$  is of the order  $O(\epsilon)$  at the surface boundary  $z = \eta$ , and so is its derivative with respect to  $z$ . The function is therefore of the order  $O(\epsilon)$  along the whole depth, that is :

$$\nu_{v,0} \frac{\partial u}{\partial z} = O(\epsilon) \quad \text{on } \Omega_t. \quad (38)$$

Since  $\nu_{v,0}$  is independent of the ration  $\epsilon$ , we can conclude that  $\frac{\partial u}{\partial z} = O(\epsilon)$  and therefore :

$$u(x, y, z, t) = \bar{u}(x, y, t) + O(\epsilon). \quad (39)$$

Equation (38) has two important consequences. First, it shows that the friction coefficient  $\alpha$  is necessarily also of the first order in  $\epsilon$ . Indeed, from boundary condition (22)<sub>2</sub>, we have that  $\nu_{v,0} \frac{\partial u}{\partial z} = \alpha \|\mathbf{u}\| u + O(\epsilon^2)$  at  $z = z_r$ . Thus, since  $\nu_{v,0} \frac{\partial u}{\partial z} = O(\epsilon)$  on  $\Omega_t$ , we have that  $\alpha = O(\epsilon)$ . In the following we will assume that

$$\alpha = \epsilon \alpha_0. \quad (40)$$

On the other hand equation (38) shows that the third viscous term in the momentum equation (35) is second order in  $\epsilon$ .

Furthermore, from (25) we know that

$$p(x, y, z, t) = p_a + G(\eta - z) + O(\epsilon). \quad (41)$$

Using (39) and (41) in the three-dimensional momentum equation (20)<sub>1</sub> we can write:

$$\begin{aligned} \frac{1}{\epsilon} \frac{\partial}{\partial z} \left( \nu_{v,0} \frac{\partial u}{\partial z} \right) &= \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} + O(\epsilon) \\ &= \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z} + G \frac{\partial \eta}{\partial x} + O(\epsilon) \\ &= \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + G \frac{\partial \eta}{\partial x} + O(\epsilon). \end{aligned} \quad (42)$$

Note that for the last step we use the fact that, from continuity,

$$\bar{u} \frac{\partial w}{\partial z} = -\bar{u} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\bar{u} \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + O(\epsilon).$$

On the other hand the vertically-integrated momentum equation (24) gives

$$\begin{aligned} h \left\{ \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + G \frac{\partial h}{\partial x} \right\} + \bar{u} \left\{ \frac{\partial h}{\partial t} + \frac{\partial h \bar{u}}{\partial x} + \frac{\partial h \bar{v}}{\partial y} \right\} \\ = -\alpha_0 \|\mathbf{u}(z_r)\| u(z_r) + O(\epsilon), \end{aligned}$$

and using the vertically-averaged continuity equation (30),

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + G \frac{\partial h}{\partial x} = - \frac{\alpha_0 \|\mathbf{u}(z_r)\| |u(z_r)|}{h} + O(\epsilon).$$

Replacing this expression in (42) we have that

$$\frac{1}{\epsilon} \frac{\partial}{\partial z} \left( \nu_{v,0} \frac{\partial u}{\partial z} \right) = - \frac{\alpha_0 \|\mathbf{u}(z_r)\| |u(z_r)|}{h} + O(\epsilon).$$

Let us now integrate this expression from the bottom reference depth  $z = z_r$  to  $z$ :

$$\frac{1}{\epsilon} \nu_{v,0} \frac{\partial u}{\partial z} = \frac{1}{\epsilon} (\nu_{v,0} \frac{\partial u}{\partial z})|_{z=z_r} - \frac{\alpha_0 \|\mathbf{u}(z_r)\| |u(z_r)| (z - z_r)}{h} + O(\epsilon).$$

Using boundary condition (22)<sub>2</sub> we get:

$$\frac{1}{\epsilon} \nu_{v,0} \frac{\partial u}{\partial z} = \alpha_0 \|\mathbf{u}(z_r)\| |u(z_r)| \left( 1 - \frac{z - z_r}{h} \right) + O(\epsilon),$$

so that

$$\frac{\partial u}{\partial z} = \epsilon \alpha_0 \|\mathbf{u}(z_r)\| |u(z_r)| \left( \frac{\eta - z}{h \nu_{v,0}} \right) + O(\epsilon^2).$$

We vertically-integrate again this expression from the bottom reference depth  $z = z_r$  to  $z$ , yielding

$$\begin{aligned} u &= u(z_r) + \epsilon \alpha_0 \|\mathbf{u}(z_r)\| |u(z_r)| \int_{z_r}^z \frac{\eta - \delta}{h \nu_{v,0}} d\delta + O(\epsilon^2) \\ &= u(z_r) \left( 1 + \frac{\epsilon \alpha_0 \|\mathbf{u}(z_r)\|}{h} \int_{z_r}^z \frac{\eta - \delta}{\nu_{v,0}} d\delta \right) + O(\epsilon^2). \end{aligned} \quad (43)$$

Integrating on the vertical and dividing by the water depth  $h$  we obtain:

$$\bar{u} = \frac{1}{h} \int_{z_r}^{\eta} u dz = u(z_r) (1 + \epsilon \alpha_0 \|\mathbf{u}(z_r)\| m(\nu_{v,0})) + O(\epsilon^2), \quad (44)$$

where for the sake of simplicity we have denoted

$$m(\nu_{v,0}) = \frac{1}{h^2} \int_{z_r}^{\eta} \int_{z_r}^z \frac{\eta - \delta}{\nu_{v,0}} d\delta dz.$$

Equation (44) leads us to two important results. On one hand, it gives us some information about the Boussinesq coefficient  $\beta$ . Indeed, from (43) we deduce that

$$u^2 = u^2(z_r) \left( 1 + \frac{2\epsilon \alpha_0 \|\mathbf{u}(z_r)\|}{h} \int_{z_r}^z \frac{\eta - \delta}{\nu_{v,0}} d\delta \right) + O(\epsilon^2),$$

and therefore:

$$\begin{aligned} \frac{1}{h} \int_{z_r}^{\eta} u^2 dz &= u^2(z_r) (1 + 2\epsilon \alpha_0 \|\mathbf{u}(z_r)\| m(\nu_{v,0})) + O(\epsilon^2) \\ &= \bar{u}^2 (1 - 2\epsilon \alpha_0 \|\mathbf{u}(z_r)\| m(\nu_{v,0})) (1 + 2\epsilon \alpha_0 \|\mathbf{u}(z_r)\| m(\nu_{v,0})) + O(\epsilon^2) \\ &= \bar{u}^2 + O(\epsilon^2). \end{aligned}$$

Thus

$$\bar{u}^2 = \bar{u}^2 + O(\epsilon^2), \quad (45)$$

which means that, up to the second order in  $\epsilon$ , the Boussinesq coefficient  $\beta$  only depends on the transversal variations of the velocity  $u$  (and not on its vertical variations). Indeed:

$$\beta = \frac{1}{A} \int_{l_1}^{l_2} \int_{z_r}^{\eta} \frac{u^2}{\bar{u}^2} dz dy = A \frac{\int_{l_1}^{l_2} h \bar{u}^2 dy}{\left( \int_{l_1}^{l_2} h \bar{u} dy \right)^2} + O(\epsilon^2).$$

On the other hand equation (44) allows to model the friction term, that is:

$$\int_{l_1}^{l_2} \alpha_0 \|\mathbf{u}(z_r)\| u(z_r) dy. \quad (46)$$

For this purpose we need to express the velocity at  $z = z_r$  with respect to the vertically-averaged velocity  $\bar{u}$ . From (44) we have that

$$|u(z_r)| = \frac{|\bar{u}|}{1 + \epsilon \alpha_0 \|\mathbf{u}(z_r)\| m(\nu_{v,0})} + O(\epsilon^2). \quad (47)$$

In addition, assumption (33) gives:

$$\|\mathbf{u}(z_r)\| = \sqrt{u(z_r)^2 + v(z_r)^2} = |u(z_r)| + O(\epsilon^2). \quad (48)$$

Using this expression of  $\|\mathbf{u}(z_r)\|$  in (47) yields:

$$|u(z_r)| = \frac{|\bar{u}|}{1 + \epsilon \alpha_0 |u(z_r)| m(\nu_{v,0})} + O(\epsilon^2), \quad (49)$$

and the friction term writes:

$$\begin{aligned} \alpha_0 \|\mathbf{u}(z_r)\| u(z_r) &= \alpha_0 |u(z_r)| u(z_r) + O(\epsilon^2) \\ &= \frac{\alpha_0 |\bar{u}| \bar{u}}{(1 + \epsilon \alpha_0 |u(z_r)| m(\nu_{v,0}))^2} + O(\epsilon^2) \\ &= \frac{\alpha_0 |\bar{u}| \bar{u}}{1 + 2\epsilon \alpha_0 |u(z_r)| m(\nu_{v,0})} + O(\epsilon^2). \end{aligned}$$

Furthermore, using (49) we can write that:

$$\begin{aligned} 1 + 2\epsilon \alpha_0 |u(z_r)| m(\nu_{v,0}) &= 1 + \frac{2\epsilon \alpha_0 |u(z_r)| m(\nu_{v,0})}{1 + \epsilon \alpha_0 |u(z_r)| m(\nu_{v,0})} \\ &= 1 + 2\epsilon \alpha_0 |u(z_r)| m(\nu_{v,0}) (1 - \epsilon \alpha_0 |u(z_r)| m(\nu_{v,0})) + O(\epsilon^2) \\ &= 1 + 2\epsilon \alpha_0 |u(z_r)| m(\nu_{v,0}) + O(\epsilon^2). \end{aligned}$$

Neglecting the  $O(\epsilon^2)$  term, we finally obtain an approximation of the friction term (46) which is independent of the velocity at  $z = z_r$ , that is:

$$-\frac{1}{\epsilon} \int_{l_1}^{l_2} \alpha |u(z_r)| u(z_r) dy = -\frac{1}{\epsilon} \int_{l_1}^{l_2} \frac{\alpha |\bar{u}| \bar{u}}{1 + 2\alpha m(\nu_{v,0}) |\bar{u}|} dy. \quad (50)$$

In this way we have overcome the initial difficulty and we use expression (50) to model the friction term in the momentum equation (35).

### 3.5 The section averaged shallow water model

We have derived a section-averaged shallow water model which is an approximation of the second order in  $\epsilon$  of the initial three-dimensional free surface flow problem (1) with boundary conditions (6)-(9). Switching to the dimensional variables, this model writes:

$$\left\{ \begin{aligned} \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \beta \frac{Q^2}{A} \right) + g \int_{l_1}^{l_2} h \frac{\partial \eta}{\partial x} dy &= - \int_{l_1}^{l_2} \frac{\alpha |\bar{u}| \bar{u}}{1 + c_\alpha} dy \\ &+ 3 \frac{\partial}{\partial x} \left( \int_{l_1}^{l_2} \int_{z_r}^{\eta} \frac{\mu_e}{\rho} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz dy \right) \\ &+ \frac{\partial}{\partial x} \left( \int_{l_1}^{l_2} \int_{z_r}^{\eta} \frac{\mu_h}{\rho} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dz dy \right) \\ \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \end{aligned} \right. \quad (51)$$

where

$$c_\alpha = \frac{2 \alpha |\bar{u}|}{h^2} \int_{z_r}^{\eta} \int_b^z \frac{\rho (\eta - \delta)}{\mu_v} d\delta dz \quad (52)$$

acts as a correction to the classical one-dimensional friction term, and  $\beta$  is the momentum correction coefficient

$$\beta = A \frac{\int_{l_1}^{l_2} h \bar{u}^2 dy}{\left( \int_{l_1}^{l_2} h \bar{u} dy \right)^2}. \quad (53)$$

This model results of a direct asymptotic derivation from the three-dimensional free surface flow equations. In addition, this derivation is very general since it is valid for flows with arbitrary cross-section, non-constant, turbulent viscosity and non-linear boundary conditions. Thus, we expect the coupling of such a reduced model to a three-dimensional model to be easier and yield better results.

## 4 Computation of the corrected friction term

In this section we give an explicit expression of the friction correction to use in the laminar case and with a parabolic turbulence model for the vertical eddy viscosity.

### 4.1 The laminar case

We first consider the case where a constant vertical viscosity  $\mu_v$  is used. Note that in order to be consistent with our analysis, its adimensional value  $\nu_v = \frac{\mu_v}{\rho UL}$  must be  $O(\epsilon)$ . In that case we have that

$$\int_b^{\eta} \int_b^z \frac{\rho (\eta - \delta)}{\mu_v} d\delta dz = \frac{\rho}{\mu_v} \frac{h^3}{3},$$

and therefore the correction (52) of the friction term writes:

$$c_\alpha = \frac{2}{3} \frac{\rho}{\mu_v} \alpha h |\bar{u}|. \quad (54)$$

Note that in this case we retrieve a friction correction term which is very similar to the one presented by Gerbeau *et al.* in Ref. [7]. Indeed, starting from a linear friction boundary condition, Gerbeau *et al.* obtain the correction:

$$c_f = \frac{1}{3} \frac{\rho}{\mu_v} f h, \quad (55)$$

where  $f$  is the constant friction coefficient considered in their model. In our analysis, we have started from nonlinear boundary conditions, which explains the difference in the friction correction for this special case. If repeating our derivation under the hypothesis of [7], exactly the same correction is obtained.

If the flow is homogeneous in the  $y$ -direction and has a rectangular-cross-section,  $\bar{u} = \frac{Q}{A}$  and the friction term in (51) writes

$$- \frac{\alpha |Q|}{h^2 l \left( 1 + \frac{2}{3} \frac{\rho}{\mu_v} \alpha \frac{|Q|}{l} \right)} Q. \quad (56)$$

## 4.2 Parabolic model for the vertical eddy viscosity

Let us now consider a turbulence model, see e.g. [11], which assumes a parabolic distribution of the vertical eddy viscosity over the water depth, vanishing on the bottom:

$$\frac{\mu_v}{\rho} = \kappa u^*(z - b) \left( 1 - \frac{(z - b)}{h + \Delta z_r} \right), \quad (57)$$

where  $\kappa$  is the von Karman constant and  $u^*$  the friction velocity, defined by  $u^* = \sqrt{\|\tau_{\mathbf{b}}\|/\rho}$ ,  $\tau_{\mathbf{b}}$  being the bottom stress. Note that  $h + \Delta z_r$  is the real water height. A simple dimensional analysis shows that we have  $\nu_v = \frac{\mu_v}{\rho UL} = O(\epsilon)$  as expected.

In the case of a uniform flow in an open channel, this turbulence model allows to retrieve a logarithmic profile of the tangential velocity component of the form:

$$\frac{u_t}{u^*} = \frac{1}{\kappa} \ln\left(\frac{z - b}{\Delta z_0}\right), \quad (58)$$

where  $u_t$  is the tangential velocity component and  $\Delta z_0$  is the distance to the wall at which the velocity is zero. Note that in this case, from boundary condition (7) at  $z = z_r$ , we can retrieve the value of the friction coefficient  $\alpha$  for which the profile of the tangential velocity is actually (58), that is:

$$\alpha = \kappa^2 \frac{\left( 1 - \frac{\Delta z_r}{h + \Delta z_r} \right)}{\left( \ln\left(\frac{\Delta z_r}{\Delta z_0}\right) \right)^2}. \quad (59)$$

In the general case, for strongly three-dimensional and non-uniform flows, it can be assumed that the tangential velocity has a logarithmic profile of the form (58) inside a boundary layer near the wall. The parabolic viscosity model (57) can therefore also be applied to general flows,

together with definition (59) of the friction coefficient.

The asymptotic analysis in section 3.4 has shown that the friction coefficient  $\alpha$  is of the first order in  $\epsilon$ . Following definition (59), this gives a condition on the ratio between the distance to the wall  $\Delta z_r$  at which the friction condition is imposed, and the distance to the wall  $\Delta z_0$  at which the tangential velocity is zero. It will be checked in the following that the values we consider for this coefficient in section 5 are indeed compatible with a reasonable choice of  $\epsilon$ .

Let us give an expression of the friction correction term with this turbulence model. By analytical computation we have that

$$\int_b^\eta \int_b^z \frac{\rho(\eta - \delta)}{\mu_v} d\delta dz = \frac{\Delta z_r (h + \Delta z_r)}{\kappa u^*} \left( \left( \frac{h + \Delta z_r}{\Delta z_r} \right) \ln \left( \frac{h + \Delta z_r}{\Delta z_r} \right) - h \right).$$

The friction correction (52) then writes:

$$c_\alpha = \frac{2\alpha |\bar{u}|}{\kappa u^*} \left( \frac{(h + \Delta z_r)^2}{h^2} \ln \left( 1 + \frac{h}{\Delta z_r} \right) - \frac{\Delta z_r (h + \Delta z_r)}{h} \right).$$

Coming back to the adimensional variables and using (44), we have that:

$$\frac{\alpha |\hat{u}|}{\hat{u}^*} = \frac{\epsilon \alpha_0 |\hat{u}|}{\hat{u}^*} = \frac{\epsilon \alpha_0 |u(\hat{z}_r)|}{\hat{u}^*} + O(\epsilon^2),$$

where the “hat” denotes here the adimensional variables. Therefore, up to the second order in  $\epsilon$ ,

$$\frac{\alpha |\bar{u}|}{u^*} = \frac{\alpha |u(z_r)|}{u^*}.$$

Now, considering that  $z_r$  is near enough to the wall to be inside the logarithmic layer, we can use equations (58) and (59) to obtain:

$$\frac{\alpha |u(z_r)|}{u^*} = \frac{\alpha}{\kappa} \ln \left( \frac{\Delta z_r}{\Delta z_0} \right) = \sqrt{\alpha \left( 1 - \frac{\Delta z_r}{\Delta z_0} \right)}.$$

We finally retrieve the following expression of the friction correction:

$$c_\alpha = \frac{2\sqrt{\alpha}}{\kappa} \sqrt{\left( 1 - \frac{\Delta z_r}{\Delta z_0} \right)} \left( \frac{(h + \Delta z_r)^2}{h^2} \ln \left( 1 + \frac{h}{\Delta z_r} \right) - \frac{\Delta z_r (h + \Delta z_r)}{h} \right). \quad (60)$$

If the flow is homogeneous in the  $y$ -direction and has a rectangular-cross-section,  $\bar{u} = \frac{Q}{A}$  and the friction term in (51) writes

$$-\frac{\alpha |Q| Q}{h^2 l \left( 1 + \frac{2\sqrt{\alpha}}{\kappa} \sqrt{\left( 1 - \frac{\Delta z_r}{\Delta z_0} \right)} \left( \frac{(h + \Delta z_r)^2}{h^2} \ln \left( 1 + \frac{h}{\Delta z_r} \right) - \frac{\Delta z_r (h + \Delta z_r)}{h} \right) \right)}. \quad (61)$$

## 5 Comparison of the three-dimensional and the section-averaged solutions in the case of flows with rectangular cross-section

Our aim is now to illustrate the accuracy gain achieved by taking into account the correction of the friction term in the section-averaged model. In addition, we want to compare the friction term derived here to one of the empirical closures widely used in computational hydraulics (see e.g. [2]). For this purpose, we restrict ourselves to the case of rectangular cross-section open channels, for which steady state solutions can be computed analytically. Note that these flows are representative of the main physical features of river flows and are commonly used as a first benchmark in many hydraulics applications.

In this case the water depth  $h$  is constant along the  $y$ -direction and, denoting by  $l$  the width of the river, the section area is  $A = lh$ . In addition we suppose  $\mu_e = \mu_h = 0$ . The section averaged shallow water model then writes in the more classical form:

$$\begin{cases} \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \beta \frac{Q^2}{A} \right) + gA \frac{\partial \eta}{\partial x} = - \int_{l_1}^{l_2} \frac{\alpha |\bar{u}|}{1 + c_\alpha} \bar{u} dy \\ \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0. \end{cases} \quad (62)$$

Note that the Boussinesq term (53) then reduces to  $\beta = l \frac{\int_{l_1}^{l_2} \bar{u}^2 dy}{\left( \int_{l_1}^{l_2} \bar{u} dy \right)^2}$ .

We emphasize the fact that in this particular case we obtain the classical section-averaged equations,[3] with a correction of the friction term.

**Remark 5.1** *If the flow is homogeneous in the  $y$ -direction, we have that  $\bar{u} = \frac{Q}{A}$  and therefore the friction term writes*

$$- \frac{\alpha |Q| l}{A^2 (1 + c_\alpha)} Q.$$

*Without correction the friction term reduces to*

$$- \frac{\alpha |Q| l}{A^2} Q,$$

*which is the expression of the friction in the classical section-averaged shallow water equations.*

We choose a three-dimensional test case with an analytic solution, to be compared to the analytic solution of the section-averaged model with and without friction correction. The test case consists of a steady state turbulent flow in a channel with a slight slope  $i_F$ , as illustrated in figure 2.

We take the channel as the reference configuration  $-(x, y, z)$  in figure 2 – and we suppose that  $\mu_h = \mu_e = 0$ . The flow is steady and uniform in the  $x$ -direction, and the free surface is perfectly parallel to the bottom, that is:

$$\nabla \eta = \left( \frac{\partial b}{\partial x}, 0 \right)^T, \quad \mathbf{U} = (u, 0, 0)^T. \quad (63)$$

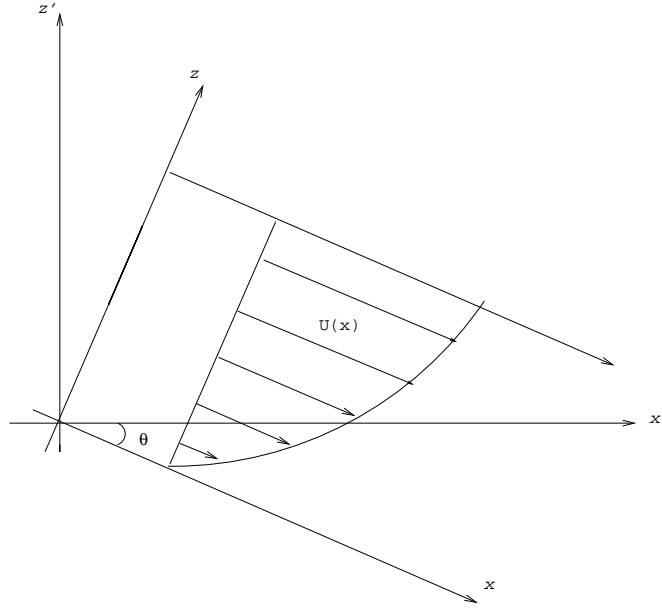


Figure 2: Uniform flow in a chanel with slope

Rewriting the three-dimensional RANS equations (1) in the new reference configuration, and considering (63), we retrieve the following system:

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = g \sin \theta + (\cos^2 \theta - \sin^2 \theta)^2 \frac{\partial}{\partial z} \left( \frac{\mu_v}{\rho} \frac{\partial u}{\partial z} \right) \quad (64)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = -g \cos \theta + 2 \cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta) \frac{\partial}{\partial z} \left( \frac{\mu_v}{\rho} \frac{\partial u}{\partial z} \right), \quad (65)$$

where  $\theta$  is the angle of the slope. The boundary conditions on the free surface are:

$$p = p_a \quad \text{and} \quad \frac{\mu_v}{\rho} \frac{\partial u}{\partial z} = 0, \quad (66)$$

and near the bottom at  $z = z_r$ :

$$\frac{\mu_v}{\rho} \frac{\partial u}{\partial z} = \alpha \psi(\theta) |u| u, \quad (67)$$

where  $\psi(\theta) = \frac{1}{(\cos^2 \theta - \sin^2 \theta)^2}$ .

Since  $u$  does not depend on  $x$ , we deduce from (65) that  $\frac{\partial p}{\partial z}$  is independent of  $x$ , and therefore also the pressure  $p$ . Thus  $\frac{\partial p}{\partial x} = 0$  and equation (64) reduces to:

$$\frac{\partial}{\partial z} \left( \frac{\mu_v}{\rho} \frac{\partial u}{\partial z} \right) = -g \phi(\theta), \quad (68)$$

where  $\phi(\theta) = \frac{\sin \theta}{(\cos^2 \theta - \sin^2 \theta)^2}$ . Integrating (68) from an arbitrary elevation  $z$  to the free surface  $\eta$ , and using boundary condition (66) we obtain:

$$\frac{\partial u}{\partial z} = g \phi(\theta) \frac{\rho(\eta - z)}{\mu_v}. \quad (69)$$



Integrating now (69) from  $z$  to the reference depth  $z_r$  near the bottom we obtain the following expression of the velocity:

$$u = u(z_r) + g\phi(\theta) \int_{z_r}^z \frac{\rho(\eta - \delta)}{\mu_v} d\delta. \quad (70)$$

This expression can be vertically-integrated on the entire water column in order to retrieve an expression of the flow. Indeed,

$$Q = \int_{l_1}^{l_2} \int_{z_r}^{\eta} u dz dy = \int_{l_1}^{l_2} \left( h u(z_r) + g\phi(\theta) \int_{z_r}^{\eta} \int_{z_r}^z \frac{\rho(\eta - \delta)}{\mu_v} d\delta dz \right) dy,$$

and since the flow is homogeneous in the  $y$ -direction:

$$Q = A u(z_r) + gl\phi(\theta) \int_{z_r}^{\eta} \int_{z_r}^z \frac{\rho(\eta - \delta)}{\mu_v} d\delta dz.$$

Let us now retrieve an expression of the velocity near the bottom at  $z = z_r$ . From (69) with  $z = z_r$  we have that

$$\frac{\mu_v}{\rho} \frac{\partial u}{\partial z} \Big|_{z=z_r} = g\phi(\theta)h.$$

Using boundary condition (67) we obtain:

$$\alpha|u(z_r)|u(z_r)\psi(\theta) = gh\phi(\theta),$$

and since in the particular case we are considering the velocity is always positive, we have that:

$$u(z_r) = \sqrt{\frac{gh \sin \theta}{\alpha}}. \quad (71)$$

Finally we have derived the following expression of the flow:

$$q = \frac{Q}{l} = h \sqrt{\frac{gh \sin \theta}{\alpha}} + g\phi(\theta) \int_{z_r}^{\eta} \int_{z_r}^z \frac{\rho(\eta - \delta)}{\mu_v} d\delta dz, \quad (72)$$

which is an analytic solution of the three-dimensional problem considered in this section.

This three-dimensional solution is to be compared with the analytic solution of the section-averaged model (62) with and without friction correction. In the particular case considered here we can easily derive the following analytic solution to the section-averaged equations:

$$q = h \sqrt{\frac{gh i_F}{\alpha} (1 + c_\alpha)}. \quad (73)$$

Note that if the correction of the friction term is not taken into account in the section-averaged model, the analytic solution is:

$$q = h \sqrt{\frac{gh i_F}{\alpha}}. \quad (74)$$

Since  $c_\alpha$  and  $\mu_v$  can depend on the flow rate  $q$ , equations (72), (73) and (74) yield an implicit relation between  $q$  and  $h$ . We have solved this relation for different values of the water depth  $h$ , in order to compare the analytic solutions of the different models. We have made several comparisons that we describe next, and for which we have used a density  $\rho = 1\text{kg/m}^3$  and a

value of the slope  $i_F = 10^{-4}$ .

First, we have compared the solutions in the laminar case, with a constant vertical viscosity  $\mu = 0.01$ . In this case, consistently with the discussion in section 4.1, we have used a constant friction coefficient  $\alpha$  given by the Chézy law

$$\alpha = g/\chi^2, \quad (75)$$

where  $g$  is the gravity and  $\chi$  is the Chézy coefficient. Figures 3 and 4 show the profile of the analytical water depth  $h$  as a function of the flow  $q$  for  $\chi = 66.5 \text{ m}^{1/2}/\text{s}$  and  $\chi = 30 \text{ m}^{1/2}/\text{s}$ , respectively. The solid line corresponds to the three-dimensional solution given by (72), the starred line corresponds to the solution to the section-averaged model with friction correction given by (73), whereas the dotted line corresponds to the solution without friction correction (74).

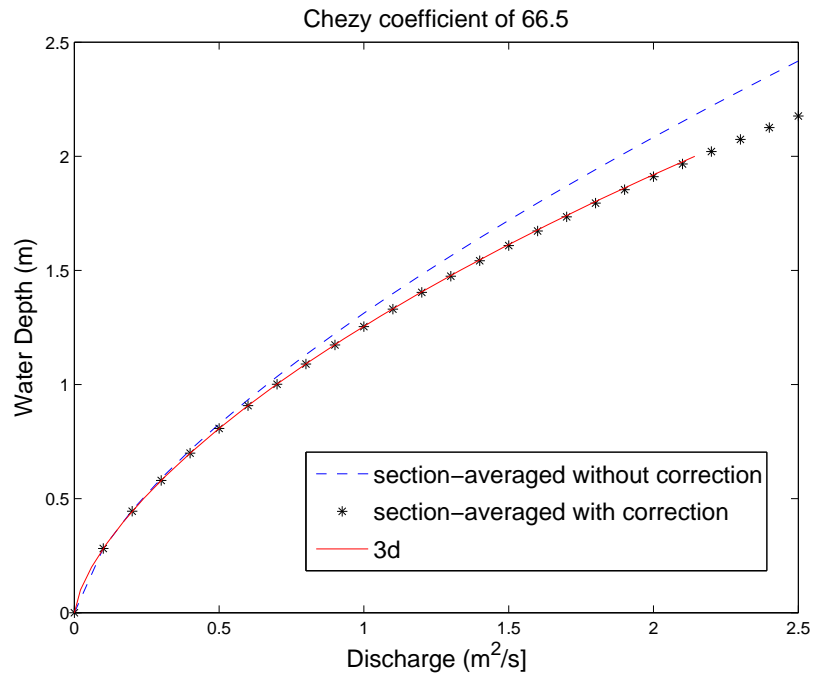


Figure 3: Analytic solutions for the three-dimensional problem (solid line), the section-averaged problem with friction correction (starred line) and without correction (dashed line). Laminar case with  $\nu = 0.01$  and with  $\chi = 66.5$ .

We have then compared the solution of the different models in the turbulent case, using the parabolic model (57) for the turbulent vertical viscosity, and the friction coefficient  $\alpha$  given by definition (59). For this case we have chosen  $\Delta z_0 = 10^{-5} \text{ m}$  and  $\Delta z_r = 0.03h \text{ m}$ , but similar results can be obtained with a wide range of values for these layer amplitudes. Figure 5 shows the profile of the analytic water depth  $h$  as a function of the flow  $q$  for the different models.

As we can see, the analytic solution of the section-averaged model is much closer to the three-dimensional solution when the friction correction is taken into account. This is true when

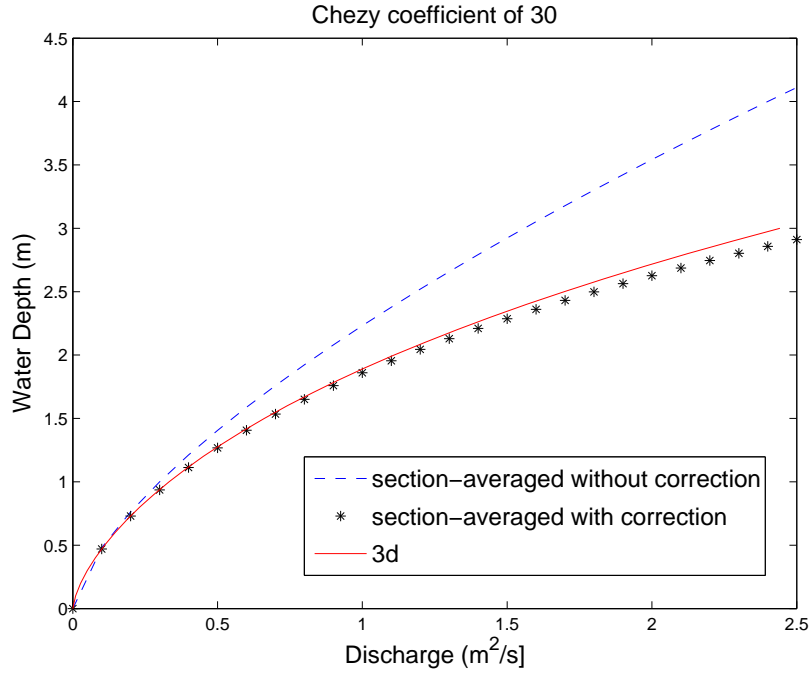


Figure 4: Analytic solutions for the three-dimensional problem (solid line), the section-averaged problem with friction correction (starred line) and without correction (dashed line). Laminar case with  $\nu = 0.01$  and with  $\chi = 30$ .

taking a constant vertical viscosity, as well as when using the parabolic turbulence model. The results obtained in this test case, which is a relevant regime for river hydraulics, confirm that the classical friction term in the section-averaged shallow water equations should be corrected as defined in (51).

However, in hydraulic engineering, the classical section-averaged model is often used with empirical closures that yield a friction coefficient  $\alpha$  depending on the fluid depth. For instance, in the case of uniform flows, the friction coefficient  $\alpha$  used in the section-averaged equations can be derived from the Chézy law (75) by some assumption on the turbulent velocity profile. In the case of a logarithmic velocity profile and assuming the standard values for the von Karman constant, one can derive the following formula for the Chézy coefficient:

$$\chi = 7.83 \ln \left( \frac{h + \Delta z_r}{e \Delta z_0} \right), \quad (76)$$

where we recall that  $\Delta z_0$  is the distance at which the velocity is supposed to be zero, and  $\Delta z_r$  is the reference distance to the wall at which the wall law condition is imposed. Carrying out the same derivation for the three-dimensional equations yields instead the following formula for the Chézy coefficient:

$$\chi = 7.83 \ln \left( \frac{2 \Delta z_r}{e \Delta z_0} \right). \quad (77)$$

Our aim is now to compare, in the case of uniform steady state flows, the solutions of the different models: 1) the three-dimensional model, 2) the section averaged model with and 3)

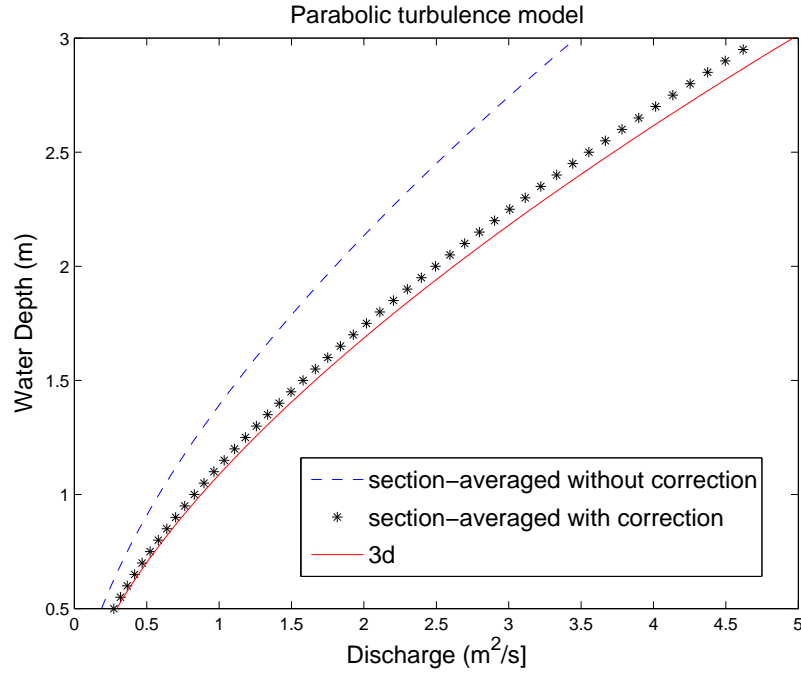


Figure 5: Analytic solutions for the three-dimensional problem (solid line), the section-averaged problem with friction correction (starred line) and without correction (dashed line). Case with a parabolic turbulent and  $\alpha$  given by (59).

without friction correction, all using the analytic value of the friction coefficient  $\alpha$  given by (59), and 4) the solution of the section averaged model using the empirical friction closure (76). Of course we consider the parabolic model (57) for the vertical viscosity.

Let us first compare the values of the friction coefficient  $\alpha$  using the different formulas. Figure 6 shows the profile as a function of the water depth of, respectively, the three-dimensional analytic friction coefficient  $\alpha$  defined by (59) (solid line), the friction coefficient resulting from the empirical closure for the three-dimensional model (starred line), given by (75) with a Chézy coefficient computed using formula (77), and the coefficient used for the section-averaged model (dashed line), with a Chézy coefficient computed using formula (76). As we can see, the friction coefficient resulting from the empirical closure for the three-dimensional model is quasi-identical to the three-dimensional analytic value, whereas the coefficient used for the section-averaged model is significantly different.

Let us now compare the solutions for the different models using the different values of the friction coefficient  $\alpha$ . Figure 7 shows the profile of the analytical water depth  $h$  as a function of the flow  $q$ . As in figure 5, the solid line corresponds to the solution (72) of the three-dimensional model, the starred line to the section-averaged solution (73) with friction correction, and the dashed line corresponds to the solution without friction correction (74). These three solutions have been computed using the analytic friction coefficient  $\alpha$  defined by (59). The additional profile is the dotted line, which corresponds to the section-averaged solution without friction correction (74), but with the empirical friction coefficient  $\alpha$  given by formula (76). We see

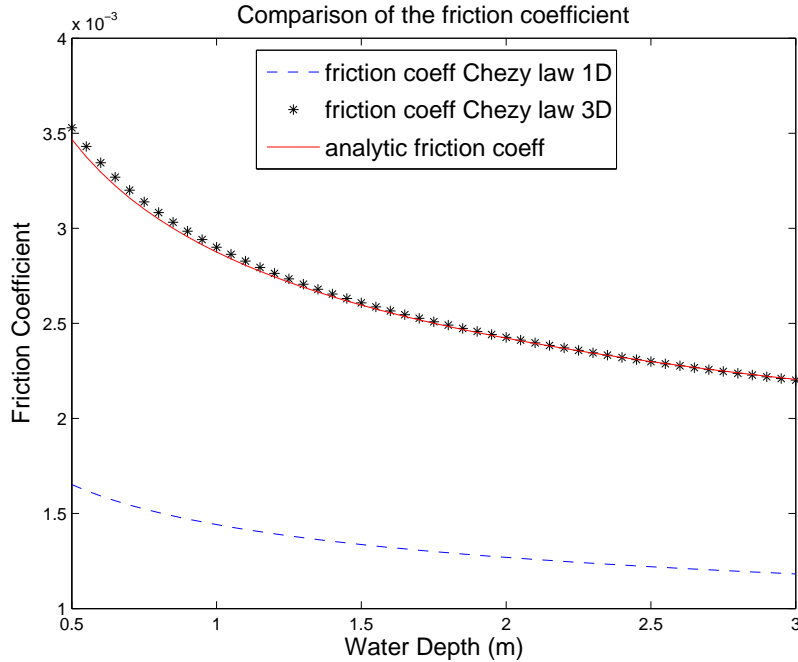


Figure 6: Values of the friction coefficient  $\alpha$  with respect to the water depth. Analytic value given by (59) (solid line), value for the three-dimensional model (starred line) given by (75) and (77), and value for the section-averaged model (dashed line), given by (75) and (76).

that this profile is identical to the solution (73) of the section-averaged model with friction correction using the friction coefficient  $\alpha$  defined by (59). We can therefore conclude that, in this case, the correction  $c_\alpha$  defined by (52) and derived in this work for the friction term of the section-averaged model, has the same effect on the solution of the model as the introduction of an empirical closure formula for the friction coefficient. However, our correction is independent of the uniformity assumptions that are instead necessary to derive formula (76). Thus, our correction can be applied in a much wider range of flows, with a rigorous justification based on the asymptotic derivation we have presented.

## 6 Conclusions

In this paper, we have extended the analysis of Ref. [7] to the three-dimensional RANS equations with anisotropic Reynolds tensor for free surface flows in arbitrary geometry. A rigorous derivation of a section-averaged system has been presented, including the effects of eddy viscosity and friction. When applied to flows with rectangular cross-section, this system is similar to the classical section-averaged shallow water equations,[11] except for the friction term. Indeed, our derivation shows that, in order to take into account effects up to the second order in the asymptotic parameter, the classical friction term should be corrected by a term which depends on the turbulent vertical viscosity. The generalized friction term obtained does not rely on local uniformity assumptions and can be computed directly from three-dimensional turbulence models, without need for local uniformity assumptions. This conclusion is in good agreement with the one achieved by Gerbeau *et al.* in Ref. [7] for two-dimensional flows with constant vis-

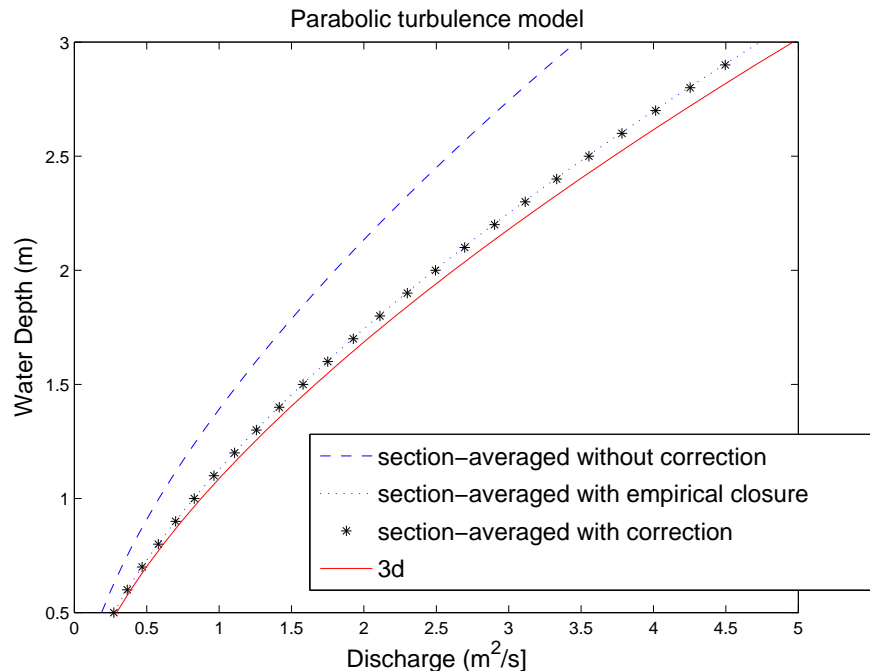


Figure 7: Analytic solutions for the three-dimensional problem (solid line), the section-averaged problem with friction correction (starred line) and without correction (dashed line). Case with a parabolic turbulent and  $\alpha$  given by (59).

cosity over a flat bottom. Indeed, if the vertical viscosity is taken constant and linear boundary conditions are considered, and when the flow is homogeneous in the transversal direction, we retrieve the same friction correction as in Ref. [7]. Our derivation provides the expression of the friction correction term in a more general case than those treated by Ref. [7], including turbulent flows, non-linear boundary conditions and three-dimensional arbitrary geometries. In particular, we compute the correction term associated to a specific model for the vertical profile of turbulent velocity. For steady state open channel flows admitting analytic solutions of the three-dimensional as well as the simplified models, we have shown that the solutions computed including our correction term are much closer to those of the three dimensional model than those of the standard shallow water model. Furthermore, we show that our formulation yields results that are very similar to those obtained including in the classical equations empirical friction closures derived in computational hydraulics. The generalized friction term resulting from the present derivation can also be interpreted as a generalization and an *a posteriori* justification of these empirical closures, that allows to avoid the assumptions on local flow uniformity on which these closures rely.

In a forthcoming work, we plan to take advantage of the present results by including the friction correction term in section averaged models such as the one proposed by Deponti *ea* in Ref. [3]. Its use is also expected to ease the coupling of three- and one-dimensional free surface models in the framework of an integrated hydrological basin model.

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