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Cauchy-Pompeiu type formulas for $\bar{\partial}$ on affine algebraic Riemann surfaces and some applications

G.M.Henkin

Abstract

We have constructed the explicit formulas $f = \hat{R}\varphi$ and $u = R_\lambda f$ for solutions of $\bar{\partial}f = \varphi$ and of $(\partial + \lambda dz_1)u = f - \mathcal{H}f$ on affine algebraic curve $V \subset \mathbb{C}^2$, where $\mathcal{H}f$ is projection of $f \in W_{1,0}^{\tilde{p}}(V)$ on the subspace of holomorphic (1,0)-forms on V , $\tilde{p} > 2$. These formulas can be interpreted as explicit versions and precisions of the Hodge-Riemann decomposition on Riemann surfaces. The main application consists in the construction of the Faddeev-Green function for $\bar{\partial}(\partial + \lambda dz_1)$ on V as the kernel of operator $R_\lambda \circ \hat{R}$. Basing on this in the separate paper [HM] we have extended from the case $X \subset \mathbb{C}$ to the case of bordered Riemann surface $X \subset V$ the R.Novikov scheme [N1] for the effective reconstruction of conductivity function σ in the equation $d(\sigma d^c U) = 0$ through the Dirichlet-to-Neumann mapping

$$U|_{bX} \mapsto \sigma d^c U|_{bX}.$$

Introduction

This paper is motivated by one of the problems of two-dimensional Electrical Impedance Tomography, namely by the question of reconstruction of conductivity function σ on a bordered two-dimensional Riemann surface X from the knowledge of Dirichlet-to-Neumann mapping $u|_{bX} \rightarrow \sigma d^c U|_{bX}$ for solutions U of the Dirichlet problem:

$$d(\sigma d^c U)|_X = 0, \quad U|_{bX} = u, \quad \text{where } d = \partial + \bar{\partial}, \quad d^c = i(\bar{\partial} - \partial).$$

For the case $X = \Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ ($z = x_1 + ix_2$) the exact reconstruction scheme was given firstly by R.Novikov [N1] under some restriction on conductivity function σ , This restriction was eliminated later by A.Nachman [Na].

The scheme consists in the following.

Let $\sigma(x) > 0$ for $x \in \bar{\Omega}$ and $\sigma \in C^{(2)}(\bar{\Omega})$. Put $\sigma(x) = 1$ for $x \in \mathbb{R}^2 \setminus \bar{\Omega}$. Substitution $\psi = \sqrt{\sigma}U$ transforms equation $d(\sigma d^c U) = 0$ into equation $dd^c \psi = \frac{d d^c \sqrt{\sigma}}{\sqrt{\sigma}} \psi$ on \mathbb{R}^2 . From L.Faddeev [F1] result (with additional arguments [BC2] and [Na]) it follows that $\forall \lambda \in \mathbb{C} \exists!$ solution $\psi(z, \lambda)$ of the above equation with asymptotics

$$\psi(z, \lambda) \cdot e^{-\lambda z} \stackrel{\text{def}}{=} \mu(z, \lambda) = 1 + o(1), \quad z \rightarrow \infty.$$

Such solution can be found from the integral equation

$$\mu(z, \lambda) = 1 + \frac{i}{2} \int_{\xi \in \Omega} g(z - \xi, \lambda) \frac{\mu(\xi, \lambda) d d^c \sqrt{\sigma}}{\sqrt{\sigma}},$$

where function

$$g(z, \lambda) = \frac{-1}{2i(2\pi)^2} \int_{w \in \mathbb{C}} \frac{e^{i(w\bar{z} + \bar{w}z)} dw \wedge d\bar{w}}{w(\bar{w} - i\lambda)}, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{C}.$$

is called the Faddeev-Green function for operator $\mu \mapsto \bar{\partial}(\partial + \lambda dz)\mu$.

From R.Novikov [N1] work it follows that function $\psi|_{b\Omega}$ can be found through Dirichlet-to-Neumann mapping by integral equation

$$\psi(z, \lambda)|_{b\Omega} = e^{\lambda z} + \int_{\xi \in b\Omega} e^{\lambda(z-\xi)} g(z - \xi, \lambda) (\hat{\Phi}\psi(\xi, \lambda) - \hat{\Phi}_0\psi(\xi, \lambda)),$$

$$\text{where } \hat{\Phi}\psi = \bar{\partial}\psi|_{b\Omega}, \quad \hat{\Phi}_0\psi = \bar{\partial}\psi_0|_{b\Omega}, \quad \psi_0|_{b\Omega} = \psi, \quad \bar{\partial}\bar{\partial}\psi|_{\Omega} = 0.$$

From works of R.Beals, R.Coifmann [BC1], P.Grinevich, S.Novikov [GN] and R.Novikov [N2] it follows that $\psi(z, \lambda)$ satisfies $\bar{\partial}$ -equation of Bers-Vekua type with respect to $\lambda \in \mathbb{C}$:

$$\frac{\partial\psi}{\partial\lambda} = b(\lambda)\bar{\psi},$$

where $\lambda \mapsto b(\lambda) \in L^{2+\varepsilon}(\mathbb{C}) \cap L^{2-\varepsilon}(\mathbb{C})$ and $\psi(z, \lambda)e^{-\lambda z} \rightarrow 1, \lambda \rightarrow \infty, \forall z \in \mathbb{C}$.

This $\bar{\partial}$ -equation combined with R.Novikov's integral equation permit to find from the Dirichlet-to-Neumann mapping, at first, $\psi|_{b\Omega}$, secondly, " $\bar{\partial}$ - scattering data $b(\lambda)$ " and then $\psi|_{\Omega}$.

Finally, conductivity $\sigma|_{\Omega}$ is found from Dirichlet-to-Neumann data by the scheme:

$$\text{DN data} \rightarrow \psi|_{b\Omega} \rightarrow \bar{\partial} - \text{scattering data} \rightarrow \psi|_{\Omega} \rightarrow \frac{d d^c \sqrt{\sigma}}{\sqrt{\sigma}}|_{\Omega}.$$

Main result

We suppose that instead of \mathbb{C} we have smooth algebraic Riemann surface V in \mathbb{C}^2 , given by equation $V = \{z \in \mathbb{C}^2 : P(z) = 0\}$, where P holomorphic polynomial of degree $d \geq 1$. Put $z_1 = w_1/w_0, z_2 = w_2/w_0$ and suppose that projective compactification \tilde{V} of V in $\mathbb{C}P^2 \supset \mathbb{C}^2$ with coordinates $w = (w_0 : w_1 : w_2)$ intersects $CP_{\infty}^1 = \{z \in \mathbb{C}P^2 : w_0 = 0\}$ transversally in d points. In order to extend the Novikov's reconstruction scheme on Riemann surface $V \subset \mathbb{C}^2$ we need, firstly, to find appropriate Faddeev type Green function for $\bar{\partial}(\partial + \lambda dz_1)$ on V . One can check that for the case $V = \mathbb{C}$ the Faddeev Green function $g(z, \lambda)$ is composition of Cauchy-Green-Pompeiu kernels for operators $f \mapsto \varphi = \bar{\partial}f$ and $u \mapsto f = (\partial + \lambda dz)u$, where u, f, φ are correspondingly function, (1,0)-form and (1,1)-form on \mathbb{C} . Namely,

$$g(z, \lambda) = \frac{-1}{i(2\pi)^2} \int_{w \in \mathbb{C}} \frac{e^{\lambda w - \bar{\lambda} \bar{w}} dw \wedge d\bar{w}}{(w + z) \cdot \bar{w}}.$$

The main purpose of this paper is to construct analogue of Faddeev's Green function on Riemann surface V . To do this we need to find explicit formulae (with appropriate estimates) $f = \hat{R}\varphi$ and $u = R_{\lambda}f$ for solutions of equations $\bar{\partial}f = \varphi$ and $(\partial + \lambda dz_1)u = f - \mathcal{H}f$ on V , where for V equipped by euclidean volume form $dd^c|z|^2$, $\varphi \in L_{1,1}^{\infty}(V) \cap L_{1,1}^1(V)$, $f \in W_{1,0}^{1,\tilde{p}}(V)$ and $u \in W^{2,\tilde{p}}(V)$, $\tilde{p} > 2$, $\mathcal{H}f$ projection of f on subspace of holomorphic (1,0)-forms on \tilde{V} .

The new formulas obtained in this paper for solution of $\bar{\partial}f = \varphi$ and $(\partial + \lambda dz_1)u = f$ on V one can interpretate as explicit versions and precisions of the Hodge-Riemann decomposition results on Riemann surfaces. We will define the Faddeev type Green function for $\bar{\partial}(\partial + \lambda dz_1)$ on V as kernel $g_\lambda(z, \xi)$ of operator $G_\lambda = R_\lambda \circ \hat{R}$.

Further results

In the separate paper [HM] using Faddeev type Green function, constructed here, we have obtained on the Riemann surface V the natural analogues of all steps of the Novikov reconstruction scheme.

The existence (and uniqueness) of solution $\mu(z, \lambda)$ of Faddeev type integral equation

$$\mu(z, \lambda) = 1 + \frac{i}{2} \int_{\xi \in V} g_\lambda(z, \xi) \frac{\mu(\xi, \lambda) d d^c \sqrt{\sigma}}{\sqrt{\sigma}}, \quad z \in V, \quad \lambda \in \mathbb{C}$$

is obtained under assumption $\sigma \in C^{(3)}(V)$.

§1. Cauchy-Pompeiu type formula on affine algebraic Riemann surface

By $L_{p,q}(V)$ we denote the space of (p,q)-forms on V with coefficients in distributions of measure type on V . By $L_{p,q}^s(V)$ we denote the space of (p,q)-forms on V with absolutely integrable in degree $s \geq 1$ coefficients with respect to euclidiene volume form on V . If $V = \mathbb{C}$ and f is a function from $L^1(\mathbb{C})$ such that $\bar{\partial}f \in L_{0,1}(\mathbb{C})$, then the generalized Cauchy formula has the following form

$$f(z) = -\frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\bar{\partial}f(\xi) \wedge d\xi}{\xi - z}, \quad z \in \mathbb{C}.$$

This formula becomes the classical Cauchy formula, when $f = 0$ on $\mathbb{C} \setminus \Omega$ and $f \in \mathcal{O}(\Omega)$, where Ω is some bounded domain with rectifiable boundary in \mathbb{C} . The generalized Cauchy formula was discovered by Pompeiu [P1] for solution of the Painleve problem, i.e. for proving of existence for totally disconnected compact sets with positive Lebesgue measure of non-zero functions $f \in \mathcal{O}(\mathbb{C} \setminus E) \cap C(\mathbb{C}) \cap L^1(\mathbb{C})$. The Cauchy-Pompeiu formula has many fundamental applications: theory of distributions (L.Schwartz), approximations problems (E.Bishop, S.Mergelyan, A.Vitushkin), corona problem (L.Carleson), pseudo-analytic functions (L.Bers, I.Vekua), inverse scattering and integrable equations (R.Beals, R.Coifman, M.Ablowitz, D.Bar Yaacov, A.Fokas)....

Motivated by applications to electrical impedance tomography we develop in this paper the Cauchy-Pompeiu type formulas on affine algebraic Riemann surfaces $V \subset \mathbb{C}^2$ and give some applications. A version of the Cauchy-Pompeiu formula on compact Riemann surface for generalization of the Riemann-Roch theorem on pseudoanalytic functions on such surfaces was found by Yu.Rodin [R].

Let \tilde{V} be smooth algebraic curve in $\mathbb{C}P^2$ given by equation

$$\tilde{V} = \{w \in \mathbb{C}P^2 : \tilde{P}(w) = 0\},$$

where \tilde{P} be homogeneous holomorphic polynomial of homogeneous coordinates

$w = (w_0 : w_1 : w_2) \in \mathbb{C}P^2$. Without restriction of generality we suppose that

- i) \tilde{V} intersects $\mathbb{C}P^1_\infty = \{w \in \mathbb{C}P^2 : w_0 = 0\}$ transversally, $\tilde{V} \cap \mathbb{C}P^1_\infty = \{a_1, \dots, a_d\}$,
 $d = \deg \tilde{P}$,
- ii) $V = \tilde{V} \setminus \mathbb{C}P^1_\infty$ is a connected curve in \mathbb{C}^2 with equation $V = \{z \in \mathbb{C}^2 : P(z) = 0\}$,
where $P(z) = \tilde{P}(1, z_1, z_2)$ such that

$$\left| \frac{\partial P / \partial z_1}{\partial P / \partial z_2} \right| \leq \text{const}(V), \quad \text{if } |z_1| \geq r_0 = \text{const}(V),$$

- iii) For any $z^* \in V$, such that $\frac{\partial P}{\partial z_2}(z^*) = 0$ we have $\frac{\partial^2 P}{\partial z_2^2}(z^*) \neq 0$.

By Hurwitz-Riemann theorem number of such points of ramification is equal to $d(d-1)$.

Let us equip V by euclidean volume form $d d^c |z|^2$.

Notation

Let us denote by $H_{0,1}^p(V)$ the subspace in $L_{0,1}^p(V)$, $1 < p < 2$, consisting from antiholomorphic forms. $\forall p \in (1, 2)$, the space $H_{0,1}^p(V)$ coincides with the space of antiholomorphic forms on V admitting antiholomorphic extension on compactification $\tilde{V} \subset \mathbb{C}P^2$. Hence, by classical Riemann-Klebsch theorem $\dim_{\mathbb{C}} H_{0,1}^p(V) = \frac{(d-1)(d-2)}{2} \quad \forall p \in (1, 2)$.

Proposition 1

There exist operators $R_1 : L_{0,1}^p(V) \rightarrow L^{\tilde{p}}(V)$ and $R_0 : L_{0,1}^p(V) \rightarrow W^{1,\tilde{p}}(V)$ and $\mathcal{H} : L_{0,1}^p(V) \rightarrow H_{0,1}^p(V)$, $1 < p < 2$, $1/\tilde{p} = 1/p - 1/2$ such that $\forall \Phi \in L_{0,1}^p(V)$ we have decomposition

$$\Phi = \bar{\partial} R \Phi + \mathcal{H}(\Phi), \quad \text{where } R = R_1 + R_0, \tag{1.1}$$

$$R_1 \Phi = \frac{1}{2\pi i} \int_{\xi \in V} \Phi(\xi) \frac{d\xi_1}{\partial \xi_2} \det \left[\frac{\partial P}{\partial \xi}(\xi), \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2} \right]. \tag{1.2}$$

Remark 1

If $p \in [1, 2)$ and $q \in (2, \infty]$ the condition $\Phi \in L_{0,1}^p(V) \cap L_{0,1}^q(V)$ implies $R\Phi \in C(\tilde{V})$.

Remark 2

For the case when $V = \mathbb{C} = \{z \in \mathbb{C}^2 : z_2 = 0\}$ Proposition 1 and Remark 1 are reduced to the classical results of Pompeiu [P1], [P2] and of Vekua [V].

Remark 3

Basing on the technique of [HP] one can construct explicit formula not only for the main part of R -operator R_1 , but for the whole operator R . It can be useful for numerical implementaions of the formulas of this paper.

Proof of Proposition 1

Let $Q(\xi, z) = \{Q_1(\xi, z), Q_2(\xi, z)\}$ be a couple of holomorphic polynomials in $\xi = (\xi_1, \xi_2)$ and $z = (z_1, z_2)$ such that

$$Q(\xi, \xi) = \frac{\partial P}{\partial \xi}(\xi) \quad \text{and} \tag{1.3}$$

$$P(\xi) - P(z) = Q_1(\xi, z)(\xi_1 - z_1) + Q_2(\xi, z)(\xi_2 - z_2) \stackrel{\text{def}}{=} \langle Q(\xi, z), \xi - z \rangle .$$

The conditions i), ii) imply that for ε_0 small enough there exists holomorphic retraction $z \rightarrow r(z)$ of the domain $\mathcal{U}_{\varepsilon_0} = \{z \in \mathbb{C}^2 : |P(z)| < \varepsilon_0\}$ on the curve V .

Put $\mathcal{U}_{\varepsilon,r} = \{z \in \mathbb{C}^2 : |P(z)| < \varepsilon, |z_1| < r\}$, where $0 < \varepsilon \leq \varepsilon_0$ and $r \geq r_0$. Put $V^c = \{z \in \mathbb{C}^2 : P(z) = c\}$, $c \in \mathbb{C}$, $|c| \leq \varepsilon_0$ and $\tilde{\Phi}(z) = \Phi(r(z))$, $z \in \mathcal{U}_{\varepsilon_0}$. The condition $\tilde{\Phi} \in L^p_{0,1}(V)$ and properties of retraction $z \rightarrow r(z)$ implies that $\bar{\partial}\tilde{\Phi} = 0$ on $\mathcal{U}_{\varepsilon_0}$ and

$$\|\tilde{\Phi}\|_{L^p(V^c)} \leq \text{const}(V) \cdot \|\Phi\|_{L^p(V)}, \quad (1.4)$$

uniformly with respect to $c : |c| \leq \varepsilon_0$. By references [H] and [Po] we can choose the following explicit solution

$$\tilde{F}_{\varepsilon,r} \text{ on } \mathcal{U}_{\varepsilon,r} \text{ of the } \bar{\partial} - \text{equation } \bar{\partial}\tilde{F}_{\varepsilon,r} = \tilde{\Phi}|_{\mathcal{U}_{\varepsilon,r}} : \quad (1.5)$$

$$\begin{aligned} \tilde{F}_{\varepsilon,r}(z) = & \left(\frac{1}{2\pi i}\right) \left\{ \int_{\xi \in \mathcal{U}_{\varepsilon,r}} \tilde{\Phi} \wedge \det \left[\frac{\bar{\xi} - \bar{z}}{|\xi - z|^2}, \bar{\partial}_{\xi} \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2} \right] \wedge d\xi_1 \wedge d\xi_2 + \right. \\ & \int_{\xi \in b\mathcal{U}_{\varepsilon,r}: |\xi_1|=r} \tilde{\Phi} \wedge \left[-\frac{(\bar{\xi}_2 - \bar{z}_2)}{(\xi_1 - z_1)|\xi - z|^2} \right] \wedge d\xi_1 \wedge d\xi_2 + \\ & \left. \int_{\xi \in b\mathcal{U}_{\varepsilon,r}: |P(\xi)|=\varepsilon} \tilde{\Phi} \wedge \det \left[\frac{\bar{\xi} - \bar{z}}{|\xi - z|^2}, \frac{Q}{P(\xi) - P(z)} \right] \wedge d\xi_1 \wedge d\xi_2 \right\}, \quad z \in \mathcal{U}_{\varepsilon,R}. \end{aligned}$$

The property (1.4) implies that for any $z \in V$ we have

$$\begin{aligned} \int_{\xi \in \mathcal{U}_{\varepsilon,r}} \tilde{\Phi} \wedge \det \left[\frac{\bar{\xi} - \bar{z}}{|\xi - z|^2}, \bar{\partial}_{\xi} \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2} \right] \wedge d\xi_1 \wedge d\xi_2 & \rightarrow 0, \quad \varepsilon \rightarrow 0 \quad \text{and} \\ \int_{\xi \in b\mathcal{U}_{\varepsilon,r}: |\xi_1|=r} \tilde{\Phi} \wedge \left[-\frac{(\bar{\xi}_2 - \bar{z}_2)}{(\xi_1 - z_1)|\xi - z|^2} \right] \wedge d\xi_1 \wedge d\xi_2 & \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad r \rightarrow \infty. \end{aligned}$$

Hence $\forall z \in V$ there exists $\lim_{\substack{\varepsilon \rightarrow 0 \\ r \rightarrow \infty}} \tilde{F}_{\varepsilon,r} = \tilde{F}(z)$, where

$$\tilde{F}(z) = -\frac{1}{2\pi i} \int_{\xi \in V} \frac{\Phi d\xi_1}{\frac{\partial P}{\partial \xi_2}(\xi)} \wedge \det \left[\frac{\bar{\xi} - \bar{z}}{|\xi - z|^2}, Q(\xi, z) \right]. \quad (1.6)$$

From (1.5), (1.6) it follows that

$$\bar{\partial}_z \tilde{F}|_V = \Phi(z). \quad (1.7)$$

Put $F_1 = R_1\Phi$. Using (1.2), (1.3), (1.6), (1.7) we obtain

$$\begin{aligned} \bar{\partial}_z F_1(z)|_V &= \frac{1}{2\pi i} \int_{\xi \in V} \Phi(\xi) \wedge \frac{d\xi_1}{\frac{\partial P}{\partial \xi_2}} \wedge \det \left[\frac{\partial P}{\partial \xi}(\xi), \bar{\partial}_z \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2} \right] = \\ & \Phi + \frac{1}{2\pi i} \int_{\xi \in V} \Phi(\xi) \wedge \frac{d\xi_1}{\frac{\partial P}{\partial \xi_2}} \wedge \frac{1}{|\xi - z|^4} \det \begin{vmatrix} \frac{\partial P}{\partial \xi_1}(\xi) & \xi_2 - z_2 \\ \frac{\partial P}{\partial \xi_2}(\xi) & -(\xi_1 - z_1) \end{vmatrix} \det \begin{vmatrix} \bar{\xi}_1 - \bar{z}_1 & d\bar{z}_1 \\ \bar{\xi}_2 - \bar{z}_2 & d\bar{z}_2 \end{vmatrix} = \\ & \Phi + K\Phi, \end{aligned}$$

where

$$K\Phi = \frac{1}{2\pi i} \int_{\xi \in V} \frac{\Phi(\xi) \wedge d\xi_1}{|\xi - z|^4} \wedge \frac{\langle \frac{\partial P}{\partial \xi}(\xi), \xi - z \rangle \cdot \langle \frac{\partial \bar{P}}{\partial \bar{\xi}}(z), \bar{\xi} - \bar{z} \rangle}{\frac{\partial P}{\partial \xi_2}(\xi) \cdot \frac{\partial \bar{P}}{\partial \bar{\xi}_2}(z)} d\bar{z}_1. \quad (1.8)$$

The estimate $R_1\Phi = F_1 \in L^{\tilde{p}}(V)$ follows from the property $\Phi \in L^p(V)$ and the following estimate of the kernel for the operator R_1

$$\left| \left(\frac{\partial P}{\partial \xi_2}(\xi) \right)^{-1} \det \left[\frac{\partial P}{\partial \xi}(\xi), \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2} \right] d\xi_1 \right| = O\left(\frac{1}{|\xi - z|} \right) \frac{|d\xi_1|}{\left| \frac{\partial P}{\partial \xi_2}(\xi) \right|},$$

where $\xi, z \in V$.

For the kernels of operators $\Phi \mapsto K\Phi$ and $\Phi \mapsto \partial_z K\Phi$ we have correspondingly estimates

$$\left| \frac{\langle \frac{\partial P}{\partial \xi}(\xi), \xi - z \rangle \cdot \langle \frac{\partial \bar{P}}{\partial \bar{\xi}}(z), \bar{\xi} - \bar{z} \rangle d\xi_1 \wedge d\bar{z}_1}{\frac{\partial P}{\partial \xi_2}(\xi) \cdot |\xi - z|^4 \cdot \frac{\partial \bar{P}}{\partial \bar{\xi}_2}(z)} \right| = O\left(\frac{1}{1 + |z|^2} \right) \frac{|d\xi_1|}{\left| \frac{\partial P}{\partial \xi_2}(\xi) \right|} \frac{|d\bar{z}_1|}{\left| \frac{\partial \bar{P}}{\partial \bar{\xi}_2}(z) \right|}, \quad (1.9)$$

$$\begin{aligned} & \left| \partial_z \frac{\langle \frac{\partial P}{\partial \xi}(\xi), \xi - z \rangle \cdot \langle \frac{\partial \bar{P}}{\partial \bar{\xi}}(z), \bar{\xi} - \bar{z} \rangle d\xi_1 \wedge d\bar{z}_1}{\frac{\partial P}{\partial \xi_2}(\xi) \cdot |\xi - z|^4 \cdot \frac{\partial \bar{P}}{\partial \bar{\xi}_2}(z)} \right| = \\ & O\left(\frac{1}{|\xi - z|} \right) \frac{|d\xi_1|}{\left| \frac{\partial P}{\partial \xi_2}(\xi) \right|} \cdot \frac{|d\bar{z}_1| \wedge |d\bar{z}_2|}{\left| \frac{\partial \bar{P}}{\partial \bar{\xi}_2}(z) \right|^2}, \end{aligned} \quad (1.10)$$

where $\xi, z \in V$. These estimates imply that $\forall \tilde{p} > 2$ and $p > 1$ we have

$$\Phi_0 \stackrel{\text{def}}{=} K\Phi \in W_{0,1}^{1,\tilde{p}}(V) \cap L_{0,1}^p(V). \quad (1.11)$$

From estimates (1.9)-(1.11) it follows that (0,1)-form $\Phi_0 = K\Phi$ on V can be considered also as (0,1)-form on compactification \tilde{V} of V in $\mathbb{C}P^2$ belonging to the spaces $L_{0,1}^{1,p}(\tilde{V}) \forall p < 2$, where \tilde{V} is equipped by the projective volume form $d d^c \ln(1 + |z|^2)$.

From the Hodge-Riemann decomposition theorem [Ho], [W] we have

$$\Phi_0 = \bar{\partial}(\bar{\partial}^* G\Phi_0) + \mathcal{H}\Phi_0, \quad (1.12)$$

where $\mathcal{H}\Phi_0 \in H_{0,1}(\tilde{V})$, G is the Hodge-Green operator for laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on \tilde{V} with properties

$$G(H_{0,1}(\tilde{V})) = 0, \quad \bar{\partial}G = G\bar{\partial}, \quad \bar{\partial}^*G = G\bar{\partial}^*.$$

Decomposition (1.12) implies that

$$\bar{\partial}^*G\Phi_0 \in W^{2,p}(\tilde{V}), \quad p \in (1, 2) \quad \text{and} \quad \mathcal{H}\Phi_0 \in H_{0,1}(\tilde{V}).$$

Returning on the affine curve V with eucliden volume form we obtain that

$$\begin{aligned} R_0\Phi & \stackrel{\text{def}}{=} \bar{\partial}^*GK\Phi|_V \in W^{1,\tilde{p}}(V), \quad \forall \tilde{p} > 2 \quad \text{and} \\ \mathcal{H}\Phi & \stackrel{\text{def}}{=} \mathcal{H}K\Phi|_V \in H_{0,1}^p(V), \quad p > 1. \end{aligned} \quad (1.13)$$

Proposition 1 is proved.

Corollary 1.1

Let $F \in L^\infty(V)$ and $\bar{\partial}F \in L_{0,1}^p(V)$, $1 < p < 2$. Then there exists $\lim_{z \rightarrow \infty} F(z) \stackrel{\text{def}}{=} F(\infty)$ such that $(F - F(\infty)) \in L^{\tilde{p}}$ and $\forall z \in V$ the following representation is valid

$$F(z) = F(\infty) + R\bar{\partial}F. \quad (1.14)$$

Proof

Put $\bar{\partial}F = \Phi$. Then by Proposition 1 we have $R\Phi \in L^{\tilde{p}}(V)$, $1/\tilde{p} = 1/p - 1/2$ and $\bar{\partial}(F - R\Phi) = \mathcal{H}\Phi$. Then the function $h = F - R\Phi$ is harmonic on V . Estimates $F \in L^\infty(V)$ and $R\Phi \in L^{\tilde{p}}(V)$ imply by the Riemann extension theorem that h can be extended until harmonic function \tilde{h} on \tilde{V} . Hence, $h = F - R\Phi = \text{const} = c$. This implies formula (1.14) with $F(\infty) = c$. From (1.2), (1.13), (1.14) we deduce that $\exists \lim_{z \rightarrow \infty} F(z) = c \stackrel{\text{def}}{=} F(\infty)$. Corollary 1.1 is proved. Corollary 1.1 admits useful reformulation.

Corollary 1.2

In the notations of Proposition 1 for any bounded function ψ on V such that $\bar{\partial}\psi \in L^p(V)$, $1 < p < 2$, the following formula is valid

$$\psi(z) = \psi(\infty) + R_0\bar{\partial}\psi + \frac{1}{2\pi i} \int_{\xi \in V} \bar{\partial}_\xi \left(\frac{\det[\frac{\partial P}{\partial \xi}(\xi), \bar{\xi} - \bar{z}] d\xi}{\frac{\partial P}{\partial \xi_2}(\xi) \cdot |\xi - z|^2} \right) \wedge \psi,$$

where $R_0\bar{\partial}\psi \in W^{1,\tilde{p}}(V)$, $1/\tilde{p} = 1/2 - 1/p$.

§2. Kernels and estimates for $\bar{\partial}f = \varphi$, $\varphi \in L_{1,1}^1(V) \cap L_{1,1}^\infty(V)$

Let φ be (1,1)-form of class $L_{1,1}^\infty(V)$ with support in $V_0 = \{z \in V : |z_1| \leq r_0\}$, where r_0 satisfies condition ii) of §1.

If $V = \mathbb{C}$ then by classical result [P1], [V] the Cauchy-Pompeiu operator

$$\varphi \mapsto \frac{1}{2\pi i} \int_{\xi \in V_0} \frac{\varphi(\xi)}{\xi - z} \stackrel{\text{def}}{=} \hat{R}\varphi$$

determines the solution $f = \hat{R}\varphi$ for equation $\bar{\partial}f = \varphi$ on \mathbb{C} with the property

$$f \in W_{1,0}^{1,\tilde{p}}(\mathbb{C}) \cap \mathcal{O}(\mathbb{C} \setminus V_0) \quad \forall \tilde{p} > 2.$$

In this section we obtain analogous result for the case of affine algebraic Riemann surface $V \subset \mathbb{C}^2$.

Let $V \setminus V_0 = \cup_{j=1}^d V_j$, where $\{V_j\}$ are connected components of $V \setminus V_0$.

Lemma 2.1

Let $\Phi = dz_1 \rfloor \varphi$ and $f = Fdz_1 = (R\Phi)dz_1$, where R is the operator from Proposition 1. Then

- i) $\Phi \in L_{0,1}^p(V_0)$, $p \in [1, 2)$, $\Phi = 0$ on $V \setminus V_0$
- ii) $F|_{V_0} \in W^{1,p}(V_0) \forall p \in (1, 2)$, $f \in W_{1,0}^{1,\tilde{p}}(V) \forall \tilde{p} \in (2, \infty)$,
 $\bar{\partial}F = \Phi - \mathcal{H}\Phi$, where \mathcal{H} is the operator from Proposition 1,
 $\bar{\partial}f = \varphi - (\mathcal{H}\Phi) \wedge dz_1$,
 $\|F\|_{L^{\tilde{p}}(V)} \leq \text{const}(V, p) \|\Phi\|_{L^p(V)}$, $1/\tilde{p} = 1/p - 1/2$
- iii) if, in addition, $\varphi \in W^{1,\infty}(V)$, then $f \in W^{2,\tilde{p}}(V)$.

Proof

- i) The property $\Phi|_{V \setminus V_0} = 0$ follows from $\varphi|_{V \setminus V_0} = 0$.

Put $V_0^\pm = \{z \in V_0 : \pm |\frac{\partial P}{\partial z_2}| \geq \pm |\frac{\partial P}{\partial z_1}|\}$. The definition $\Phi = dz_1 \rfloor \varphi$ implies that

$$\begin{aligned} \Phi|_{V_j^+} &= \Phi^+ d\bar{z}_1, \quad \text{where } \Phi^+ \in L^\infty(V_0^+) \text{ and} \\ \Phi|_{V_j^-} &= \Phi^- d\bar{z}_2 / (\partial z_1 / \partial z_2), \quad \text{where } \Phi^- \in L^\infty(V_0^-). \end{aligned} \tag{2.1}$$

Properties (2.1) imply that $\Phi \in L_{0,1}^p(V_0) \forall p \in (1, 2)$.

- ii) The equalities $\bar{\partial}F = \Phi - \mathcal{H}\Phi$ and $\bar{\partial}f = \varphi - (\mathcal{H}\Phi) \wedge dz_1$ follow from Proposition 1 and definitions $\Phi = dz_1 \rfloor \varphi$ and $f = Fdz_1$. The inclusions $F \in L^{\tilde{p}}(V)$ and $F|_{V_0} \in W^{1,p}(V_0)$ follow from the formula $F = R\Phi$ and Proposition 1. The inclusion $f \in W^{1,\tilde{p}}(V)$ follows from equalities $\bar{\partial}f = \varphi - (\mathcal{H}\Phi) \wedge dz_1$, $f = Fdz_1$ and Proposition 1.
- iii) if, in addition, $\varphi \in W^{1,\infty}(V)$, then $f \in W^{2,\tilde{p}}(V)$. It follows from equalities above with $\varphi \in W^{1,\infty}(V)$ and $\text{supp } \varphi \subset V_0$.

Lemma 2.2

$\forall g \in H_{0,1}^p(V) \exists h \in L_{1,0}^p(\tilde{V})$ ($1 \leq p < 2$) unique up to holomorphic (1,0)-forms on \tilde{V} such that

$$\bar{\partial}h|_{\tilde{V}} = g dz_1. \tag{2.2}$$

Proof

For any $g \in H_{0,1}^p(V)$ the (1,1)-form $g \wedge dz_1$ determines the current G on \tilde{Y} by the equality

$$\langle G, \chi \rangle \stackrel{\text{def}}{=} \lim_{R \rightarrow \infty} \sum_{j=1}^d \left[\int_{V_j} (\chi - \chi_j(\infty)) g dz_1 + \chi_j(\infty) \int_{\{z \in V_j: |z_1| < R\}} g \wedge dz_1 \right],$$

where $\chi \in C^{(\varepsilon)}(\tilde{V})$, $\varepsilon > 0$ and $\chi_j(\infty) = \lim_{\substack{z \in V_j \\ z \rightarrow \infty}} \chi(z)$.

By the Serre duality [S] the current G is $\bar{\partial}$ -exact on \tilde{V} iff

$$\langle G, 1 \rangle = \lim_{R \rightarrow \infty} \int_{\{z \in V: |z_1| \leq R\}} g \wedge dz_1 = 0. \tag{2.3}$$

Let us check (2.3). We have

$$\int_{\{z \in V: |z_1| \leq r\}} g \wedge dz_1 = - \int_{\{z \in V: |z_1| = r\}} z_1 \wedge g.$$

Putting into the right-hand side of this equality $w_1 = 1/z_1$, we obtain

$$\int_{\{z \in V: |z_1| \leq r\}} g \wedge dz_1 = - \sum_{j=1}^d \int_{|w_1|=1/r} g_j(\bar{w}_1) \frac{d\bar{w}_1}{w_1} = 0.$$

The last equality follows from the properties

$$g_j(\bar{w}_1) d\bar{w}_1 = g|_{V_j \cap \{|w_1| \leq 1/r\}} \quad \text{and} \quad \bar{g}_j \in \mathcal{O}(D(0, 1/r)).$$

Hence, by (2.3) there exists $h \in L_{1,0}^1(\tilde{V})$ such that equality (2.2) is valid in the sense of currents. Moreover, any solution of (2.2) automatically belongs to $L_{1,0}^p(\tilde{V})$, $1 < p < 2$. Such solution h of (2.2) is unique up to holomorphic (1,0)-forms on \tilde{V} because conditions $h \in L_{1,0}^p(\tilde{V})$ and $\bar{\partial}h = 0$ on V imply that h extends as a holomorphic (1,0)-form on \tilde{V} .

Notation

Let $\mathcal{H}^\perp : H_{0,1}^p(V) \rightarrow L_{1,0}^p(\tilde{V})$ ($1 < p < 2$) be the operator defined by formula $g \mapsto \mathcal{H}^\perp g$, where $\mathcal{H}^\perp g$ be the unique solution h of (2.2) in $L_{1,0}^p(\tilde{V})$ with the property

$$\int_{\tilde{V}} h \wedge \tilde{g} = 0 \quad \forall \tilde{g} \in H_{0,1}^p(V).$$

Lemma 2.2 proves the existence and uniqueness of $H^\perp g \in L_{1,0}^p(\tilde{V})$ for any $g \in H_{0,1}^p(V)$.

Proposition 2

Let R be operator defined by formula (1.1) and \mathcal{H} be operator defined by formula (1.13). For any (1,1)-form $\varphi \in L_{1,1}^1(V) \cap L_{1,1}^\infty(V)$ with support in V_0 put

$$\hat{R}\varphi = R^1\varphi + R^0\varphi, \tag{2.4}$$

where

$$R^1\varphi = (R(dz_1] \varphi)) dz_1, \quad R^0\varphi = \mathcal{H}^\perp \circ \mathcal{H}(dz_1] \varphi).$$

Then

$$\bar{\partial} \hat{R}\varphi = \varphi, \tag{2.5},$$

$$f = Fdz_1 = \hat{R}\varphi \in W^{1,\tilde{p}}(V) \quad \forall \tilde{p} \in (2, \infty), \quad F|_{V_0} \in W^{1,p}(V_0) \quad \forall p \in (1, 2)$$

$$\text{and } f|_{V_j} = \sum_{k=1}^{\infty} \frac{c_k^{(j)}}{z_1^k} dz_1, \quad \text{if } |z_1| \geq r_0. \tag{2.6}$$

Proof

The properties (2.5) and $f = \hat{R}\varphi \in W^{1,\tilde{p}}(V)$ follow from Lemmas 2.1, 2.2. The properties (2.5) and $\varphi|_{V \setminus V_0} = 0$ imply analyticity of f on $V \setminus V_0$. Development (2.6) follows from analyticity of $f|_{V \setminus V_0}$ and inclusion $f \in L^{\tilde{p}}(V \setminus V_0)$.

§3. Kernels and estimates for $(\partial + \lambda dz_1)u = f$, $f \in W_{1,0}^{1,\tilde{p}}(V)$

Let $f = Fdz_1$ be (1,0)-form as in Proposition 2, i.e. $F|_{V_0} \in W^{1,p}(V_0) \forall p \in (1, 2)$ and $f \in W_{1,0}^{1,\tilde{p}}(V) \forall \tilde{p} \in (2, \infty)$.

If $V = \mathbb{C}$ then equation $\partial u + \lambda u dz_1 = f$ was also introduced by Pompeiu [P2]. One can check that this equation can be solved by the explicit formula:

$$e^{\lambda z - \bar{\lambda} \bar{z}} u(z) = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{e^{\lambda \xi - \bar{\lambda} \bar{\xi}} f(\xi) d\bar{\xi}}{\xi - \bar{z}} \stackrel{\text{def}}{=} \\ \lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{\{\xi \in \mathbb{C}: |\xi| < r\}} \frac{e^{\lambda \xi - \bar{\lambda} \bar{\xi}} f(\xi) d\bar{\xi}}{\xi - \bar{z}}.$$

For Riemann surface $V = \{z \in \mathbb{C}^2 : P(z) = 0\}$ we will obtain further the following generalization of this formula.

Proposition 3

Let $e_\lambda(\xi) = e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1}$ and

$$\overline{R_1(\bar{e}_\lambda \bar{f})} \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_{\{\xi \in V: |\xi| < r\}} e_\lambda(\xi) f(\xi) \frac{d\bar{\xi}_1}{\partial \bar{P} / \partial \bar{\xi}_2} \det \left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \frac{\xi - z}{|\xi - z|^2} \right].$$

Let

$$u = R_\lambda f = R_\lambda^1 f + R_\lambda^0 f, \quad (3.1)$$

where $R_\lambda^1 f = e_{-\lambda}(z) \cdot \overline{R_1(\bar{e}_\lambda \bar{f})} + e_{-\lambda}(z) \cdot \overline{R_0(\bar{e}_\lambda \bar{f})}$, R_1, R_0 - operators from Proposition 1. Let $\mathcal{H}f \stackrel{\text{def}}{=} \overline{\mathcal{H}f}$, where \mathcal{H} operator from Proposition 1.

Then $\forall \lambda \neq 0$:

- i) $(\partial + \lambda dz_1)R_\lambda f = f - \mathcal{H}(f)$
- ii)

$$\|u\|_{L^{\tilde{p}}(V)} \leq \text{const}(V, p) \cdot \min\left(\frac{1}{\sqrt{|\lambda|}}, \frac{1}{|\lambda|}\right) (\|F\|_{L^{\tilde{p}}(V)} + \|\partial F\|_{L^p(V)}),$$

$$\|\partial u\|_{L^{\tilde{p}}(V)} \leq \text{const}(V, p) \cdot \|\partial F\|_{L^p(V)}, \quad \text{where } 1/\tilde{p} = 1/p - 1/2$$

- iii)

$$\|(1 + |z_1|)u\|_{L^\infty(V)} \leq \frac{\text{const}(V, \tilde{p})}{\sqrt{|\lambda|}} \|F\|_{L^{\tilde{p}}(V)}, \quad \forall \tilde{p} > 2,$$

$$\|(1 + |z_1|)\frac{\partial u}{\partial z_1}\|_{L^\infty(V)} \leq \text{const}(V, \tilde{p}) \sqrt{|\lambda|} \|F\|_{L^{\tilde{p}}(V)}, \quad \forall \tilde{p} > 2$$

iv) if, in addition, $f \in W^{2, \tilde{p}}(V)$, then

$$\left\| \frac{\partial^2 u}{\partial z_1^2} \right\|_{L^{\tilde{p}}(V)} \leq \text{const}(V, \varepsilon, p) \cdot |\lambda|^{\tilde{p}/(\tilde{p}+1-\varepsilon)} (\|F\|_{L^{\tilde{p}}(V)} + \|\partial F\|_{L^p(V)}),$$

where $1/\tilde{p} = 1/p - 1/2$, $\varepsilon > 0$.

Remark 1

Proposition 3 is still valid, if we replace the condition $f \in W^{1, \tilde{p}}(V)$ by $f - c(\lambda) dz_1 \in W_{1,0}^{1, \tilde{p}}(V)$ and in statements ii) and iii) we replace u by $u + \frac{c(\lambda)}{\lambda}$.

Remark 2

Part ii) of Proposition 3 gives a natural generalization of Lemma 1.2 from [Na] for the case of Riemann surfaces.

Proof of Proposition 3

i)

$$\begin{aligned} (\partial + \lambda dz_1) R_\lambda f &= (\partial + \lambda dz_1) e_{-\lambda}(z) \cdot \overline{R(\bar{e}_\lambda \bar{f})} = \\ &= \partial(e_{-\lambda}(z)) \cdot \overline{R(\bar{e}_\lambda \bar{f})} + e_{-\lambda}(z) \partial(\overline{R(\bar{e}_\lambda \bar{f})}) + \lambda dz_1 e_{-\lambda}(z) \cdot \overline{R(\bar{e}_\lambda \bar{f})} = \\ &= (-\lambda dz_1 + \lambda dz_1) e_{-\lambda}(z) \cdot \overline{R(\bar{e}_\lambda \bar{f})} + \\ &= e_{-\lambda}(z) \cdot e_\lambda(z) (f + \overline{\mathcal{H}f}) = f + \overline{\mathcal{H}f} \stackrel{\text{def}}{=} f + \mathcal{H}f, \end{aligned}$$

where we have used equality (1.1) from Proposition 1.

iii) Let $r \geq r_0$. Let $\chi_\pm \in C^{(1)}(V)$ be such that $\chi_+ + \chi_- \equiv 1$ on V ,

$\text{supp } \chi_+ \subset \{\xi \in V : |\xi_1| < 2r\}$, $\text{supp } \chi_- \subset \{\xi \in V : |\xi_1| \geq r\}$, $|\partial \chi_\pm| = O(1/r)$. We have $u = u_+ + u_-$, where

$$u_\pm(z) = R_\lambda(\chi_\pm f). \quad (3.1)_\pm$$

Using properties $f \in L^\infty(V)$, $|e_\lambda| \equiv 1$ and equality $\partial u_+ = \chi_+ F dz_1 - \lambda u_+ dz_1$, we obtain for u_+ and $\frac{\partial u_\pm}{\partial z_1}$ the estimates:

$$\begin{aligned} \|(1 + |z|)u_+(z)\|_{L^\infty(V)} &= O(r)\|f\|_{L^\infty(V)} \\ \|(1 + |z|)\frac{\partial u_+}{\partial z_1}(z)\|_{L^\infty(V)} &= O(\lambda r)\|f\|_{L^\infty(V)}. \end{aligned} \quad (3.2)$$

In order to estimate u_- we transform expression (3.1)₋ using development (2.6) for $f|_{V_j}$ and we integrate by part. We obtain

$$\begin{aligned} u_-(z) &= R_\lambda \chi_- f = R_\lambda^1 \chi_- f + R_\lambda^0 \chi_- f = \\ &= -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{\xi \in V} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} (d\chi_-) F \wedge d\bar{\xi}_1 \det\left[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2} + \\ &= \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \sum_j \int_{\xi \in V_j} e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \chi_- \left(\sum_{k=1}^{\infty} k \frac{c_k^{(j)}}{\xi_1^{k+1}} \right) \frac{d\xi_1 \wedge d\bar{\xi}_1 \det\left[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2} + \\ &= \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{\xi \in V} e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \chi_- F \partial_\xi \left(\frac{\det\left[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z\right] d\bar{\xi}_1}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2} \right) + e_{-\lambda}(z) \bar{R}_0(e_\lambda \chi_- f), \end{aligned} \quad (3.3)$$

where operator $R_0 = \bar{\partial}^* GK$ is defined by (1.13). Using Corollary 1.2 we have, in addition,

$$\begin{aligned} & -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{\xi \in V} e^{\lambda\xi_1 - \bar{\lambda}\bar{\xi}_1} \chi_- F \partial_\xi \left(\frac{\det[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z] d\bar{\xi}_1}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2} \right) = \\ & \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} e_\lambda(z) \chi_-(z) F(z) - \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \bar{R}_0(\partial(e_\lambda \chi_- F)) = \\ & \frac{1}{2\pi i} \frac{1}{\lambda} \chi_-(z) F(z) - \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \bar{R}_0(\partial(e_\lambda \chi_- F)). \end{aligned}$$

Putting last equality in (3.3) and using the properties $|e_\lambda| \equiv 1$, $|d\chi_-| = O(1/r)$, $\partial u_- = \chi_- F dz_1 - \lambda u_- dz_1$ and $R_0 = \bar{\partial}^* GK$, we obtain

$$\begin{aligned} \|(1 + |z_1|)u_-\|_{L^\infty(V)} &= O\left(\frac{1}{|\lambda|r}\right) \|F\|_{L^{\bar{p}}(V)} + \|(1 + |z_1|)\bar{R}_0(e_\lambda \chi_- f)\|_{L^\infty(V)} + \\ \frac{1}{2\pi|\lambda|} \|(1 + |z_1|)\bar{R}_0\partial(e_\lambda \chi_- F)\|_{L^\infty(V)} &\leq O\left(\frac{1}{|\lambda|r}\right) \|F\|_{L^{\bar{p}}(V)} \quad (3.4) \\ \text{and } \|(1 + |z_1|)\frac{\partial u_-}{\partial z_1}\|_{L^\infty(V)} &= O(1/r) \|F\|_{L^{\bar{p}}(V)}. \end{aligned}$$

Estimates (3.2) and (3.4) imply

$$\begin{aligned} \|(1 + |z_1|)u\|_{L^\infty(V)} &= O\left(r + \frac{1}{|\lambda|r}\right) \|F\|_{L^{\bar{p}}(V)}, \quad (3.5) \\ \text{and } \|(1 + |z_1|)\frac{\partial u}{\partial z_1}\|_{L^\infty(V)} &= O(|\lambda|r + 1/r) \|F\|_{L^{\bar{p}}(V)}, \quad \forall \bar{p} > 2. \end{aligned}$$

Putting in (3.5) $r = r_0/\sqrt{|\lambda|}$ we obtain iii).

ii) For proving ii) let us put $r = r_0$ and transform (3.1)₊ for u_+ in the following

$$\begin{aligned} u_+(z) &= R_\lambda \chi_+ f = \\ & -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{|\xi_1| \leq r} \frac{e^{\lambda\xi_1 - \bar{\lambda}\bar{\xi}_1} d\chi_+ F \wedge d\bar{\xi}_1 \det[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z]}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2} - \\ & \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{|\xi_1| \leq r} \frac{e^{\lambda\xi_1 - \bar{\lambda}\bar{\xi}_1} \chi_+ \partial F \wedge d\xi_1 \det[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z]}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2} - \quad (3.6) \\ & \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{|\xi_1| \leq r} e^{\lambda\xi_1 - \bar{\lambda}\bar{\xi}_1} \chi_+ F \partial \left(\frac{\det[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z] d\bar{\xi}_1}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2} \right) + e_{-\lambda}(z) \bar{R}_0(e_\lambda \chi_+ f), \end{aligned}$$

where operator $R_0 = \bar{\partial}^* GK$. Using the last expression for $u_+(z)$, the property $F|_{V_0} \in W^{1,p}(V_0)$ and Corollary 1.2 we obtain

$$\|u_+\|_{L^{\bar{p}}(V)} = O(1/\lambda) \|F\|_{W^{1,p}(V_0)}. \quad (3.7)$$

This inequality together with (3.4) and statement iii) proves the first part of statement ii). Formula $u = R_\lambda f$ implies $\partial_z u = f - \lambda dz_1 u$. From this and from already obtained estimates for u we deduce the second part of statement ii):

$$\|\partial u\|_{L^{\bar{p}}(V)} \leq \text{const}(V, p) \|\partial F\|_{L^p}.$$

iv) Formula $\frac{\partial u_+}{\partial z_1} = \chi_+ F - \lambda u_+$ implies

$$\frac{\partial^2 u_+}{\partial z_1^2} = \left(\frac{\partial \chi_+}{\partial z_1} F + \chi_+ \frac{\partial F}{\partial z_1} \right) - \lambda(\chi_+ F - \lambda u_+). \quad (3.8)$$

From (3.6) for u_+ we deduce:

$$\begin{aligned} \lambda u_+ - \chi_+ F &= \\ \frac{e_{-\lambda}(z)}{2\pi i} \int_{|\xi_1| \leq r} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \frac{\partial \chi_+}{\partial \xi_1} d\xi_1 F d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2} + \\ \frac{e_{-\lambda}(z)}{2\pi i} \int_{|\xi_1| \leq r} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \chi_+ \frac{\partial F}{\partial \xi_1} d\xi_1 \wedge d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2} - \\ \frac{e_{-\lambda}(z)}{2\pi i} \bar{R}_0(\partial(e_\lambda \chi_+ F)) + \lambda e_{-\lambda}(z) \bar{R}_0(e_\lambda \chi_+ f) &= J_1^+ + J_2^+ + J_3^+, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} J_1^+(z) &= \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{|\xi_1| \leq r} \frac{e^{-\bar{\lambda} \bar{\xi}_1} \frac{\partial \chi_+}{\partial \xi_1} de^{\lambda \xi_1} \wedge F d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2}, \\ J_2^+(z) &= \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{|\xi_1| \leq r} \frac{e^{-\bar{\lambda} \bar{\xi}_1} \chi_+ \frac{\partial F}{\partial \xi_1} de^{\lambda \xi_1} \wedge d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2}, \\ J_3^+(z) &= -\frac{e_{-\lambda}(z)}{2\pi i} \bar{R}_0(\partial(e_\lambda \chi_+ F)) + \lambda e_{-\lambda}(z) \bar{R}_0(e_\lambda \chi_+ f). \end{aligned}$$

By integration by part we obtain for $J_1^+(z)$:

$$J_1^+(z) = J_{11}^+(z) + J_{12}^+(z) + J_{13}^+(z), \quad (3.10)$$

where

$$\begin{aligned} J_{11}^+(z) &= -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{|\xi_1| \leq r} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \frac{\partial^2 \chi_+}{\partial \xi_1^2} d\xi_1 \wedge F \wedge d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2}, \\ J_{12}^+(z) &= -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{|\xi_1| \leq r} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \frac{\partial \chi_+}{\partial \xi_1} \frac{\partial F}{\partial \xi_1} d\xi_1 \wedge d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2}, \\ J_{13}^+(z) &= -\frac{1}{2\pi i} \frac{1}{\lambda} e_{-\lambda}(z) e_\lambda(z) \frac{\partial \chi_+}{\partial \xi_1}(z) F(z) + e_{-\lambda}(z) \frac{1}{\lambda} \bar{R}_0(\partial(e_\lambda \frac{\partial \chi_+}{\partial \xi_1} F)). \end{aligned}$$

For $J_2^+(z)$ we have

$$J_2^+(z) = J_{21}^+(z) + J_{22}^+(z), \quad (3.11)$$

where

$$J_{21}^+(z) = \frac{e_{-\lambda}(z)}{2\pi i} \int_{\{\xi: \left| \frac{\partial \xi_1}{\partial \xi_2} \right| \leq r\}} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \kappa_\rho \frac{\partial F}{\partial \xi_2} d\xi_2 \wedge d\bar{\xi}_2 \det \left[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z \right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2},$$

$$J_{22}^+(z) = \frac{e_{-\lambda}(z)}{2\pi i} \int_{\{\xi: \left| \frac{\partial \xi_1}{\partial \xi_2} \right| \geq r\}} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} (1 - \kappa_\rho) \chi_+ \frac{\partial F}{\partial \xi_1} d\xi_1 \wedge d\bar{\xi}_2 \det \left[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z \right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2},$$

where κ_ρ is a smooth function such that

$$\kappa_\rho = 1 \quad \text{if} \quad \left| \frac{\partial \xi_1}{\partial \xi_2} \right| < \frac{\rho}{2}, \quad \text{supp } \kappa_\rho \subset \{\xi : \left| \frac{\partial \xi_1}{\partial \xi_2} \right| < \rho\}, \quad |d\chi_\rho| = O(1/\rho)$$

with $\rho \leq \min(1, r_0)$.

Using in (3.10), (3.11) Lemma 2.1 ii) we obtain following estimates for J_{11}^+ , J_{12}^+ , J_{13}^+ :

$$\begin{aligned} \|(1 + |z_1|)J_{11}^+(z)\|_{L^\infty(V)} &= O(1/|\lambda|) \|F\|_{L^{\tilde{p}}(V_0)}, \quad \forall \tilde{p} > 2, \\ \|(1 + |z_1|)J_{12}^+(z)\|_{L^{\tilde{p}}(V)} &= O(1/|\lambda|) \|\partial F\|_{L^p(V_0)}, \quad p < 2, \quad 1/\tilde{p} = 1/p - 1/2, \\ \|(1 + |z_1|)J_{13}^+(z)\|_{L^{\tilde{p}}(V)} &= O(1/|\lambda|) \|F\|_{L^{\tilde{p}}(V_0)}, \quad \forall \tilde{p} > 2. \end{aligned} \quad (3.12)$$

For J_{21}^+ we obtain

$$|J_{21}^+(z)| \leq O(1) \int_{\{\xi: \left| \frac{\partial \xi_1}{\partial \xi_2} \right| \leq \rho\}} \frac{\left| \frac{\partial F}{\partial \xi_1} \cdot \frac{\partial \xi_1}{\partial \xi_2} \right| \cdot |d\xi_2 \wedge d\bar{\xi}_2|}{\left| \frac{\partial \xi_1}{\partial \xi_2} \right| \cdot |\xi - z|}.$$

By Lemma 2.1 ii) the function $\frac{\partial F}{\partial \xi_1}(\xi) \cdot \frac{\partial \xi_1}{\partial \xi_2}(\xi)$ belongs to $L^{\tilde{p}}(V_0)$ $\forall \tilde{p} > 2$. For further estimate of $J_{21}(z)$ we need the following statement.

Lemma 3.1

Put

$$J(z) = \int_{\{\xi \in \mathbb{C}: |\xi| < \rho\}} \frac{\psi(\xi) d\xi \wedge d\bar{\xi}}{|\xi| \cdot |\xi - z|}, \quad z \in \mathbb{C},$$

where $\psi \in L^p(V_0)$, $p > 1$. Then $\forall \varepsilon > 0$ and $\forall \tilde{p} > 2$ we have the estimate

$$\|J(z)\|_{L^{\tilde{p}}(\mathbb{C})} \leq O(\rho^{(2-2\varepsilon\tilde{p})/\tilde{p}}) \cdot \|\psi\|_{L^{(1+\varepsilon)/\varepsilon}(V_0)}.$$

Proof

Noting $\|\psi\|_\varepsilon = \|\psi\|_{L^{(1+\varepsilon)/\varepsilon}(V_0)}$ we obtain from expression for $J(z)$ the following estimate

$$\begin{aligned}
|J(z)| &\leq \left(\int_{|\xi| \leq \rho} \frac{|d\xi \wedge d\bar{\xi}|}{|\xi|^{1+\varepsilon} |\xi - z|^{1+\varepsilon}} \right)^{1/(1+\varepsilon)} \cdot \|\psi\|_\varepsilon \leq \\
&O\left(\int_{r=0}^{\rho} \frac{dr}{r^\varepsilon} \int_0^1 \frac{d\varphi}{(|r - |z|| + |z|\varphi)^{1+\varepsilon}} \right)^{1/(1+\varepsilon)} \cdot \|\psi\|_\varepsilon \leq \\
&O\left(\int_{r=0}^{\rho} \frac{dr}{r^\varepsilon} \frac{1}{|z|^{1+\varepsilon}} \int_0^1 \frac{d\varphi}{(|\frac{r}{|z|} - 1| + \varphi)^{1+\varepsilon}} \right)^{1/(1+\varepsilon)} \cdot \|\psi\|_\varepsilon \leq \\
&O\left(\int_{r=0}^{\rho} \frac{dr}{r^\varepsilon} \frac{1}{|z|} \left(\frac{1}{|r - |z||^\varepsilon} - \frac{1}{(|r - |z|| + |z|)^\varepsilon} \right) \right)^{1/(1+\varepsilon)} \cdot \|\psi\|_\varepsilon.
\end{aligned}$$

From the last estimate we deduce

$$\begin{aligned}
|J(z)| &\leq O\left(\frac{1}{|z|} \left(\int_0^{|z|} \frac{dr}{r^\varepsilon |z|^\varepsilon} + \int_{|z|}^{\rho} \frac{\varepsilon |z|}{r^\varepsilon r^\varepsilon} \right) \right)^{1/(1+\varepsilon)} \|\psi\|_\varepsilon, \quad \text{if } |z| \leq \rho, \\
\text{and } |J(z)| &\leq O\left(\frac{1}{|z|} \int_0^{\rho} \frac{dr}{r^\varepsilon |z|^\varepsilon} \right)^{1/(1+\varepsilon)} \|\psi\|_\varepsilon, \quad \text{if } |z| \geq \rho.
\end{aligned}$$

These equalities imply

$$\begin{aligned}
|J(z)| &\leq O\left(\left(\frac{1}{|z|} \right)^{2\varepsilon/(1+\varepsilon)} \right) \|\psi\|_\varepsilon, \quad \text{if } |z| \leq \rho, \\
|J(z)| &\leq O\left(\left(\frac{1}{|z|} \rho^{(1-\varepsilon)/(1+\varepsilon)} \right) \right) \|\psi\|_\varepsilon, \quad \text{if } |z| \geq \rho.
\end{aligned}$$

Putting $|z| = t$ we obtain finally that

$$\|J\|_{L^{\tilde{p}}(\mathbb{C})} \leq O\left(\int_0^\infty \frac{dt}{t^{2\varepsilon\tilde{p}/(1+\varepsilon)-1}} + \rho^{\frac{1-\varepsilon}{1+\varepsilon}\tilde{p}} \int_\rho^\infty \frac{dt}{t^{\tilde{p}-1}} \right)^{1/\tilde{p}} \|\psi\|_\varepsilon \leq O\left(\rho^{\frac{2-\varepsilon\tilde{p}}{\tilde{p}}} \right) \|\psi\|_\varepsilon.$$

Lemma 3.1 is proved.

From (3.11) and Lemma 3.1 we obtain

$$\|J_{21}^+\|_{L^{\tilde{p}}(V_0)} \leq O\left(\rho^{\frac{2-\varepsilon\tilde{p}}{\tilde{p}}} \right) \cdot \|f\|_{W^{1,1+1/\varepsilon}(V_0)}. \quad (3.13)$$

For estimate of $J_{22}^+(z)$ we put

$$J_{22}^+(z) = J_{22}^1(z) + J_{22}^2(z) + J_{22}^3(z), \quad (3.14)$$

where

$$\begin{aligned}
J_{22}^1(z) &= -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{\left|\frac{\partial \xi_1}{\partial \xi_2}\right| \geq \rho} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \partial((1 - \kappa_\rho) \chi_+) \frac{\partial F}{\partial \xi_1} d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2}, \\
J_{22}^2(z) &= -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{\left|\frac{\partial \xi_1}{\partial \xi_2}\right| \geq \rho} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} (1 - \kappa_\rho) \chi_+ \frac{\partial^2 F}{\partial \xi_1 \partial \xi_2} d\xi_2 \wedge d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2}, \\
J_{22}^3(z) &= -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int_{\left|\frac{\partial \xi_1}{\partial \xi_2}\right| \geq \rho} e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} (1 - \kappa_\rho) \chi_+ \frac{\partial F}{\partial \xi_1} \partial \frac{d\bar{\xi}_2 \det\left[\frac{\partial \bar{P}}{\partial \xi}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \xi_1}(\xi) \cdot |\xi - z|^2}.
\end{aligned}$$

Using the property $f \in W^{1, \tilde{p}}(V)$ i.e. $\frac{\partial F}{\partial \xi_2} d\xi_2 \in L^{\tilde{p}}(V_0)$ and Lemma 2.1 ii) we obtain for J_{22}^1 :

$$\|(1 + |z|) J_{22}^1\|_{L^\infty(V)} \leq O\left(\frac{1}{|\lambda| \rho^2}\right) \cdot \|f\|_{W^{1, \tilde{p}}(V)}. \quad (3.15)$$

Using Corollary 1.2 we obtain for J_{22}^3 :

$$J_{22}^3(z) = \frac{e_{-\lambda}(z) e_\lambda(z)}{2\pi i} \frac{1}{\lambda} (1 - \kappa_\rho) \chi_+(z) \frac{\partial F}{\partial z_1}(z) - \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \bar{R}_0 \left(\partial(e_\lambda(1 - \chi_\rho) \chi_+ \frac{\partial F}{\partial \xi_1}) \right),$$

which implies

$$\|(1 + |z|) J_{22}^3\|_{L^{\tilde{p}}(V)} \leq O(1/|\lambda| \rho^2) \|f\|_{W^{1, \tilde{p}}(V)}.$$

Using the property $f \in W^{2, \tilde{p}}(V)$ i.e. $\frac{\partial^2 F}{\partial \xi_2^2} d\xi_2 \in L^{\tilde{p}}(V_0) \forall \tilde{p} > 2$, from Lemma 2.1 iii) (under condition $\varphi \in C^{(1)}(V)$) we obtain for J_{22}^2 :

$$\|(1 + |z|) J_{22}^2\|_{L^\infty(V)} \leq O\left(\frac{1}{|\lambda| \rho}\right) \cdot \|f\|_{W^{2, \tilde{p}}(V)}, \quad \tilde{p} > 2. \quad (3.16)$$

Hence, for $\lambda u_+ - \chi_+ F = J_1^+ + J_2^+ + J_3^+$ we have:

$$\|\lambda u_+ - \chi_+\|_{L^{1, \tilde{p}}(V)} \leq \left[O\left(\frac{1}{|\lambda|}\right) + O(\rho^{2/\tilde{p}}) + O\left(\frac{1}{|\lambda| \rho^2}\right) + O\left(\frac{1}{|\lambda| \rho}\right) \right] \cdot \|f\|_{W^{2, \tilde{p}}(V)}.$$

We have used here that $f \in W^{2, \tilde{p}}(V)$ implies $f \in W^{1, \infty}(V_0)$, $\tilde{p} > 2$. Putting in this estimate $\rho = (1/|\lambda|)^{1/(2+2/\tilde{p})}$ we obtain

$$\|\lambda u_+ - \chi_+\|_{L^{1, \tilde{p}}(V)} \leq O(|\lambda|^{-(2/\tilde{p})/(2+2/\tilde{p})}) \cdot \|f\|_{W^{2, \tilde{p}}(V)}. \quad (3.17)$$

$$\left\| \frac{\partial^2}{\partial z_1^2} \right\|_{L^{1, \tilde{p}}(V)} \leq O(|\lambda|^{(2/(2+2/\tilde{p}))}) \cdot \|f\|_{W^{2, \tilde{p}}(V_0)}, \quad \tilde{p} > 2.$$

Let us estimate at the end $\frac{\partial^2 u_-}{\partial z_1^2}$. Formula $\frac{\partial u_-}{\partial z_1} = \chi_- F - \lambda u_-$ implies

$$\frac{\partial^2 u_-}{\partial z_1^2} = \left(\frac{\partial \chi_-}{\partial z_1} F + \chi_- \frac{\partial F}{\partial z_1} \right) - \lambda (\chi_- F - \lambda u_-).$$

From (3.3) for $u_-(z)$ we obtain $\lambda u_- - \chi_- F = J_1^- + J_2^- + J_3^-$, where

$$\begin{aligned} F|_{V_j}(\xi) &= \sum_{k=1}^{\infty} \frac{c_k^{(j)}}{\xi_1^k}, \quad j = 1, \dots, d \quad \text{and} \\ J_1^-(z) &= -\frac{e_{-\lambda}(z)}{2\pi i} \sum_j \int_{\{\xi \in V_j: |\xi_1| \geq r\}} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \frac{\partial \chi_-}{\partial \xi_1} F \wedge d\xi_1 \wedge d\bar{\xi}_1 \det \left[\frac{\partial \bar{P}}{\partial \xi}(1), \xi - z \right]}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2}, \\ J_2^-(z) &= \frac{e_{-\lambda}(z)}{2\pi i} \sum_j \int_{\{\xi \in V_j: |\xi_1| \geq r\}} \frac{e^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \chi_- \frac{\partial F}{\partial \xi_1} \wedge d\xi_1 \wedge d\bar{\xi}_1 \det \left[\frac{\partial \bar{P}}{\partial \xi}(1), \xi - z \right]}{\frac{\partial \bar{P}}{\partial \xi_2}(\xi) \cdot |\xi - z|^2}, \\ J_3^-(z) &= \lambda e_{-\lambda}(z) \bar{R}_0(e_\lambda \chi_- f) - \frac{e_{-\lambda}(z)}{2\pi i} \bar{R}_0(e_\lambda \chi_- f). \end{aligned} \tag{3.18}$$

From (3.18) we obtain like in (3.10), (3.12), but much simpler, that

$$\|J_j^-\|_{L^\infty(V)} = O\left(\frac{1}{|\lambda|}\right) \cdot \|F\|_{L^{\bar{p}}(V_0)}, \quad j = 1, 2, 3.$$

Hence, for $\|\lambda u_- - \chi_- F\|$ and for $\frac{\partial^2 u_-}{\partial z_1^2}$ we have

$$\|\lambda u_- - \chi_- F\|_{L^\infty(V)} = O(1/|\lambda|) \cdot \|F\|_{L^{\bar{p}}(V_0)} \quad \text{and} \quad \left\| \frac{\partial^2 u_-}{\partial z_1^2} \right\|_{L^\infty(V)} \leq O(1) \|F\|_{L^{\bar{p}}(V_0)}.$$

Proposition 3 is proved.

§4. Faddeev type Green function for $\bar{\partial}(\partial + \lambda dz_1)u = \varphi$ and further results

Let \hat{R} be operator defined by formula (2.4) and R_λ be operator defined by formula (3.1).

Proposition 4

Let $\varphi \in L_{1,1}^\infty(V)$ with support in $V_0 = \{z \in V : |z_1| \leq r_0\}$, where r_0 satisfies the condition of §1. Then for $u = G_\lambda \varphi \stackrel{\text{def}}{=} R_\lambda \circ \hat{R} \varphi$, where $\lambda \neq 0$, we have

- i) $\bar{\partial}(\partial + \lambda dz_1)u = \varphi$ on V ,
- ii)

$$\|u\|_{L^{\bar{p}}(V)} \leq \text{const}(V_0, \tilde{p}) \min(1/\sqrt{|\lambda|}, 1/|\lambda|) \|\varphi\|_{L_{1,1}^\infty(V_0)}, \quad \tilde{p} > 2,$$

$$\left\| \frac{\partial u}{\partial z_1} \right\|_{L^{\bar{p}}(V)} \leq \text{const}(V_0, \tilde{p}) \min(\sqrt{|\lambda|}, 1) \|\varphi\|_{L_{1,1}^\infty(V_0)}, \quad \tilde{p} > 2,$$

iii) under additional condition $\varphi \in W_{1,1}^{1,\infty}(V)$ we have also

$$\left\| \frac{\partial^2 u}{\partial z_1^2} \right\|_{L^{\tilde{p}}(V)} \leq \text{const}(V_0, \tilde{p}) (|\lambda|^{1/(1+1/\tilde{p})}) \|\varphi\|_{L_{1,1}^{1,\infty}(V_0)}, \quad \tilde{p} > 2.$$

Proof

By Proposition 2 we have

$$f = F dz_1 = \hat{R}\varphi \in W^{1,\tilde{p}}(V) \quad \forall \tilde{p} \in (2, \infty), \quad F|_{V_0} \in W^{1,p}(V_0) \quad \forall p \in (1, 2).$$

Propositions 2 and 3 imply that $u = R_\lambda \circ \hat{R}\varphi \in W^{1,\tilde{p}}(V)$. Let us check now statement i) of Proposition 4. From Proposition 3 i) we obtain

$$(\partial + \lambda dz_1)u = (\partial + \lambda dz_1)R_\lambda \circ \hat{R}\varphi = \hat{R}\varphi + \mathcal{H}(\hat{R}\varphi). \quad (4.1)$$

From (4.1) and Proposition 2 we obtain

$$\bar{\partial}(\partial + \lambda dz_1)Ru = \varphi + \bar{\partial}(\mathcal{H}(\hat{R}\varphi)) = \varphi,$$

where we have used that $\mathcal{H}(\hat{R}\varphi) \in H_{1,0}(\tilde{V})$. For proving property 4 ii) it is sufficient to remark that it follows from Proposition 3 ii), iii). Property 4 iii) follows (under additional condition $\varphi \in W^{1,\infty}(V)$) from Proposition 3 iv).

Definition

We define the Faddeev type Green function for $\bar{\partial}(\partial + \lambda dz_1)$ on V as a kernel $g_\lambda(z, \xi)$ of the operator $G_\lambda = R_\lambda \circ \hat{R}$.

Definition

Let form $q \in C_{1,1}(\tilde{V})$ and support q be contained in V_0 . The function $\psi(z, \lambda)$, $z \in V$, $\lambda \in \mathbb{C}$, will be called the Faddeev type function associated with q , if $\forall \lambda \in \mathbb{C}$ the function $\mu = \psi(z, \lambda)e^{-\lambda z}$ satisfies the properties:

$$\begin{aligned} \mu &\in L^\infty(V), \quad \bar{\partial}\mu \in L^{\tilde{p}}(V) \quad \forall \tilde{p} > 2, \\ \bar{\partial}(\partial + \lambda dz_1)\mu &= q\mu \quad \text{and} \quad \lim_{\substack{z \rightarrow \infty \\ z \in V}} \mu(z, \lambda) = 1. \end{aligned}$$

Basing on the Faddeev type Green function $g_\lambda(z, \xi)$ and on Proposition 4, in separate paper [HM], we extend the Novikov reconstruction scheme from the case $X \subset \mathbb{C}$ to the case of bordered Riemann surface $X \subset V$ on Riemann surface. We have obtained in [HM], in particular, the following results.

1. Let $\sigma \in C^{(3)}(V)$, $\sigma > 0$ on V and $\sigma = \text{const}$ on $V \setminus X \subset V \setminus V_0$. Then $\forall \lambda \in \mathbb{C}$ there exists unique Faddeev type function $\psi(z, \lambda)$ associated with $q = \frac{i}{2} \frac{d d^c \sqrt{\sigma}}{\sqrt{\sigma}}$. Such a function can be found from the Faddeev type integral equation

$$\psi(z, \lambda) = e^{\lambda z_1} + \frac{i}{2} \int_{\xi \in X} e^{\lambda(z_1 - \xi_1)} g_\lambda(z, \xi) \frac{\psi(z, \lambda) d d^c \sqrt{\sigma}}{\sqrt{\sigma}}.$$

2. For any domain X with smooth boundary on V such that $X \supseteq \bar{V}_0$, $\bar{X} \subset V$ and $\forall \lambda \neq 0$ the restriction $\psi|_{bX}$ can be found through Dirichlet-to-Neumann mapping on bX by the Fredholm integral equation

$$\psi(z, \lambda)|_{bX} = e^{\lambda z_1} + \int_{\xi \in bX} e^{\lambda(z_1 - \xi_1)} g_\lambda(z, \xi) (\hat{\Phi}\psi(\xi, \lambda) - \hat{\Phi}_0\psi(\xi, \lambda)),$$

where

$$\hat{\Phi}\psi = \bar{\partial}\psi|_{bX}, \quad \hat{\Phi}_0\psi = \bar{\partial}\psi_0|_{bX}, \quad dd^c\psi_0|_X = 0, \quad \psi_0|_{bX} = \psi|_{bX}.$$

3. The Faddeev type function $\psi(z, \lambda)$ satisfies $\bar{\partial}$ - equation of the Bers-Vekua-Beals-Coifman type with respect to $\lambda \in \mathbb{C}$:

$$\frac{\partial\psi(z, \lambda)}{\partial\lambda} = b(\lambda)\bar{\psi}(z, \lambda), \quad \text{where } b(\lambda) = \lim_{\substack{z \rightarrow \infty \\ z \in V}} \lim_{\bar{z}_1} \frac{\bar{z}_1}{\lambda} e^{(\lambda z_1 - \bar{\lambda} \bar{z}_1)} \frac{\partial\mu}{\partial\bar{z}_1}(z, \lambda).$$

The function $b(\lambda)$ "nonphysical scattering data" can be found from Dirichlet-to-Neumann data on bX through $\psi|_{bX}$. Finally, as in plane case, the whole function $\psi(z, \lambda)$ and, as a consequence, the form $dd^c\sqrt{\sigma}/\sqrt{\sigma}$ can be found through $b(\lambda)$ by the Fredholm type integral equation with Cauchy kernel with respect to λ . This integral equation is of Fredholm type because of estimates from Proposition 4 ii), iii).

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