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# Phase Transition in the 1d Random Field Ising Model with long range interaction. \*

Marzio Cassandro <sup>1</sup> Enza Orlandi <sup>2</sup> and Pierre Picco <sup>3</sup>

**Abstract** We study one-dimensional Ising spin systems with ferromagnetic, long-range interaction which decays as  $|i - j|^{-2+\alpha}$ ,  $\alpha \in (\frac{1}{2}, 1)$ , in the presence of external random fields. We assume that the random fields are given by a collection of independent identically distributed real random variables, subgaussian with mean zero. We show that for temperature and strength of the randomness (variance) small enough, with  $IP = 1$  with respect to the distribution of the random fields, there are at least two distinct extremal Gibbs measures.

## 1 Introduction

It is well known that for one dimensional bounded spin system with long range interactions one has existence of thermodynamics, see [12], when the interaction of any single site with the rest of the spins in lattice  $\mathbb{Z}$  is finite. For two body interactions which decays as  $|i - j|^{-2+\alpha}$  this means that  $\alpha < 1$ . When  $\alpha < 0$  there is only one Gibbs states [14,6,7] and the free energy is analytic in the thermodynamic parameters, see [8]. For Ising ferromagnetic models with two body long range interaction and  $0 < \alpha < 1$ , Dyson [9] proved the existence of phase transition by using a hierarchical model and Griffiths inequalities. For the  $\alpha = 0$  case, the existence of a phase transition was proved by Fröhlich and Spencer [11] by means of a sophisticated Peierls argument, the discontinuity of the magnetization at the critical temperature, the so-called Thouless effect, by Aizenman, Chayes, Chayes, & Newman [1].

More recently the notion of contour introduced in [11] was implemented in [5], by giving a graphical description of the spin configurations more explicit and better suited for further generalizations. The case studied in [5] covers the regime  $0 \leq \alpha \leq (\log 3 / \log 2) - 1$ . Applying Griffiths inequalities the result implies the existence of a phase transition in the full interval  $0 \leq \alpha < 1$ .

A natural extension of this analysis is its application to disordered systems. One of the simplest prototype models for disordered spin systems is obtained by adding random magnetic fields, say gaussian independent identically distributed with mean zero and finite variance. The problem of (lower) critical dimension for the  $d$ -dimensional Random Field Ising Model was very challenging at the end of the eighties since the physical literature predicted conflicting results. For the short range interaction the problem was rigorously solved by two complementary articles, Bricmont & Kupiainen [4] and Aizenman & Wehr [2]. In [4] a renormalisation group argument was used to show that if  $d \geq 3$  and the variance of the random magnetic field is small enough then almost surely there are at least two distinct Gibbs states (the plus and the minus Gibbs states). In [2] it was proved that for  $d \leq 2$ , almost surely there is an unique Gibbs state. The guide lines of these proofs are suggested by an heuristic argument due to Imry & Ma [13].

In the long-range one-dimensional setting the Imry & Ma argument is the following: the deterministic cost to create a run of  $-1$  in an interval of length  $L$  with respect to the state made of  $+1$  at each site,

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is of order  $L^\alpha$ , while the cumulative effect of the random field inside this interval is just  $L^{1/2}$ . So when  $0 \leq \alpha \leq 1/2$  the randomness is dominant and there is no phase transition. This has been rigorously proved by Aizenman and Wehr [2]. They show that the Gibbs state is unique for almost all realization of the randomness.

When  $1/2 < \alpha < 1$ , the above Imry & Ma argument suggests the existence of a phase transition since the deterministic part is dominant with respect to the random part as in the case of the three-dimensional random field Ising model. However a rigorous result in this direction was missing.

In this paper we study the random field one-dimensional Ising model with long range interaction  $|i - j|^{-2+\alpha}$ ,  $1/2 < \alpha < 1$ . We assume that the random field  $h[\omega] := \{h_i[\omega], i \in \mathbb{Z}\}$  is given by a collection of independent random variables, with mean zero and symmetrically distributed. We take  $h_i = \pm 1$  with  $p = \frac{1}{2}$  and we introduce the strength parameter  $\theta$ . However one could take different distributions, for example gaussian distribution with mean zero and variance  $\theta^2$ , or subgaussian. In fact what is just needed is that  $E(e^{th_1}) \leq e^{c\theta^2 t^2}$  for some positive constant  $c$  and for all  $t \in \mathbb{R}$ . We prove that for  $\frac{1}{2} < \alpha < 1$  the situation is analogous to the three-dimensional short range random field Ising model: for temperature and variance of the randomness small enough, with  $\mathbb{P} = 1$  with respect to the randomness, there exist at least two distinct infinite volume Gibbs states, namely the  $\mu^+[\omega]$  and the  $\mu^-[\omega]$  Gibbs states. The proof is based on the representation of the system in term of the contours as defined in [5]. A Peierls argument is obtained by using the lower bound of the deterministic part of the cost to erase a contour as done in [5] and controlling of the contribution of the stochastic part. This control is done applying an exponential Markov inequality and the so-called Yurinski's martingale difference sequences method. We do not use any coarse-grained contours as in [4], a fact that simplifies the proof.

## 2 Model, notations and main results

### 2.1. The model and the main results

Let  $(\Omega, \mathcal{B}, \mathbb{P})$  be a probability space on which we define  $h \equiv \{h_i\}_{i \in \mathbb{Z}}$ , a family of independent, identically distributed Bernoulli random variables with  $\mathbb{P}[h_i = +1] = \mathbb{P}[h_i = -1] = 1/2$ . The spin configuration space is  $\mathcal{S} \equiv \{-1, +1\}^{\mathbb{Z}}$ . If  $\sigma \in \mathcal{S}$  and  $i \in \mathbb{Z}$ ,  $\sigma_i$  represents the value of the spin at site  $i$ . The pair interaction among spins is given by  $J(|i - j|)$  defined as following<sup>1</sup>:

$$J(n) = \begin{cases} J(1) \gg 1 \\ \frac{1}{n^{2-\alpha}} & \text{if } n > 1, \quad \text{with } \alpha \in [0, 1). \end{cases}$$

For  $\Lambda \subseteq \mathbb{Z}$  we set  $\mathcal{S}_\Lambda = \{-1, +1\}^\Lambda$ ; its elements are denoted by  $\sigma_\Lambda$ ; also, if  $\sigma \in \mathcal{S}$ ,  $\sigma_\Lambda$  denotes its restriction to  $\Lambda$ . Given  $\Lambda \subset \mathbb{Z}$  finite and a realization of the magnetic fields, the Hamiltonian in the volume  $\Lambda$ , with  $\tau = \pm 1$  boundary conditions, is the random variable on  $(\Omega, \mathcal{A}, \mathbb{P})$  given by

$$H^\tau(\sigma_\Lambda)[\omega] = H_0^\tau(\sigma_\Lambda) + \theta G(\sigma_\Lambda)[\omega] \quad (2.1)$$

where

$$H_0^\tau(\sigma_\Lambda) := \frac{1}{2} \sum_{(i,j) \in \Lambda \times \Lambda} J(|i - j|)(1 - \sigma_i \sigma_j) + \sum_{i \in \Lambda} \sum_{j \in \Lambda^c} J(|i - j|)(1 - \tau \sigma_i), \quad (2.2)$$

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<sup>1</sup> The condition  $J(1) \gg 1$  is essential to apply the results of [5], reported in Subsection 2.2.

and

$$G(\sigma_\Lambda)[\omega] := - \sum_{i \in \Lambda} h_i[\omega] \sigma_i. \quad (2.3)$$

In the following we drop the  $\omega$  from the notation. The corresponding *Gibbs measure* on the finite volume  $\Lambda$ , at inverse temperature  $\beta > 0$  and  $+$  boundary condition is then a random variable with value on the space of probability measures on  $\mathcal{S}_\Lambda$  defined by

$$\mu_\Lambda^+(\sigma_\Lambda) = \frac{1}{Z_\Lambda^+} \exp\{-\beta H^+(\sigma_\Lambda)\} \quad \sigma_\Lambda \in \mathcal{S}_\Lambda, \quad (2.4)$$

where  $Z_\Lambda^+$  is the normalization factor. Using FKG inequalities, one can construct with  $\mathbb{P} = 1$  the infinite volume Gibbs measure  $\mu^+[\omega]$  as limits of local specifications with homogeneous plus boundary conditions along any deterministic sequence of increasing and absorbing finite volumes  $\Lambda_n$ . Of course the same holds with minus boundary conditions, see for example Theorem 7.2.2 in [3] or Theorem IV.6.5 in [10]. The main results are the following.

**Theorem 2.1** *Let  $\alpha \in (\frac{1}{2}, \frac{\ln 3}{\ln 2} - 1)$ . There exist positive  $\theta_0 := \theta_0(\alpha) > 0$  and  $\beta_0 := \beta_0(\alpha) > 0$  so that for  $0 < \theta \leq \theta_0$  and  $\beta \geq \beta_0$  there exists  $\Omega_1 \subset \Omega$  such that*

$$\mathbb{P}[\Omega_1] \geq 1 - e^{-\frac{\bar{b}}{200}}, \quad (2.5)$$

and for any  $\omega \in \Omega_1$ ,

$$\mu^+(\{\sigma_0 = -1\})[\omega] < e^{-\frac{\bar{b}}{200}} \quad (2.6)$$

where

$$\bar{b} = \min\left(\frac{\beta\zeta}{4}, \frac{\zeta^2}{2^{10}\theta^2}\right) \quad (2.7)$$

and

$$\zeta = \zeta(\alpha) = 1 - 2(2^\alpha - 1) > 0. \quad (2.8)$$

This result is an immediate consequence of the following theorem and the Markov inequality.

**Theorem 2.2** *Let  $\alpha \in (\frac{1}{2}, \frac{\ln 3}{\ln 2} - 1)$ . There exist positive  $\theta_0 := \theta_0(\alpha) > 0$  and  $\beta_0 := \beta_0(\alpha) > 0$  so that for  $0 < \theta \leq \theta_0$  and  $\beta \geq \beta_0$*

$$\mathbb{E} \left[ \mu^+ \left( \frac{1 - \sigma_0}{2} \right) \right] \leq e^{-\frac{\bar{b}}{100}}, \quad (2.9)$$

where  $\bar{b}$  is the quantity defined in (2.7).

**Remark:** If we prove phase transition for  $\alpha \in (\frac{1}{2}, \frac{\ln 3}{\ln 2} - 1)$  then FKG inequalities imply that the result holds for  $\alpha \in (\frac{1}{2}, 1)$ .

**Remark:** Since the translation invariant,  $\mathcal{B}$  measurable event  $A \equiv \{\exists i \in \mathbb{Z} : \mu^+[\omega](\sigma_i = +1) > 1 - e^{-\frac{\bar{b}}{200}}\}$  has strictly positive probability, see (2.5) and (2.6), by ergodicity  $\mathbb{P}[A] = 1$ . Therefore almost surely the two extremal Gibbs states  $\mu^\pm[\omega]$  are distinct.

## 2.2. Geometrical description of the spin configurations

We will follow the geometrical description of the spin configuration presented in [5] and use the same notations. We will consider homogeneous boundary conditions, i.e the spins in the boundary conditions are either all +1 or all -1. Actually we will restrict ourself to + boundary conditions and consider spin configurations  $\sigma = \{\sigma_i, i \in \mathbb{Z}\} \in \mathcal{X}_+$  so that  $\sigma_i = +1$  for all  $|i|$  large enough.

In one dimension an interface at  $(x, x+1)$  means  $\sigma_x \sigma_{x+1} = -1$ . Due to the above choice of the boundary conditions, any  $\sigma \in \mathcal{X}_+$  has a finite, even number of interfaces. The precise location of the interface is immaterial and this fact has been used to choose the interface points as follows: For all  $x \in \mathbb{Z}$  so that  $(x, x+1)$  is an interface take the location of the interface to be a point inside the interval  $[x + \frac{1}{2} - \frac{1}{100}, x + \frac{1}{2} + \frac{1}{100}]$ , with the property that for any four distinct points  $r_i, i = 1, \dots, 4$   $|r_1 - r_2| \neq |r_3 - r_4|$ . This choice is done once for all so that the interface between  $x$  and  $x+1$  is uniquely fixed. Draw from each one of these interfaces points two lines forming respectively an angle of  $\frac{\pi}{4}$  and of  $\frac{3}{4}\pi$  with the  $\mathbb{Z}$  line. We have thus a bunch of growing  $\vee$ - lines each one emanating from an interface point. Once two  $\vee$ - lines meet, they are frozen and stop their growth. The other two lines emanating from the the same interface points are erased. The  $\vee$ - lines emanating from others points keep growing. The collision of the two lines is represented graphically by a triangle whose basis is the line joining the two interfaces points and whose sides are the two segment of the  $\vee$ - lines which meet. The choice done of the location of the interface points ensure that collisions occur one at a time so that the above definition is unambiguous. In general there might be triangles inside triangles. The endpoints of the triangles are suitable coupled pairs of interfaces points. The graphical representation just described maps each spin configuration in  $\mathcal{X}_+$  to a set of triangles.

**Notation** *Triangles will be usually denoted by  $T$ , the collection of triangles constructed as above by  $\{\underline{T}\}$  and we will write*

$$|T| = \text{cardinality of } T \cap \mathbb{Z} = \text{mass of } T,$$

and by  $\text{supp}(T) \subset \mathbb{R}$  the basis of the triangle.

We have thus represented a configuration  $\sigma \in \mathcal{X}_+$  as a collection of  $\underline{T} = (T_1, \dots, T_n)$ . The above construction defines a one to one map from  $\mathcal{X}_+$  onto  $\{\underline{T}\}$ . It is easy to see that a triangle configuration  $\underline{T}$  belongs to  $\{\underline{T}\}$  iff for any pair  $T$  and  $T'$  in  $\underline{T}$

$$\text{dist}(T, T') \geq \min\{|T|, |T'|\}. \quad (2.10)$$

We say that two collections of triangles  $\underline{S}'$  and  $\underline{S}$  are compatible and we denote it by  $\underline{S}' \simeq \underline{S}$  iff  $\underline{S}' \cup \underline{S} \in \{\underline{T}\}$  (i.e. there exists a configuration in  $\mathcal{X}_+$  such that its corresponding collection of triangles is the collection made of all triangles that are in  $\underline{S}'$  or in  $\underline{S}$ .) By an abuse of notation, we write

$$H_0^+(\underline{T}) = H_0^+(\sigma), \quad G(\sigma(\underline{T}))[\omega] = G(\sigma)[\omega], \quad \sigma \in \mathcal{X}_+ \iff \underline{T} \in \{\underline{T}\}.$$

**Definition 2.3 The energy difference** *Given two compatible collections of triangles  $\underline{S} \simeq \underline{T}$ , we denote*

$$H^+(\underline{S}|\underline{T}) := H^+(\underline{S} \cup \underline{T}) - H^+(\underline{T}). \quad (2.11)$$

Let  $\underline{T} = (T_1, \dots, T_n)$  with  $|T_i| \leq |T_{i+1}|$  then using (2.11) one has

$$H^+(\underline{T}) = H^+(T_1|\underline{T} \setminus T_1) + H^+(\underline{T} \setminus T_1). \quad (2.12)$$

The following Lemma proved in Lemma 2.1 in [5] gives a lower bound on the cost to “erase” triangles sequentially starting from the smallest ones.

**Lemma 2.4** For  $\alpha \in (0, \frac{\ln 3}{\ln 2} - 1)$ , for  $\zeta := \zeta(\alpha)$  defined by

$$\zeta(\alpha) = 1 - 2(2^\alpha - 1) > 0, \quad (2.13)$$

one has

$$H_0^+(T_1 | \underline{T} \setminus T_1) \geq \zeta |T_1|^\alpha, \quad (2.14)$$

and by iteration, for any  $1 \leq i \leq n$

$$H_0^+(\cup_{\ell=1}^i T_\ell | \underline{T} \setminus [\cup_{\ell=1}^i T_\ell]) \geq \zeta \sum_{\ell=1}^i |T_\ell|^\alpha. \quad (2.15)$$

The estimate (2.15) involves contributions coming from the full set of triangles associated to a given spin configuration, starting from the triangle having the smallest mass. To implement a Peierls bound in our set up we need to “localize” the estimates to compute the weight of a triangle or of a finite set of triangles in a generic configuration. In order to do this [5] introduced the notion of contours as clusters of nearby triangles sufficiently far away from all other triangles.

**Contours** A contour  $\Gamma$  is a collection  $\underline{T}$  of triangles related by a hierarchical network of connections, controlled by a positive number  $C$ , see (2.16), under which all the triangles of a contour become mutually connected. We denote by  $T(\Gamma)$  the triangle whose basis is the smallest interval which contains all the triangles of the contour, the right and left endpoints of  $T(\Gamma) \cap \mathbb{Z}$  are denoted by  $x_\pm(\Gamma)$ . We denote  $|\Gamma|$  the mass of the contour  $\Gamma$

$$|\Gamma| = \sum_{T \in \Gamma} |T|$$

i.e.  $|\Gamma|$  is the sum of the masses of all the triangles belonging to  $\Gamma$ . In [5] it has been defined an algorithm  $\mathcal{R}(\underline{T})$  on  $\{\underline{T}\}$  which associates to any configuration  $\underline{T}$  a configuration  $\{\Gamma_j\}$  of contours with the following properties.

**P.0** Let  $\mathcal{R}(\underline{T}) = (\Gamma_1, \dots, \Gamma_n)$ ,  $\Gamma_i = \{T_{j,i}, 1 \leq j \leq k_i\}$ , then  $\underline{T} = \{T_{j,i}, 1 \leq i \leq n, 1 \leq j \leq k_i\}$

**P.1** Contours are well separated from each other. Any pair  $\Gamma \neq \Gamma'$  verifies one of the following alternatives.

$$T(\Gamma) \cap T(\Gamma') = \emptyset$$

i.e.  $[x_-(\Gamma), x_+(\Gamma)] \cap [x_-(\Gamma'), x_+(\Gamma')] = \emptyset$ , in which case

$$dist(\Gamma, \Gamma') := \min_{T \in \Gamma, T' \in \Gamma'} dist(T, T') > C \{|\Gamma|^3, |\Gamma'|^3\} \quad (2.16)$$

where  $C$  is a positive number. If

$$T(\Gamma) \cap T(\Gamma') \neq \emptyset,$$

then either  $T(\Gamma) \subset T(\Gamma')$  or  $T(\Gamma') \subset T(\Gamma)$ ; moreover, supposing for instance that the former case is verified, (in which case we call  $\Gamma$  an inner contour) then for any triangle  $T'_i \in \Gamma'$ , either  $T(\Gamma) \subset T'_i$  or  $T(\Gamma) \cap T'_i = \emptyset$  and

$$dist(\Gamma, \Gamma') > C|\Gamma|^3, \quad \text{if } T(\Gamma) \subset T(\Gamma'). \quad (2.17)$$

**P.2 Independence.** Let  $\{\underline{T}^{(1)}, \dots, \underline{T}^{(k)}\}$ , be  $k > 1$  configurations of triangles;  $\mathcal{R}(\underline{T}^{(i)}) = \{\Gamma_j^{(i)}, j = 1, \dots, n_i\}$  the contours of the configurations  $\underline{T}^{(i)}$ . Then if any distinct  $\Gamma_j^{(i)}$  and  $\Gamma_{j'}^{(i')}$  satisfies **P.1**,

$$\mathcal{R}(\underline{T}^{(1)}, \dots, \underline{T}^{(k)}) = \{\Gamma_j^{(i)}, j = 1, \dots, n_i; i = 1, \dots, k\}.$$

In [5] it has been proved that not only **P.0**, **P.1** and **P.2** can be actually implemented by some algorithm  $\mathcal{R}$ , but also that such algorithm is unique and therefore there is a bijection between families of triangles and contours. Next we report the estimates proven in [5] which are essential in this paper.

**Theorem 2.5** [5] *Let  $\alpha \in (0, \frac{\ln 3}{\ln 2} - 1)$  and the constant  $C$  in the definition of the contours, see (2.16), be so large that*

$$\sum_{m \geq 1} \frac{4m}{[Cm]^3} \leq \frac{1}{2}, \quad (2.18)$$

where  $[x]$  denotes the integer part of  $x$ . For any  $\underline{T} \in \{\underline{T}\}$ , let  $\Gamma_0 \in \mathcal{R}(\underline{T})$  be a contour,  $\underline{S}^{(0)}$  the triangles in  $\Gamma_0$  and  $\zeta(\alpha)$  as in (2.13) Then

$$H^+(\underline{S}^{(0)} | \underline{T} \setminus \underline{S}^{(0)}) \geq \frac{\zeta}{2} |\Gamma_0|^\alpha, \quad (2.19)$$

where

$$|\Gamma_0|^\alpha := \sum_{T \in \Gamma_0} |T|^\alpha. \quad (2.20)$$

**Theorem 2.6** [5] *For any  $\gamma > 0$  there exists  $C_0(\gamma)$  so that for  $b \geq C_0(\gamma)$  and for all  $m > 0$*

$$\sum_{0 \in \Gamma | \Gamma| = m} w_b^\gamma(\Gamma) \leq 2me^{-bm^\gamma}, \quad (2.21)$$

where

$$w_b^\gamma(\Gamma) := \prod_{T \in \Gamma} e^{-b|T|^\gamma}. \quad (2.22)$$

In the sequel, it is convenient to identify in each contour  $\Gamma$  the families of triangles having the same mass.

**Definition 2.7**

$$\Gamma = \{\underline{T}^{(0)}, \underline{T}^{(1)}, \dots, \underline{T}^{(k_\Gamma)}\}$$

where for  $\ell = 0, \dots, k_\Gamma$ ,  $\underline{T}^{(\ell)} := \{T_1^{(\ell)}, T_2^{(\ell)}, \dots, T_{n_\ell}^{(\ell)}\}$ , and each triangle of the family  $\underline{T}^{(\ell)}$  has the same mass, i.e. for all  $i \in \{1, \dots, n_\ell\}$ ,  $|T_i^{(\ell)}| = \Delta_\ell$  for  $\Delta_\ell \in \mathbb{N}$ . According to (2.20)

$$|\Gamma|^\rho = \sum_{\ell=0}^{k_\Gamma} |\underline{T}^{(\ell)}|^\rho, \quad |\underline{T}^{(\ell)}|^\rho = \sum_{T \in \underline{T}^{(\ell)}} |T|^\rho = n_\ell \Delta_\ell^\rho, \quad \rho \in \mathbb{R}^+. \quad (2.23)$$

### 3 Proof of Theorem 2

A necessary condition to have  $\sigma_0 = -1$  is that the site zero is contained in the support of some contour  $\Gamma$  so that

$$\mu_\Lambda^+(\sigma_0 = -1) \leq \mu_\Lambda^+(\{\exists \Gamma : 0 \in \Gamma\}) \leq \sum_{\Gamma \ni 0} \mu_\Lambda^+(\Gamma). \quad (3.1)$$

By definition, see (2.4),

$$\mu_\Lambda^+(\Gamma)[\omega] := \frac{1}{Z_\Lambda^+[\omega]} \sum_{\underline{T}: \underline{T} \simeq \Gamma} e^{-\beta H^+(\underline{T} \cup \Gamma)[\omega]}, \quad (3.2)$$

where  $\sum_{\underline{T}: \underline{T} \simeq \Gamma}$  means that the sum is over all families of triangles compatible with the contour  $\Gamma$ . Recalling (2.2) and (2.11), picking up a  $j$  such that  $0 \leq j \leq k_\Gamma$ , we write

$$H_0^+(\underline{T} \cup \Gamma) = H_0^+(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})) + H_0^+(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)}) | (\cup_{\ell=0}^j \underline{T}^{(\ell)})). \quad (3.3)$$

Using (2.19) and recalling (2.23)

$$H_0^+(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)}) | (\cup_{\ell=0}^j \underline{T}^{(\ell)})) \geq \frac{\zeta}{2} \sum_{\ell=0}^j n_\ell |\Delta_\ell|^\alpha. \quad (3.4)$$

Therefore

$$\mu_\Lambda^+(\Gamma) \leq e^{-\beta \frac{\zeta}{2} \sum_{\ell=0}^j n_\ell |\Delta_\ell|^\alpha} \frac{1}{Z_\Lambda^+} \sum_{\underline{T}: \underline{T} \simeq \Gamma} e^{-\beta H_0^+(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})) + \beta \theta G(\sigma(\underline{T} \cup \Gamma))[\omega]}. \quad (3.5)$$

Multiplying and dividing (3.5) by

$$\sum_{\underline{T}: \underline{T} \simeq \Gamma} e^{-\beta H_0^+(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})) + \beta \theta G(\sigma(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})))[\omega]}, \quad (3.6)$$

and noting that  $\sum_{\underline{T}: \underline{T} \simeq \Gamma} 1 \leq \sum_{\underline{T}: \underline{T} \simeq \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})} 1$ , one reconstructs  $\mu_\Lambda^+(\Gamma \setminus [\cup_{\ell=0}^j \underline{T}^{(\ell)}])$ , so that

$$\mu_\Lambda^+(\Gamma) \leq e^{-\frac{\beta \zeta}{2} \sum_{\ell=0}^j |\underline{T}^{(\ell)}|^\alpha} \mu_\Lambda^+(\Gamma \setminus [\cup_{\ell=0}^j \underline{T}^{(\ell)}]) e^{\beta F_j[\omega]} \quad (3.7)$$

where we set

$$F_j[\omega] := \frac{1}{\beta} \ln \left\{ \frac{\sum_{\underline{T}: \underline{T} \simeq \Gamma} e^{-\beta H_0^+(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})) + \beta \theta G(\sigma(\underline{T} \cup \Gamma))[\omega]}}{\sum_{\underline{T}: \underline{T} \simeq \Gamma} e^{-\beta H_0^+(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})) + \beta \theta G(\sigma(\underline{T} \cup \Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})))[\omega]}} \right\}. \quad (3.8)$$

We define for each  $\Gamma$  the partition:  $\Omega = \cup_{j=-1}^{k_\Gamma} B_j$  where for  $j \in \{0, \dots, k_\Gamma - 1\}$

$$B_j = B_j(\Gamma) := \{\omega : F_j[\omega] \leq \frac{\zeta}{4} \sum_{\ell=0}^j |\underline{T}^{(\ell)}|^\alpha, \text{ and } \forall i > j, F_i[\omega] > \frac{\zeta}{4} \sum_{\ell=0}^i |\underline{T}^{(\ell)}|^\alpha\}, \quad (3.9)$$

$$B_{k_\Gamma} = B_{k_\Gamma}(\Gamma) := \{\omega : F_{k_\Gamma}[\omega] \leq \frac{\zeta}{4} \sum_{\ell=0}^{k_\Gamma} |\underline{T}^{(\ell)}|^\alpha\}, \quad (3.10)$$

and

$$B_{-1} = B_{-1}(\Gamma) := \{\omega : \forall i > -1; F_i[\omega] > \frac{\zeta}{4} \sum_{\ell=0}^i |\underline{T}^{(\ell)}|^\alpha\}. \quad (3.11)$$

The relevant properties of the partition are given in the following lemma, whose proof is given in Section 4.

**Lemma 3.1** For  $-1 \leq j \leq k_\Gamma$ ,

$$\mathbb{E} [\mathbb{1}_{B_j}] \leq e^{-\frac{\zeta^2}{2^{10}\theta^2} \sum_{\ell=j+1}^{k_\Gamma} |\underline{T}^{(\ell)}|^{2\alpha-1}}. \quad (3.12)$$

with the convention that an empty sum is zero.

Writing

$$\mu_\Lambda^+(\Gamma) = \sum_{j=-1}^{k_\Gamma} \mu_\Lambda^+(\Gamma) \mathbb{1}_{\{B_j\}}$$

and using Lemma 3.1, we obtain

$$\begin{aligned} \mathbb{E} [\mu_\Lambda^+(\Gamma)] &= \sum_{j=-1}^{k_\Gamma} \mathbb{E} [\mu_\Lambda^+(\Gamma) \mathbb{1}_{\{B_j\}}] \\ &\leq \sum_{j=-1}^{k_\Gamma} e^{-\frac{\beta\zeta}{4} \sum_{\ell=0}^j |\underline{T}^{(\ell)}|^\alpha} e^{-c\frac{\zeta^2}{\theta^2} \sum_{\ell=j+1}^{k_\Gamma} |\underline{T}^{(\ell)}|^{2\alpha-1}} \\ &\leq (k_\Gamma + 1) e^{-\bar{b} \sum_{\ell=0}^{k_\Gamma} |\underline{T}^{(\ell)}|^{2\alpha-1}}, \end{aligned} \quad (3.13)$$

where  $\bar{b} := \min(\frac{\beta\zeta}{4}, \frac{\zeta^2}{2^{10}\theta^2})$ .

Recalling (2.22), one has

$$\mathbb{E} [\mu_\Lambda^+(\{\emptyset \in \Gamma\})] \leq \sum_{\Gamma \ni \emptyset} (k_\Gamma + 1) w_b^{2\alpha-1}(\Gamma) = \sum_{m \geq 3} (m+1) \sum_{\emptyset \in \Gamma: |\Gamma|=m} w_b^{2\alpha-1}(\Gamma). \quad (3.14)$$

Using (2.21), after a few lines computation one gets (2.9) ■

## 4 Probabilistic estimates

Let  $h = h[\omega]$  be a realization of the random magnetic fields and  $A \subset \mathbb{Z}$ . Define

$$(S_A h)_i = \begin{cases} -h_i, & \text{if } i \in A; \\ h_i, & \text{otherwise,} \end{cases} \quad (4.1)$$

and denote  $h[S_A \omega] \equiv S_A h[\omega]$ . Recalling (2.3), it is easy to see that

$$G(\sigma(\underline{T} \cup \Gamma \setminus \underline{T}^{(0)}))[\omega] = G(\sigma(\underline{T} \cup G))[S_{\underline{T}^{(0)}} \omega]. \quad (4.2)$$

In general

$$G(\sigma(\underline{T} \cup \Gamma \setminus \cup_{\ell=0}^i \underline{T}^{(\ell)}))[\omega] = G(\sigma(\underline{T} \cup \Gamma))[S_{D_i} \omega] \quad (4.3)$$

where

$$D_i \subset \cup_{\ell=0}^i \text{supp}(\underline{T}^{(\ell)}) \quad (4.4)$$

is the non-empty set so that

$$S_{D_i} = S_{\underline{T}^{(i)}} S_{\underline{T}^{(i-1)}} \dots S_{\underline{T}^{(1)}} S_{\underline{T}^{(0)}}. \quad (4.5)$$

When all the triangles in  $(\underline{T}^{(\ell)}, \ell = 0, \dots, j)$  have disjoint supports (4.4) becomes an equality. In general there are triangles inside triangles and in this case the inclusion in (4.4) is strict.

By construction the  $F_j[\omega]$  defined in (3.8) are such that

$$F_j(h(D_j^c), h(D_j)) = -F_j(h(D_j^c), -h(D_j)), \quad j \in 0, \dots, k_\Gamma, \quad (4.6)$$

where for a set  $A \subset \mathbb{Z}$ , we denote by  $h(A) = \{h_i : i \in A\}$ . Therefore one gets that  $\mathbb{IE}[F_j] = 0$ .

**Proof of Lemma 3.1** Set

$$A_i := \frac{\zeta}{4} \sum_{\ell=0}^i |\underline{T}^{(\ell)}|^\alpha. \quad (4.7)$$

We have  $\mathbb{IP}[B_j] \leq \mathbb{IP}[\forall i > j; F_i[\omega] > A_i]$ . Let  $\lambda_i$  for  $i = j+1, \dots, k_\Gamma$  be positive parameters, by exponential Markov inequality we have

$$\mathbb{IP}[\forall i > j : F_i[\omega] \geq A_i] \leq e^{-\sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell A_\ell} \mathbb{IE} \left[ e^{\sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell F_\ell} \right]. \quad (4.8)$$

Set

$$F[\omega] := \sum_{i=j+1}^{k_\Gamma} \lambda_i F_i[\omega]. \quad (4.9)$$

It remains to estimate  $\mathbb{IE}[e^F]$ . Note that  $F[\omega]$  depends on all the random fields on  $\Lambda$ . Let  $N$  be the number of sites in  $\Lambda$ . To avoid involved notations, we define a bijection  $\Pi$  from  $\Lambda$  to  $\{1, \dots, N\}$  as follows: first pick up all the  $n_0 \Delta_0$  sites in  $\text{supp}(\underline{T}^{(0)})$  and put them consecutively in  $N, \dots, N - n_0 \Delta_0 + 1$  (keeping them in the same order as they are for definiteness). Then pick up the sites in  $\text{supp}(\underline{T}^{(1)})$  that are not in  $\text{supp}(\underline{T}^{(0)})$  and put them consecutively starting at  $N - n_0 \Delta_0$  until they are exhausted. Note that if no triangles of size  $\Delta_0$  are within triangle of size  $\Delta_1$ ,  $\Pi$  maps  $\text{supp}(\underline{T}^{(0)}) \cup \text{supp}(\underline{T}^{(1)})$  onto  $\{N, \dots, N - n_0 \Delta_0 - n_1 \Delta_1 + 1\}$ ; otherwise  $\Pi$  maps  $\text{supp}(\underline{T}^{(0)}) \cup \text{supp}(\underline{T}^{(1)})$  onto a proper subset of  $\{N, \dots, N - n_0 \Delta_0 - n_1 \Delta_1 + 1\}$ . One can iterate this procedure until all the sites of the support of  $\Gamma$  are exhausted. As above, for all  $1 \leq j \leq k_\Gamma - 1$ , if all triangles considered are disjoint  $\Pi$  maps  $\cup_{\ell=0}^{j+1} \text{supp}(\underline{T}^{(\ell)})$  onto  $\{N, N-1, \dots, N - M_{j+1} + 1\}$  where  $M_{j+1} = \sum_{\ell=0}^{j+1} n_\ell \Delta_\ell$ , otherwise on a proper subset of it. Then one can pick up all the remaining sites of  $\Lambda$  and continue as above. The  $\Pi$  so defined induces a bijection from the random magnetic fields indexed by  $\Lambda$  to a family of random variables  $(h_1, \dots, h_N)$  by  $(\Pi h)_i := h_{\Pi i}, \forall i \in \Lambda$ . Using this bijection, one can work with the random variables  $(h_i, 1 \leq i \leq N)$ . Define the family of increasing  $\sigma$ -algebra:

$$(\emptyset, \Omega) = \Sigma_0 \subset \Sigma_1 = \sigma(h_1) \subset \Sigma_2 = \sigma(h_1, h_2) \subset \dots \subset \Sigma_N = \sigma(h_1, h_2, \dots, h_N)$$

and  $\Delta_k(F) = \mathbb{IE}[F|\Sigma_k] - \mathbb{IE}[F|\Sigma_{k-1}]$  the associated martingale difference sequences. We have

$$\mathbb{IE}[F|\Sigma_N] = F; \quad \mathbb{IE}[F|\Sigma_0] = \mathbb{IE}[F] = 0, \quad F = \sum_{k=1}^N \Delta_k(F).$$

Remark that

$$\mathbb{IE}[F_{j+1}|\Sigma_i] = 0 \quad \forall i \in \{1, \dots, N - M_{j+1}\} \quad (4.10)$$

since by (4.6)

$$F_{j+1}(h(D_{j+1}^c), h(D_{j+1})) = -F_{j+1}(h(D_{j+1}^c), -h(D_{j+1}))$$

and

$$\Pi h(\cup_{\ell=0}^{j+1} \underline{T}^{(\ell)}) \subset \{h_{N-M_{j+1}}, \dots, h_N\}.$$

Note also that

$$\mathbb{IE}[F|\Sigma_i] = 0 \quad \forall i \in \{1, \dots, N - M_{k_\Gamma}\}. \quad (4.11)$$

To estimate  $\mathbb{IE}[e^F]$  one uses the fact that  $\mathbb{IE}[e^F] = \mathbb{IE}[e^{\sum_{k=1}^{N-1} \Delta_k(F)} \mathbb{IE}[e^{\Delta_N(F)}|\Sigma_{N-1}]]$ . With self explained notations, using Jensen inequality one has

$$\mathbb{IE}[e^{\Delta_N(F)}|\Sigma_{N-1}] = \int e^{\Delta_N(F)} \mathbb{IP}(dh_N) \leq \int e^{[F(h_{<N}, h_N) - F(h_{<N}, \tilde{h}_N)]} \mathbb{IP}(dh_N) \mathbb{IP}(d\tilde{h}_N). \quad (4.12)$$

We then expand the exponential in the right hand side of (4.12). All the odd powers but the constant one vanish. For the even power one uses simply, see (4.9),

$$\left| F(h_{<N}, h_N) - F(h_{<N}, \tilde{h}_N) \right| \leq 2\theta |h_N - \tilde{h}_N| \sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell. \quad (4.13)$$

Then estimating  $|h_N - \tilde{h}_N| \leq 2$  and  $2^{(2n-1)^+} (2n!)^{-1} \leq (n!)^{-1}$  to re-sum the series one gets

$$\mathbb{IE}[e^{\Delta_N}|\Sigma_{N-1}] \leq e^{16\theta^2 (\sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell)^2}. \quad (4.14)$$

In the case of subgaussian variable one just performs all the integration instead of using  $|h_N - \tilde{h}_N| \leq 2$ . This will modify the result by a constant different from 16. To iterate, one uses again the Jensen inequality obtaining

$$\begin{aligned} \mathbb{IE}[e^{\Delta_{N-1}}|\Sigma_{N-2}] &= \int e^{\Delta_{N-1}(F)} \mathbb{IP}(dh_{N-1}) \\ &\leq \int e^{\int [F(h_{<N-1}, h_{N-1}, \hat{h}_N) - F(h_{<N-1}, \tilde{h}_{N-1}, \hat{h}_N)]} \mathbb{IP}(d\hat{h}_N) \mathbb{IP}(dh_{N-1}) \mathbb{IP}(d\tilde{h}_{N-1}). \end{aligned}$$

It is clear that the random variable

$$\int \left[ F(h_{<N-1}, h_{N-1}, \hat{h}_N) - F(h_{<N-1}, \tilde{h}_{N-1}, \hat{h}_N) \right] \mathbb{IP}(d\hat{h}_N)$$

is a symmetric ones under  $\mathbb{IP}(dh_{N-1})\mathbb{IP}(d\tilde{h}_{N-1})$  and satisfies an estimate as (4.13) from which one gets

$$\mathbb{IE}[e^{\Delta_{N-1}(F)}|\Sigma_{N-2}] \leq e^{16\theta^2 (\sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell)^2}. \quad (4.15)$$

So iterating one gets  $\mathbb{IE}[e^{\Delta_k(F)}|\Sigma_{k-1}] \leq e^{4\theta^2 (\sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell)^2}$  for  $k \in \{N, N-1, \dots, N - M_{j+1}\}$ . When  $k = N - M_{j+1} - 1$ , a new fact happens. Using (4.10) for  $i = N - M_{j+1} \equiv m$  and computing

$$\Delta_m(F) = \sum_{i=j+1}^{k_\Gamma} \lambda_i (\mathbb{IE}[F_i|\Sigma_m] - \mathbb{IE}[F_i|\Sigma_{m-1}]) \quad (4.16)$$

one obtains that the term corresponding to  $i = j + 1$  in the sum gives zero contribution. Therefore, in this case, one has

$$\left| \int \left[ F(h_{<m}, h_m, \hat{h}_{>m}) - F(h_{<m}, \tilde{h}_m, \hat{h}_{>m}) \right] \mathbb{IP}(\hat{h}_{>m}) \right| \leq 4\theta \sum_{\ell=j+2}^{k_\Gamma} \lambda_\ell. \quad (4.17)$$

Iterating this procedure one gets

$$\mathbb{E} [e^F] \leq e^{16\theta^2 \left\{ \left( \sum_{\ell=0}^{j+1} n_\ell \Delta_\ell \right) \left( \sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell \right)^2 + n_{j+2} \Delta_{j+2} \left( \sum_{\ell=j+2}^{k_\Gamma} \lambda_\ell \right)^2 + \dots + n_{k_\Gamma} \Delta_{k_\Gamma} (\lambda_{k_\Gamma})^2 \right\}}. \quad (4.18)$$

The estimate (4.18) suggests to set for  $\ell = j+1, \dots, k_\Gamma$

$$\mu_\ell \equiv \sum_{n=\ell}^{k_\Gamma} \lambda_n \quad (4.19)$$

and the constraints ( $\lambda_i \geq 0, j+1 \leq i \leq k_\Gamma$ ) become  $\mu_\ell$  decreasing with  $\ell$ . The first exponent in (4.8) can be written

$$- \sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell A_\ell = -\frac{\zeta}{4} \mu_{j+1} \left( \sum_{\ell=0}^j n_\ell \Delta_\ell^\alpha \right) - \sum_{\ell=j+1}^{k_\Gamma} \frac{\zeta}{4} \mu_\ell n_\ell \Delta_\ell^\alpha, \quad (4.20)$$

and the exponent in (4.18) as

$$16\theta^2 (\mu_{j+1})^2 \left( \sum_{\ell=0}^j n_\ell \Delta_\ell \right) + \sum_{\ell=j+1}^{k_\Gamma} 16\theta^2 (\mu_\ell)^2 n_\ell \Delta_\ell. \quad (4.21)$$

Denote

$$f(\mu_\ell) \equiv -\frac{\zeta}{4} \mu_\ell \Delta_\ell^\alpha + 16\theta^2 \mu_\ell^2 \Delta_\ell \quad \ell = j+1, \dots, k_\Gamma$$

Choose  $\mu_\ell = \bar{\mu}_\ell$  where

$$\bar{\mu}_\ell = \frac{1}{4 \times 32} \frac{\zeta}{\theta^2 \Delta_\ell^{1-\alpha}}, \quad (4.22)$$

is the minimizer of  $f(\mu_\ell)$ . Note that it is decreasing in  $\ell$  and

$$f(\bar{\mu}_\ell) = -\frac{\zeta^2 \Delta_\ell^{2\alpha-1}}{2^{10} \theta^2}. \quad (4.23)$$

Therefore collecting together the last sum in (4.20) and the one in (4.21) we get

$$- \sum_{\ell=j+1}^{k_\Gamma} \frac{\zeta}{4} \bar{\mu}_\ell n_\ell \Delta_\ell^\alpha + \sum_{\ell=j+1}^{k_\Gamma} 16\theta^2 (\bar{\mu}_\ell)^2 n_\ell \Delta_\ell = -\frac{\zeta^2}{2^{10} \theta^2} \sum_{\ell=j+1}^{k_\Gamma} n_\ell \Delta_\ell^{2\alpha-1} = -\frac{\zeta^2}{2^{10} \theta^2} \sum_{\ell=j+1}^{k_\Gamma} |\underline{\mathbf{I}}^{(\ell)}|^{2\alpha-1}. \quad (4.24)$$

Taking in account (4.20) and (4.21) one has

$$-\frac{\zeta}{4} \bar{\mu}_{j+1} \left( \sum_{\ell=0}^j n_\ell \Delta_\ell^\alpha \right) + 16\theta^2 (\bar{\mu}_{j+1})^2 \left( \sum_{\ell=0}^j n_\ell \Delta_\ell \right) = - \sum_{\ell=0}^j n_\ell \left( \frac{\zeta}{4} \bar{\mu}_{j+1} \Delta_\ell^\alpha - 16\theta^2 \Delta_\ell (\bar{\mu}_{j+1})^2 \right). \quad (4.25)$$

One can check easily that for all  $0 \leq \ell \leq j$  one has

$$\left( \frac{\zeta}{4} \bar{\mu}_{j+1} \Delta_\ell^\alpha - 16\theta^2 \Delta_\ell (\bar{\mu}_{j+1})^2 \right) = \frac{\zeta^2 \Delta_\ell}{2^9 \theta^2 \Delta_{j+1}^{1-\alpha}} \left( \frac{1}{\Delta_\ell^{1-\alpha}} - \frac{1}{2 \Delta_{j+1}^{1-\alpha}} \right) > 0 \quad (4.26)$$

since by construction  $\Delta_\ell < \Delta_{j+1}$  for  $0 \leq \ell \leq j$ . Putting together (4.8), (4.18), (4.24), and (4.25) one gets (3.12). ■

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