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Abstract

We consider the approximation of the unsteady Stokes equations in a time dependent domain when the motion of the domain is given. More precisely, we apply the finite element method to an Arbitrary Lagrangian Eulerian (A.L.E.) formulation of the system. Our main results state the convergence of the solutions of the semi-discretized (with respect to the space variable) and of the fully-discrete problems towards the solutions of the Stokes system.

1 Introduction

In this work we consider the discretization of a system of partial differential equations which describes the motion of a viscous incompressible fluid in a time dependent domain. More precisely we consider the Stokes system written in a bounded domain $\Omega_t \subset \mathbb{R}^2$ which depends on time $t \in (0,T)$. We want to approximate this system by considering an Arbitrary Lagrangian Eulerian (A.L.E.) formulation for the problem and by using the finite element method.

In many problems and applications one has to work with a fluid written in a moving domain. It is generally the case for fluid-structure interaction problems like the displacement of fishes or of submarines or like the motion of blood in the arteries, etc. Several numerical techniques have been proposed in the literature to overcome the difficulty due to the time dependent domain: see, for instance, [26], [38], [39], [9], [19], [20], [32], [36], [4], [29]. Here we consider the Arbitrary Lagrangian Eulerian (A.L.E.) method which main idea consists in moving in a convenient way the mesh in order to follow the motion of the domain, instead of re-meshing at each step time (which leads to a too expensive computation). If the deformation of the domain is not too important, it is possible to keep the regularity properties of the initial grid. This method have been proposed and studied by many authors: [12], [14], [23], [15], [30], [16], [27], [13], [17].

For many fluid-structure interaction problems, the motion of the domain is time dependent but is also an unknown of the problem and the equations for the fluid have to be coupled with some equations for the structure. For instance, if we deal with the motion of rigid bodies into a viscous incompressible fluid, the problem can be modeled by the coupling between the Navier-Stokes equations (corresponding to the fluid part) and ordinary differential equations (corresponding to the rigid bodies). The problem could even be

more complicated if the structure is deformable and although many authors (see, for instance, [3], [5], [11], [8]) have tackled the well-posedness of such systems, there are still many open questions (even for deriving a model with "good" properties). In the present paper, we tackle the problem when the motion of the domain is given. Moreover, to split the difficulties, we focus on the Stokes equations, neglecting the non linear term of the Navier-Stokes system. Despite these hypotheses, the problem remains complicated since we have to consider a mixed formulation in a time dependent domain, which is completely new with respect to the recent literature.

Let us briefly recall some reference about the numerical convergence for the Stokes/Navier-Stokes equations and the fluid-structure interaction problems. In the case of a fixed domain, and for the Navier-Stokes equations, the Lagrange-Galerkin method has been proposed and analyzed in [33]. In [37], the author has proved optimal error estimates for the Lagrange-Galerkin mixed finite element approximation of Navier-Stokes equations in a velocity/pressure formulation. We also mention the work of Achdou and Guermond [1], where convergence analysis of a finite element projection/Lagrange-Galerkin method for the incompressible Navier-Stokes equations is done. In the case where the domain is time dependent but given, the convergence analysis for the ALE method has been considered by [15], [30], [16], in the case of the advection-diffusion equation instead of the Stokes or the Navier-Stokes equations. Finally, when the domain is time dependent but unknown, few results exist in the literature: Grandmont, Guimet and Maday (in [21]) deal with the case of one dimensional problem discretized by using the ALE formulation. In [36] the authors have proved the convergence of a numerical method based on the use of characteristics and on finite elements with a fixed mesh for a two dimensional fluid-rigid body problem.

Let us describe more precisely our problem. For a given $T > 0$, and for each $t \in [0, T]$, we consider a bounded polyhedral convex domain Ω_t in $\mathbb{R}^2$. We set

$$Q_T = \left\{ (\mathbf{x}, t) \in \mathbb{R}^3 \mid \mathbf{x} \in \Omega_t,\ t \in (0, T) \right\}.$$

The Stokes system in the domain Ω_t, $t \in (0, T)$ can be written as follows:

$$\left\{ \begin{array}{rcll} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } Q_T, \\ \operatorname{div} \mathbf{u} &=& 0 & \text{in } Q_T, \\ \mathbf{u} &=& 0 & \text{on } \partial\Omega_t,\ t \in (0, T), \\ \mathbf{u}(0) &=& \mathbf{u}_0 & \text{in } \Omega_0. \end{array} \right. \tag{1.1}$$

In these equations, $\mathbf{u} = (u_1, u_2)$ is the velocity of the fluid, its density is assumed to be equal to 1, $\nu > 0$ is its constant kinematic viscosity and p is its pressure; $\mathbf{f} = (f_1, f_2)$ represents a density of body forces per unit mass (for instance, gravity).

It can be proved that the system (1.1) is well-posed provided that Q_T and the data ($\mathbf{f}$ and $\mathbf{u}_0$) are smooth enough. The difficulty in this proof, which comes from the fact that the domain is moving on time, has been overcome by several works. We mention, among others, the paper of Ôtani and Yamada [31] and the work of Inoue and Wakimoto [25]. In the last one, the equations (1.1) are recast on a cylindrical space time domain introducing a suitable diffeomorphism. A result of existence of a weak solution is obtained also in Salvi [35], [34] through an elliptic regularization, under weaker hypotheses on the regularity of the domain boundary than in the previously cited paper.

The paper is organized as follows. In the next section we deal with the ALE formulation of the Stokes system and we state our main results. The first result given in Theorem 2.1 consists in the convergence of a semi-discretization scheme with respect to the space variable and the second one (Theorem 2.2) states an error estimate for a fully-discrete formulation. Section 3 is devoted to some preliminary results useful to prove our main theorems. In Section 4 we introduce the projections on the finite element spaces and we prove some estimates for their time derivative on the ALE frame. The fifth section is devoted to the proof of the first main result and finally, in Section 6 we prove the second main result.

2 Statement of the main results

2.1 The ALE formulation of the Stokes equations

Let first give some assumptions on the non cylindrical domain Q_T. We assume that there exists a mapping $\mathbf{X} \in H^1(0,T;W^{2,\infty}(\Omega_0)^2)$ such that for each $t \in (0,T)$, the mapping

$$\begin{array}{rcl} \mathbf{X}_t \colon \Omega_0 & \longrightarrow & \Omega_t, \\ \mathbf{y} & \longmapsto & \mathbf{X}(\mathbf{y},t), \end{array} \tag{2.1}$$

is invertible and $\mathbf{X}_t^{-1} \in W^{1,\infty}(\Omega_t)^2$. In the literature, $\mathbf{y} \in \Omega_0$ is called the ALE coordinate, and $\mathbf{x} \in \Omega_t$ the spatial (or Eulerian) coordinate.

Using the transformation $\mathbf{X}$, we can write the ALE formulation of (1.1). To achieve this, we introduce the following notation: first, we denote by $\mathbf{w}$ the domain velocity, which is defined by

$$\mathbf{w} \colon Q_T \longrightarrow \mathbb{R}^2, \qquad \mathbf{w}(\mathbf{x},t) = \frac{\partial \mathbf{X}}{\partial t}(\mathbf{X}_t^{-1}(\mathbf{x}),t). \tag{2.2}$$

Then we use the notation $\left.\dfrac{dv}{dt}\right|_{\mathbf{Y}}$ for the time derivative on the ALE frame which is defined as follows: for any function $v \colon Q_T \to \mathbb{R}$ regular enough and defined on the Eulerian frame, we set

$$\begin{array}{rccl} \left.\dfrac{dv}{dt}\right|_{\mathbf{Y}} \colon & Q_T & \longrightarrow & \mathbb{R}, \\ & (\mathbf{x},t) & \longmapsto & \left.\dfrac{dv}{dt}\right|_{\mathbf{Y}}(\mathbf{x},t) = \dfrac{\partial v}{\partial t}(\mathbf{x},t) + \mathbf{w}(\mathbf{x},t)\cdot\nabla v(\mathbf{x},t). \end{array} \tag{2.3}$$

Using this definition, we obtain that the Stokes system (1.1) is equivalent to the following system ("ALE formulation of (1.1)"):

$$\left\{ \begin{array}{rcll} \left.\dfrac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}} - \nu\Delta\mathbf{u} + \nabla p - (\mathbf{w}\cdot\nabla)\,\mathbf{u} &=& \mathbf{f} & \text{in } Q_T, \\ \operatorname{div}\mathbf{u} &=& 0 & \text{in } Q_T, \\ \mathbf{u} &=& 0 & \text{on } \partial\Omega_t,\ t\in(0,T), \\ \mathbf{u}(0) &=& \mathbf{u}_0 & \text{in } \Omega_0. \end{array} \right. \tag{2.4}$$

It may be noticed that the main difference with the original formulation (1.1) is the appearance of the convective-type term due to the domain movement.

In order to write the ALE weak formulation of problem (2.4) we need some results on the time derivatives of integrals on moving domains. This kind of results will be developed in details in Section 3. Using these results, we get the following mixed weak formulation:

Find $\mathbf{u} \colon Q_T \to \mathbb{R}^2$ and $p \colon Q_T \to \mathbb{R}$ such that for each $t \in (0,T)$, $\mathbf{u}(\cdot,t) \in H_0^1(\Omega_t)^2$, $p(\cdot,t) \in L_0^2(\Omega_t)$ and the following system holds:

$$\left\{ \begin{array}{l} \dfrac{d}{dt}\displaystyle\int_{\Omega_t} \mathbf{u}\cdot(\mathbf{v}\circ\mathbf{X}_t^{-1})\,d\mathbf{x} + \int_{\Omega_t} \nabla\mathbf{u} \colon \nabla(\mathbf{v}\circ\mathbf{X}_t^{-1})\,d\mathbf{x} \\ \qquad - \displaystyle\int_{\Omega_t} \operatorname{div}\,(\mathbf{w}\otimes\mathbf{u})\cdot(\mathbf{v}\circ\mathbf{X}_t^{-1})\,d\mathbf{x} - \int_{\Omega_t} p\operatorname{div}\,(\mathbf{v}\circ\mathbf{X}_t^{-1})\,d\mathbf{x} \\ \qquad\qquad = \displaystyle\int_{\Omega_t} \mathbf{f}\cdot(\mathbf{v}\circ\mathbf{X}_t^{-1})\,d\mathbf{x} \qquad \forall \mathbf{v} \in H_0^1(\Omega_0)^2, \\ \displaystyle\int_{\Omega_t} (q\circ\mathbf{X}_t^{-1})\operatorname{div}\mathbf{u}\,d\mathbf{x} = 0 \qquad \forall q \in L^2(\Omega_0), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot) \quad \text{in } \Omega_0, \end{array} \right. \tag{2.5}$$

where for any open set $\Omega \subset \mathbb{R}^2$, we have denoted by $L^2_0(\Omega)$ the classical pressure space, that is:

$$L^2_0(\Omega) = \left\{ f \in L^2(\Omega) \;\Big|\; \int_\Omega f(\mathbf{x})\, d\mathbf{x} = 0 \right\}.$$

Let us also introduce the classical space of free divergence fields associated to Stokes problem, defined by

$$H^1_{0,\sigma}(\Omega) = \left\{ \mathbf{u} \in H^1_0(\Omega)^2 \mid \operatorname{div} \mathbf{u} = 0 \right\}.$$

Since we deal with the mixed formulation (2.5), it is natural to assume the following uniform "inf-sup" condition:

$$\inf_{p \in L^2_0(\Omega_t)} \sup_{\mathbf{v} \in H^1_0(\Omega_t)^2} \frac{\displaystyle\int_{\Omega_t} p \operatorname{div} \mathbf{v}\, d\mathbf{x}}{\|\mathbf{v}\|_{H^1(\Omega_t)^2}\, \|p\|_{L^2(\Omega_t)}} \geqslant \beta, \tag{2.6}$$

where β is a positive constant which does not depend on time. The "inf-sup" condition was introduced independently by Babuška [2] and Bezzi [7]. Notice that a sufficient condition to guarantee (2.6) is that the deformation of Ω_t is "small". More precisely, there exists a constant $\alpha > 0$ depending only on Ω_0 such that if

$$\|\mathbf{X} - \mathrm{Id}\|_{L^\infty(\Omega_0 \times (0,T))^2} + \|\nabla \mathbf{X} - \mathrm{Id}\|_{L^\infty(\Omega_0 \times (0,T))^4} < \alpha, \tag{2.7}$$

then (2.6) holds true. It is important to remark that the assumption (2.7) is quite natural: indeed, in practice, the ALE formulation can not be used to discretize a problem when the deformation are too big and it is usually necessary to re-mesh the domain to preserve the regularity of the mesh (see, [28] for instance)

2.2 Semi-discretization scheme and statement of the first main result

In order to discretize our problem with respect to the space variable, we introduce two finite element spaces of Hood-Taylor type; these spaces depend on time since our problem is written on the domain Ω_t.

Let h denote a discretization parameter, with $0 < h < 1$. At initial time $t = 0$, we consider a quasi-uniform triangulation $\mathcal{T}_{h,0}$ of Ω_0, as defined, for instance, in Brenner and Scott [6, p.106]. We also assume that there is no triangle of $\mathcal{T}_{h,0}$ with two edges on $\partial\Omega_0$. These assumptions on $\mathcal{T}_{h,0}$ will be assumed throughout this paper.

For any $t \in [0,T]$, we consider a discretization of the mapping $\mathbf{X}_t$ by means of piecewise linear Lagrangian finite elements, denoted by $\mathbf{X}_{h,t}$:

$$\begin{array}{rcl} \mathbf{X}_{h,t} \colon \Omega_0 & \longrightarrow & \Omega_t, \\ \mathbf{y} & \longmapsto & \mathbf{X}_{h,t}(\mathbf{y}). \end{array}$$

We assume that $\mathbf{X}_{h,t}$ is smooth and invertible. Let $\mathcal{T}_{h,t}$ be the image of $\mathcal{T}_{h,0}$ under the discrete ALE mapping $\mathbf{X}_{h,t}$.

We associate to this triangulation two classical approximation spaces used in the mixed finite element methods for the Stokes system. The first space, classically used for the approximation of the velocity field in the mixed statement of the Stokes system, is denoted by $W_{h,t}$ and is composed with the $\mathscr{P}_2$-finite elements associated to $\mathcal{T}_{h,t}$. More precisely:

$$W_{h,t} \;=\; \left\{ v_h \in H^1_0(\Omega_t) \mid v_{h|_K} \in \mathscr{P}_2(K) \quad \forall K \in \mathcal{T}_{h,t} \right\},$$

where $\mathscr{P}_n(K)$ is the set of polynomials on K of degree less than or equal to n.

The second space, classically used for the approximation of the pressure in mixed formulations of the Stokes system, is denoted by $M_{h,t}$ and is composed with the $\mathscr{P}_1$-finite elements associated to $\mathcal{T}_{h,t}$, that is,

$$M_{h,t} \;=\; \Big\{ q_h \in H^1(\Omega_t) \mid q_{h|_K} \in \mathscr{P}_1(K) \quad \forall K \in \mathcal{T}_{h,t} \Big\}.$$

We also consider the space

$$M^0_{h,t} \;=\; M_{h,t} \cap L^2_0(\Omega_t).$$

Since Ω_0 is a polyhedral convex domain and $\mathbf{X}_{h,t}$ is piecewise linear and smooth, we can characterize the spaces $W_{h,t}$ and $M_{h,t}$ as follows:

$$W_{h,t} \;=\; \Big\{ v_h \circ \mathbf{X}^{-1}_{h,t} \mid v_h \in W_{h,0} \Big\}, \tag{2.8}$$

$$M_{h,t} \;=\; \Big\{ q_h \circ \mathbf{X}^{-1}_{h,t} \mid q_h \in M_{h,0} \Big\}. \tag{2.9}$$

As in the previous subsection, we consider $\mathbf{w}_h$ the velocity field associated to the discrete ALE mapping:

$$\mathbf{w}_h(\mathbf{x},t) = \frac{\partial \mathbf{X}_{h,t}}{\partial t}\left(\mathbf{X}^{-1}_{h,t}(\mathbf{x})\right).$$

Using this discrete velocity field, we can introduce the time derivative on the discrete ALE frame as follows: for any $v\colon Q_T \longrightarrow \mathbb{R}$ smooth enough, we define

$$\left.\frac{dv}{dt}\right|^h_{\mathbf{Y}}(\mathbf{x},t) = \frac{\partial v}{\partial t}(\mathbf{x},t) + \mathbf{w}_h(\mathbf{x},t)\cdot\nabla v(\mathbf{x},t). \tag{2.10}$$

Now, using the weak ALE formulation (2.5) and the definitions above, we can derive a semi-discrete version of our problem. For any $h \in (0,1)$ we denote by $\mathbf{u}_h$ and p_h the solution of the following problem:

Find $\mathbf{u}_h$ and p_h such that $\mathbf{u}_h(\cdot,0) = \mathbf{u}_{h,0}$ and for any $t \in (0,T)$, $\mathbf{u}_h(\cdot,t) \in (W_{h,t})^2$, $p_h(\cdot,t) \in M^0_{h,t}$ and the following system holds

$$\left\{\begin{array}{l} \dfrac{d}{dt}\displaystyle\int_{\Omega_t} \mathbf{u}_h \cdot \big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t}\big)\, d\mathbf{x} + \nu \int_{\Omega_t} \nabla \mathbf{u}_h : \nabla\big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t}\big)\, d\mathbf{x} \\ \qquad\qquad - \displaystyle\int_{\Omega_t} \operatorname{div}\,(\mathbf{w}_h \otimes \mathbf{u}_h)\cdot\big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t}\big)\, d\mathbf{x} - \int_{\Omega_t} p_h \operatorname{div}\big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t}\big)\, d\mathbf{x} \\ \qquad\qquad\qquad = \widetilde{I}_{h,t}\Big(\mathbf{f}(t)\cdot\big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t}\big)\Big) \qquad \forall \mathbf{v}_h \in (W_{h,0})^2, \\ \displaystyle\int_{\Omega_t} \big(q_h \circ \mathbf{X}^{-1}_{h,t}\big) \operatorname{div} \mathbf{u}_h\, d\mathbf{x} = 0 \qquad \forall q_h \in M_{h,0}, \end{array}\right. \tag{2.11}$$

where $\mathbf{u}_{h,0}$ is a finite element approximation of the initial data $\mathbf{u}_0$. In the third line we have used the notation $\widetilde{I}_{h,t}(F)$ to denote a numerical quadrature formula for the integral $\displaystyle\int_{\Omega_t} F(\mathbf{x})\, d\mathbf{x}$. In the rest of paper, we assume that the quadrature formula is exact for the continuous functions in Ω_t, whose restriction of each triangle is polynomial of degree less than or equal to 4. Using this fact, each integral of the above numerical scheme can be replaced by the numerical integration formula.

To get the convergence of the numerical scheme, it essential to assume that the discrete ALE mapping $\mathbf{X}_h$ approximates $\mathbf{X}$ in some sense. More precisely, we assume that the following error estimate holds true.

$$\|\mathbf{X}_t - \mathbf{X}_{h,t}\|_{L^\infty(\Omega_0)^2} + h\, \|\nabla(\mathbf{X}_t - \mathbf{X}_{h,t})\|_{L^\infty(\Omega_0)^4} \le C\, h^2 |\ln h|\, \|\mathbf{X}_t\|_{W^{2,\infty}(\Omega_0)^2}. \tag{2.12}$$

For more details about the construction of a mapping $\mathbf{X}_h$ satisfying such an estimate, we refer the reader to Gastaldi [16]. We can notice that if we assume $\mathbf{w}(t) \in W^{2,\infty}(\Omega_t)^2$, then the following error estimate on the domain velocity holds true (for more details, see Gastaldi [16]): for all $t \in (0,T)$,

$$\|\mathbf{w}(t) - \mathbf{w}_h(t)\|_{L^\infty(\Omega_t)^2} + h\|\nabla(\mathbf{w}(t) - \mathbf{w}_h(t))\|_{L^\infty(\Omega_t)^4} \leq Ch^2|\ln h|\ \|\mathbf{w}(t)\|_{W^{2,\infty}(\Omega_t)^2}. \tag{2.13}$$

The other important hypothesis to obtain the convergence of our scheme is that the triangulation $\mathcal{T}_{h,t}$ remains non-degenerate with the time (see [6, pp.106-107]): we assume that there exists $\rho > 0$ such that

$$\operatorname{diam} B_K \geq \rho h \operatorname{diam} K \quad \forall K \in \mathcal{T}_{h,t} \tag{2.14}$$

for all $t \in [0,T]$ and for all $h \in (0,1]$, where B_K is the largest disk contained in K. In practice, this hypothesis holds only for a small time interval, especially when one deals with great deformations. If we assume that $\mathcal{T}_{h,0}$ is non-degenerate, that the deformation is small enough (see (2.7)) and that the approximation $\mathbf{X}_h$ is close to $\mathbf{X}$ (see (2.12)), then for h small enough, we can prove that (2.14) holds true.

We are now in position to state the first main result of the paper:

Theorem 2.1. *Suppose that the above assumptions on $\mathcal{T}_{h,t}$ and on $\mathbf{X}_h$ hold true and that (2.6) is satisfied. Let also assume that the solution $(\mathbf{u}, p)$ of the problem (2.4) and the data $\mathbf{w}$, $\mathbf{f}$ satisfy the following properties:*

$$\left.\begin{array}{l} \mathbf{u} \in L^\infty(0,T;H^3(\Omega_t)^2 \cap H^1_{0,\sigma}(\Omega_t)),\ \left.\dfrac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}} \in L^2(0,T;H^2(\Omega_t)^2),\ \mathbf{u}(0) \in H^3(\Omega_t)^2, \\ p \in L^\infty(0,T;H^2(\Omega_t) \cap L^2_0(\Omega_t)),\ \left.\dfrac{dp}{dt}\right|_{\mathbf{Y}} \in L^2(0,T;H^1(\Omega_t)),\ p(0) \in H^2(\Omega_t), \\ \mathbf{w} \in L^\infty(0,T;W^{2,\infty}(\Omega_t)^2),\ \mathbf{f} \in L^2(0,T;W^{2,q}(\Omega_t)^2),\ \text{for some } q > 2. \end{array}\right\} \tag{2.15}$$

Then there exists a constant $C > 0$, independent of h, such that the solution $(\mathbf{u}_h, p_h)$ of the semi-discretization problem (2.11) satisfies

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + \nu\|\nabla(\mathbf{u} - \mathbf{u}_h)\|^2_{L^2(0,T;L^2(\Omega_t)^4)} &\leq \|\mathbf{u}_0 - \mathbf{u}_{h,0}\|^2_{L^2(\Omega_0)^2} \\ + Ch^2|\ln h|^2 \Bigg[\|\mathbf{f}\|^2_{L^2(0,T;W^{2,q}(\Omega_t)^2)} &+ \|\mathbf{u}\|^2_{L^\infty(0,T;H^3(\Omega_t)^2)} + \left\| \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}} \right\|^2_{L^2(0,T;H^2(\Omega_t)^2)} \\ &+ \|p\|^2_{L^\infty(0,T;H^2(\Omega_t))} + \left\| \left.\frac{dp}{dt}\right|_{\mathbf{Y}} \right\|^2_{L^2(0,T;H^1(\Omega_t))} \Bigg]. \end{aligned} \tag{2.16}$$

2.3 The fully-discrete formulation and statement of the second main result

In order to discretize our problem with respect to the time variable, let us denote by $\Delta t > 0$ the time step and $t_n = n\Delta t$, for $n = 0, \ldots, N$, where N is such that $t_N \leq T$ and $t_{N+1} > T$.

In the fully-discrete problem, we will consider a piecewise linear interpolation in time of the domain deformation. Thus, the domain velocity is constant on each interval (t_n, t_{n+1}) and at time $t = t_{n+1}$ is given by:

$$\mathbf{w}^*_{h,n,n+1}(\mathbf{x}) = \frac{1}{\Delta t}\left[\mathbf{x} - \mathbf{X}_{h,t_n}\left(\mathbf{X}^{-1}_{h,t_{n+1}}(\mathbf{x})\right)\right] \quad \forall \mathbf{x} \in \Omega_{t_{n+1}},$$

for all $n \in \{0, \ldots, N-1\}$.

With the above definitions, we can introduce the fully-discrete problem, using an implicit Euler scheme, as follows:

Find $\{\mathbf{u}_h^n\}$ and $\{p_h^n\}$ such that $\mathbf{u}_h^0 = \mathbf{u}_{h,0}$ and for any $n = 0, \dots, N-1$, one has that $\mathbf{u}_h^{n+1} \in (W_{h,t_{n+1}})^2$, $p_h^{n+1} \in M^0_{h,t_{n+1}}$ and the following system holds:

$$\begin{cases} \displaystyle\int_{\Omega_{t_{n+1}}} \mathbf{u}_h^{n+1} \cdot (\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}})\, d\mathbf{x} - \int_{\Omega_{t_n}} \mathbf{u}_h^{n} \cdot (\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n}})\, d\mathbf{x} \\ \qquad +\nu\Delta t \displaystyle\int_{\Omega_{t_{n+1}}} \nabla \mathbf{u}_h^{n+1} : \nabla(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}})\, d\mathbf{x} \\ \qquad\qquad -\Delta t \displaystyle\int_{\Omega_{t_{n+1}}} \mathrm{div}\, \left(\mathbf{w}^*_{h,n,n+1} \otimes \mathbf{u}_h^{n+1}\right) \cdot (\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}})\, d\mathbf{x} \\ \qquad\qquad\qquad -\Delta t \displaystyle\int_{\Omega_{t_{n+1}}} p_h^{n+1}\, \mathrm{div}\, (\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}})\, d\mathbf{x} \\ \qquad\qquad\qquad\qquad = \Delta t \widetilde{I}_{h,t_{n+1}} \left(\mathbf{f}(t_{n+1}) \cdot (\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}})\right) \qquad \forall \mathbf{v}_h \in (W_{h,0})^2, \\ \displaystyle\int_{\Omega_{t_{n+1}}} (q_h \circ \mathbf{X}^{-1}_{h,t_{n+1}})\, \mathrm{div}\, \mathbf{u}_h^{n+1}\, d\mathbf{x} = 0 \qquad \forall q_h \in M_{h,0}. \end{cases} \tag{2.17}$$

In the sequel, we state the second main result of this paper, which gives the error estimate in the approach given by the ALE method for the Stokes problem in time depending domain. More precisely, we have the following theorem:

Theorem 2.2. *Suppose that the assumptions of Theorem 2.1 hold true. Let also assume that*

$$\frac{\partial^2 \mathbf{X}_h}{\partial t^2} \in L^\infty(0,T; L^\infty(\Omega_0)^2) \quad \textit{and} \quad \left.\frac{d\mathbf{f}}{dt}\right|_{\mathbf{Y}} \in L^2(0,T; L^2(\Omega_t)^2). \tag{2.18}$$

Then, there exists a positive constant C, independent of h and Δt, such that for all sufficiently small Δt and h, we have the following error estimate:

$$\begin{aligned} &\|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \nu\ \Delta t\ \sum_{i=1}^{n+1} \|\nabla\left(\mathbf{u}(t_i) - \mathbf{u}_h^i\right)\|^2_{L^2(\Omega_{t_i})^4} \\ &\quad \le \|\mathbf{u}_0 - \mathbf{u}_{h,0}\|^2_{L^2(\Omega_0)^2} + C\left(\frac{h^4}{\Delta t^2} + h^4\right)\left(\|\mathbf{u}\|^2_{L^\infty(0,T;H^3(\Omega_t)^2)} + \|p\|^2_{L^\infty(0,T;H^2(\Omega_t))}\right) \\ &\qquad + C\Delta t^2 \sup_{s\in(0,T)} \left\| \left[\frac{\partial^2 \mathbf{X}_h}{\partial s^2}(s)\right]^2 \right\|_{L^\infty(\Omega_0)^2} \|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + C\Delta t\ h^4 \sum_{i=1}^{n+1} \|\mathbf{f}(t_i)\|^2_{W^{2,q}(\Omega_{t_i})^2} \\ &\qquad\quad + C\Delta t^2 \int_0^{t_{n+1}} \left(\left\| \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) \right\|^2_{H^1(\Omega_t)^2} + \|\mathbf{u}(t)\|^2_{H^2(\Omega_t)^2} + \left\| \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) \right\|^2_{L^2(\Omega_t)} \right. \\ &\qquad\qquad \left. + \|p(t)\|^2_{H^1(\Omega_t)} + \left\| \left.\frac{d\mathbf{f}}{dt}\right|_{\mathbf{Y}}(t) \right\|^2_{L^2(\Omega_t)^2} + \|\mathbf{f}(t)\|^2_{H^1(\Omega_t)^2} \right) dt. \end{aligned} \tag{2.19}$$

Remark 2.3. In particular, if there exists a fixed constant $C_0 > 0$ such that $h \le C_0\Delta t$ and $\|\mathbf{u}(0) - \mathbf{u}_{h,0}\|_{L^2(\Omega_0)^2} \le C_0 h$, we have that

$$\|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \nu\ \Delta t\ \sum_{i=1}^{n+1} \|\nabla\left(\mathbf{u}(t_i) - \mathbf{u}_h^i\right)\|^2_{L^2(\Omega_{t_i})^4} \le C\Delta t^2.$$

Remark 2.4. Let us observe that the condition $h \le C_0\Delta t$ is quite natural for the convergence of mixed schemes. For instance, in [33] the convergence is obtained for $h \le C_0\Delta t$ and in [37] for $h^2 \le C_0\Delta t \le C_1 h^\sigma$ and $\sigma > 1/2$ (with h and Δt small enough).

Remark 2.5. The regularity assumption (2.18) on $\mathbf{X}_h$ is quite natural in the case of a time depending operator, in order to obtain the fully error estimate (2.19) given above in the Theorem 2.2. If we use the construction of $\mathbf{X}_h$ and its continuous counterpart $\mathbf{X}$, given in Gastaldi [16], it is clear that this regularity with respect to t is strictly related with the displacement of the boundary.

3 Preliminary results

This section is devoted to some preliminary results which will be useful to prove Theorem 2.1 and Theorem 2.2. These results are either easy to prove or classical and, for this reason, we shall omit all the proofs in what follows.

Let us first recall the following classical result (see, for instance [22, pp.19-20]). In the context of ALE formulations, this result has been also presented in [15].

Proposition 3.1. *Consider Ω_1 and Ω_2 two bounded open subsets of $\mathbb{R}^2$ and assume that $X \in W^{1,\infty}(\Omega_1)$. Suppose also that $X : \Omega_1 \to \Omega_2$ is invertible and such that $X^{-1} \in W^{1,\infty}(\Omega_2)$. Then for any $u \in H^1(\Omega_2)$ we have that $u \circ X \in H^1(\Omega_1)$.*

This proposition justify the mixed formulation (2.5) and will be used throughout the paper.

Since we have to deal with integrals on moving domain in this problem, we give also some useful formulas for the time derivative of integrals on moving domains. First of all, we recall the Reynolds transport formula, that is, let $\psi(\mathbf{x}, t)$ be a smooth function defined on Q_T. Then for any open subdomain $V_t \subseteq \Omega_t$ such that $V_t = \mathbf{X}_t(V_0)$ with $V_0 \subseteq \Omega_0$, we have that

$$\frac{d}{dt}\int_{V_t} \psi(\mathbf{x},t)\, d\mathbf{x} = \int_{V_t} \left(\frac{\partial \psi}{\partial t} + \nabla\psi \cdot \mathbf{w} + \psi \ \mathrm{div}\, \mathbf{w} \right) d\mathbf{x} = \int_{V_t} \left(\left.\frac{d\psi}{dt}\right|_{\mathbf{Y}} + \psi \ \mathrm{div}\, \mathbf{w} \right) d\mathbf{x}$$

(see, for instance, Gurtin [24]).

Furthermore, since for any $\boldsymbol{\chi} : \Omega_0 \to \mathbb{R}^2$ we have that $\left.\dfrac{d}{dt}\left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right)\right|_{\mathbf{Y}} = 0$, it is not difficult to prove the following lemma, which is a consequence of the above formula.

Lemma 3.2. *Let assume that $\boldsymbol{\varphi}\colon Q_T \to \mathbb{R}^2$, $\psi\colon Q_T \to \mathbb{R}$ and $\boldsymbol{\chi} : \Omega_0 \to \mathbb{R}^2$ are smooth functions. Then we have the following relations:*

$$\frac{d}{dt}\int_{\Omega_t} \left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) \cdot \boldsymbol{\varphi} d\mathbf{x} = \int_{\Omega_t} \left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) \cdot \left(\left.\frac{d\boldsymbol{\varphi}}{dt}\right|_{\mathbf{Y}} + \boldsymbol{\varphi} \mathrm{div}\, \mathbf{w} \right) d\mathbf{x}, \tag{3.1}$$

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} \nabla\boldsymbol{\varphi} : \nabla\left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right)\, d\mathbf{x} = \int_{\Omega_t} \Bigg[\nabla\left(\left.\frac{d\boldsymbol{\varphi}}{dt}\right|_{\mathbf{Y}} \right) : \nabla\left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) + \nabla\boldsymbol{\varphi} : \nabla\left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) \mathrm{div}\, \mathbf{w} \\ - \left(\left(\nabla\mathbf{w} + \nabla\mathbf{w}^T\right) \nabla\boldsymbol{\varphi} \right) : \nabla\left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) \Bigg] d\mathbf{x},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} \left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) \mathrm{div}\, \boldsymbol{\varphi}\, d\mathbf{x} = \int_{\Omega_t} \Bigg[\left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) \mathrm{div} \left(\left.\frac{d\boldsymbol{\varphi}}{dt}\right|_{\mathbf{Y}} \right) + \left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) \mathrm{div}\, \boldsymbol{\varphi} \mathrm{div}\, \mathbf{w} \\ - \left(\boldsymbol{\chi} \circ \mathbf{X}_t^{-1}\right) \ \nabla\mathbf{w} : \nabla\boldsymbol{\varphi}^T \Bigg] d\mathbf{x},\end{aligned} \tag{3.3}$$

$$\frac{d}{dt}\int_{\Omega_t}\psi\mathrm{div}\,\left(\boldsymbol{\chi}\circ\mathbf{X}_t^{-1}\right)\,d\mathbf{x}=\int_{\Omega_t}\left[\left.\frac{d\psi}{dt}\right|_{\mathbf{Y}}\mathrm{div}\,\left(\boldsymbol{\chi}\circ\mathbf{X}_t^{-1}\right)+\psi\mathrm{div}\,\left(\boldsymbol{\chi}\circ\mathbf{X}_t^{-1}\right)\mathrm{div}\,\mathbf{w}\right.$$
$$\left.-\,\psi\,\nabla\mathbf{w}:\nabla\left(\boldsymbol{\chi}\circ\mathbf{X}_t^{-1}\right)^T\right]d\mathbf{x}. \quad (3.4)$$

It is well-known (see, for instance, [18]) that the mixed formulation (2.11) is a well-posed problem, provided that the spaces $W_{h,t}$, $M^0_{h,t}$ and the bilinear form

$$b(p_h,\mathbf{v}_h):=\int_{\Omega_t}p_h\,\mathrm{div}\,\mathbf{v}_h\,d\mathbf{x}$$

satisfy the Brezzi-Babuška (*inf-sup*) condition. The fact that this *inf-sup* condition is satisfied in our case, at each time $t\in(0,T)$, follows from the choice of the finite element used. That is, at each time $t\in(0,T)$, there exists a positive constant β_t such that

$$\inf_{p_h\in M^0_{h,t}}\sup_{\mathbf{v}_h\in(W_{h,t})^2}\frac{\displaystyle\int_{\Omega_t}p_h\,\mathrm{div}\,\mathbf{v}_h\,d\mathbf{x}}{\|\mathbf{v}_h\|_{H^1(\Omega_t)^2}\,\|p_h\|_{L^2(\Omega_t)}}\geqslant\beta_t.$$

In fact, if h is small enough, we can choose a constant β^* independent of t instead of β_t in the above inequality. More precisely, we have the following result.

Theorem 3.3. *Assume that* (2.6) *and* (2.14) *hold true. Then there exist two positive constant h^* and β^* such that for all $t\in(0,T)$ and for all $h\in(0,h^*)$,*

$$\inf_{p_h\in M^0_{h,t}}\sup_{\mathbf{v}_h\in(W_{h,t})^2}\frac{\displaystyle\int_{\Omega_t}p_h\,\mathrm{div}\,\mathbf{v}_h\,d\mathbf{x}}{\|\mathbf{v}_h\|_{H^1(\Omega_t)^2}\,\|p_h\|_{L^2(\Omega_t)}}\geqslant\beta^*. \quad (3.5)$$

This theorem can be easily proved by using (2.14) and (2.6) and by following the proof of Theorem 10.6.6 in [6]. Therefore, we omit the proof of the preceding theorem.

4 Estimates of the projection on the finite element spaces

One of the key ingredient in the proof of our convergence results is the introduction of a projection on the finite element space $(W_{h,t})^2\times M^0_{h,t}$ of the exact problem solution

$$(\mathbf{u},p)\in\left[H^{s+1}(\Omega_t)^2\cap H^1_0(\Omega_t)^2\right]\times\left[H^s(\Omega_t)\cap L^2_0(\Omega_t)\right]$$

(with s a real number $s\geq 1$).

Proposition 4.1. *Suppose that $s\geq 1$ is a real number. If $\mathbf{u}(t)\in H^{s+1}(\Omega_t)^2\cap H^1_0(\Omega_t)^2$ and $p(t)\in H^s(\Omega_t)\cap L^2_0(\Omega_t)$, then there exists an unique couple $(\mathbf{U}(t),P(t))$ in $(W_{h,t})^2\times M^0_{h,t}$ such that*

$$\begin{cases}\nu\displaystyle\int_{\Omega_t}\nabla\left(\mathbf{U}(t)-\mathbf{u}(t)\right):\nabla\mathbf{v}_h\,d\mathbf{x}-\int_{\Omega_t}\left(P(t)-p(t)\right)\mathrm{div}\,\mathbf{v}_h\,d\mathbf{x}=0 & \forall\mathbf{v}_h\in(W_{h,t})^2,\\ \displaystyle\int_{\Omega_t}q_h\,\mathrm{div}\,\left(\mathbf{U}(t)-\mathbf{u}(t)\right)\,d\mathbf{x}=0 & \forall q_h\in M_{h,t}.\end{cases} \quad (4.1)$$

Moreover, there exists a positive constant $C>0$, independent of h and t, such that

$$\|\mathbf{u}(t)-\mathbf{U}(t)\|_{H^1(\Omega_t)^2}+\|p(t)-P(t)\|_{L^2(\Omega_t)}\leq Ch^r\left(\|\mathbf{u}(t)\|_{H^{r+1}(\Omega_t)^2}+\|p(t)\|_{H^r(\Omega_t)}\right), \quad (4.2)$$

for all r such that $1\leq r\leq\min(2,s)$.

The proof of this Proposition is a direct consequence of Theorem 1.1 from Girault and Raviart (see [18, p.114]) and of Theorem 3.3.

Remark 4.2. Due to Proposition 3.1, the problem (4.1) is equivalent to the following one:

$$\begin{cases} \nu \displaystyle\int_{\Omega_t} \nabla\left(\mathbf{U}(t)-\mathbf{u}(t)\right) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ \qquad\qquad - \displaystyle\int_{\Omega_t} \left(P(t)-p(t)\right) \operatorname{div}\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} = 0 \qquad \forall \mathbf{v}_h \in (W_{h,0})^2, \\ \displaystyle\int_{\Omega_t} \big(q_h \circ \mathbf{X}_{h,t}^{-1}\big) \operatorname{div}\left(\mathbf{U}(t)-\mathbf{u}(t)\right)\, d\mathbf{x} = 0 \qquad \forall q_h \in M_{h,0}. \end{cases} \tag{4.3}$$

In order to prove our main results, we need some estimates of the time derivatives on the ALE frame for the projections introduced above. More precisely, we get the following theorem:

Theorem 4.3. *Assume that* $\mathbf{u}\colon Q_T \longrightarrow \mathbb{R}^2$, $p\colon Q_T \longrightarrow \mathbb{R}$ *satisfy*

$$\mathbf{u}(t) \in H^3(\Omega_t)^2 \cap H^1_{0,\sigma}(\Omega_t), \quad p(t) \in H^2(\Omega_t) \cap L^2_0(\Omega_t), \qquad \text{for all } t \in (0,T).$$

Let consider the projection $(\mathbf{U}(t), P(t))$ *onto* $(W_{h,t})^2 \times M^0_{h,t}$ *of* $(\mathbf{u}(t), p(t))$, *defined in Proposition 4.1. We assume that*

$$\mathbf{w}(t) \in W^{2,\infty}(\Omega_t)^2, \quad \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) \in H^2(\Omega_t)^2, \quad \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) \in H^1(\Omega_t). \tag{4.4}$$

Then there exists a positive constant C, *independent of* h, *such that*

$$\begin{aligned} &\left\| \left.\frac{d\mathbf{u}}{dt}\right|^h_{\mathbf{Y}}(t) - \left.\frac{d\mathbf{U}}{dt}\right|^h_{\mathbf{Y}}(t) \right\|_{H^1(\Omega_t)^2} + \left\| \left.\frac{dp}{dt}\right|^h_{\mathbf{Y}}(t) - \left.\frac{dP}{dt}\right|^h_{\mathbf{Y}}(t) \right\|_{L^2(\Omega_t)} \\ &\qquad \le Ch|\ln h| \left(\|\mathbf{u}(t)\|_{H^3(\Omega_t)^2} + \left\| \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) \right\|_{H^2(\Omega_t)^2} + \|p(t)\|_{H^2(\Omega_t)} + \left\| \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) \right\|_{H^1(\Omega_t)} \right). \end{aligned} \tag{4.5}$$

Proof. Using (3.2)–(3.4) we differentiate with respect to t the both equations of (4.3), then we obtain: for all $\mathbf{v}_h \in (W_{h,0})^2$ and $q_h \in M_{h,0}$,

$$\begin{aligned} \nu \int_{\Omega_t} \nabla \left(\left.\frac{d\mathbf{U}}{dt}\right|^h_{\mathbf{Y}}(t) - \left.\frac{d\mathbf{u}}{dt}\right|^h_{\mathbf{Y}}(t) \right) &: \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ - \int_{\Omega_t} \left(\left.\frac{dP}{dt}\right|^h_{\mathbf{Y}}(t) - \left.\frac{dp}{dt}\right|^h_{\mathbf{Y}}(t) \right) &\operatorname{div}\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ = -\nu \int_{\Omega_t} \nabla\left(\mathbf{U}(t)-\mathbf{u}(t)\right) &: \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big) \operatorname{div} \mathbf{w}_h(t)\, d\mathbf{x} \\ + \nu \int_{\Omega_t} \left(\nabla \mathbf{w}_h(t) + \nabla \mathbf{w}_h(t)^T\right) &\nabla\left(\mathbf{U}(t)-\mathbf{u}(t)\right) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ + \int_{\Omega_t} \left(P(t)-p(t)\right) &\operatorname{div}\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big) \operatorname{div} \mathbf{w}_h(t)\, d\mathbf{x} \\ - \int_{\Omega_t} \left(P(t)-p(t)\right) &\nabla \mathbf{w}_h(t) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)^T\, d\mathbf{x}, \end{aligned} \tag{4.6a}$$

and

$$\int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \mathrm{div} \left(\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^{h}(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}^{h}(t) \right) d\mathbf{x}$$
$$= -\int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \mathrm{div}\, \left(\mathbf{U}(t) - \mathbf{u}(t)\right) \mathrm{div}\, \mathbf{w}_h(t)\, d\mathbf{x}$$
$$+ \int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \nabla \mathbf{w}_h(t) : \nabla \left(\mathbf{U}(t) - \mathbf{u}(t)\right)^T \, d\mathbf{x}. \quad (4.6b)$$

Now, we recall that

$$\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}^{h}(t) = \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) + \left(\left(\mathbf{w}_h(t) - \mathbf{w}(t)\right) \cdot \nabla\right) \mathbf{u}(t), \quad (4.7)$$
$$\left.\frac{dp}{dt}\right|_{\mathbf{Y}}^{h}(t) = \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) + \left(\mathbf{w}_h(t) - \mathbf{w}(t)\right) \cdot \nabla p(t), \quad (4.8)$$

therefore, the system (4.6a)–(4.6b) can be written as follows: for all $\mathbf{v}_h \in (W_{h,0})^2$ and $q_h \in M_{h,0}$,

$$\nu \int_{\Omega_t} \nabla \left(\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^{h}(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) \right) : \nabla \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}$$
$$- \int_{\Omega_t} \left(\left.\frac{dP}{dt}\right|_{\mathbf{Y}}^{h}(t) - \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) \right) \mathrm{div} \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}$$
$$= -\nu \int_{\Omega_t} \nabla \left(\mathbf{U}(t) - \mathbf{u}(t)\right) : \nabla \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) \mathrm{div}\, \mathbf{w}_h(t)\, d\mathbf{x}$$
$$+ \nu \int_{\Omega_t} \left(\nabla \mathbf{w}_h(t) + \nabla \mathbf{w}_h(t)^T\right) \nabla \left(\mathbf{U}(t) - \mathbf{u}(t)\right) : \nabla \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}$$
$$+ \int_{\Omega_t} \left(P(t) - p(t)\right) \mathrm{div} \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) \mathrm{div}\, \mathbf{w}_h(t)\, d\mathbf{x}$$
$$- \int_{\Omega_t} \left(P(t) - p(t)\right) \nabla \mathbf{w}_h(t) : \nabla \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right)^T d\mathbf{x}$$
$$+ \nu \int_{\Omega_t} \nabla \left[\left(\left(\mathbf{w}_h(t) - \mathbf{w}(t)\right) \cdot \nabla\right) \mathbf{u}(t) \right] : \nabla \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}$$
$$- \int_{\Omega_t} \left(\mathbf{w}_h(t) - \mathbf{w}(t)\right) \cdot \nabla p(t) \, \mathrm{div} \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x} \quad (4.9a)$$

and

$$\int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \mathrm{div} \left(\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^{h}(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) \right) d\mathbf{x}$$
$$= -\int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \mathrm{div}\, \left(\mathbf{U}(t) - \mathbf{u}(t)\right) \mathrm{div}\, \mathbf{w}_h(t)\, d\mathbf{x}$$
$$+ \int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \nabla \mathbf{w}_h(t) : \nabla \left(\mathbf{U}(t) - \mathbf{u}(t)\right)^T \, d\mathbf{x}$$
$$+ \int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \mathrm{div} \left[\left(\left(\mathbf{w}_h(t) - \mathbf{w}(t)\right) \cdot \nabla\right) \mathbf{u}(t) \right] d\mathbf{x}. \quad (4.9b)$$

On the other hand, we have that $\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) \in H^2(\Omega_t)^2 \cap H_0^1(\Omega_t)^2$ and $\left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) \in H^1(\Omega_t)$. In order to project them onto $(W_{h,t})^2 \times M_{h,t}^0$, since $\left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) \notin L_0^2(\Omega_t)$, we need to introduce an auxiliary function $\hat{p}_1$ defined by

$$\hat{p}_1(t) = \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) - \lambda,$$

where $\lambda = \dfrac{1}{|\Omega_t|}\displaystyle\int_{\Omega_t} \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t)\, d\mathbf{x}$.

Let us note that the equation (4.9a) is also true if we change $\left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t)$ by $\hat{p}_1(t)$.

Now, since $\hat{p}_1(t) \in H^1(\Omega_t) \cap L_0^2(\Omega_t)$, we can consider the projection $\big(\mathbf{U}_1(t), P_1(t)\big) \in (W_{h,t})^2 \times M_{h,t}^0$ of $\left(\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t), \hat{p}_1(t)\right)$, which are solutions of the following well-defined problem:

$$\begin{cases} \nu \displaystyle\int_{\Omega_t} \nabla\left(\mathbf{U}_1(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ \qquad\qquad - \displaystyle\int_{\Omega_t} \left(P_1(t) - \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) + \lambda\right) \operatorname{div}\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big) d\mathbf{x} = 0 \quad \forall \mathbf{v}_h \in (W_{h,0})^2, \\ \displaystyle\int_{\Omega_t} \big(q_h \circ \mathbf{X}_{h,t}^{-1}\big) \operatorname{div}\left(\mathbf{U}_1(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right) d\mathbf{x} = 0 \quad \forall q_h \in M_{h,0}. \end{cases} \tag{4.10}$$

From Proposition 4.1 and Proposition 3.1, we have that

$$\begin{aligned} \left\|\mathbf{U}_1(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right\|_{H^1(\Omega_t)^2} + \left\|P_1(t) - \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) + \lambda\right\|_{L^2(\Omega_t)} \\ \leq Ch\left(\left\|\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right\|_{H^2(\Omega_t)^2} + \left\|\left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t)\right\|_{H^1(\Omega_t)}\right). \end{aligned} \tag{4.11}$$

Subtracting (4.10) from the system obtained by (4.9a) and (4.9b), we get the following problem: for all $\mathbf{v}_h \in (W_{h,0})^2$ and $q_h \in M_{h,0}$,

$$\begin{aligned} &\nu \int_{\Omega_t} \nabla\left(\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^{h}(t) - \mathbf{U}_1(t)\right) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ &\quad - \int_{\Omega_t} \left(\left.\frac{dP}{dt}\right|_{\mathbf{Y}}^{h}(t) - P_1(t)\right) \operatorname{div}\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ &\qquad = -\nu \int_{\Omega_t} \nabla\left(\mathbf{U}(t) - \mathbf{u}(t)\right) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big) \operatorname{div} \mathbf{w}_h(t)\, d\mathbf{x} \\ &\qquad\quad + \nu \int_{\Omega_t} \left(\nabla \mathbf{w}_h(t) + \nabla \mathbf{w}_h(t)^T\right) \nabla\left(\mathbf{U}(t) - \mathbf{u}(t)\right) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ &\qquad\quad + \int_{\Omega_t} \left(P(t) - p(t)\right) \operatorname{div}\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big) \operatorname{div} \mathbf{w}_h(t)\, d\mathbf{x} \\ &\qquad\quad - \int_{\Omega_t} \left(P(t) - p(t)\right) \nabla \mathbf{w}_h(t) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)^T d\mathbf{x} \\ &\qquad\quad + \nu \int_{\Omega_t} \nabla\big[\left(\left(\mathbf{w}_h(t) - \mathbf{w}(t)\right) \cdot \nabla\right) \mathbf{u}(t)\big] : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \\ &\qquad\quad - \int_{\Omega_t} \left(\mathbf{w}_h(t) - \mathbf{w}(t)\right) \cdot \nabla p(t) \operatorname{div}\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big)\, d\mathbf{x} \end{aligned} \tag{4.12a}$$

and

$$\int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \operatorname{div} \left(\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^{h}(t) - \mathbf{U}_1(t) \right) d\mathbf{x}$$
$$= -\int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \operatorname{div} \left(\mathbf{U}(t) - \mathbf{u}(t)\right) \operatorname{div} \mathbf{w}_h(t)\, d\mathbf{x}$$
$$+ \int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \nabla \mathbf{w}_h(t) : \nabla \left(\mathbf{U}(t) - \mathbf{u}(t)\right)^T \, d\mathbf{x}$$
$$+ \int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \operatorname{div} \left[\left(\left(\mathbf{w}_h(t) - \mathbf{w}(t)\right) \cdot \nabla \right) \mathbf{u}(t) \right] d\mathbf{x}. \quad (4.12b)$$

In the system (4.12a)–(4.12b), we can change $\left.\frac{dP}{dt}\right|_{\mathbf{Y}}^{h}(t)$ by the corresponding zero mean value projection $\hat{P}_1(t)$ defined by

$$\hat{P}_1(t) = \left.\frac{dP}{dt}\right|_{\mathbf{Y}}^{h}(t) - \lambda_h,$$

where $\lambda_h = \dfrac{1}{|\Omega_t|} \displaystyle\int_{\Omega_t} \left.\frac{dP}{dt}\right|_{\mathbf{Y}}^{h}(t)\, d\mathbf{x}$. By using this zero mean value projection and Remark 1.3 from Girault and Raviart (see [18, p.117]), we have that

$$\left\| \left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^{h}(t) - \mathbf{U}_1(t) \right\|_{H^1(\Omega_t)^2} + \left\| \left.\frac{dP}{dt}\right|_{\mathbf{Y}}^{h}(t) - \lambda_h - P_1(t) \right\|_{L^2(\Omega_t)}$$
$$\leq C \left\|\nabla \mathbf{w}_h(t)\right\|_{L^\infty(\Omega_t)^4} \left(\left\|\nabla \left(\mathbf{U}(t) - \mathbf{u}(t)\right)\right\|_{L^2(\Omega_t)^4} + \left\|P(t) - p(t)\right\|_{L^2(\Omega_t)} \right)$$
$$+ C \left\|\mathbf{w}_h(t) - \mathbf{w}(t)\right\|_{W^{1,\infty}(\Omega_t)^2} \left(\left\|\mathbf{u}(t)\right\|_{H^2(\Omega_t)^2} + \left\|p(t)\right\|_{H^1(\Omega_t)} \right), \quad (4.13)$$

where the constant $C > 0$ is independent of h and t. Therefore, using (2.13) and (4.2) (with $s = r = 2$), the estimate (4.13) becomes

$$\left\| \left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^{h}(t) - \mathbf{U}_1(t) \right\|_{H^1(\Omega_t)^2} + \left\| \left.\frac{dP}{dt}\right|_{\mathbf{Y}}^{h}(t) - \lambda_h - P_1(t) \right\|_{L^2(\Omega_t)}$$
$$\leq Ch|\ln h| \ \left\|\mathbf{w}(t)\right\|_{W^{2,\infty}(\Omega_t)^2} \left(\left\|\mathbf{u}(t)\right\|_{H^3(\Omega_t)^2} + \left\|p(t)\right\|_{H^2(\Omega_t)} \right). \quad (4.14)$$

By (4.11) and (4.14) it follows that

$$\left\| \left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^{h}(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) \right\|_{H^1(\Omega_t)^2} + \left\| \left.\frac{dP}{dt}\right|_{\mathbf{Y}}^{h}(t) - \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) \right\|_{L^2(\Omega_t)}$$
$$\leq Ch|\ln h| \left(\left\|\mathbf{u}(t)\right\|_{H^3(\Omega_t)^2} + \left\| \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t) \right\|_{H^2(\Omega_t)^2} \right.$$
$$\left. + \left\|p(t)\right\|_{H^2(\Omega_t)} + \left\| \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t) \right\|_{H^1(\Omega_t)} \right) + C|\lambda - \lambda_h|. \quad (4.15)$$

In order to estimate the term $|\lambda - \lambda_h|$, let us remark that

$$\int_{\Omega_t} p(t)\, d\mathbf{x} = \int_{\Omega_t} P(t)\, d\mathbf{x} = 0 \qquad \forall t \in (0, T),$$

then by differentiating with respect to t we get

$$\lambda = -\frac{1}{|\Omega_t|}\int_{\Omega_t} p(t)\mathrm{div}\,\mathbf{w}(t)\,d\mathbf{x} \qquad \text{and } \lambda_h = -\frac{1}{|\Omega_t|}\int_{\Omega_t} P(t)\mathrm{div}\,\mathbf{w}_h(t)\,d\mathbf{x}.$$

Hence,

$$|\lambda-\lambda_h| \;\leq\; \frac{1}{|\Omega_t|}\|p(t)-P(t)\|_{L^2(\Omega_t)}\|\nabla\mathbf{w}_h(t)\|_{L^2(\Omega_t)^4} + \frac{1}{|\Omega_t|}\|p(t)\|_{L^2(\Omega_t)}\|\nabla(\mathbf{w}_h(t)-\mathbf{w}(t))\|_{L^2(\Omega_t)^4}.$$

This inequality together with (2.13) and (4.2) yields to

$$|\lambda-\lambda_h| \leq Ch|\ln h|(\|\mathbf{u}(t)\|_{H^3(\Omega_t)^2} + \|p(t)\|_{H^2(\Omega_t)}).$$

Using this estimate, (4.15) becomes

$$\begin{aligned}
&\left\|\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^h(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right\|_{H^1(\Omega_t)^2} + \left\|\left.\frac{dP}{dt}\right|_{\mathbf{Y}}^h(t) - \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t)\right\|_{L^2(\Omega_t)} \\
&\qquad \leq Ch|\ln h|\left(\|\mathbf{u}(t)\|_{H^3(\Omega_t)^2} + \left\|\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right\|_{H^2(\Omega_t)^2} + \|p(t)\|_{H^2(\Omega_t)} + \left\|\left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t)\right\|_{H^1(\Omega_t)}\right). \quad (4.16)
\end{aligned}$$

Therefore, the estimate (4.5) is a direct consequence of (4.16). In fact, we have that

$$\begin{aligned}
&\left\|\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^h(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}^h(t)\right\|_{H^1(\Omega_t)^2} + \left\|\left.\frac{dP}{dt}\right|_{\mathbf{Y}}^h(t) - \left.\frac{dp}{dt}\right|_{\mathbf{Y}}^h(t)\right\|_{L^2(\Omega_t)} \\
&\qquad \leq \left\|\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^h(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right\|_{H^1(\Omega_t)^2} + \left\|\left.\frac{dp}{dt}\right|_{\mathbf{Y}}^h(t) - \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t)\right\|_{L^2(\Omega_t)} \\
&\qquad\qquad + \|\left((\mathbf{w}_h(t)-\mathbf{w}(t))\cdot\nabla\right)\mathbf{u}(t)\|_{H^1(\Omega_t)^2} + \|(\mathbf{w}_h(t)-\mathbf{w}(t))\cdot\nabla p(t)\|_{L^2(\Omega_t)} \\
&\qquad \leq \left\|\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^h(t) - \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right\|_{H^1(\Omega_t)^2} + \left\|\left.\frac{dp}{dt}\right|_{\mathbf{Y}}^h(t) - \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t)\right\|_{L^2(\Omega_t)} \\
&\qquad\qquad + C\,\|\mathbf{w}_h(t)-\mathbf{w}(t)\|_{W^{1,\infty}(\Omega_t)^2}\,\left(\|\mathbf{u}(t)\|_{H^2(\Omega_t)^2} + \|p(t)\|_{H^1(\Omega_t)}\right),
\end{aligned}$$

and we conclude by combining (2.13) and (4.16).

□

5 Proof of first main result

In this section we prove Theorem 2.1 by using the results of the previous section.

Since $(\mathbf{u},p)$ is solution of (2.4), then we have

$$\left\{\begin{array}{l}
\dfrac{d}{dt}\displaystyle\int_{\Omega_t}\mathbf{u}(t)\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} + \nu\int_{\Omega_t}\nabla\mathbf{u}(t):\nabla\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} \\
\qquad\qquad - \displaystyle\int_{\Omega_t}\mathrm{div}\left(\mathbf{w}_h(t)\otimes\mathbf{u}(t)\right)\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} - \int_{\Omega_t}p(t)\,\mathrm{div}\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)\,d\mathbf{x} \\
\qquad\qquad\qquad = \displaystyle\int_{\Omega_t}\mathbf{f}(t)\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} \qquad \forall\mathbf{v}_h\in(W_{h,0})^2, \\
\displaystyle\int_{\Omega_t}\left(q_h\circ\mathbf{X}_{h,t}^{-1}\right)\mathrm{div}\,\mathbf{u}(t)d\mathbf{x} = 0 \qquad \forall q_h\in M_{h,0}, \\
\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega_0.
\end{array}\right. \quad (5.1)$$

Subtracting (2.11) from (5.1) and introducing the projections $\mathbf{U}(t) \in (W_{h,t})^2,\ P(t) \in M^0_{h,t}$ of the exact solutions $\mathbf{u}(t)$, $p(t)$ defined in Proposition 4.1, we obtain

$$\left\{\begin{array}{l} \dfrac{d}{dt}\displaystyle\int_{\Omega_t} (\mathbf{u}(t)-\mathbf{u}_h(t))\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} + \nu\int_{\Omega_t}\nabla\left(\mathbf{U}(t)-\mathbf{u}_h(t)\right):\nabla\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} \\ \qquad -\displaystyle\int_{\Omega_t}\operatorname{div}\left[\mathbf{w}_h(t)\otimes\left(\mathbf{u}(t)-\mathbf{u}_h(t)\right)\right]\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} \\ \qquad -\displaystyle\int_{\Omega_t}\left(P(t)-p_h(t)\right)\operatorname{div}\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} \\ \qquad =\displaystyle\int_{\Omega_t}\mathbf{f}(t)\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} - \widetilde{I}_{h,t}\left(\mathbf{f}(t)\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)\right) \qquad \forall\mathbf{v}_h\in(W_{h,0})^2, \\ \displaystyle\int_{\Omega_t}\left(q_h\circ\mathbf{X}_{h,t}^{-1}\right)\operatorname{div}\left(\mathbf{U}(t)-\mathbf{u}_h(t)\right)d\mathbf{x}=0 \qquad \forall q_h\in M_{h,0}, \\ \mathbf{u}(0)-\mathbf{u}_h(0)=\mathbf{u}_0-\mathbf{u}_{h,0} \qquad \text{in } \Omega_0. \end{array}\right.$$

For the time derivative of the first integral, we apply formula (3.1) to obtain

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega_t}\left[\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}^h(t)-\left.\frac{d\mathbf{u}_h}{dt}\right|_{\mathbf{Y}}^h(t)\right]\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x}+\nu\int_{\Omega_t}\nabla(\mathbf{U}(t)-\mathbf{u}_h(t))\cdot\nabla\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} \\ \qquad -\displaystyle\int_{\Omega_t}\left(\mathbf{w}_h(t)\cdot\nabla\right)\left(\mathbf{u}(t)-\mathbf{u}_h(t)\right)\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x}-\int_{\Omega_t}\left(P(t)-p_h(t)\right)\operatorname{div}\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x} \\ \qquad =\displaystyle\int_{\Omega_t}\mathbf{f}(t)\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)d\mathbf{x}-\widetilde{I}_{h,t}\left(\mathbf{f}(t)\cdot\left(\mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1}\right)\right) \qquad \forall\mathbf{v}_h\in(W_{h,0})^2, \\ \displaystyle\int_{\Omega_t}\left(q_h\circ\mathbf{X}_{h,t}^{-1}\right)\operatorname{div}\left(\mathbf{U}(t)-\mathbf{u}_h(t)\right)d\mathbf{x}=0 \qquad \forall q_h\in M_{h,0}, \\ \mathbf{u}(0)-\mathbf{u}_h(0)=\mathbf{u}_0-\mathbf{u}_{h,0} \qquad \text{in } \Omega_0. \end{array}\right. \tag{5.2}$$

Using Proposition 3.1, we can choose in the above system the test functions $(\mathbf{v}_h, q_h)$ such that

$$\begin{aligned} \mathbf{v}_h\circ\mathbf{X}_{h,t}^{-1} &= \mathbf{U}(t)-\mathbf{u}_h(t)\in(W_{h,t})^2, \\ q_h\circ\mathbf{X}_{h,t}^{-1} &= P(t)-p_h(t)\in M_{h,t}. \end{aligned}$$

Then it follows that

$$\begin{aligned} \int_{\Omega_t}\left[\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}^h(t)-\left.\frac{d\mathbf{u}_h}{dt}\right|_{\mathbf{Y}}^h(t)\right]\cdot(\mathbf{U}(t)-\mathbf{u}_h(t))d\mathbf{x}+\nu\int_{\Omega_t}\left|\nabla(\mathbf{U}(t)-\mathbf{u}_h(t))\right|^2d\mathbf{x} \\ -\int_{\Omega_t}\left(\mathbf{w}_h(t)\cdot\nabla\right)\left(\mathbf{u}(t)-\mathbf{u}_h(t)\right)\cdot\left(\mathbf{U}(t)-\mathbf{u}_h(t)\right)d\mathbf{x} \\ =\int_{\Omega_t}\mathbf{f}(t)\cdot\left(\mathbf{U}(t)-\mathbf{u}_h(t)\right)d\mathbf{x}-\widetilde{I}_{h,t}\left(\mathbf{f}(t)\cdot\left(\mathbf{U}(t)-\mathbf{u}_h(t)\right)\right) \end{aligned} \tag{5.3}$$

On the other hand, due to the Reynolds formula, it can be checked that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{U}(t)-\mathbf{u}_h(t)\|^2_{L^2(\Omega_t)^2} = \int_{\Omega_t}\left[\left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^h(t)-\left.\frac{d\mathbf{u}_h}{dt}\right|_{\mathbf{Y}}^h(t)\right]\cdot\left(\mathbf{U}(t)-\mathbf{u}_h(t)\right)d\mathbf{x} \\ -\int_{\Omega_t}(\mathbf{w}_h(t)\cdot\nabla)(\mathbf{U}(t)-\mathbf{u}_h(t))\cdot(\mathbf{U}(t)-\mathbf{u}_h(t))\,d\mathbf{x}. \end{aligned}$$

Combining this identity with (5.3), we obtain that:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{U}(t)-\mathbf{u}_h(t)\|^2_{L^2(\Omega_t)^2}+\nu\|\nabla(\mathbf{U}(t)-\mathbf{u}_h(t))\|^2_{L^2(\Omega_t)^4}=\sum_{i=1}^3 T_i, \tag{5.4}$$

where the terms T_1, T_2 and T_3 are defined as follows:

$$\begin{aligned} T_1 &= -\int_{\Omega_t} \left[\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}^h (t) - \left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{Y}}^h (t) \right] \cdot (\mathbf{U}(t) - \mathbf{u}_h(t)) d\mathbf{x}, \\ T_2 &= \int_{\Omega_t} (\mathbf{w}_h(t) \cdot \nabla) (\mathbf{u}(t) - \mathbf{U}(t)) \cdot (\mathbf{U}(t) - \mathbf{u}_h(t)) \, d\mathbf{x}, \\ T_3 &= \int_{\Omega_t} \mathbf{f}(t) \cdot (\mathbf{U}(t) - \mathbf{u}_h(t)) \, d\mathbf{x} - \widetilde{I}_{h,t} (\mathbf{f}(t) \cdot (\mathbf{U}(t) - \mathbf{u}_h(t))) . \end{aligned}$$

Now, let us estimate separately each term. Due to the Cauchy-Schwarz inequality and Theorem 4.3 we get that the first term is bounded as follows:

$$T_1 \leq Ch|\ln h| \left(\|\mathbf{u}(t)\|_{H^3(\Omega_t)^2} + \left\| \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}} (t) \right\|_{H^2(\Omega_t)^2} + \|p(t)\|_{H^2(\Omega_t)} + \left\| \left.\frac{dp}{dt}\right|_{\mathbf{Y}} (t) \right\|_{H^1(\Omega_t)} \right) \cdot \|\mathbf{U}(t) - \mathbf{u}_h(t)\|_{L^2(\Omega_t)^2}.$$

The next term can be bounded using the Cauchy-Schwarz inequality, the estimates (2.13), (4.2) and we obtain that

$$T_2 \leq Ch^2 \|\mathbf{w}(t)\|_{W^{2,\infty}(\Omega_t)^2} \left(\|\mathbf{u}(t)\|_{H^3(\Omega_t)^2} + \|p(t)\|_{H^2(\Omega_t)} \right) \|\mathbf{U}(t) - \mathbf{u}_h(t)\|_{L^2(\Omega_t)^2}.$$

Now, let us estimate T_3. Using the fact that $\Omega_t = \bigcup_{K \in \mathcal{T}_{h,t}} K$ we can write

$$T_3 = \int_{\Omega_t} \mathbf{f}(t) \cdot (\mathbf{U}(t) - \mathbf{u}_h(t)) \, d\mathbf{x} - \widetilde{I}_{h,t} (\mathbf{f}(t) \cdot (\mathbf{U}(t) - \mathbf{u}_h(t))) = \sum_{K \in \mathcal{T}_{h,t}} E_K (\mathbf{f}(t) \cdot (\mathbf{U}(t) - \mathbf{u}_h(t))),$$

where E_K represents the quadrature error on triangle K. To estimate this term, we apply Theorem 4.1.5 from Ciarlet [10, p. 195] and we obtain that for any $q > 2$,

$$T_3 \leq Ch^2 \sum_{K \in \mathcal{T}_{h,t}} |K|^{1/2 - 1/q} \|\mathbf{f}(t)\|_{W^{2,q}(K)^2} \|\mathbf{U}(t) - \mathbf{u}_h(t)\|_{H^1(K)^2}.$$

Combining the above inequality and the Hölder inequality $\left(\text{with } \frac{1}{2} + \frac{1}{p} + \frac{1}{q} = 1\right)$, it follows that

$$\begin{aligned} T_3 &\leq Ch^2 \left(\sum_{K \in \mathcal{T}_{h,t}} |K|^{\frac{1}{2} - \frac{1}{q} \; p} \right)^{1/p} \left(\sum_{K \in \mathcal{T}_{h,t}} \|\mathbf{f}(t)\|^q_{W^{2,q}(K)^2} \right)^{1/q} \left(\sum_{K \in \mathcal{T}_{h,t}} \|\mathbf{U}(t) - \mathbf{u}_h(t)\|^2_{H^1(K)^2} \right)^{1/2} \\ &\leq Ch^2 \|\mathbf{f}(t)\|_{W^{2,q}(\Omega_t)^2} \|\mathbf{U}(t) - \mathbf{u}_h(t)\|_{H^1(\Omega_t)^2}. \end{aligned}$$

By using all previous bounds and the Poincaré inequality, (5.4) becomes

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\mathbf{U}(t) - \mathbf{u}_h(t)\|^2_{L^2(\Omega_t)^2} + \nu \|\nabla(\mathbf{U}(t) - \mathbf{u}_h(t))\|^2_{L^2(\Omega_t)^4} \\ &\qquad \leq Ch|\ln h| \left[\|\mathbf{u}(t)\|_{H^3(\Omega_t)^2} + \left\| \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}} (t) \right\|_{H^2(\Omega_t)^2} + \|p(t)\|_{H^2(\Omega_t)} + \left\| \left.\frac{dp}{dt}\right|_{\mathbf{Y}} (t) \right\|_{H^1(\Omega_t)} \right. \\ &\qquad\qquad \left. + \|\mathbf{f}(t)\|_{W^{2,q}(\Omega_t)^2} \right] \|\nabla(\mathbf{U}(t) - \mathbf{u}_h(t))\|_{L^2(\Omega_t)^4}. \end{aligned}$$

Now, integrating the above inequality, from 0 to t, we get

$$\begin{aligned}\frac{1}{2}\|\mathbf{U}(t)-\mathbf{u}_h(t)\|^2_{L^2(\Omega_t)^2}+\nu\int_0^t\|\nabla(\mathbf{U}(s)-\mathbf{u}_h(s))\|^2_{L^2(\Omega_s)^4}ds\\ \leq\frac{1}{2}\|\mathbf{U}(0)-\mathbf{u}_h(0)\|^2_{L^2(\Omega_0)^2}+Ch|\ln h|\int_0^t\left[\|\mathbf{u}(s)\|_{H^3(\Omega_s)^2}+\left\|\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(s)\right\|_{H^2(\Omega_s)^2}\right.\\ \left.+\|p(s)\|_{H^2(\Omega_s)}+\left\|\left.\frac{dp}{dt}\right|_{\mathbf{Y}}(s)\right\|_{H^1(\Omega_s)}+\|\mathbf{f}(s)\|_{W^{2,q}(\Omega_s)^2}\right]\|\nabla(\mathbf{U}(s)-\mathbf{u}_h(s))\|_{L^2(\Omega_s)^4}ds,\end{aligned}$$

then, due to the inequality $ab\leq\dfrac{5}{2\nu}a^2+\dfrac{\nu}{10}b^2\ \forall a,b\in\mathbb{R}$, we obtain that for all $t\in(0,T)$,

$$\begin{aligned}\frac{1}{2}\|\mathbf{U}(t)-\mathbf{u}_h(t)\|^2_{L^2(\Omega_t)^2}+\frac{\nu}{2}\int_0^t\|\nabla(\mathbf{U}(s)-\mathbf{u}_h(s))\|^2_{L^2(\Omega_s)^4}ds\\ \leq\frac{1}{2}\|\mathbf{U}(0)-\mathbf{u}_h(0)\|^2_{L^2(\Omega_0)^2}+Ch^2|\ln h|^2\int_0^t\left[\|\mathbf{u}(s)\|^2_{H^3(\Omega_s)^2}+\left\|\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(s)\right\|^2_{H^2(\Omega_s)^2}\right.\\ \left.+\|p(s)\|^2_{H^2(\Omega_s)}+\left\|\left.\frac{dp}{dt}\right|_{\mathbf{Y}}(s)\right\|^2_{H^1(\Omega_s)}+\|\mathbf{f}(s)\|^2_{W^{2,q}(\Omega_s)^2}\right]ds.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\|\mathbf{U}-\mathbf{u}_h\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)}+\frac{\nu}{2}\|\nabla(\mathbf{U}-\mathbf{u}_h)\|^2_{L^2(0,T;L^2(\Omega_t)^4)}\\ \leq\frac{1}{2}\|\mathbf{U}(0)-\mathbf{u}_h(0)\|^2_{L^2(\Omega_0)^2}+Ch^2|\ln h|^2\left[\|\mathbf{u}\|^2_{L^2(0,T;H^3(\Omega_t)^2)}+\left\|\left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}\right\|^2_{L^2(0,T;H^2(\Omega_t)^2)}\right.\\ \left.+\|p\|^2_{L^2(0,T;H^2(\Omega_t))}+\left\|\left.\frac{dp}{dt}\right|_{\mathbf{Y}}\right\|^2_{L^2(0,T;H^1(\Omega_t))}+\|\mathbf{f}\|^2_{L^2(0,T;W^{2,q}(\Omega_t)^2)}\right].\quad(5.5)\end{aligned}$$

In order to obtain the estimation (2.16), let us first observe that

$$\begin{aligned}\frac{1}{4}\|\mathbf{u}-\mathbf{u}_h\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)}+\frac{\nu}{4}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2(0,T;L^2(\Omega_t)^4)}\leq\frac{1}{2}\|\mathbf{u}-\mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)}\\ +\frac{\nu}{2}\|\nabla(\mathbf{u}-\mathbf{U})\|^2_{L^2(0,T;L^2(\Omega_t)^4)}+\frac{1}{2}\|\mathbf{U}-\mathbf{u}_h\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)}+\frac{\nu}{2}\|\nabla(\mathbf{U}-\mathbf{u}_h)\|^2_{L^2(0,T;L^2(\Omega_t)^4)}.\quad(5.6)\end{aligned}$$

On the other hand, since (4.2) holds true for each $t\in(0,T)$, we get that

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}-\mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)}+\frac{\nu}{2}\|\nabla(\mathbf{u}-\mathbf{U})\|^2_{L^2(0,T;L^2(\Omega_t)^4)}\\ \leq Ch^4\left(\|\mathbf{u}\|^2_{L^\infty(0,T;H^3(\Omega_t)^2)}+\|p\|^2_{L^\infty(0,T;H^2(\Omega_t))}\right)\quad(5.7)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\|\mathbf{U}(0)-\mathbf{u}_h(0)\|^2_{L^2(\Omega_0)^2}&\leq\|\mathbf{u}(0)-\mathbf{U}(0)\|^2_{L^2(\Omega_0)^2}+\|\mathbf{u}(0)-\mathbf{u}_h(0)\|^2_{L^2(\Omega_0)^2}\\ &\leq Ch^4\left(\|\mathbf{u}(0)\|^2_{H^3(\Omega_0)^2}+\|p(0)\|^2_{H^2(\Omega_0)}\right)+\|\mathbf{u}(0)-\mathbf{u}_h(0)\|^2_{L^2(\Omega_0)^2}.\quad(5.8)\end{aligned}$$

By using (5.5)–(5.8), we get the result stated in Theorem 2.1.

6 Proof of second main result

In this section, we will analyze the full discretization of the problem (2.5) given in (2.11). We will prove that the numerical solution converges to the exact solution of the problem, when the discretization parameters Δt and h go to zero, if a compatibility condition between Δt and h is fulfilled.

6.1 Proof of Theorem 2.2

We remark that the approximation error $\mathbf{u}(t_{n+1}) - \mathbf{U}(t_{n+1})$ is well known, and is given in the estimate (4.2). For this reason, we will study the following error:

$$\mathbf{e}_h^{n+1} = \mathbf{U}(t_{n+1}) - \mathbf{u}_h^{n+1} \qquad \forall n = 0, \dots, N-1. \tag{6.1}$$

Since $(\mathbf{u}, p)$ is solution of (2.4), we have that

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{\Omega_t} \mathbf{u}(t) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x} + \nu \int_{\Omega_t} \nabla \mathbf{u}(t) : \nabla\left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x} \\ \qquad - \displaystyle\int_{\Omega_t} p(t) \mathrm{div}\left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x} - \int_{\Omega_t} \mathrm{div}\left(\mathbf{w}_h(t) \otimes \mathbf{u}(t)\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x} \\ \qquad = \displaystyle\int_{\Omega_t} \mathbf{f}(t) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x} \qquad \forall \mathbf{v}_h \in (W_{h,0})^2, \\ \displaystyle\int_{\Omega_t} \left(q_h \circ \mathbf{X}_{h,t}^{-1}\right) \mathrm{div}\, \mathbf{u}(t)\, d\mathbf{x} = 0 \qquad \forall q_h \in M_{h,0}, \\ \mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega_0. \end{cases} \tag{6.2}$$

Then, integrating the first equation of the above system from t_n to t_{n+1}, we get

$$\begin{aligned} &\int_{\Omega_{t_{n+1}}} \mathbf{u}(t_{n+1}) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right) d\mathbf{x} - \int_{\Omega_{t_n}} \mathbf{u}(t_n) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_n}^{-1}\right) d\mathbf{x} \\ &\qquad + \nu \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \nabla \mathbf{u}(t) : \nabla\left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}\, dt - \int_{t_n}^{t_{n+1}} \int_{\Omega_t} p(t) \mathrm{div}\left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}\, dt \\ &\qquad\qquad - \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \mathrm{div}\left(\mathbf{w}_h(t) \otimes \mathbf{u}(t)\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}\, dt \\ &\qquad\qquad\qquad = \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \mathbf{f}(t) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}\, dt \qquad \forall \mathbf{v}_h \in (W_{h,0})^2. \end{aligned}$$

The previous identity could be rewritten similarly to the numerical equations as follows:

$$\begin{aligned} &\int_{\Omega_{t_{n+1}}} \mathbf{u}(t_{n+1}) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right) d\mathbf{x} - \int_{\Omega_{t_n}} \mathbf{u}(t_n) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_n}^{-1}\right) d\mathbf{x} \\ &\qquad + \Delta t\, \nu \int_{\Omega_{t_{n+1}}} \nabla \mathbf{u}(t_{n+1}) : \nabla\left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right) d\mathbf{x} - \Delta t \int_{\Omega_{t_{n+1}}} p(t_{n+1}) \mathrm{div}\left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right) d\mathbf{x} \\ &\qquad\qquad - \Delta t \int_{\Omega_{t_{n+1}}} \mathrm{div}\left(\mathbf{w}_h(t_{n+1}) \otimes \mathbf{u}(t_{n+1})\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right) d\mathbf{x} \\ &\qquad\qquad\qquad = \Delta t \int_{\Omega_{t_{n+1}}} \mathbf{f}(t_{n+1}) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right) d\mathbf{x} + \sum_{i=1}^{4} Q_i \qquad \forall \mathbf{v}_h \in (W_{h,0})^2, \end{aligned} \tag{6.3}$$

where Q_i $(i = 1, \ldots, 4)$ are the differences between the time integrals and the numerical approximations given by the right point integration formula. That is,

$$Q_1 = \nu\,\Delta t \int_{\Omega_{t_{n+1}}} \nabla \mathbf{u}(t_{n+1}) : \nabla \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) d\mathbf{x} - \nu \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \nabla \mathbf{u}(t) : \nabla \big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t} \big)\, d\mathbf{x}\, dt, \quad (6.4)$$

$$Q_2 = -\Delta t \int_{\Omega_{t_{n+1}}} \text{div } (\mathbf{w}_h(t_{n+1}) \otimes \mathbf{u}(t_{n+1})) \cdot \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) d\mathbf{x} + \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \text{div } (\mathbf{w}_h(t) \otimes \mathbf{u}(t)) \cdot \big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t} \big)\, d\mathbf{x}\, dt, \quad (6.5)$$

$$Q_3 = -\Delta t \int_{\Omega_{t_{n+1}}} p(t_{n+1}) \text{div } \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) d\mathbf{x} + \int_{t_n}^{t_{n+1}} \int_{\Omega_t} p(t) \text{div } \big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t} \big)\, d\mathbf{x}\, dt, \quad (6.6)$$

$$Q_4 = -\Delta t \int_{\Omega_{t_{n+1}}} \mathbf{f}(t_{n+1}) \cdot \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) d\mathbf{x} + \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \mathbf{f}(t) \cdot \big(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t} \big)\, d\mathbf{x}\, dt. \quad (6.7)$$

Using the projections of $\mathbf{u}(t_{n+1})$ and $p(t_{n+1})$, denoted by $\mathbf{U}(t_{n+1}) \in (W_{h,t_{n+1}})^2$ and $P(t_{n+1}) \in M^0_{h,t_{n+1}}$, and defined in (4.3), the problem (6.2) can be written as follows:

$$\begin{cases} \displaystyle\int_{\Omega_{t_{n+1}}} \mathbf{u}(t_{n+1}) \cdot \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) d\mathbf{x} \\ \qquad \displaystyle - \int_{\Omega_{t_n}} \mathbf{u}(t_n) \cdot \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_n} \right) d\mathbf{x} + \Delta t\ \nu \int_{\Omega_{t_{n+1}}} \nabla \mathbf{U}(t_{n+1}) : \nabla \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) d\mathbf{x} \\ \qquad \displaystyle -\Delta t \int_{\Omega_{t_{n+1}}} \text{div } (\mathbf{w}_h(t_{n+1}) \otimes \mathbf{u}(t_{n+1})) \cdot \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) d\mathbf{x} \\ \qquad \displaystyle -\Delta t \int_{\Omega_{t_{n+1}}} P(t_{n+1}) \text{div } \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) d\mathbf{x} \\ \qquad \displaystyle = \Delta t \int_{\Omega_{t_{n+1}}} \mathbf{f}(t_{n+1}) \cdot \left(\mathbf{v}_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right)\ d\mathbf{x} + \sum_{i=1}^{4} Q_i \qquad \forall \mathbf{v}_h \in (W_{h,0})^2, \\ \displaystyle\int_{\Omega_{t_{n+1}}} \left(q_h \circ \mathbf{X}^{-1}_{h,t_{n+1}} \right) \text{div } \mathbf{U}(t_{n+1})\, d\mathbf{x} = 0 \qquad \forall q_h \in M_{h,0}, \\ \mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega_0. \end{cases} \quad (6.8)$$

The preceding system allows us to compare directly the numerical solution with the exact one: by subtracting (6.8) and (2.17) we get

$$
\begin{cases}
\displaystyle\int_{\Omega_{t_{n+1}}} \left(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} - \int_{\Omega_{t_n}} \left(\mathbf{u}(t_n) - \mathbf{u}_h^{n}\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_n}^{-1}\right)\, d\mathbf{x} \\
\qquad +\Delta t\ \nu \displaystyle\int_{\Omega_{t_{n+1}}} \nabla \left(\mathbf{U}(t_{n+1}) - \mathbf{u}_h^{n+1}\right) : \nabla \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad -\Delta t \displaystyle\int_{\Omega_{t_{n+1}}} \mathrm{div}\ \left(\mathbf{w}_h(t_{n+1}) \otimes \mathbf{u}(t_{n+1})\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad +\Delta t \displaystyle\int_{\Omega_{t_{n+1}}} \mathrm{div}\ \left(\mathbf{w}_{h,n,n+1}^{*} \otimes \mathbf{u}_h^{n+1}\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad -\Delta t \displaystyle\int_{\Omega_{t_{n+1}}} \left(P(t_{n+1}) - p_h^{n+1}\right) \mathrm{div}\ \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad = \displaystyle\sum_{i=1}^{4} Q_i + \Delta t \int_{\Omega_{t_{n+1}}} \mathbf{f}(t_{n+1}) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad -\Delta t \widetilde{I}_{h,t_{n+1}} \left(\mathbf{f}(t_{n+1}) \cdot (\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1})\right) \qquad \forall \mathbf{v}_h \in (W_{h,0})^2, \\
\displaystyle\int_{\Omega_{t_{n+1}}} \left(q_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right) \mathrm{div}\ \left(\mathbf{U}(t_{n+1}) - \mathbf{u}_h^{n+1}\right)\, d\mathbf{x} = 0 \qquad \forall q_h \in M_{h,0}, \\
\mathbf{u}(0) - \mathbf{u}_h^0 = \mathbf{u}_0 - \mathbf{u}_{h,0} \qquad \text{in } \Omega_0.
\end{cases}
\tag{6.9}
$$

We note that in the previous problem there are two convective terms, with the velocities $\mathbf{w}_h$ and $\mathbf{w}_{h,n,n+1}^{*}$. In order to compare these two velocities, we use the definition of $\mathbf{w}_{h,n,n+1}^{*}$, and therefore we get

$$
\mathbf{w}_h(\mathbf{x}, t_{n+1}) = \mathbf{w}_{h,n,n+1}^{*}(\mathbf{x}) + \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (s - t_n) \frac{\partial^2 \mathbf{X}_h}{\partial s^2} \left(\mathbf{X}_{h,t_{n+1}}^{-1}(\mathbf{x}), s\right)\, ds. \tag{6.10}
$$

Gathering this identity and (6.9), then by using the notation (6.1) it follows the following system:

$$
\begin{cases}
\displaystyle\int_{\Omega_{t_{n+1}}} \left(\mathbf{u}(t_{n+1}) - \mathbf{U}(t_{n+1})\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad - \displaystyle\int_{\Omega_{t_n}} \left(\mathbf{u}(t_n) - \mathbf{U}(t_n)\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_n}^{-1}\right)\, d\mathbf{x} \\
\qquad + \displaystyle\int_{\Omega_{t_{n+1}}} \mathbf{e}_h^{n+1} \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} - \int_{\Omega_{t_n}} \mathbf{e}_h^{n} \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_n}^{-1}\right)\, d\mathbf{x} \\
\qquad +\Delta t\ \nu \displaystyle\int_{\Omega_{t_{n+1}}} \nabla \mathbf{e}_h^{n+1} : \nabla \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad -\Delta t \displaystyle\int_{\Omega_{t_{n+1}}} \mathrm{div}\ \left(\mathbf{w}_{h,n,n+1}^{*} \otimes \mathbf{e}_h^{n+1}\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad - \displaystyle\int_{\Omega_{t_{n+1}}} \mathrm{div}\ \left(\int_{t_n}^{t_{n+1}} (s - t_n) \frac{\partial^2 \mathbf{X}_h}{\partial s^2} \left(\mathbf{X}_{h,t_{n+1}}^{-1}(\mathbf{x}), s\right)\, ds \otimes \mathbf{u}(t_{n+1})\right) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad -\Delta t \displaystyle\int_{\Omega_{t_{n+1}}} \left(P(t_{n+1}) - p_h^{n+1}\right) \mathrm{div}\ \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad = \displaystyle\sum_{i=1}^{4} Q_i + \Delta t \int_{\Omega_{t_{n+1}}} \mathbf{f}(t_{n+1}) \cdot \left(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right)\, d\mathbf{x} \\
\qquad -\Delta t \widetilde{I}_{h,t_{n+1}} \left(\mathbf{f}(t_{n+1}) \cdot (\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1})\right) \qquad \forall \mathbf{v}_h \in (W_{h,0})^2, \\
\displaystyle\int_{\Omega_{t_{n+1}}} \left(q_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}\right) \mathrm{div}\, \mathbf{e}_h^{n+1}\, d\mathbf{x} = 0 \qquad \forall q_h \in M_{h,0}, \\
\mathbf{u}(0) - \mathbf{u}_h^0 = \mathbf{u}_0 - \mathbf{u}_{h,0} \qquad \text{in } \Omega_0.
\end{cases}
\tag{6.11}
$$

In the above system, we choose the following test functions:

$$\begin{aligned} \mathbf{v}_h &= \mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \in (W_{h,0})^2\,, \\ q_h &= \left(P(t_{n+1}) - p_h^{n+1}\right) \circ \mathbf{X}_{h,t_{n+1}} \in M_{h,0} \end{aligned}$$

and we get

$$\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \nu\ \Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4} = \sum_{j=1}^{9} R_j, \tag{6.12}$$

where the right hand side is given by

$$R_1 = \int_{\Omega_{t_n}} \mathbf{e}_h^n \cdot \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,t_n}^{-1}\right) d\mathbf{x} + \Delta t \int_{\Omega_{t_{n+1}}} \text{div}\ \left(\mathbf{w}^*_{h,n,n+1} \otimes \mathbf{e}_h^{n+1}\right) \cdot \mathbf{e}_h^{n+1}\, d\mathbf{x}, \tag{6.13}$$

$$\begin{aligned} R_2 &= \int_{\Omega_{t_n}} (\mathbf{u}(t_n) - \mathbf{U}(t_n)) \cdot \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,t_n}^{-1}\right) d\mathbf{x} \\ &\qquad - \int_{\Omega_{t_{n+1}}} (\mathbf{u}(t_{n+1}) - \mathbf{U}(t_{n+1})) \cdot \mathbf{e}_h^{n+1}\, d\mathbf{x}, \end{aligned} \tag{6.14}$$

$$R_3 = -\Delta t \int_{\Omega_{t_{n+1}}} \text{div}\ \left(\mathbf{w}^*_{h,n,n+1} \otimes (\mathbf{u}(t_{n+1}) - \mathbf{U}(t_{n+1}))\right) \cdot \mathbf{e}_h^{n+1}\, d\mathbf{x}, \tag{6.15}$$

$$R_4 = \int_{\Omega_{t_{n+1}}} \text{div}\ \left(\int_{t_n}^{t_{n+1}} (s - t_n) \frac{\partial^2 \mathbf{X}_h}{\partial s^2} \left(\mathbf{X}_{h,t_{n+1}}^{-1}(\mathbf{x}), s\right)\ ds \otimes \mathbf{u}(t_{n+1})\right) \cdot \mathbf{e}_h^{n+1}\, d\mathbf{x}, \tag{6.16}$$

$$\begin{aligned} R_5 &= \nu\, \Delta t \int_{\Omega_{t_{n+1}}} \nabla \mathbf{u}(t_{n+1}) : \nabla \mathbf{e}_h^{n+1}\, d\mathbf{x} \\ &\qquad - \nu \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \nabla \mathbf{u}(t) : \nabla\left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,t}^{-1}\right)\, d\mathbf{x}\, dt, \end{aligned} \tag{6.17}$$

$$\begin{aligned} R_6 &= -\Delta t \int_{\Omega_{t_{n+1}}} \text{div}\ (\mathbf{w}_h(t_{n+1}) \otimes \mathbf{u}(t_{n+1})) \cdot \mathbf{e}_h^{n+1}\, d\mathbf{x} \\ &\qquad + \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \text{div}\ (\mathbf{w}_h(t) \otimes \mathbf{u}(t)) \cdot \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}\, dt, \end{aligned} \tag{6.18}$$

$$R_7 = -\Delta t \int_{\Omega_{t_{n+1}}} p(t_{n+1}) \text{div}\, \mathbf{e}_h^{n+1}\, d\mathbf{x} + \int_{t_n}^{t_{n+1}} \int_{\Omega_t} p(t) \text{div}\, \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,t}^{-1}\right) d\mathbf{x}\, dt, \tag{6.19}$$

$$R_8 = -\Delta t \int_{\Omega_{t_{n+1}}} \mathbf{f}(t_{n+1}) \cdot \mathbf{e}_h^{n+1}\, d\mathbf{x} + \int_{t_n}^{t_{n+1}} \int_{\Omega_t} \mathbf{f}(t) \cdot \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,t}^{-1}\right)\, d\mathbf{x}\, dt, \tag{6.20}$$

$$R_9 = \Delta t \int_{\Omega_{t_{n+1}}} \mathbf{f}(t_{n+1}) \cdot \mathbf{e}_h^{n+1}\, d\mathbf{x} - \Delta t \widetilde{I}_{h,t_{n+1}} \left(\mathbf{f}(t_{n+1}) \cdot \mathbf{e}_h^{n+1}\right). \tag{6.21}$$

The estimates of the terms R_i $(i = 1, \ldots, 9)$ are very technical, and we prefer to postpone their proof to Subsection 6.2. For the sake of completeness, in the sequel we present the results obtained, nevertheless, the precise results are stated in Lemmas 6.1–6.4 below. We get that

$$R_1 \leq \frac{1}{2}\|\mathbf{e}_h^n\|^2_{L^2(\Omega_{t_n})^2} + \frac{1}{2}\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \frac{1}{2}\Delta t\ \gamma \|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2}, \tag{6.22}$$

where

$$\begin{aligned} \gamma = \max_{n=0,\ldots,N-1} \Big[&\sup_{t\in(t_n,t_{n+1})} \|\text{div}\, \mathbf{w}_h(t)\|_{L^\infty(\Omega_t)} \sup_{t\in(t_n,t_{n+1})} \left\|J_{\mathbf{X}_{h,t}}\right\|_{L^\infty(\Omega_0)} \left\|J_{\mathbf{X}^{-1}_{h,t_{n+1}}}\right\|_{L^\infty(\Omega_{t_{n+1}})} \\ &+ \|\text{div}\, \mathbf{w}^*_{h,n,n+1}\|_{L^\infty(\Omega_{t_{n+1}})}\Big]. \end{aligned}$$

Furthermore,

$$R_2 \leq C\frac{1}{\Delta t}\|\mathbf{u}-\mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + \frac{1}{18}\nu\Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4}, \tag{6.23}$$

$$R_3 \leq C\Delta t\ \|\mathbf{u}-\mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + \frac{1}{18}\nu\Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4}, \tag{6.24}$$

$$R_4 \leq C\Delta t^3 \sup_{s\in(t_n,t_{n+1})}\left\|\left[\frac{\partial^2 \mathbf{X}_h}{\partial s^2}(s)\right]^2\right\|_{L^\infty(\Omega_0)} \|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + \frac{1}{18}\nu\Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4}. \tag{6.25}$$

In addition,

$$R_5 \leq C\Delta t^2\int_{t_n}^{t_{n+1}}\left(\|\mathbf{u}(t)\|^2_{H^1(\Omega_t)^2} + \left\|\left.\frac{d\mathbf{u}}{dt}\right|^h_{\mathbf{Y}}(t)\right\|^2_{H^1(\Omega_t)^2}\right)dt + \frac{1}{18}\nu\Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4}, \tag{6.26}$$

$$R_6 \leq C\Delta t^2\int_{t_n}^{t_{n+1}}\left(\|\mathbf{u}(t)\|^2_{H^1(\Omega_t)^2} + \left\|\left.\frac{d\mathbf{u}}{dt}\right|^h_{\mathbf{Y}}(t)\right\|^2_{L^2(\Omega_t)^2}\right)dt + \frac{1}{9}\nu\Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4}, \tag{6.27}$$

$$R_7 \leq C\Delta t^2\int_{t_n}^{t_{n+1}}\left(\|p(t)\|^2_{L^2(\Omega_t)} + \left\|\left.\frac{dp}{dt}\right|^h_{\mathbf{Y}}(t)\right\|^2_{L^2(\Omega_t)}\right)dt + \frac{1}{18}\nu\Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4}, \tag{6.28}$$

$$R_8 \leq C\Delta t^2\int_{t_n}^{t_{n+1}}\left(\|\mathbf{f}(t)\|^2_{L^2(\Omega_t)^2} + \left\|\left.\frac{d\mathbf{f}}{dt}\right|^h_{\mathbf{Y}}(t)\right\|^2_{L^2(\Omega_t)^2}\right)dt + \frac{1}{18}\nu\Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4} \tag{6.29}$$

and

$$R_9 \leq C\Delta t\ h^4\|\mathbf{f}(t_{n+1})\|^2_{W^{2,q}(\Omega_{t_{n+1}})^2} + \frac{1}{18}\nu\Delta t\|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4}. \tag{6.30}$$

By using these estimates of R_i $(i=1,\ldots,9)$ in (6.12), we obtain

$$\begin{aligned}
\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \nu\ \Delta t\ \|\nabla \mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4} &\leq \Delta t\ \gamma\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} \\
&+ \|\mathbf{e}_h^{n}\|^2_{L^2(\Omega_{t_n})^2} + C\left(\frac{1}{\Delta t}+\Delta t\right)\ \|\mathbf{u}-\mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + C\Delta t\ h^4\|\mathbf{f}(t_{n+1})\|^2_{W^{2,q}(\Omega_{t_{n+1}})^2} \\
&+ C\Delta t^3 \sup_{s\in(t_n,t_{n+1})}\left\|\left[\frac{\partial^2 \mathbf{X}_h}{\partial s^2}(s)\right]^2\right\|_{L^\infty(\Omega_0)} \|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} \\
&+ C\Delta t^2\int_{t_n}^{t_{n+1}}\left(\|\mathbf{u}(t)\|^2_{H^1(\Omega_t)^2} + \left\|\left.\frac{d\mathbf{u}}{dt}\right|^h_{\mathbf{Y}}(t)\right\|^2_{H^1(\Omega_t)^2} + \|p(t)\|^2_{L^2(\Omega_t)}\right. \\
&\left. + \left\|\left.\frac{dp}{dt}\right|^h_{\mathbf{Y}}(t)\right\|^2_{L^2(\Omega_t)} + \|\mathbf{f}(t)\|^2_{L^2(\Omega_t)^2} + \left\|\left.\frac{d\mathbf{f}}{dt}\right|^h_{\mathbf{Y}}(t)\right\|^2_{L^2(\Omega_t)^2}\right)dt.
\end{aligned}$$

In order to obtain the global error, we sum over n, that is,

$$\begin{aligned}
\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \nu\ \Delta t\ \sum_{i=1}^{n+1} \|\nabla \mathbf{e}_h^i\|^2_{L^2(\Omega_{t_i})^4} &\leq \|\mathbf{e}_h^0\|^2_{L^2(\Omega_0)^2} + \Delta t\ \gamma \sum_{i=1}^{n+1} \|\mathbf{e}_h^i\|^2_{L^2(\Omega_{t_i})^2} \\
&+ Cn\left(\frac{1}{\Delta t} + \Delta t\right)\ \|\mathbf{u} - \mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + C\Delta t\ h^4 \sum_{i=1}^{n+1} \|\mathbf{f}(t_i)\|^2_{W^{2,q}(\Omega_{t_i})^2} \\
&+ Cn\Delta t^3 \sup_{s\in(0,T)} \left\| \left[\frac{\partial^2 \mathbf{X}_h}{\partial s^2}(s)\right]^2 \right\|_{L^\infty(\Omega_0)} \|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} \\
&+ C\Delta t^2 \int_0^{t_{n+1}} \left(\|\mathbf{u}(t)\|^2_{H^1(\Omega_t)^2} + \left\| \frac{d\mathbf{u}}{dt}\Big|^h_{\mathbf{Y}}(t) \right\|^2_{H^1(\Omega_t)^2} \right. \\
&\left. + \|p(t)\|^2_{L^2(\Omega_t)} + \left\| \frac{dp}{dt}\Big|^h_{\mathbf{Y}}(t) \right\|^2_{L^2(\Omega_t)} + \|\mathbf{f}(t)\|^2_{L^2(\Omega_t)^2} + \left\| \frac{d\mathbf{f}}{dt}\Big|^h_{\mathbf{Y}}(t) \right\|^2_{L^2(\Omega_t)^2} \right) dt.
\end{aligned}$$

By applying the discrete Gronwall lemma, we get

$$\begin{aligned}
\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \nu\ \Delta t\ \sum_{i=1}^{n+1} \|\nabla \mathbf{e}_h^i\|^2_{L^2(\Omega_{t_i})^4} &\leq C_1 \|\mathbf{e}_h^0\|^2_{L^2(\Omega_0)^2} \\
&+ CC_1T\left(\frac{1}{\Delta t^2} + 1\right)\ \|\mathbf{u} - \mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + CC_1\Delta t\ h^4 \sum_{i=1}^{n+1} \|\mathbf{f}(t_i)\|^2_{W^{2,q}(\Omega_{t_i})^2} \\
&+ CC_1T\Delta t^2 \sup_{s\in(0,T)} \left\| \left[\frac{\partial^2 \mathbf{X}_h}{\partial s^2}(s)\right]^2 \right\|_{L^\infty(\Omega_0)} \|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} \\
&+ CC_1\Delta t^2 \int_0^{t_{n+1}} \left(\|\mathbf{u}(t)\|^2_{H^1(\Omega_t)^2} + \left\| \frac{d\mathbf{u}}{dt}\Big|^h_{\mathbf{Y}}(t) \right\|^2_{H^1(\Omega_t)^2} \right. \\
&\left. + \|p(t)\|^2_{L^2(\Omega_t)} + \left\| \frac{dp}{dt}\Big|^h_{\mathbf{Y}}(t) \right\|^2_{L^2(\Omega_t)} + \|\mathbf{f}(t)\|^2_{L^2(\Omega_t)^2} + \left\| \frac{d\mathbf{f}}{dt}\Big|^h_{\mathbf{Y}}(t) \right\|^2_{L^2(\Omega_t)^2} \right) dt, \quad (6.31)
\end{aligned}$$

where the constant C_1 is given by

$$C_1 = \exp\left(t_{n+1}\frac{\gamma}{1-\gamma\Delta t}\right).$$

In the previous estimate, we will introduce the continuous ALE derivatives using the identities (4.7), (4.8) and

$$\left.\frac{d\mathbf{f}}{dt}\right|^h_{\mathbf{Y}}(t) \quad = \quad \left.\frac{d\mathbf{f}}{dt}\right|_{\mathbf{Y}}(t) + ((\mathbf{w}_h(t) - \mathbf{w}(t))\cdot\nabla)\,\mathbf{f}(t).$$

Therefore, the estimate (6.31) becomes

$$\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \nu\ \Delta t\ \sum_{i=1}^{n+1} \|\nabla \mathbf{e}_h^i\|^2_{L^2(\Omega_{t_i})^4} \leq C\|\mathbf{e}_h^0\|^2_{L^2(\Omega_0)^2}$$
$$+ C\left(\frac{1}{\Delta t^2}+1\right)\ \|\mathbf{u}-\mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + C\Delta t\ h^4 \sum_{i=1}^{n+1} \|\mathbf{f}(t_i)\|^2_{W^{2,q}(\Omega_{t_i})^2}$$
$$+ C\Delta t^2 \sup_{s\in(0,T)} \left\| \left[\frac{\partial^2 \mathbf{X}_h}{\partial s^2}(s)\right]^2 \right\|_{L^\infty(\Omega_0)} \|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)}$$
$$+ C\Delta t^2 \int_0^{t_{n+1}} \left(\|\mathbf{u}(t)\|^2_{H^2(\Omega_t)^2} + \left\| \left.\frac{d\mathbf{u}}{dt}\right|_{\mathbf{Y}}(t)\right\|^2_{H^1(\Omega_t)^2} \right.$$
$$\left. + \|p(t)\|^2_{H^1(\Omega_t)} + \left\| \left.\frac{dp}{dt}\right|_{\mathbf{Y}}(t)\right\|^2_{L^2(\Omega_t)} + \|\mathbf{f}(t)\|^2_{H^1(\Omega_t)^2} + \left\| \left.\frac{d\mathbf{f}}{dt}\right|_{\mathbf{Y}}(t)\right\|^2_{L^2(\Omega_t)^2} \right) dt. \quad (6.32)$$

This inequality gives us the numerical error $\mathbf{U}(t_{n+1}) - \mathbf{u}_h^{n+1}$. In order to obtain the complete error, we observe that

$$\|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \nu\ \Delta t\ \sum_{i=1}^{n+1} \|\nabla\left(\mathbf{u}(t_i) - \mathbf{u}_h^i\right)\|^2_{L^2(\Omega_{t_i})^4}$$
$$\leq \frac{1}{2}\|\mathbf{u}(t_{n+1}) - \mathbf{U}(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})^2} + \frac{1}{2}\nu\ \Delta t\ \sum_{i=1}^{n+1} \|\nabla\left(\mathbf{u}(t_i) - \mathbf{U}(t_i)\right)\|^2_{L^2(\Omega_{t_i})^4}$$
$$+ \frac{1}{2}\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2} + \frac{1}{2}\nu\ \Delta t\ \sum_{i=1}^{n+1} \|\nabla \mathbf{e}_h^i\|^2_{L^2(\Omega_{t_i})^4}.$$

Combining the previous inequalities and using (4.2), we conclude the proof of the second main result of this paper.

6.2 Some additional estimates

In this subsection, we derive estimates on R_i $(i = 1, \ldots, 9)$ which have been used in the proof of Theorem 2.2.

Lemma 6.1. *Suppose that the assumptions of Theorem 2.2 hold true. Then, the terms R_1 and R_2 defined in* (6.13) *and* (6.14) *satisfy* (6.22), *respectively* (6.23).

Proof. By using the Cauchy-Schwarz inequality, we have that

$$R_1 \leq \frac{1}{2}\|\mathbf{e}_h^n\|^2_{L^2(\Omega_{t_n})^2} + \frac{1}{2}\|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}^{-1}_{h,t_n}\|^2_{L^2(\Omega_{t_n})^2} + \Delta t \int_{\Omega_{t_{n+1}}} \mathrm{div}\ \left(\mathbf{w}^*_{h,n,n+1}\otimes \mathbf{e}_h^{n+1}\right)\cdot \mathbf{e}_h^{n+1}\ d\mathbf{x},$$

then, integrating twice by parts, we obtain

$$R_1 \leq \frac{1}{2}\|\mathbf{e}_h^n\|^2_{L^2(\Omega_{t_n})^2} + \frac{1}{2}\|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}^{-1}_{h,t_n}\|^2_{L^2(\Omega_{t_n})^2} + \frac{1}{2}\ \Delta t\ \int_{\Omega_{t_{n+1}}} |\mathbf{e}_h^{n+1}|^2 \mathrm{div}\,\mathbf{w}^*_{h,n,n+1} d\mathbf{x}. \quad (6.33)$$

In order to transform the second term in the right hand side, we use the Reynolds formula:

$$\frac{d}{dt}\int_{\Omega_t}\left|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}_{h,t}^{-1}\right|^2 d\mathbf{x}=\int_{\Omega_t}\left|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}_{h,t}^{-1}\right|^2\operatorname{div}\mathbf{w}_h(t)d\mathbf{x}.$$

Therefore, integrating from t_n to t_{n+1}, we get

$$\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2}-\left\|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}_{h,t_n}^{-1}\right\|^2_{L^2(\Omega_{t_n})^2}$$
$$=\int_{t_n}^{t_{n+1}}\left(\int_{\Omega_t}\left|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}_{h,t}^{-1}\right|^2\operatorname{div}\mathbf{w}_h(t)\,d\mathbf{x}\right)\,dt.$$

By combining the above equation with (6.33), we get

$$R_1\le\frac{1}{2}\|\mathbf{e}_h^n\|^2_{L^2(\Omega_{t_n})^2}+\frac{1}{2}\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2}+\frac{1}{2}\Delta t\int_{\Omega_{t_{n+1}}}|\mathbf{e}_h^{n+1}|^2\operatorname{div}\mathbf{w}^*_{h,n,n+1}d\mathbf{x}$$
$$-\frac{1}{2}\int_{t_n}^{t_{n+1}}\left(\int_{\Omega_t}\left|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}_{h,t}^{-1}\right|^2\operatorname{div}\mathbf{w}_h(t)\,d\mathbf{x}\right)\,dt.$$

Hence, we get that

$$R_1\le\frac{1}{2}\|\mathbf{e}_h^n\|^2_{L^2(\Omega_{t_n})^2}+\frac{1}{2}\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2}+\frac{1}{2}\Delta t\,\|\operatorname{div}\mathbf{w}^*_{h,n,n+1}\|_{L^\infty(\Omega_{t_{n+1}})}\,\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2}$$
$$+\frac{1}{2}\sup_{t\in(t_n,t_{n+1})}\|\operatorname{div}\mathbf{w}_h(t)\|_{L^\infty(\Omega_t)}\int_{t_n}^{t_{n+1}}\|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}_{h,t}^{-1}\|^2_{L^2(\Omega_t)^2}\,dt.\quad(6.34)$$

In order to bound the last integral, let us remark that, due to the change of variable $y=\mathbf{X}_{h,t_{n+1}}\left(\mathbf{X}_{h,t}^{-1}(x)\right)$, we have that

$$\left\|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}_{h,t}^{-1}\right\|^2_{L^2(\Omega_t)^2}\le\left\|J_{\mathbf{X}_{h,t}}\right\|_{L^\infty(\Omega_0)}\left\|J_{\mathbf{X}^{-1}_{h,t_{n+1}}}\right\|_{L^\infty(\Omega_{t_{n+1}})}\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2}.\quad(6.35)$$

Let us observe that

$$\left\|J_{\mathbf{X}_{h,t}}\right\|_{L^\infty(\Omega_0)}=\left\|\det\left(\mathbf{J}_{\mathbf{X}_{h,t}}\right)\right\|_{L^\infty(\Omega_0)}\le C\,\|\mathbf{J}_{\mathbf{X}_t}\|^2_{L^\infty(\Omega_0)^4}+Ch|\ln h|\,\|\mathbf{X}_t\|_{W^{2,\infty}(\Omega_0)^2}.$$

Thus, there exists C_1 depending on $\mathbf{X}$ and $h_0>0$ such that

$$\left\|J_{\mathbf{X}_{h,t}}\right\|_{L^\infty(\Omega_0)}\le C_1\qquad\forall t\in[0,T],\ \forall h\in(0,h_0).\quad(6.36)$$

We can prove in a similar way that there exists C_2 depending on $\mathbf{X}$ and $h_0>0$ such that

$$\left\|J_{\mathbf{X}^{-1}_{h,t}}\right\|_{L^\infty(\Omega_t)}\le C_2\qquad\forall t\in[0,T],\ \forall h\in(0,h_0).\quad(6.37)$$

From (6.35), (6.36) and (6.37), we obtain

$$\left\|\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\circ\mathbf{X}_{h,t}^{-1}\right\|^2_{L^2(\Omega_t)^2}\le C_1C_2\|\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^2}.\quad(6.38)$$

Combining the above inequality with (6.34) we get (6.22).

Let us estimate the term R_2. The Cauchy-Schwarz inequality together with (6.38) yields to

$$R_2 \leq C\|\mathbf{u}-\mathbf{U}\|_{L^\infty(0,T;L^2(\Omega_t)^2)}\|\mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^2}.$$

To conclude, it is enough to use the Poincaré inequality and that

$$ab \leq \frac{2}{9}a^2 + \frac{1}{18}b^2 \quad \forall a,b \in \mathbb{R}. \tag{6.39}$$

□

Lemma 6.2. *Suppose that the assumptions of Theorem 2.2 hold true. Then, the terms R_3 and R_4 defined in* (6.15) *and* (6.16) *satisfy* (6.24), *respectively* (6.25).

Proof. To estimate R_3, first we integrate by parts:

$$R_3 = -\Delta t \int_{\Omega_{t_{n+1}}} \left(\mathbf{w}^*_{h,n,n+1}\cdot\nabla\right)\mathbf{e}_h^{n+1}\cdot\left(\mathbf{u}(t_{n+1})-\mathbf{U}(t_{n+1})\right)\,d\mathbf{x}.$$

Then, by the Cauchy-Schwarz inequality and (6.39), we obtain

$$R_3 \leq \frac{4}{\nu}\Delta t\ \|\mathbf{w}^*_{h,n,n+1}\|^2_{L^\infty(\Omega_{t_{n+1}})^2}\|\mathbf{u}-\mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega_t)^2)} + \frac{1}{18}\nu\Delta t\ \|\nabla\mathbf{e}_h^{n+1}\|^2_{L^2(\Omega_{t_{n+1}})^4},$$

which implies the estimate (6.24).

Let us estimate the term R_4. First, we integrate by parts and use the Einstein notation:

$$\begin{aligned} R_4 &= -\int_{\Omega_{t_{n+1}}}\left(\int_{t_n}^{t_{n+1}}(s-t_n)\frac{\partial^2\mathbf{X}_h}{\partial s^2}\left(\mathbf{X}^{-1}_{h,t_{n+1}}(\mathbf{x}),s\right)\,ds\cdot\nabla\right)\mathbf{e}_h^{n+1}\cdot\mathbf{u}(t_{n+1})\,d\mathbf{x} \\ &= -\int_{t_n}^{t_{n+1}}(s-t_n)\int_{\Omega_{t_{n+1}}}\left[\frac{\partial^2\mathbf{X}_h}{\partial s^2}\left(\mathbf{X}^{-1}_{h,t_{n+1}}(\mathbf{x}),s\right)\right]_j\frac{\partial(\mathbf{e}_h^{n+1})_i}{\partial x_j}\ u_i(t_{n+1})\,d\mathbf{x}\,ds. \end{aligned}$$

In order to write the integral in the domain Ω_0, we use the change of variable $\mathbf{X}^{-1}_{h,t_{n+1}}(\mathbf{x}) = \mathbf{y}\in\Omega_0$, then it follows that

$$R_4 = -\int_{t_n}^{t_{n+1}}(s-t_n)\int_{\Omega_0}\left[\frac{\partial^2\mathbf{X}_h}{\partial s^2}(\mathbf{y},s)\right]_j\frac{\partial\left(\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\right)_i}{\partial y_k}\cdot \frac{\partial\left[\mathbf{X}^{-1}_{h,t_{n+1}}\right]_k}{\partial x_j}\left(\mathbf{u}(t_{n+1})\circ\mathbf{X}_{h,t_{n+1}}\right)_i J_{\mathbf{X}_{h,t_{n+1}}}\,d\mathbf{y}\,ds.$$

Applying the Cauchy-Schwarz inequality, we have

$$R_4 \leq \int_{t_n}^{t_{n+1}}|s-t_n|\left(\int_{\Omega_0}\left|\frac{\partial\left(\mathbf{e}_h^{n+1}\circ\mathbf{X}_{h,t_{n+1}}\right)_i}{\partial y_k}\frac{\partial y_k}{\partial x_j}\right|^2 J_{\mathbf{X}_{h,t_{n+1}}}d\mathbf{y}\right)^{1/2} \cdot\left(\int_{\Omega_0}\left[\frac{\partial^2\mathbf{X}_h}{\partial s^2}(\mathbf{y},s)\right]^2_j\left(\mathbf{u}(t_{n+1})\circ\mathbf{X}_{h,t_{n+1}}\right)^2_i J_{\mathbf{X}_{h,t_{n+1}}}\,d\mathbf{y}\right)^{1/2}ds,$$

and therefore,

$$R_4 \leq \int_{t_n}^{t_{n+1}}|s-t_n|\ \|\nabla\mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^4}\left\|\left[\frac{\partial^2\mathbf{X}_h}{\partial s^2}(s)\right]^2\right\|^{1/2}_{L^\infty(\Omega_0)^2}\|\mathbf{u}(t_{n+1})\|_{L^2(\Omega_{t_{n+1}})^2}\,ds.$$

By using the Cauchy-Schwarz inequality in time, it follows that

$$R_4 \leq \frac{\Delta t^2}{\sqrt{3}} \left(\sup_{s\in(t_n,t_{n+1})} \left\| \left[\frac{\partial^2 \mathbf{X}_h}{\partial s^2}(s) \right]^2 \right\|_{L^\infty(\Omega_0)^2} \right)^{1/2} \|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega_t)^2)} \, \|\nabla \mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^4},$$

which yields (6.25).

□

Lemma 6.3. *Suppose that the assumptions of Theorem 2.2 hold true. Then, the terms R_5 to R_8 defined in* (6.17)–(6.20) *satisfy* (6.26)–(6.29)*, respectively.*

Proof. In order to simplify matters, let us start our proof by studying the terms Q_1 to Q_4 defined in (6.4)–(6.7), which are basically the same as the terms R_5 to R_8, but written for a general test function $\mathbf{v}_h \in (W_{h,0})^2$.

We will begin by rewriting the term Q_1 as follows:

$$\begin{aligned} Q_1 &= \nu \int_{t_n}^{t_{n+1}} \left[\int_{\Omega_{t_{n+1}}} \nabla \mathbf{u}(t_{n+1}) : \nabla(\mathbf{v}_h \circ \mathbf{X}_{h,t_{n+1}}^{-1}) \, d\mathbf{x} - \int_{\Omega_t} \nabla \mathbf{u}(t) : \nabla\big(\mathbf{v}_h \circ \mathbf{X}_{h,t}^{-1}\big) \, d\mathbf{x} \right] \, dt \\ &= \nu \int_{t_n}^{t_{n+1}} \left[\int_t^{t_{n+1}} \left(\frac{d}{ds} \int_{\Omega_s} \nabla \mathbf{u}(s) : \nabla(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1}) \, d\mathbf{x} \right) \, ds \right] \, dt. \end{aligned}$$

Due to Lemma 3.2, it follows that

$$\begin{aligned} Q_1 = \nu \int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \Bigg\{ \int_{\Omega_s} \Big[\nabla\Big(\frac{d\mathbf{u}}{dt}\Big|_{\mathbf{Y}}^h (s) \Big) : \nabla(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1}) + \nabla \mathbf{u}(s) : \nabla(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1}) \mathrm{div}\, \mathbf{w}_h(s) \\ - \big(\big(\nabla \mathbf{w}_h(s) + \nabla \mathbf{w}_h(s)^T \big) \nabla \mathbf{u}(s) \big) : \nabla(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1}) \Big] \, d\mathbf{x} \Bigg\} \, ds \Bigg] \, dt. \quad (6.40) \end{aligned}$$

Similarly, we deduce that

$$\begin{aligned} Q_2 = -\int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \Bigg\{ \int_{\Omega_s} \Big[\mathrm{div}\Big(\frac{d\mathbf{w}_h}{dt}\Big|_{\mathbf{Y}}^h (s) \otimes \mathbf{u}(s) \Big) + \mathrm{div}\,(\mathbf{w}_h(s) \otimes \mathbf{u}(s)) \mathrm{div}\, \mathbf{w}_h(s) \\ + \mathrm{div}\Big(\mathbf{w}_h(s) \otimes \frac{d\mathbf{u}}{dt}\Big|_{\mathbf{Y}}^h (s) \Big) \Big] \cdot \Big(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1} \Big) \, d\mathbf{x} \Bigg\} \, ds \Bigg] \, dt, \quad (6.41) \end{aligned}$$

$$\begin{aligned} Q_3 = -\int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \Bigg\{ \int_{\Omega_s} \Big[\frac{dp}{dt}\Big|_{\mathbf{Y}}^h (s) \mathrm{div}\,(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1}) + p(s) \mathrm{div}\,(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1}) \mathrm{div}\, \mathbf{w}_h(s) \\ - p(s) \nabla \mathbf{w}_h(s) : \nabla(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1}) \Big] \, d\mathbf{x} \Bigg\} \, ds \Bigg] \, dt, \quad (6.42) \end{aligned}$$

$$Q_4 = -\int_{t_n}^{t_{n+1}} \left[\int_t^{t_{n+1}} \left\{ \int_{\Omega_s} \Big(\frac{d\mathbf{f}}{dt}\Big|_{\mathbf{Y}}^h (s) + \mathbf{f}(s) \mathrm{div}\, \mathbf{w}_h(s) \Big) \cdot \Big(\mathbf{v}_h \circ \mathbf{X}_{h,s}^{-1} \Big) \, d\mathbf{x} \right\} \, ds \right] \, dt. \quad (6.43)$$

In order to obtain the estimates (6.26)–(6.29), let us choose in the terms Q_i, for all $i = 1, \ldots, 4$, the following test function:

$$\mathbf{v}_h = \mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \in (W_{h,0})^2.$$

Therefore, we have that

$$R_5 = \nu \int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \Bigg\{ \int_{\Omega_s} \Big[\nabla \Big(\left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (s) \Big) : \nabla (\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1}) \\ + \nabla \mathbf{u}(s) : \nabla (\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1}) \mathrm{div}\, \mathbf{w}_h(s) \\ - \left(\left(\nabla \mathbf{w}_h(s) + \nabla \mathbf{w}_h(s)^T \right) \nabla \mathbf{u}(s) \right) : \nabla (\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1}) \Big] \, d\mathbf{x} \Bigg\} \, ds \Bigg] \, dt.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$R_5 \leq \nu \int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \left(\left\| \nabla \Big(\left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (s) \Big) \right\|_{L^2(\Omega_s)^4} + 3 \|\nabla \mathbf{w}_h(s)\|_{L^\infty(\Omega_s)^4} \|\nabla \mathbf{u}(s)\|_{L^2(\Omega_s)^4} \right) \\ \cdot \left\| \nabla \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1} \right) \right\|_{L^2(\Omega_s)^4} \, ds \Bigg] \, dt. \quad (6.44)$$

By using the following inequality

$$\left\| \nabla \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,t}^{-1} \right) \right\|_{L^2(\Omega_t)^4}^2 \leq 2^4 \left\| \mathbf{J}_{\mathbf{X}_{h,t}} \right\|_{L^\infty(\Omega_0)^4}^2 \left\| J_{\mathbf{X}_{h,t}^{-1}} \right\|_{L^\infty(\Omega_t)} \\ \cdot \left\| \mathbf{J}_{\mathbf{X}_{h,t_{n+1}}^{-1}} \right\|_{L^\infty(\Omega_{t_{n+1}})^4}^2 \left\| J_{\mathbf{X}_{h,t_{n+1}}} \right\|_{L^\infty(\Omega_0)} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^4}^2, \quad (6.45)$$

then (6.44) yields to

$$R_5 \leq C \int_{t_n}^{t_{n+1}} \Bigg[\int_{t_n}^{t_{n+1}} \left(\left\| \nabla \Big(\left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (s) \Big) \right\|_{L^2(\Omega_s)^4} + 3 \|\nabla \mathbf{w}_h(s)\|_{L^\infty(\Omega_s)^4} \|\nabla \mathbf{u}(s)\|_{L^2(\Omega_s)^4} \right) \\ \cdot \left\| \nabla \mathbf{e}_h^{n+1} \right\|_{L^2(\Omega_{t_{n+1}})^4} \, ds \Bigg] \, dt.$$

By the Cauchy-Schwarz inequality and (2.13), the previous estimation becomes

$$R_5 \leq C \Delta t^{3/2} \left[\int_{t_n}^{t_{n+1}} \left(\left\| \nabla \left(\left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (t) \right) \right\|_{L^2(\Omega_t)^4} + \|\nabla \mathbf{u}(t)\|_{L^2(\Omega_t)^4} \right)^2 dt \right]^{\frac{1}{2}} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^4},$$

and therefore, due to (6.39), we get

$$R_5 \leq C \Delta t^2 \int_{t_n}^{t_{n+1}} \left(\left\| \nabla \left(\left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (t) \right) \right\|_{L^2(\Omega_t)^4}^2 + \|\nabla \mathbf{u}(t)\|_{L^2(\Omega_t)^4}^2 \right) dt + \frac{1}{18} \nu \Delta t \, \|\nabla \mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^4}^2,$$

which completes the proof of the estimate (6.26).

On the other hand, we have that

$$R_6 = - \int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \Bigg\{ \int_{\Omega_s} \Bigg[\mathrm{div} \left(\left. \frac{d\mathbf{w}_h}{dt} \right|_{\mathbf{Y}}^h (s) \otimes \mathbf{u}(s) \right) + \mathrm{div} \left(\mathbf{w}_h(s) \otimes \mathbf{u}(s) \right) \mathrm{div}\, \mathbf{w}_h(s) \\ + \mathrm{div} \left(\mathbf{w}_h(s) \otimes \left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (s) \right) \Bigg] \cdot \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1} \right) \, d\mathbf{x} \Bigg\} \, ds \Bigg] \, dt,$$

then, integrating by parts, it follows that

$$
\begin{aligned}
R_6 = \int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \left(\int_{\Omega_s} \left(\left. \frac{d\mathbf{w}_h}{dt} \right|_{\mathbf{Y}}^h (s) \cdot \nabla \right) \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1} \right) \cdot \mathbf{u}(s)\, d\mathbf{x} \right) ds \Bigg] dt \\
+ \int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \left(\int_{\Omega_s} \left(\mathbf{w}_h(s) \cdot \nabla \right) \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1} \right) \cdot \left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (s)\, d\mathbf{x} \right) ds \Bigg] dt \\
- \int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \left(\int_{\Omega_s} [\operatorname{div} \mathbf{w}_h(s)]^2 \mathbf{u}(s) \cdot \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1} \right) d\mathbf{x} \right) ds \Bigg] dt \\
- \int_{t_n}^{t_{n+1}} \Bigg[\int_t^{t_{n+1}} \left(\int_{\Omega_s} \operatorname{div} \mathbf{w}_h(s) \left(\mathbf{w}_h(s) \cdot \nabla \right) \mathbf{u}(s) \cdot \left(\mathbf{e}_h^{n+1} \circ \mathbf{X}_{h,t_{n+1}} \circ \mathbf{X}_{h,s}^{-1} \right) d\mathbf{x} \right) ds \Bigg] dt.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality, (6.35) and (6.45), we get

$$
\begin{aligned}
R_6 \leq C \int_{t_n}^{t_{n+1}} \Bigg[\int_{t_n}^{t_{n+1}} \Bigg(\left\| \left. \frac{d\mathbf{w}_h}{dt} \right|_{\mathbf{Y}}^h (s) \right\|_{L^\infty(\Omega_s)^2} \|\mathbf{u}(s)\|_{L^2(\Omega_s)^2} \\
+ \|\mathbf{w}_h(s)\|_{L^\infty(\Omega_s)^2} \left\| \left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (s) \right\|_{L^2(\Omega_s)^2} \Bigg) \|\nabla \mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^4}\, ds \Bigg] dt \\
+ C \int_{t_n}^{t_{n+1}} \Bigg[\int_{t_n}^{t_{n+1}} \Bigg(\|\nabla \mathbf{w}_h(s)\|_{L^\infty(\Omega_s)^4}^2 \|\mathbf{u}(s)\|_{L^2(\Omega_s)^2} \\
+ \|\nabla \mathbf{w}_h(s)\|_{L^\infty(\Omega_s)^4} \|\mathbf{w}_h(s)\|_{L^\infty(\Omega_s)^2} \|\nabla \mathbf{u}(s)\|_{L^2(\Omega_s)^4} \Bigg) \|\mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^2}\, ds \Bigg] dt.
\end{aligned}
$$

Using the estimate (2.13), the hypothesis (2.18) and the Cauchy-Schwarz inequality, it follows that

$$
\begin{aligned}
R_6 \leq C\Delta t^{3/2} \Bigg[\int_{t_n}^{t_{n+1}} \left(\|\mathbf{u}(t)\|_{L^2(\Omega_t)^2} + \left\| \left. \frac{d\mathbf{u}}{dt} \right|_{\mathbf{Y}}^h (t) \right\|_{L^2(\Omega_t)^2} \right)^2 dt \Bigg]^{1/2} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^4} \\
+ C\Delta t^{3/2} \left[\int_{t_n}^{t_{n+1}} \left(\|\mathbf{u}(t)\|_{L^2(\Omega_t)^2} + \|\nabla \mathbf{u}(t)\|_{L^2(\Omega_t)^4} \right)^2 dt \right]^{1/2} \|\mathbf{e}_h^{n+1}\|_{L^2(\Omega_{t_{n+1}})^2},
\end{aligned}
$$

then, by the Poincaré inequality and (6.39), we get (6.27).

Estimates (6.28) and (6.29) can be obtained in a similar way, then we skip their derivation. □

Lemma 6.4. *Suppose that the assumptions of Theorem 2.2 hold true. Then, the term R_9 defined in* (6.21) *satisfies* (6.30).

Proof. First of all, we observe that this term is similar with T_3, which has been estimated in the proof of Theorem 2.1. Hence, we are going to proceed similarly. We have that

$$
\begin{aligned}
R_9 &= \Delta t \int_{\Omega_{t_{n+1}}} \mathbf{f}(t_{n+1}) \cdot \mathbf{e}_h^{n+1}\, d\mathbf{x} - \Delta t \widetilde{I}_{h,t_{n+1}} \left(\mathbf{f}(t_{n+1}) \cdot \mathbf{e}_h^{n+1} \right) \\
&= \Delta t \sum_{K \in \mathcal{T}_{h,t_{n+1}}} E_K \left(\mathbf{f}(t_{n+1}) \cdot \mathbf{e}_h^{n+1} \right).
\end{aligned}
$$

In order to obtain this error, we use Theorem 4.1.5 from Ciarlet [10, p.195], then for any $q > 2$,

$$
R_9 \leq C\Delta t\ h^2 \sum_{K \in \mathcal{T}_{h,t_{n+1}}} |K|^{1/2-1/q} \|\mathbf{f}(t_{n+1})\|_{W^{2,q}(K)^2} \|\mathbf{e}_h^{n+1}\|_{H^1(K)^2}.
$$

Now, applying the Hölder inequality $\left(\text{with } \frac{1}{2}+\frac{1}{p}+\frac{1}{q}=1\right)$, we get

$$\begin{aligned} R_9 &\leq C\Delta t\, h^2 \left(\sum_K |K|^{\ \frac{1}{2}-\frac{1}{q}\ p}\right)^{\frac{1}{p}} \left(\sum_K \|\mathbf{f}(t_{n+1})\|^q_{W^{2,q}(K)^2}\right)^{\frac{1}{q}} \left(\sum_K \|\mathbf{e}_h^{n+1}\|^2_{H^1(K)^2}\right)^{\frac{1}{2}} \\ &\leq C\Delta t\, h^2 \|\mathbf{f}(t_{n+1})\|_{W^{2,q}(\Omega_{t_{n+1}})^2} \|\mathbf{e}_h^{n+1}\|_{H^1(\Omega_{t_{n+1}})^2}. \end{aligned}$$

To conclude, we combine the above relations with the Poincaré inequality and with (6.39). □

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References

[1] Y. Achdou and J.-L. Guermond, *Convergence analysis of a finite element projection/Lagrange-Galerkin method for the incompressible Navier-Stokes equations*, SIAM J. Numer. Anal., 37 (2000), pp. 799–826 (electronic).

[2] I. Babuška, *The finite element method with Lagrangian multipliers*, Numer. Math., 20 (1972/73), pp. 179–192.

[3] H. Beirão da Veiga, *On the existence of strong solutions to a coupled fluid-structure evolution problem*, J. Math. Fluid Mech., 6 (2004), pp. 21–52.

[4] S. Bertoluzza, M. Ismail, and B. Maury, *The fat boundary method: semi-discrete scheme and some numerical experiments*, in Domain decomposition methods in science and engineering, vol. 40 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2005, pp. 513–520.

[5] M. Boulakia, *Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid*, J. Math. Pures Appl. (9), 84 (2005), pp. 1515–1554.

[6] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer-Verlag, New York, second ed., 2002.

[7] F. Brezzi, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 8 (1974), pp. 129–151.

[8] A. Chambolle, B. Desjardins, M. J. Esteban, and C. Grandmont, *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate*, J. Math. Fluid Mech., 7 (2005), pp. 368–404.

[9] Y. C. Chang, T. Y. Hou, B. Merriman, and S. Osher, *A level set formulation of Eulerian interface capturing methods for incompressible fluid flows*, J. Comput. Phys., 124 (1996), pp. 449–464.

[10] P. G. Ciarlet, *The finite element method for elliptic problems*, vol. 40 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].

[11] B. Desjardins, M. J. Esteban, C. Grandmont, and P. Le Tallec, *Weak solutions for a fluid-elastic structure interaction model*, Rev. Mat. Complut., 14 (2001), pp. 523–538.

[12] J. Donea, *Arbitrary Lagrangian Eulerian Methods. Computational Methods for Transient Analysis*, vol. 1 of Computational Methods in Mechanics, North-Holland, Elsevier, 1983.

[13] F. Duarte, R. Gormaz, and S. Natesan, *Arbitrary Lagrangian-Eulerian method for Navier-Stokes equations with moving boundaries*, Comput. Methods Appl. Mech. Engrg., 193 (2004), pp. 4819–4836.

[14] C. Farhat, M. Lesoinne, and N. Maman, *Mixed explicit/implicit time integration of coupled aeroelastic problems: three-field formulation, geometric conservation and distributed solution*, Internat. J. Numer. Methods Fluids, 21 (1995), pp. 807–835. Finite element methods in large-scale computational fluid dynamics (Tokyo, 1994).

[15] L. Formaggia and F. Nobile, *A stability analysis for the arbitrary Lagrangian Eulerian formulation with finite elements*, East-West J. Numer. Math., 7 (1999), pp. 105–131.

[16] L. Gastaldi, *A priori error estimates for the arbitrary Lagrangian Eulerian formulation with finite elements*, East-West J. Numer. Math., 9 (2001), pp. 123–156.

[17] V. Girault, H. López, and B. Maury, *One time-step finite element discretization of the equation of motion of two-fluid flows*, Numer. Methods Partial Differential Equations, 22 (2006), pp. 680–707.

[18] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations*, vol. 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1986. Theory and algorithms.

[19] R. Glowinski, T.-W. Pan, and J. Périaux, *A fictitious domain method for external incompressible viscous flow modeled by Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 112 (1994), pp. 133–148. Finite element methods in large-scale computational fluid dynamics (Minneapolis, MN, 1992).

[20] ——, *A fictitious domain method for unsteady incompressible viscous flow modelled by Navier-Stokes equations*, in Domain decomposition methods in science and engineering (Como, 1992), vol. 157 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1994, pp. 421–431.

[21] C. Grandmont, V. Guimet, and Y. Maday, *Numerical analysis of some decoupling techniques for the approximation of the unsteady fluid structure interaction*, Math. Models Methods Appl. Sci., 11 (2001), pp. 1349–1377.

[22] P. Grisvard, *Elliptic problems in nonsmooth domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[23] H. Guillard and C. Farhat, *On the significance of the geometric conservation law for flow computations on moving meshes*, Comput. Methods Appl. Mech. Engrg., 190 (2000), pp. 1467–1482.

[24] M. E. Gurtin, *An introduction to continuum mechanics*, vol. 158 of Mathematics in Science and Engineering, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.

[25] A. Inoue and M. Wakimoto, *On existence of solutions of the Navier-Stokes equation in a time dependent domain*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24 (1977), pp. 303–319.

[26] A. Masud and T. J. R. Hughes, *A space-time Galerkin/least-squares finite element formulation of the Navier-Stokes equations for moving domain problems*, Comput. Methods Appl. Mech. Engrg., 146 (1997), pp. 91–126.

[27] B. Maury, *Characteristics ALE method for the unsteady 3D Navier-Stokes equations with a free surface*, Int. J. Comp. Fluid Dyn., **6** (1996), pp. 175–188.

[28] B. Maury, *Direct simulations of 2D fluid-particle flows in biperiodic domains*, J. Comput. Phys., 156 (1999), pp. 325–351.

[29] B. Maury, *A fat boundary method for the Poisson problem in a domain with holes*, J. Sci. Comput., 16 (2001), pp. 319–339.

[30] F. Nobile, *Numerical approximation of fluid-structure interaction problems with application to haemodynamics*, Ph.D Thesis, Ecole Polytechnique Fédérale de Lausanne, Suisse, 2001.

[31] M. Ôtani and Y. Yamada, *On the Navier-Stokes equations in noncylindrical domains: an approach by the subdifferential operator theory*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 25 (1978), pp. 185–204.

[32] C. S. Peskin, *Numerical analysis of blood flow in the heart*, J. Computational Phys., 25 (1977), pp. 220–252.

[33] O. Pironneau, *On the transport-diffusion algorithm and its applications to the Navier-Stokes equations*, Numer. Math., 38 (1981/82), pp. 309–332.

[34] R. Salvi, *On the existence of weak solutions of a nonlinear mixed problem for the Navier-Stokes equations in a time dependent domain*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 32 (1985), pp. 213–221.

[35] ——, *The exterior nonstationary problem for the Navier-Stokes equations ireference [13]n regions with moving boundaries*, J. Math. Soc. Japan, 42 (1990), pp. 495–509.

[36] J. San Martín, J.-F. Scheid, T. Takahashi, and M. Tucsnak, *Convergence of the Lagrange-Galerkin method for the equations modelling the motion of a fluid-rigid system*, SIAM J. Numer. Anal., 43 (2005), pp. 1536–1571 (electronic).

[37] E. Süli, *Convergence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations*, Numer. Math., 53 (1988), pp. 459–483.

[38] T. E. Tezduyar, M. Behr, and J. Liou, *A new strategy for finite element computations involving moving boundaries and interfaces—the deforming-spatial-domain/space-time procedure. I. The concept and the preliminary numerical tests*, Comput. Methods Appl. Mech. Engrg., 94 (1992), pp. 339–351.

[39] T. E. Tezduyar, M. Behr, S. Mittal, and J. Liou, *A new strategy for finite element computations involving moving boundaries and interfaces—the deforming-spatial-domain/space-time procedure. II. Computation of free-surface flows, two-liquid flows, and flows with drifting cylinders*, Comput. Methods Appl. Mech. Engrg., 94 (1992), pp. 353–371.