



**HAL**  
open science

## Anisotropic developments for homogeneous shear flows

Claude Cambon, Robert Rubinstein

► **To cite this version:**

Claude Cambon, Robert Rubinstein. Anisotropic developments for homogeneous shear flows. *Physics of Fluids*, 2006, 18, pp.085106. 10.1063/1.2265012 . hal-00274852

**HAL Id: hal-00274852**

**<https://hal.science/hal-00274852>**

Submitted on 15 Jun 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Anisotropic developments for homogeneous shear flows

Claude Cambon

LMFA, UMR 5509, École Centrale de Lyon, 69134 Ecully Cedex, France

Robert Rubinstein

Computational Aerosciences Branch, NASA Langley Research Center, Hampton, Virginia 23681

(Received 6 February 2006; accepted 3 July 2006; published online 21 August 2006)

The general decomposition of the spectral correlation tensor  $R_{ij}(\mathbf{k})$  by Cambon *et al.* [J. Fluid Mech. **202**, 295 (1989); **337**, 303 (1997)] into *directional* and *polarization* components is applied to the representation of  $R_{ij}(\mathbf{k})$  by spherically averaged quantities. The decomposition splits the deviatoric part  $H_{ij}(k)$  of the spherical average of  $R_{ij}(\mathbf{k})$  into directional and polarization components  $H_{ij}^{(e)}(k)$  and  $H_{ij}^{(z)}(k)$ . A self-consistent representation of the spectral tensor is constructed in terms of these spherically averaged quantities. The directional and polarization components must be treated independently: representation of the spectral tensor using the spherical average  $H_{ij}(k)$  alone proves to be inconsistent with Navier-Stokes dynamics. In particular, a spectral tensor consistent with a prescribed Reynolds stress is not unique. Since spherical averaging entails a loss of information, the description of an anisotropic correlation tensor by spherical averages is limited to weak departures from isotropy. The degree of anisotropy permitted is restricted by realizability requirements. More general descriptions can be given using a higher-order expansion of the spectral tensor. Directionality is described by a conventional expansion in spherical harmonics, but polarization requires an expansion in special tensorial quantities generated by irreducible representations of the rotation group  $SO^3$ . These expansions are considered in more detail in the special case of axial symmetry. © 2006 American Institute of Physics. [DOI: 10.1063/1.2265012]

### I. INTRODUCTION

The most basic statistical property of the fluctuating velocity field in a turbulent flow is its single-time two-point correlation tensor

$$\tilde{R}_{ij}(\mathbf{x}, \mathbf{x}'; t) = \langle u'_i(\mathbf{x}, t) u'_j(\mathbf{x}', t) \rangle. \quad (1)$$

In homogeneous turbulence, a simpler description is possible by the second-order spectral tensor  $R_{ij}(\mathbf{k}, t)$ , which is a function of the wave-vector argument  $\mathbf{k}$ .<sup>1,2</sup> The dependence of  $\mathbf{R}$  on the entire wave vector  $\mathbf{k}$  and the consequent angle dependence is involved in various important dynamical properties such as redistribution of energy by the “rapid” pressure-strain process.<sup>3</sup>

Many simplified descriptions of the wave-vector dependence of the correlation tensor  $\mathbf{R}$  have been proposed.<sup>4-7</sup> They share the generic form

$$R_{ij}(\mathbf{k}, t) = U(k, t) P_{ij}(\hat{\mathbf{k}}) + BU(k, t) H_{pq}(k, t) \hat{k}_p \hat{k}_q P_{ij}(\hat{\mathbf{k}}) + CU(k, t) P_{in}(\hat{\mathbf{k}}) P_{jm}(\hat{\mathbf{k}}) H_{nm}(k, t). \quad (2)$$

Complete explanation of the notation will be given later; for now, we stress the essential point that the anisotropic properties of the correlation are described by a single tensor function  $H(k, t)$  that depends only on the wave number  $k = |\mathbf{k}|$ .

These models are revisited here by comparison with an exact decomposition of the spectral tensor<sup>8-10</sup> into terms that represent distinct properties: *directional* and *polarization* anisotropy. This decomposition has both a physical and a geometrical basis, which we review. Equation (2) proves to con-

strain directional and polarization anisotropy in ways that may be inconsistent with Navier-Stokes dynamics. Examples are given of anisotropic flows that cannot be described by a model of this form. Instead, we propose a new model in which directionality and polarization are unconstrained. The consequent description of anisotropy by *two* tensor functions of wave number  $k$  is shown to be consistent with the dynamics.

Since spherical averaging must suppress some of the  $\mathbf{k}$  dependence of the correlation tensor, it is anticipated that only relatively weak anisotropy is well characterized by spherical averages, and in fact, realizability conditions prove to constrain the degree of anisotropy that can be so characterized. These limitations can be overcome in principle by more complex expansions. These expansions are considered from the viewpoint of the representation theory of the rotation group  $SO^3$ , which has become prominent in many recent investigations.<sup>11,12</sup> Directional anisotropy is described by an expansion in conventional spherical harmonics, however polarization anisotropy proves to require a more complex tensorial generalization. The lowest-order correction to the description by spherical averages is considered in some detail in the special case of axisymmetric kinematics.

These results on the description of anisotropy can be applied in several different ways. First, they have applications in turbulence modeling where special expressions for the spectral tensor have long been used to evaluate pressure-strain correlations in second moment closures<sup>4,5</sup> using expressions such as Eq. (2). They can also be useful in analyzing data taken in anisotropic flows, where the full

information available from direct numerical simulations or experimental measurements is only poorly characterized by simple single-point statistics such as the Reynolds stresses.<sup>13,14</sup> In another direction, we can use the refined description to improve data *synthesis*, as in methods such as kinematic simulation,<sup>15</sup> in which turbulent fluctuations are replaced by synthesized fields with given statistics. Applying refined anisotropy descriptions should improve the predictions when such methods are applied to anisotropic flows.<sup>16</sup>

The paper is organized as follows. General anisotropic correlation functions are discussed in Sec. II without approximations. The fundamental ideas of directional and polarization anisotropy are introduced. The description of anisotropy by spherically averaged tensors is introduced in Sec. III. The undesirable effects of not distinguishing directionality and polarization are described. Two examples of weak shear effects—sheared isotropic turbulence at short times and the small scales in sheared turbulence at arbitrary times—are analyzed in Sec. IV, and shown to require a description in terms of directional and polarization anisotropy. Section V discusses the parametrization of spectral anisotropy by single-point moments. Realizability constraints on anisotropic models are discussed in Sec. VI. They make precise the limitation of these models to weak anisotropy. Section VII considers how the restriction to weak anisotropy can be mitigated by more accurate, higher-order expansions based on representation theory of the rotation group  $SO^3$ . Explicit formulas are given in the special case of axial symmetry. Section VIII summarizes the main results. An appendix presents some basic results on anisotropic turbulence with helicity.

## II. EXACT RELATIONSHIP FOR ARBITRARY ANISOTROPIC SECOND-ORDER STATISTICS

The decomposition of a second-rank tensor into an isotropic trace component and a deviator will be generalized to the correlation tensor, taking into account two special features: the solenoidal property

$$k_i R_{ij}(\mathbf{k}, t) = R_{ij}(\mathbf{k}, t) k_j = 0 \quad (3)$$

that follows from the incompressibility of the velocity field, and the dependence on the vector argument  $\mathbf{k}$ . In what follows, the helicity of the velocity field will be assumed to vanish; accordingly, the correlation tensor has the symmetries  $R_{ij}(\mathbf{k}, t) = R_{ji}(\mathbf{k}, t) = R_{ij}(-\mathbf{k}, t)$ . The extension of the kinematics to helical turbulence is considered in the Appendix. We recall that if the (nonhelical) correlation tensor is isotropic, then elementary arguments<sup>1</sup> show that it is proportional to the special tensor

$$P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j, \quad (4)$$

where  $k = |\mathbf{k}|$  and  $\hat{k}_i = k_i/k$  is the unit vector along  $\mathbf{k}$ . We also recall the elementary property

$$P_{ij}(\mathbf{k}) k_i = P_{ij}(\mathbf{k}) k_j = 0, \quad (5)$$

which states that  $\mathbf{P}$  is solenoidal, and

$$P_{im}(\mathbf{k}) P_{mj}(\mathbf{k}) = P_{ij}(\mathbf{k}), \quad (6)$$

which states that  $\mathbf{P}$  is a projection. The geometric meaning of Eqs. (5) and (6) is that at any vector  $\mathbf{k}$ ,  $\mathbf{P}(\mathbf{k})$  is the projection onto the plane perpendicular to  $\mathbf{k}$ . We will also use the obvious results

$$P_{mm}(\mathbf{k}) = P_{mn}(\mathbf{k}) P_{mn}(\mathbf{k}) = 2. \quad (7)$$

To begin, note from Eq. (3) that 0 is always an eigenvalue of  $\mathbf{R}(\mathbf{k}, t)$ , and that  $\mathbf{k}$  itself is the corresponding eigenvector. It follows that in any frame centered at  $\mathbf{k}$  in which  $\hat{\mathbf{k}}$  is one of the basis vectors,  $\mathbf{R}(\mathbf{k}, t)$  can be represented as the matrix

$$\mathbf{R} = \begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (8)$$

characterized by exactly three real scalars, where again we remark that the absence of helicity implies that  $\mathbf{R}$  is symmetric in any basis. A trace-deviator decomposition in the plane normal to  $\mathbf{k}$  yields

$$\mathbf{R} = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} d & b & 0 \\ b & -d & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (9)$$

where

$$e = \frac{1}{2}(a + c), \quad d = \frac{1}{2}(a - c). \quad (10)$$

In this frame,  $\mathbf{P}$ , as the projection onto the plane perpendicular to  $\hat{\mathbf{k}}$ , is represented by the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (11)$$

Introducing the two independent symmetric matrices

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (12)$$

we therefore have

$$\mathbf{R} = e\mathbf{P} + d\mathbf{M}_1 + b\mathbf{M}_2. \quad (13)$$

In terms of the complex quantities,

$$Z = d + ib, \quad \mathbf{M} = \mathbf{M}_1 - i\mathbf{M}_2, \quad (14)$$

Eq. (13) can be written alternatively as

$$\mathbf{R} = e\mathbf{P} + \Re(Z\mathbf{M}). \quad (15)$$

An obvious coordinate system in which  $\hat{\mathbf{k}}$  is a basis vector at every  $\mathbf{k}$  is the spherical coordinate system in  $\mathbf{k}$  space, in this context called the Craya-Herring frame.<sup>2</sup> For the application of this coordinate system to explicit expressions for  $\mathbf{M}_1$  and  $\mathbf{M}_2$  in terms of the helical mode decomposition using the (complex) eigenvectors of rotations about  $\hat{\mathbf{k}}$ , see Cambon and Jacquin<sup>8</sup> and Waleffe.<sup>17</sup> Equations (13) and (15) express

$\mathbf{R}$  in terms of the minimal number of scalars: the three real quantities  $b(\mathbf{k}, t)$ ,  $d(\mathbf{k}, t)$ ,  $e(\mathbf{k}, t)$  or equivalently,  $e(\mathbf{k}, t)$  and the complex scalar  $Z(\mathbf{k}, t)$ .

Defining

$$\mathbf{R}^{\text{pol}} = d\mathbf{M}_1 + b\mathbf{M}_2 = \begin{bmatrix} d & b & 0 \\ b & -d & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (16)$$

the decomposition in Eq. (13),

$$\mathbf{R} = e\mathbf{P} + \mathbf{R}^{\text{pol}} \quad (17)$$

is characterized by the properties

$$e = \frac{1}{2}\mathbf{R}:\mathbf{P} = \frac{1}{2}\text{tr } \mathbf{R}, \quad \mathbf{R}^{\text{pol}}:\mathbf{P} = 0. \quad (18)$$

Because these properties are independent of the coordinate system, we can also arrive at Eq. (17) by coordinate-free arguments. Thus, define the projection of  $\mathbf{R}$  along  $\mathbf{P}$  by

$$\mathbf{R}^{\text{P}} = \frac{1}{2}(\mathbf{R}:\mathbf{P})\mathbf{P}, \quad (19)$$

where the factor of 1/2 is due to Eq. (7). The operation so defined is a projection because  $[\mathbf{R}^{\text{P}}]^{\text{P}} = \mathbf{R}^{\text{P}}$ . Accordingly, the decomposition

$$\mathbf{R} = \frac{1}{2}(\mathbf{R}:\mathbf{P})\mathbf{P} + \left[ \mathbf{R} - \frac{1}{2}(\mathbf{R}:\mathbf{P})\mathbf{P} \right] \quad (20)$$

coincides with Eq. (17) after introducing the definition equation (18) of  $e$  and replacing Eq. (16) by the coordinate-free definition

$$\mathbf{R}^{\text{pol}} = \mathbf{R} - \frac{1}{2}(\mathbf{R}:\mathbf{P})\mathbf{P}. \quad (21)$$

The polarization tensor is geometrically very simple:  $\mathbf{R}^{\text{pol}}$  has one zero eigenvalue because it is solenoidal, and since  $\mathbf{R}$  and  $\mathbf{R}^{\text{pol}}$  are both solenoidal,

$$\text{tr } \mathbf{R}^{\text{pol}} = \mathbf{R}^{\text{pol}}:\mathbf{P} = 0. \quad (22)$$

Accordingly, its characteristic polynomial is simply

$$p(\lambda) = \lambda^3 - \left( \frac{1}{2}\mathbf{R}^{\text{pol}}:\mathbf{R}^{\text{pol}} \right) \lambda \quad (23)$$

which also follows directly from the explicit expression Eq. (16). It follows that the eigenvalues of  $\mathbf{R}^{\text{pol}}$  are  $0, \pm \sqrt{\frac{1}{2}\mathbf{R}^{\text{pol}}:\mathbf{R}^{\text{pol}}}$ . Since the eigenvalues of  $e\mathbf{P}$  are obviously just  $e, e, 0$ , the eigenvalues of  $\mathbf{R}$  are  $e \pm \sqrt{\frac{1}{2}\mathbf{R}^{\text{pol}}:\mathbf{R}^{\text{pol}}}, 0$ . The realizability of  $\mathbf{R}$  is therefore simply the condition

$$e \geq \sqrt{\frac{1}{2}\mathbf{R}^{\text{pol}}:\mathbf{R}^{\text{pol}}}. \quad (24)$$

Note from Eqs. (14) and (16) that

$$ZZ^* = \frac{1}{2}\mathbf{R}^{\text{pol}}:\mathbf{R}^{\text{pol}}, \quad (25)$$

where the “star” denotes complex conjugation. Thus, although the scalars  $d = \Re Z$  and  $b = \Im Z$  are coordinate-dependent, the magnitude  $|Z|$  is a geometric invariant.

Equation (20) is a straightforward generalization of the trace-deviator decomposition in which the trace, the projection along  $\delta_{ij}$ , is replaced by the operation of projection along  $\mathbf{P}$ .

The general decomposition equation (17) can be rewritten in a form that isolates the purely isotropic part by projecting  $e(\mathbf{k}, t)$  onto its spherical average,

$$U(k, t) = \frac{1}{4\pi k^2} \oint_{S_k} e(\mathbf{k}, t) d^2\mathbf{k}, \quad (26)$$

where  $\oint_{S_k}(\dots) d^2\mathbf{k}$  denotes integration over a spherical shell of radius  $k$ . Note that since trivially  $U(k, t) = (1/4\pi k^2) \oint_{S_k} U(k, t) d^2\mathbf{k}$ , Eq. (26) does define a projection. Defining

$$\mathcal{E}(\mathbf{k}, t) = e(\mathbf{k}, t) - U(k, t) \quad (27)$$

and

$$\mathbf{R}^{\text{dir}}(\mathbf{k}, t) = \mathcal{E}(\mathbf{k}, t)\mathbf{P}(\hat{\mathbf{k}}) \quad (28)$$

we have

$$\begin{aligned} \mathbf{R}(\mathbf{k}, t) &= \underbrace{U(k, t)\mathbf{P}(\hat{\mathbf{k}})}_{\text{Isotropic part}} + \underbrace{\mathbf{R}^{\text{dir}}(\mathbf{k}, t)}_{\text{Directional anisotropy}} \\ &+ \underbrace{\mathbf{R}^{\text{pol}}(\mathbf{k}, t)}_{\text{Polarization anisotropy}}. \end{aligned} \quad (29)$$

The decomposition of the correlation tensor in Eq. (29) has a simple but important geometrical significance: recall that the decomposition of a symmetric second rank tensor into its trace and deviator is invariant under rotation of the coordinate axes, which transforms the isotropic trace term into itself and the deviator into another deviator. The decomposition in Eq. (29) has a similar property with respect to simultaneous transformation by rotation of the tensor components and the wave vector  $\mathbf{k}$ : because the components in the decomposition have intrinsic characterizations in terms of their eigenvalues, such transformations map a polarization tensor into another polarization tensor, a directionality tensor into another directionality tensor, and the isotropic part into itself. These properties provide an elementary example of the  $\text{SO}^3$  decomposition that will be considered in detail later.

### III. DESCRIPTION BY SPHERICAL AVERAGES

The most apparent practical difficulty in formulating a complete description of spectral anisotropy lies in the  $\mathbf{k}$  dependence of the correlation tensor. Consequently, the search for simpler descriptions generally begins by considering spherically averaged quantities.<sup>18–20</sup> Whereas spherical integration of a scalar quantity would remove all information about anisotropy, the weighting by tensors like  $\mathbf{P}$  in fact permits retention of some nontrivial anisotropic properties on spherical averaging of  $\mathbf{R}$ . Nevertheless, such a description of spectral anisotropy is limited to relatively weak anisotropy, since spherical averages can give only partial information about  $\mathbf{k}$  dependence. This section presents a complete analysis of anisotropy at the level of spherical averages; even this partial description proves to be nontrivial.

The essential idea is to treat directionality and polarization anisotropy separately. Given any correlation tensor  $\mathbf{R}(\mathbf{k}, t)$ , we can construct two natural tensor measures of anisotropy that depend only on  $k$ : they are defined by

$$2E(k, t)\mathbf{H}^{(e)}(k, t) = \oint_{S_k} \mathbf{R}^{\text{dir}}(\mathbf{k}, t) d^2\mathbf{k}, \quad (30)$$

$$2E(k,t)H^{(z)}(k,t) = \oint_{S_k} R^{pol}(\mathbf{k},t)d^2\mathbf{k}, \quad (31)$$

where

$$E(k,t) = \oint_{S_k} e(\mathbf{k},t)d^2\mathbf{k} = 4\pi k^2 U(k) \quad (32)$$

is the energy spectrum.<sup>21</sup> The notation  $H^{(e,z)}$  follows Cambon and Jacquin,<sup>8</sup> and is motivated by the characterization in Sec. II of directional anisotropy by the scalar  $e(\mathbf{k},t)$  and of polarization anisotropy by the complex scalar  $Z(\mathbf{k},t)$ . It is also convenient to introduce  $H = H^{(e)} + H^{(z)}$  noting that

$$\begin{aligned} 2E(k,t)H(k,t) &= \oint_{S_k} [R^{dir}(\mathbf{k},t) + R^{pol}(\mathbf{k},t)]d^2\mathbf{k} \\ &= \oint_{S_k} [R(\mathbf{k},t) - U(k,t)P(\mathbf{k})]d^2\mathbf{k} \end{aligned} \quad (33)$$

so that  $H$  represents the complete anisotropic part of the correlation. Obviously,

$$\begin{aligned} 2E(k,t)\text{tr} H^{(e)}(k,t) &= \oint_{S_k} \text{tr} R^{dir}(\mathbf{k},t)d^2\mathbf{k} \\ &= \oint_{S_k} (e - U)\text{tr} P(\mathbf{k})d^2\mathbf{k} = 0 \end{aligned} \quad (34)$$

and, in view of Eq. (22),

$$2E(k,t)\text{tr} H^{(z)}(k,t) = \oint_{S_k} \text{tr} R^{pol}(\mathbf{k},t)d^2\mathbf{k} = 0 \quad (35)$$

so that both of  $H^{(e,z)}$  are trace-free.

We wish to construct a modeled correlation tensor that depends only on the spherical averages  $H^{(e,z)}$ . The discussion in Sec. II motivates constructing  $R^{dir}$  and  $R^{pol}$  separately. To begin, note that  $R^{dir}$  depends linearly on  $H^{(e)}$  and is proportional to  $P$ . The simplest assumption consistent with these properties is

$$R^{dir}(\mathbf{k},t) = AU(k,t)[H^{(e)}(k,t):P(\hat{\mathbf{k}})]P(\hat{\mathbf{k}}) \quad (36)$$

with an undetermined constant  $A$ . Equivalently, in terms of the solenoidal tensor  $PH^{(e)}P$ , Eq. (36) sets  $R^{dir} = A[PH^{(e)}P]^{dir}$ . The constant  $A$  should be chosen to be consistent with the definition equation (30); the spherical average of each side of Eq. (36) gives

$$2E(k,t)H^{(e)}(k,t) = -\frac{2}{15}AE(k,t)H^{(e)}(k,t) \quad (37)$$

so that  $A = -15$ .

The treatment of the polarization tensor is somewhat less straightforward.  $R^{pol}$  must be solenoidal and linear in  $H^{(z)}$ . These requirements suggest the form  $R^{pol} = PH^{(z)}P$ . But in addition, we must take into account that  $R^{pol}:P = 0$ . A general form consistent with all constraints is therefore

$$\begin{aligned} R_{ij}^{pol}(\mathbf{k},t) &= BU(k,t)\left[P_{im}(\hat{\mathbf{k}})H_{nm}^{(z)}(k,t)P_{nj}(\hat{\mathbf{k}})\right. \\ &\quad \left.- \frac{1}{2}H_{pq}^{(z)}(k,t)P_{pq}(\mathbf{k})P_{ij}(\mathbf{k})\right] \end{aligned} \quad (38)$$

with an undetermined constant  $B$ . Again, in terms of the

solenoidal tensor  $PH^{(z)}P$ , Eq. (38) sets  $R^{pol} = B[PH^{(z)}P]^{pol}$ . Spherical averaging as in Eq. (37) gives

$$2E(k,t)H^{(z)}(k,t) = \frac{1}{5}BE(k,t)H^{(z)}(k,t) \quad (39)$$

so that  $B = 5$ .

Combining the results of Eqs. (36) and (38), we obtain the required representation

$$\begin{aligned} R_{ij}(\mathbf{k},t) &= U(k,t)P_{ij}(\hat{\mathbf{k}}) + 15U(k,t)P_{ij}(\hat{\mathbf{k}})H_{pq}^{(e)}(k,t)P_{pq}(\hat{\mathbf{k}}) \\ &\quad + 5U(k,t)\left[P_{in}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}})H_{nm}^{(z)}(k,t)\right. \\ &\quad \left.- \frac{1}{2}P_{mn}(\hat{\mathbf{k}})H_{nm}^{(z)}(k,t)P_{ij}(\hat{\mathbf{k}})\right] \end{aligned} \quad (40)$$

or equivalently,

$$\begin{aligned} R_{ij}(\mathbf{k},t) &= U(k,t)P_{ij}(\hat{\mathbf{k}}) - 15U(k,t)P_{ij}(\hat{\mathbf{k}})H_{pq}^{(e)}(k,t)\hat{k}_p\hat{k}_q \\ &\quad + 5U(k,t)\left[P_{in}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}})H_{nm}^{(z)}(k,t)\right. \\ &\quad \left.+ \frac{1}{2}P_{ij}(\hat{\mathbf{k}})H_{pq}^{(z)}(k,t)\hat{k}_p\hat{k}_q\right]. \end{aligned} \quad (41)$$

This equation is the main result of this paper. It shows that a self-consistent description of *weak* anisotropy without arbitrary constants is possible using independent descriptors of directionality and polarization by tensor functions of the wave number  $k$  alone. The restriction to weak anisotropy is due to the description by spherical averages alone, as noted at the beginning of this section. The ‘‘self-consistency’’ of the result is understood in two ways: first, that the spherical averages of both sides of Eq. (41) are equal; and second, that if

$$\begin{aligned} R_{ij}(\mathbf{k},t) &= U(k,t)P_{ij}(\hat{\mathbf{k}}) - 15U(k,t)P_{ij}(\hat{\mathbf{k}})A_{pq}(k,t)\hat{k}_p\hat{k}_q \\ &\quad + 5U(k,t)\left[P_{in}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}})B_{nm}(k,t)\right. \\ &\quad \left.+ \frac{1}{2}P_{ij}(\hat{\mathbf{k}})B_{pq}(k,t)\hat{k}_p\hat{k}_q\right] \end{aligned} \quad (42)$$

with arbitrary trace-free  $A$  and  $B$ , then this construction will lead to Eq. (41) with  $H^{(e)} = A$  and  $H^{(z)} = B$ .

Note that since  $\text{tr} H^{(e)} = 0$ , the quantity  $H_{pq}^{(e)}(k,t)\hat{k}_p\hat{k}_q$  that appears in Eq. (41) is a second-order spherical harmonic: after choosing a polar axis and introducing spherical coordinates, it would be expressed in terms of Legendre functions in the standard way. The term in brackets containing  $H^{(z)}$  is a tensor analog of a spherical harmonic; we will refer to *scalar spherical harmonics* (SSH) and *tensor spherical harmonics* (TSH) henceforth. From this viewpoint, Eq. (41) states the lowest-order terms in expansions of  $R^{dir}$  and  $R^{pol}$ , respectively, in scalar and tensor harmonics, or in irreducible representations of the rotation group  $SO^3$ . This connection will be developed further in Sec. VII.

We next ask whether a solenoidal correlation function can be constructed consistent with  $H$  alone. Proceeding as before, we set

$$R^{dir} = AU(k,t)(H:P)P, \quad (43)$$

$$R^{pol} = BU(k,t)\left[PHP - \frac{1}{2}(H:P)P\right].$$

We already note that these equations implicitly impose some relation between  $R^{dir}$  and  $R^{pol}$  and therefore cannot be entirely satisfactory. On spherical averaging, we find



$$\mathbf{H} = \left(\frac{1}{15}A + \frac{1}{5}B\right)\mathbf{H}. \quad (44)$$

The solution is not unique: it is

$$A = 15 + 3a, \quad B = -a, \quad (45)$$

where  $a$  can be a function  $a=a(k, t)$ . Thus, the spectral tensor is

$$\begin{aligned} R_{ij}(\mathbf{k}, t) = & U(k, t)P_{ij}(\hat{\mathbf{k}}) + \left(1 + \frac{1}{5}a\right) \\ & \times U(k, t)H_{pq}(\mathbf{k}, t)P_{pq}(\hat{\mathbf{k}})P_{ij}(\hat{\mathbf{k}}) - \frac{1}{5}aU(k, t) \\ & \times \left[ P_{im}(\hat{\mathbf{k}})P_{jn}(\mathbf{k})R_{mn}(\hat{\mathbf{k}}, t) \right. \\ & \left. - \frac{1}{2}H_{pq}(\mathbf{k}, t)P_{pq}(\hat{\mathbf{k}})P_{ij}(\hat{\mathbf{k}}) \right] \end{aligned} \quad (46)$$

and

$$\mathbf{H}^{(e)} = \left(1 + \frac{2}{5}a\right)\mathbf{H}, \quad \mathbf{H}^{(z)} = -\frac{2}{5}a\mathbf{H}, \quad (47)$$

which implies the proportionality

$$\frac{2}{5}a\mathbf{H}^{(e)} = -\left(1 + \frac{2}{5}a\right)\mathbf{H}^{(z)}. \quad (48)$$

Equation (46) coincides with the proposal of Cambon *et al.*<sup>4</sup> to connect the spectral tensor to its spherical average. Other more empirical models in which anisotropy is parametrized by a tensor function of wave number  $k$  have been proposed.<sup>5</sup> It will be shown in the next section that no single choice of the quantity  $a$  can be adequate in all cases.

#### IV. APPLICATION: SHORT- AND LONG-TIME BEHAVIOR OF WEAKLY SHEARED TURBULENCE

This section will give examples of anisotropic flows that are known to depart only weakly from isotropy, and will verify that the correlation tensor is indeed described by Eq. (41).

The first example is homogeneous shear flow with isotropic initial conditions treated by rapid distortion theory (RDT) in the short-time limit. Using the notation of Cambon and Scott,<sup>19</sup> we begin with the general RDT equation,

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t}(\mathbf{k}, t) = & -\frac{dk_n}{dt} \frac{\partial}{\partial k_n} R_{ij}(\mathbf{k}, t) - M_{in}(\mathbf{k})R_{nj}(\mathbf{k}, t) \\ & - M_{jn}(\mathbf{k})R_{in}(\mathbf{k}, t), \end{aligned} \quad (49)$$

where  $M_{ij} = (\delta_{im} - 2\hat{k}_i\hat{k}_m)A_{mj}$ ,  $A_{im} = \partial\bar{U}_i/\partial x_m$  is the mean velocity gradient, and the wave vector  $k_n$  satisfies  $dk_i/dt = -A_{ji}k_j$ . We will consider evolution away from an isotropic initial condition  $\mathbf{R}(\mathbf{k}, 0) = U(k)\mathbf{P}(\hat{\mathbf{k}})$  at very short times, when the effects of the mean shear remain weak.

Consider the first-order Taylor series expansion  $\mathbf{R}(\mathbf{k}, t) = U(k)\mathbf{P} + t\dot{\mathbf{R}}(\mathbf{k}, 0)$ . Evaluating Eq. (49) at  $t=0$  using the isotropic initial condition leads easily to

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t}(\mathbf{k}, 0) = & A_{pn}\hat{k}_p\hat{k}_n k U'(k)P_{ij}(\hat{\mathbf{k}}) + U(k)\hat{k}_i\hat{k}_p A_{pn}P_{nj}(\hat{\mathbf{k}}) \\ & + U(k)\hat{k}_j\hat{k}_p A_{pn}P_{ni}(\hat{\mathbf{k}}) - U(k)A_{in}P_{nj}(\hat{\mathbf{k}}) \\ & - U(k)A_{jn}P_{ni}(\hat{\mathbf{k}}) \\ = & S_{pn}\hat{k}_p\hat{k}_n k U'(k)P_{ij}(\hat{\mathbf{k}}) - U(k)P_{ip}(\mathbf{k})P_{nj}(\mathbf{k})S_{pn}, \end{aligned} \quad (50)$$

where  $S_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$  is the strain rate. It follows that

$$R_{ij}^{\text{dir}}(\mathbf{k}, t) = \frac{1}{2}tS_{pq}\hat{k}_p\hat{k}_q[kU'(k) + U(k)]P_{ij}(\mathbf{k}), \quad (51)$$

$$R_{ij}^{\text{pol}}(\mathbf{k}, t) = -tU(k)\left[P_{im}(\mathbf{k})P_{jn}(\mathbf{k})S_{mn} + \frac{1}{2}S_{pq}\hat{k}_p\hat{k}_qP_{ij}(\mathbf{k})\right]. \quad (52)$$

We remark that the  $U'$  term comes from the  $dk_i/dt$  effect in the RDT equations: it is a conservative linear energy transfer mechanism in  $\mathbf{k}$  space, which therefore appears as a directionality effect; polarization effects instead arise when energy is transferred between different tensor components of the correlation. We see from Eqs. (51) and (52) that both effects are relevant. Spherical integration gives

$$\mathbf{H}^{(e)}(k, t) = -\frac{1}{15}\left(-1 + \frac{k}{E} \frac{dE}{dk}\right)\mathbf{S}t \quad \text{and} \quad \mathbf{H}^{(z)}(k, t) = -\frac{2}{5}\mathbf{S}t. \quad (53)$$

This leads to  $a(k) = 15/[5 + (k/E)dE/dk]$  in Eq. (47).

It is useful to supplement this short-time analysis by long-time nonlinear analysis.<sup>22-24</sup> These computations begin with a closure model for shear turbulence,

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t}(\mathbf{k}, t) = & -\frac{dk_n}{dt} \frac{\partial}{\partial k_n} R_{ij}(\mathbf{k}, t) - M_{in}(\mathbf{k})R_{nj}(\mathbf{k}, t) \\ & - M_{jn}(\mathbf{k})R_{in}(\mathbf{k}, t) + T_{ij}(\mathbf{k}, t), \end{aligned} \quad (54)$$

where  $\mathbf{T}$  is a theory-dependent closure for the nonlinear energy transfer.<sup>6,7,22-24</sup> Whereas Eq. (49) considers a limit in which nonlinearity can be neglected in comparison to the linear shear mechanisms, in these calculations the opposite assumption is made, namely that in some range of small scales, the mean shear may be treated as a weak perturbation of isotropic turbulence: the calculation itself will identify an appropriate small parameter. The most recent such analysis<sup>6,7</sup> overcomes some limitations of previous work, and introduces a Lagrangian viewpoint, with important analytical and conceptual advantages. Yoshida *et al.*<sup>7</sup> refer to this calculation as ‘‘linear response theory’’ for turbulence, since it is accomplished by linearizing about an isotropic nonlinear state. As in the RDT problem above, anisotropy is therefore weak, although for an entirely different reason, and we can again expect Eq. (41) to describe the spectral tensor.

General kinematic considerations<sup>4,5</sup> lead, in the notation of Ishihara *et al.*,<sup>6</sup> to

$$R_{ij}(\hat{\mathbf{k}}) = U(k)P_{ij}(\mathbf{k}) + b(k)P_{ij}(\hat{\mathbf{k}})S_{pq}\hat{k}_p\hat{k}_q + 2a'(k)P_{in}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}})S_{nm} \quad (55)$$

from which we immediately deduce

$$R_{ij}^{\text{dir}} = (b - a')(S_{pq}\hat{k}_p\hat{k}_q)P_{ij}(\hat{\mathbf{k}}), \quad (56)$$

$$R_{ij}^{\text{pol}} = 2a' \left[ P_{im}(\hat{\mathbf{k}})P_{jn}(\hat{\mathbf{k}})S_{mn} + \frac{1}{2}S_{pq}\hat{k}_p\hat{k}_q P_{ij}(\hat{\mathbf{k}}) \right],$$

which has exactly the same structure as Eqs. (51) and (52) except for the scalar functions of  $k$ ; consequently,

$$H^{(e)} = -\frac{1}{15}\theta^{(e)}\mathbf{S}, \quad H^{(z)} = -\frac{2}{5}\theta^{(z)}\mathbf{S}, \quad (57)$$

where  $\theta^{(e)}(k) = [b(k) - a'(k)]/U(k)$  and  $\theta^{(z)}(k) = 2a'(k)/U(k)$  are time scales. In a Kolmogorov inertial range, we will have (again using the notation of Ishihara *et al.*<sup>6</sup>)

$$\theta^{(e)} = (A - B)\epsilon^{-1/3}k^{-2/3}, \quad \theta^{(z)} = A\epsilon^{-1/3}k^{-2/3}, \quad (58)$$

where  $A$  and  $B$  are universal constants. In this case,

$$\frac{2}{5}AH^{(e)} = \frac{1}{15}(B - A)H^{(z)} \quad (59)$$

and the anisotropic part of the spectrum satisfies

$$EH^{(e)} \sim EH^{(z)} \sim k^{-7/3}, \quad (60)$$

the scaling suggested by dimensional analysis.<sup>25</sup>

Equations (53), (57), and (58) reveal important qualitative differences between these two calculations. In RDT, Eq. (53) implies that anisotropy is significant at all scales of motion, whereas in the nonlinear calculation, Eqs. (57) and (58) show that the descriptors of anisotropy  $H^{(e,z)}$  both approach zero in the limit  $k \rightarrow \infty$  of small scales of motion. The present analysis has treated these regimes independently: an interesting problem is to understand how these two regimes coexist and merge in homogeneous shear flow.

Yoshida *et al.*<sup>7</sup> find theoretical values from spectral closure theory  $A \approx -0.16$  and  $B \approx -0.40$ . Experimental and DNS measurements give values closer to  $A \approx -0.12$  and  $B \approx -0.009$ . It is found that making realistic corrections to the theoretical values to account for the finite inertial range in the measurements results in much closer agreement. However, for our purposes, the actual values are not so crucial; the important observation is that a weakly anisotropic spectral model based on a single tensor  $\mathbf{H}$  imposes some fixed proportionality  $H^{(e)} = \lambda H^{(z)}$  as shown by Eq. (48); such models cannot be consistent with *both* the short- and the long-time results equations (53) and (59). On the other hand, the model equation (41) can be consistent with both limits. Note also that since neither  $A$  nor  $A - B$  is approximately zero, neither directional nor polarization anisotropy can be neglected in this problem.

In the case of purely rotational strains, where  $\mathbf{A}$  is antisymmetric and hence the strain  $\mathbf{S}$  vanishes, the first-order contribution to the correlation tensor computed in Eqs. (53) and (56) vanishes because the antisymmetric tensor  $\mathbf{A}$  cannot contribute to the symmetric correlation tensor. Nevertheless, rotating turbulence provides another example of how directionality and polarization must be separated in general. The inviscid RDT equation<sup>8</sup> for this problem implies

$$\mathbf{R}^{\text{dir}}(\mathbf{k}, t) = \mathbf{R}^{\text{dir}}(\mathbf{k}, 0), \quad \mathbf{R}^{\text{pol}}(\mathbf{k}, t) = \exp(4 \mathbf{1} \Omega \hat{k}_3 t) \mathbf{R}^{\text{pol}}(\mathbf{k}, 0), \quad (61)$$

which yields

$$H^e(k, t) = H^{(e)}(k, 0), \quad H^{(z)}(k, t) \rightarrow 0. \quad (62)$$

We see then that the kinematics of turbulence under rapid rotation is dominated by directional anisotropy alone. Spherically averaged polarization can be neglected even in the presence of nonlinearity, but directional anisotropy can be created by nonlinearity, even if it is initially zero.<sup>8,10,13</sup>

### V. PARAMETRIZATION BY SINGLE-POINT MOMENTS

Turbulence statistics are very unlikely to require description by functions  $H_{ij}^{(e,z)}(k)$  that are entirely distinct for different indices  $(i, j)$ ; instead, it is expected that they might be linked by scaling relations of the form  $H_{ij}^{(e,z)}(\lambda k) \sim \lambda^\alpha H_{ij}^{(e,z)}(k)$  for a single scaling exponent  $\alpha$ . A more general form of this idea is the heuristic principle, namely that quantities that transform among themselves under rotations can exhibit characteristic scaling properties.<sup>11,12</sup>

These considerations motivate seeking a parametric description of the form

$$\mathbf{H}^{(e,z)}(k, t) \propto \psi^{(e,z)}(k, t) \mathbf{b}^{(e,z)}(t) \quad (63)$$

in terms of the two single-point moments  $\mathbf{b}^{(e,z)}(t)$  defined by

$$\mathbf{b}^{(e)}(t) = \frac{\int_0^\infty E(k, t) \mathbf{H}^{(e)}(k, t) dk}{\int_0^\infty E(k, t) dk}, \quad (64)$$

$$\mathbf{b}^{(z)}(t) = \frac{\int_0^\infty E(k, t) \mathbf{H}^{(z)}(k, t) dk}{\int_0^\infty E(k, t) dk}$$

and the scalar functions  $\psi^{(e,z)}(k, t)$ ; it might even be reasonable to assume that these functions are independent of time, but this issue will not be considered here.

It is evident from Eqs. (64) and (33) that

$$\mathbf{b} = \mathbf{b}^{(e)} + \mathbf{b}^{(z)} \quad (65)$$

is the *anisotropy tensor* defined by

$$b_{ij} = \frac{\langle u'_i u'_j \rangle}{\langle u'_p u'_p \rangle} - \frac{1}{3} \delta_{ij} \quad (66)$$

as the deviatoric part of the dimensionless Reynolds stress. Note that the  $e$  and  $z$  contributions to Eq. (63) cannot be added unless  $\psi^{(e)} = \psi^{(z)}$ ; thus,  $\mathbf{H}$  depends on  $\mathbf{b}^{(e)}$  and  $\mathbf{b}^{(z)}$  separately, not on  $\mathbf{b}$ . Thus, we have

$$H(k,t) = \frac{\int_0^\infty dk E(k,t)}{\int_0^\infty dk \psi(k,t) E(k,t)} \psi(k,t) \mathbf{b}(t) \quad \text{if } \psi^{(e)} = \psi^{(z)} = \psi. \tag{67}$$

For example, consider the short-time RDT results of the previous section. Multiplying each result in Eq. (53) by  $2E$  and integrating over  $k$  gives

$$\mathbf{b}^{(e)} = \frac{2}{15} \mathbf{S}t, \quad \mathbf{b}^{(z)} = -\frac{2}{5} \mathbf{S}t, \quad \mathbf{b} = -\frac{4}{15} \mathbf{S}t, \tag{68}$$

thus

$$\mathbf{H}^{(e)} = \frac{15}{2} \left( -1 + \frac{k}{E} \frac{dE}{dk} \right) \mathbf{b}^{(e)}, \quad \mathbf{H}^{(z)} = \mathbf{b}^{(z)}. \tag{69}$$

Note, in connection with Eq. (67), that a direct connection between  $\mathbf{H}$  and  $\mathbf{b}$  cannot be established, because Eq. (69) implies that  $\psi^{(e)} \neq \psi^{(z)}$ . Equation (68) implies that

$$\mathbf{b}^{(e)} = -(1/3)\mathbf{b}^{(z)}, \tag{70}$$

or equivalently,

$$-2\mathbf{b}^{(e)} = \mathbf{b}. \tag{71}$$

The persistence of the short-time relation equation (71) at long times for irrotational mean flows has been discussed by Kassinos *et al.*<sup>26</sup> However, Eq. (69) shows directly that this rather exceptional condition does not apply to the spectral quantities  $\mathbf{H}^{(e,z)}$ .

We can also compute  $\mathbf{b}^{(e,z)}$  for the results of Yoshida *et al.*<sup>7</sup> To obtain a definite result, we will follow previous calculations of this type<sup>22-24</sup> and integrate the inertial range spectrum  $E(k) = C_k \epsilon^{2/3} k^{-5/3}$  over a range of scales  $k \geq k_0$ , where  $k_0$  is identified with the scale at which the inertial range can be considered to begin, although this integration may include scales for which the hypotheses of the perturbation theory are not valid. The result is

$$\begin{aligned} \mathbf{b}^{(e)} &= \frac{1}{30} C_k^{-1} (B - A) \epsilon^{-1/3} k_0^{-2/3} \mathbf{S}, \\ \mathbf{b}^{(z)} &= \frac{1}{10} C_k^{-1} A \epsilon^{-1/3} k_0^{-2/3} \mathbf{S}, \end{aligned} \tag{72}$$

so that

$$\mathbf{H}^{(e)} = 2C_K \left( \frac{k_0}{k} \right)^{2/3} \mathbf{b}^{(e)}, \quad \mathbf{H}^{(z)} = 4C_K \left( \frac{k_0}{k} \right)^{2/3} \mathbf{b}^{(z)}. \tag{73}$$

In comparing the results equations (68) and (72), we see that the time scale in short-time RDT is simply elapsed time  $t$ , whereas it is a turbulent time scale  $\propto \epsilon^{-1/3} k_0^{-2/3}$  in the linear response theory. We can also note the remarkable change in the relation between  $\mathbf{H}^{(e,z)}$  and  $\mathbf{b}^{(e,z)}$  between Eqs. (69) and (73): in RDT,  $\mathbf{H}^{(e)}$  is related to  $\mathbf{b}^{(e)}$  through a term containing  $dE/dk$  but  $\mathbf{H}^{(z)}$  and  $\mathbf{b}^{(z)}$  are simply equal; in the linear response theory, explicit dependence on  $dE/dk$  has disappeared, and  $\psi^{(e,z)}(k)$  both decay as  $k^{-2/3}$ .

The introduction of  $\mathbf{b}^{(e,z)}$  is a way to describe anisotropy in terms of single-point moments. An equivalent approach is the *structure tensor* formalism proposed by Kassinos *et al.*<sup>26</sup>

The connections between these approaches are discussed in detail by Salhi and Cambon.<sup>27</sup>

The result of Eq. (63) can be substituted in Eq. (41) to give a general anisotropic representation in terms of single-point quantities and two scalar functions  $\psi^{(e,z)}(k)$ ,

$$\begin{aligned} R_{ij}(\mathbf{k},t) &= U(k,t) P_{ij}(\hat{\mathbf{k}}) - 15U(k,t) P_{ij}(\hat{\mathbf{k}}) b_{pq}^{(e)}(t) \\ &\quad \times \psi^{(e)}(k,t) \hat{k}_p \hat{k}_q + 5U(k,t) \left[ P_{in}(\hat{\mathbf{k}}) P_{jm}(\hat{\mathbf{k}}) b_{nm}^{(z)}(t) \right. \\ &\quad \left. \times \psi^{(z)}(k,t) + \frac{1}{2} P_{ij}(\hat{\mathbf{k}}) b_{pq}^{(z)}(t) \psi^{(z)}(k,t) \hat{k}_p \hat{k}_q \right]. \end{aligned} \tag{74}$$

Just like the attempt to characterize anisotropy in terms of  $\mathbf{H}$  alone, an attempt to replace Eq. (74) by an expression in  $\mathbf{b}$  alone will encounter difficulties. An example of such a parametrization of the correlation tensor is the proposal of Shih *et al.*<sup>5</sup> Written in terms of directional and polarization anisotropy components, it is

$$\begin{aligned} R_{ij}(\mathbf{k}) &= U(\mathbf{k},t) P_{ij}(\hat{\mathbf{k}}) + \psi_3(\mathbf{k},t) P_{ij}(\hat{\mathbf{k}}) \left( \gamma - \frac{1}{2} \right) b_{pq} \hat{k}_p \hat{k}_q \\ &\quad + \psi_3(\mathbf{k},t) P_{in}(\hat{\mathbf{k}}) P_{jm}(\hat{\mathbf{k}}) \left( b_{nm} + \frac{1}{2} \delta_{nm} b_{pq} \hat{k}_p \hat{k}_q \right). \end{aligned} \tag{75}$$

Thus,

$$\mathbf{H}^{(e)} = \left( \gamma - \frac{1}{2} \right) 15 \frac{\psi_3(k,t)}{E(k,t)} \mathbf{b}, \quad \mathbf{H}^{(z)} = 5 \frac{\psi_3(k,t)}{E(k,t)} \mathbf{b} \tag{76}$$

so that

$$5\mathbf{H}^{(e)} = 3 \left( \gamma - \frac{1}{2} \right) \mathbf{H}^{(z)}. \tag{77}$$

Since this result fixes a definite proportionality between  $\mathbf{H}^{(e)}$  and  $\mathbf{H}^{(z)}$ , it is a special case of Eq. (46) in which the constant  $\gamma$  could be related to the parameter  $a$ . It is not necessary to give the explicit relation; the important fact is such models cannot be consistent with the equations of motion or the general kinematics. Considerations of realizability, which will be discussed later, led to the choice  $\gamma = \frac{1}{2}$ , which removes directional anisotropy entirely, and is therefore inconsistent with the problems of shear and rotating turbulence just analyzed. The important conclusion is that since the general model for  $\mathbf{R}$  in terms of the stress anisotropy  $\mathbf{b}$  in Eq. (75) must contain an arbitrary constant, *the spectral tensor cannot be uniquely reconstructed from the stress anisotropy alone.*

## VI. REALIZABILITY

We have seen that if any arbitrary correlation tensor is given, Eq. (41) provides a simplified approximation with the property that its spherical averages agree with those of the original tensor. However, we did not demonstrate that Eq. (41) in fact describes a possible correlation tensor. This issue raises the question of realizability: is the positivity condition  $R_{ij} n_i n_j \geq 0$  satisfied for arbitrary  $n_i$ ? Obviously, if the original correlation tensor is only weakly anisotropic, then its anisotropic part is a small perturbation of the isotropic part  $U(k,t) \mathbf{P}(\hat{\mathbf{k}})$ , and as this part is strictly positive ( $R_{ij} n_i n_j > 0$ ), the realizability inequality is satisfied for sufficiently small perturbations. On the other hand, there is no guarantee that if the original correlation is strongly anisotropic, the approximation in Eq. (41) represents a possible correlation.



In view of Eqs. (24) and (25), the realizability condition formulated in Sec. II can be expressed as  $e \geq |Z|$ . For simplicity, let us consider the weaker condition  $e \geq 0$ . In the decomposition of Eq. (41),  $e = U(1 - 15H_{pq}^{(e)} \hat{k}_p \hat{k}_q)$ . Thus the simple necessary realizability condition  $e > 0$  is violated if the largest positive eigenvalue of  $H_{ij}^{(e)}$  is larger than  $1/15$ : this tensor being trace-free, at least one positive eigenvalue must exist. In fact, from  $\oint_{s_k} e(\mathbf{k}, t) P_{ij}(\hat{\mathbf{k}}) d^2\mathbf{k} = 2E(k)((1/3)\delta_{ij} + H_{ij}^{(e)})$ , these eigenvalues are bounded by  $\pm 1/3$ , as they are for any deviatoric tensor derived from a definite-positive matrix. This realizability constraint quantifies precisely how “small” the anisotropy must be to admit description by Eq. (41).

Although it is weaker than the exact condition  $e \geq |Z|$ , the condition  $e \geq 0$  already proves to be very restrictive. Even if  $H^{(e)}$  is never zero in “true” anisotropic homogeneous flows, it is safer to suppress its contribution entirely than to risk problems of nonrealizability, as in the *ad hoc* models<sup>5,7</sup> discussed in the previous section. Even the value of the constant in the quasi-isotropic model of Launder *et al.*<sup>28</sup> may reflect this condition.<sup>4,5</sup>

**VII. SO<sup>3</sup> DECOMPOSITION**

The realizability constraint  $e \geq |Z|$  restricts the application of a theory based on spherical averages alone; since simply ignoring directional anisotropy is unsatisfactory in general, we will discuss the construction of more accurate anisotropic approximations to the correlation tensor using higher-order expansions based on irreducible representations of the rotation group SO<sup>3</sup>. This theory has recently been found very useful in clarifying the scaling properties of general correlation functions in turbulent flows.<sup>11,12</sup>

**A. Directional anisotropy**

The spectrum models that have been considered so far suggest how the anisotropic part of the correlation can be expanded in powers of  $\hat{\mathbf{k}}$ . For example, we can continue the development of the first term on the right side of Eq. (41) beyond the second order by writing

$$\mathcal{E}(\mathbf{k}, t) = U_{mn}^2(k, t) \hat{k}_m \hat{k}_n + U_{mnr s}^4(k, t) \hat{k}_m \hat{k}_n \hat{k}_r \hat{k}_s + \dots, \quad (78)$$

where  $U^2 = -15H^{(e)}$ . The expansion is restricted to polynomials of even order because of the parity property  $R_{ij}(-\hat{\mathbf{k}}) = R_{ij}(\hat{\mathbf{k}})$ .

Although the notation suggests that Eq. (78) proceeds in powers of  $\hat{k}_i$ , it does not do so without some restrictions on the coefficients. For example, if  $U_{mnr s}^4 = \delta_{mr} A_{rs}$ , then  $U_{mnr s}^4 k_m k_n k_r k_s = k^2 A_{rs} k_r k_s$ ; but this term could be included in the  $U^2$  term. Without presenting any details, suffice it to say that if all such redundancies are eliminated, then  $U^4$  will belong to a nine-dimensional representation of SO<sup>3</sup> on homogeneous quartic polynomials satisfying  $\nabla^2 U_{ijmn}^4 k_i k_j k_m k_n = 0$ . This is discussed in detail in standard references.<sup>29</sup> Equivalently, after choosing a polar axis  $\mathbf{n}$ , the expansion

could be described in terms of Legendre functions as a spherical harmonics decomposition following Cambon and Teissèdre,<sup>30</sup>

$$\mathcal{E}(\mathbf{k}, t) = \sum_{n=1}^{N_0} \sum_{m=-2n}^{2n} e_{2n}^m(k, t) \underbrace{P_{2n}^m(\cos \theta) \exp(im\varphi)}_{Y_{2n}^m(\theta, \varphi)}, \quad (79)$$

where  $\theta = \arccos(\mathbf{n} \cdot \hat{\mathbf{k}})$  is the polar angle and  $\varphi$  is the azimuthal angle in a system of polar-spherical coordinates with axis  $\mathbf{n}$ , and  $P_{2n}^m$  are the associated Legendre polynomials of degree  $2n$  and order  $m$ .

**B. Polarization tensor**

The analogous higher-order expansion of the second term on the right side of Eq. (41) is

$$\begin{aligned} R_{ij}^{\text{pol}}(\mathbf{k}, t) = & \frac{1}{2} [P_{im}(\hat{\mathbf{k}}) P_{jn}(\hat{\mathbf{k}}) + P_{in}(\hat{\mathbf{k}}) P_{jm}(\hat{\mathbf{k}}) \\ & - P_{ij}(\hat{\mathbf{k}}) P_{mn}(\hat{\mathbf{k}})] \\ & \times \{T_{mn}^0(k, t) + T_{mnr s}^2(k, t) \hat{k}_r \hat{k}_s + \dots\}, \quad (80) \end{aligned}$$

where  $T^0 = 5H^{(z)}$ .

The structure of the higher-order terms in the expansion of polarization is not as simple as the higher-order terms in the expansion of directional anisotropy, and the steps that make Eq. (80) an orthogonal expansion in irreducible representations are less obvious and standard than the steps leading from Eq. (78) to Eq. (79). Although  $T^0 \propto H^{(z)}$  is simply a constant trace-free second-rank tensor,  $T^2$  consists of matrices with quadratic polynomial entries; their decomposition into irreducible representations or *tensor spherical harmonics* can be summarized as follows (compare also Arad *et al.*<sup>11</sup> and Kurien and Sreenivasan<sup>12</sup> for more general tensor quantities): beginning with the construction of invariant tensors of differential operators, their action on homogeneous polynomials belonging to a representation on scalar functions generated the appropriate representations on tensors. In the case just mentioned, we find that  $T^2$  belongs to a 25-dimensional representation that is decomposed into irreducible representations of dimensions 1, 3, 5, 7, and 9. The representation of dimension 1 cannot contribute to polarization, and the representation of dimension 3 does not survive solenoidal projection. The remaining representations of dimensions 5, 7, and 9 define solenoidal TSH. In the interest of concreteness, we list the TSH that correspond to the irreducible representation of dimension 5. To compute it, we use the infinitesimal generators<sup>29</sup> of SO<sup>3</sup>:

$$\begin{aligned} L_x &= k_y \partial / \partial k_z - k_z \partial / \partial k_y, \\ L_y &= k_z \partial / \partial k_x - k_x \partial / \partial k_z, \\ L_z &= k_x \partial / \partial k_y - k_y \partial / \partial k_x. \end{aligned} \quad (81)$$

Let  $P_2^i$  denote any five harmonic quadratics, for example,  $\{k_x^2 - k_y^2, k_y^2 - k_z^2, k_x k_y, k_y k_z, k_z k_x\}$ . The representation of dimension 5 is defined by

$$N_{ij}^I = (L_i L_j + L_j L_i - \frac{1}{3} L_p L_p \delta_{ij}) P_2^I, \quad I = 1, \dots, 5. \quad (82)$$

Examples are the matrix so generated from  $k_x^2 - k_y^2$ ,

$$\begin{bmatrix} 2(k_x^2 - k_y^2) & 2k_x k_y & 3k_x k_z \\ 2k_x k_y & 2(k_x^2 - k_y^2) & -3k_y k_z \\ 3k_x k_z & -3k_y k_z & 2(k_x^2 - k_y^2) \end{bmatrix} \quad (83)$$

and two others obtained by simultaneous cyclic permutations of  $x, y, z$  and of tensor indices (note that the three tensors so generated add up to zero), and the matrix generated from  $k_x k_y$ ,

$$\begin{bmatrix} 2k_x k_y & k_x^2 + k_y^2 - 2k_z^2 & 3k_y k_z \\ k_x^2 + k_y^2 - 2k_z^2 & 2k_x k_y & 3k_x k_z \\ 3k_y k_z & 3k_x k_z & -4k_x k_y \end{bmatrix} \quad (84)$$

and two others obtained by the same cyclic permutations.

This analysis shows that it is only a coincidence that at the lowest order of anisotropy, directionality and polarization are both described by a constant trace-free second-rank tensor. At even the next order, the descriptions are quite different: as noted above, directionality is described by a homogeneous fourth-degree polynomial, but polarization is described by three distinct types of second-rank tensors with quadratic polynomial entries.

We believe that these expansions in scalar and tensor spherical harmonics, which clearly have very different mathematical origins, again underscore the difficulties of confounding directional and polarization anisotropy.

### C. Axisymmetric turbulence

The problems raised by extending the description from weak anisotropy to arbitrary anisotropy can be better understood in the comparatively simple case of axial symmetry. Let the unit vector  $\mathbf{n}$  be the axis of symmetry. In this case, any trace-free tensor function of  $k$  alone obtained by spherical averaging, or any single-point moment obtained by integration over all Fourier modes, can be expressed as  $H_{ij} = \frac{1}{2} H_n (3n_i n_j - \delta_{ij})$  in terms of the single axial component  $H_n = H_{ij} n_i n_j$ . As for the  $k$ -dependent spectra, a polar-spherical system of coordinates can be introduced, so that Eq. (79) reduces to

$$\mathcal{E}(\mathbf{k}, t) = \sum_{n=1}^{N_0} e_{2n}(k, t) P_{2n}^0(\cos \theta) \quad (85)$$

in which only terms with  $m=0$  appear. In addition, the polarization anisotropy also can be expanded as

$$Z(\mathbf{k}, t) = \sin^2 \theta \sum_{n=0}^{N_1} z_n(k, t) P_n^0(\cos \theta) \quad (86)$$

in which the complex-valued  $Z$  is defined as in Cambon and Jacquin,<sup>8</sup> using the symmetry axis  $\mathbf{n}$  to define the helical modes. In Eq. (86),  $z_n$  is real for even  $n$  and purely imaginary for odd  $n$  due to the Hermitian symmetry property  $Z(-\mathbf{k}) = Z^*(\mathbf{k})$ . If axial symmetry is understood to include invariance under reflections through planes containing the polar vector  $\mathbf{n}$ , then  $Z$  is real, and only terms of even order appear

in the expansion of  $Z$ . Restricting to even orders with  $N_1 = 2N_0$ , Eqs. (85) and (86) can be recovered from Sreenivasan and Narasimha<sup>31</sup> and from Cambon and Teissèdre.<sup>30</sup>

Note that the factor  $\sin^2 \theta$  is essential in the expansion of  $Z$  in Eq. (86) because polarization anisotropy must vanish when the wave vector is parallel to the axial vector  $\mathbf{n}$ : reference to Eq. (16) shows that  $\mathbb{R}^{\text{pol}}$  can only be axisymmetric if  $b=d=0$ . The expansion of  $Z$  in Eq. (86), therefore, is somewhat special. Assuming<sup>2</sup> a general spherical harmonics expansion for  $Z$  is not consistent with this property. At the lowest order ( $N_0=1, N_1=0$ ),  $H_{ij}^{(e)} \hat{k}_i \hat{k}_j = H_{ij}^{(z)} n_i n_j P_2^0$  with  $P_2^0 = \frac{1}{2}(3 \cos^2 \theta - 1)$ , and  $Z = \frac{3}{4} H_n^{(z)} \sin^2 \theta$ .

Since the spherical harmonic decomposition required for the expansion of directionality to higher order is entirely standard, we consider the expansion of polarization to higher order. We will construct the axisymmetric tensor spherical harmonics of the first order beyond  $\mathbb{H}^{(z)}$ , without, however, presenting derivations. We noted at the end of Sec. VII B the existence of TSH belonging to the irreducible representations of dimensions 5, 7, and 9. We find exactly one axisymmetric TSH belonging to each of these representations. That corresponding to the irreducible representation of dimension 5 is generated by the (unique) axisymmetric quadratic polynomial  $2k_z^2 - k_x^2 - k_y^2$  by the differentiation process described in Sec. VII B. The corresponding TSH is

$$\mathbf{A}_5 = \begin{bmatrix} -2k_x^2 + 4k_y^2 - 2k_z^2 & -6k_x k_y & 3k_x k_z \\ -6k_x k_y & 4k_x^2 - 2k_y^2 - 2k_z^2 & 3k_y k_z \\ 3k_x k_z & 3k_y k_z & -2k_x^2 - 2k_y^2 + 4k_z^2 \end{bmatrix} \quad (87)$$

so that

$$\mathbf{A}_5 : \mathbf{P} = 2k^{-2} [(k_x^2 + k_y^2) - 2k_z^2]. \quad (88)$$

Recalling the relation equation (25),

$$Z = k^{-4} [6(k_x^2 + k_y^2)^2 + 6(k_x^2 + k_y^2)k_z^2] = 6 \sin^2 \theta \quad (89)$$

in agreement with (86) for  $n=0$ .

The TSH belonging to the irreducible representation of dimension 7 prove to change sign under inversion through planes containing the polar axis. We simply note the (again unique) axisymmetric TSH,

$$\mathbf{A}_7 = \begin{bmatrix} 2k_x k_y & -k_x^2 + k_y^2 & -4k_y k_z \\ -k_x^2 + k_y^2 & -2k_x k_y & 4k_x k_z \\ -4k_y k_z & 4k_x k_z & 0 \end{bmatrix} \quad (90)$$

for which  $e=0$  and  $Z = 10ik^{-3} k_z (k_x^2 + k_y^2) = 10i \sin^2 \theta \cos \theta$ , in agreement with (86) for  $n=1$ . The inversion antisymmetry implies that  $Z$  is purely imaginary.

The axisymmetric TSH belonging to the irreducible representation of dimension 9 is

$$\mathbf{A}_9 = \begin{bmatrix} -4k_z^2 + 3k_x^2 + k_y^2 & 2k_x k_y & -8k_x k_z \\ 2k_x k_y & -4k_z^2 + 3k_y^2 + k_x^2 & -8k_y k_z \\ -8k_x k_z & -8k_y k_z & 8k_z^2 - 4(k_x^2 + k_y^2) \end{bmatrix} \quad (91)$$

so that

$$\mathbf{A}_9 \cdot \mathbf{P} = k^{-4} [24k_z^2(k_x^2 + k_y^2) - 3(k_x^2 + k_y^2)^2 - 8k_z^4] \quad (92)$$

and

$$\begin{aligned} Z &= k^{-4} [30k_z^2(k_x^2 + k_y^2) - 5(k_x^2 + k_y^2)^2] \\ &= 5 \sin^2 \theta (7 \cos^2 \theta - 1) \end{aligned} \quad (93)$$

in agreement with (86) provided terms with both  $n=0$  and  $n=2$  are included.

### VIII. CONCLUSIONS AND PERSPECTIVES

We summarize the main points of this paper as follows:

- (1) The description of weak anisotropy by tensor functions of wave number  $k$  alone requires *two* tensors as in Eq. (41). These tensors represent the *distinct* effects of directional anisotropy and polarization anisotropy.<sup>8-10</sup> Since a weakly anisotropic spectral model based on a single tensor  $\mathbf{H}$  imposes some fixed proportionality  $\mathbf{H}^{(e)} = \lambda \mathbf{H}^{(z)}$  as shown by Eq. (48), such models cannot be consistent with *both* the short- and the long-time results equations (53) and (59). The reduction to a single tensor function of  $k$  compromises the kinematics by introducing implicit assumptions about anisotropy, usually that directional anisotropy vanishes. We emphasize in particular that the correlation tensor cannot be uniquely and self-consistently reconstructed in terms of the Reynolds stresses alone.
- (2) Some special cases of anisotropy were considered, namely short-time response of turbulence to arbitrary strain and the long-time nonlinear response of turbulence to small strain. Polarization and directional anisotropy are related differently in each limit; consequently, a model based on a single spherical average cannot be consistent with both limits. Turbulence under rapid rotation leads to a conclusion opposite to that usually adopted in models, namely dominant directional anisotropy and vanishing polarization.
- (3) The description of anisotropy by two tensor functions  $\mathbf{H}^{(e,z)}(k, t)$  was exhibited as the lowest order in an infinite expansion in scalar and tensor spherical harmonics generated by the  $\text{SO}^3$  decomposition. In this case, polarization and directional anisotropy are both described by a traceless second-rank tensor, but the description becomes more complex at higher order.

To conclude, we note some important open issues for anisotropic turbulence. The first is the question of connecting the higher-order coefficients in SSH and TSH expansions to spherically averaged higher-order moments of the spectral tensor. A related question is whether it is possible to model the higher-order terms by tensor products of  $H^{(e)}$ ,  $H^{(z)}$ , and the mean velocity gradient  $\partial \bar{U}_i / \partial x_j$  if a mean flow is present. Recent studies in rotating<sup>13</sup> and in stably stratified turbulence<sup>14</sup> have confirmed that the anisotropy identified by the angle distribution of  $\mathcal{E}(\mathbf{k})$  can be very large, and is only reflected by small or moderate values of  $H_{ij}^e$ . In such cases, the SSH expansion of  $\mathbf{R}$  would need to be carried out to extremely high order. This expansion is unlikely to be practical; another approach to strong anisotropy using high angu-

lar resolution of the energy spectrum in the axisymmetric case has been initiated by Bellet *et al.*<sup>13</sup> and Liechtenstein *et al.*<sup>14</sup>

### APPENDIX: HELICAL ANISOTROPIC TURBULENCE

Helicity has been assumed to vanish in the present developments, so that the statistics have been assumed to be invariant under the inversion  $\mathbf{k} \rightarrow -\mathbf{k}$ . This assumption has been made largely for simplicity, but it is not difficult to extend the formalism of anisotropic kinematics without it. We briefly describe the results.

The symmetry property  $\tilde{R}_{ij}(\mathbf{x}, \mathbf{x}'; t) = \tilde{R}_{ji}(\mathbf{x}', \mathbf{x}; t)$  is a trivial consequence of the definition equation (1). It implies that in homogeneous turbulence, the spectral tensor satisfies the Hermitian symmetry condition

$$R_{ij}(\mathbf{k}) = R_{ji}(\mathbf{k})^* = R_{ji}(-\mathbf{k}). \quad (\text{A1})$$

Separating  $\mathbf{R}$  into its real and imaginary parts and applying Eq. (A1),

$$\begin{aligned} \Re\{R_{ij}(\mathbf{k})\} + i\Im\{R_{ij}(\mathbf{k})\} &= \Re\{R_{ji}(\mathbf{k})\} - i\Im\{R_{ji}(\mathbf{k})\} \\ &= \Re\{R_{ji}(-\mathbf{k})\} + i\Im\{R_{ji}(-\mathbf{k})\} \end{aligned} \quad (\text{A2})$$

thus the real part of  $\mathbf{R}$  is symmetric and invariant under inversion and its imaginary part is antisymmetric and changes sign under inversion.

The decomposition of  $\mathbf{R}$  in Eq. (9) explicitly ignored the possibility of an antisymmetric contribution, on the basis that  $\mathbf{R}$  was assumed real and symmetric. Without that restriction, we should write

$$\mathbf{R} = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} d & b^* & 0 \\ b & -d & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\text{A3})$$

where  $b = b' + ib''$  is complex. Alternatively, in terms of the matrices  $\mathbf{M}_1$  and  $\mathbf{M}_2$  of Eq. (12) and the basic antisymmetric matrix

$$\mathbf{A} = i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\text{A4})$$

we have

$$\mathbf{R} = e\mathbf{P} + d\mathbf{M}_1 + b'\mathbf{M}_2 + b''\mathbf{A} = e\mathbf{P} + \Re(Z\mathbf{M}) + \mathcal{H}\mathbf{A}, \quad (\text{A5})$$

where now  $Z = d + ib'$  and  $\mathcal{H} = b''$ . Since there can only be one linearly independent antisymmetric matrix with the property  $A_{ij}k_i = A_{ij}k_j = 0$ , we must have  $A_{ij} = \epsilon_{ijm} \hat{k}_m$ , where  $\epsilon_{ijm}$  is the alternating third-order (Ricci) tensor. Thus,

$$\Im\{R_{ij}(\mathbf{k})\} = \mathcal{H}(\mathbf{k}) \epsilon_{ijm} \hat{k}_m. \quad (\text{A6})$$

As before, for the application of the formalism of the helical mode decomposition to this problem, we refer to Cambon and Jacquin<sup>8</sup> and Cambon *et al.*<sup>10</sup>

Helicity is defined by

$$H = \frac{1}{2} \langle u_i \omega_i \rangle = -\frac{1}{2} i \epsilon_{imn} \int k_m R_{ni}(\mathbf{k}) d^3 \mathbf{k}. \quad (\text{A7})$$

It is evident that only the antisymmetric part of  $\mathbf{R}$  can contribute, so that

$$\begin{aligned} H &= \frac{1}{2} \epsilon_{imn} \int k_m \mathcal{H}(\mathbf{k}) \epsilon_{nip} \hat{k}_p d^3 \mathbf{k} \\ &= \frac{1}{2} (\delta_{ii} \delta_{mp} - \delta_{im} \delta_{ip}) \int k^{-1} k_m k_p \mathcal{H}(\mathbf{k}) d^3 \mathbf{k} \\ &= \int k \mathcal{H}(\mathbf{k}) d^3 \mathbf{k}. \end{aligned}$$

Concerning the analysis of helical anisotropic turbulence, we note that helicity is characterized by the scalar  $\mathcal{H}(\mathbf{k})$ , for which an expansion in even-order scalar spherical harmonics is possible, because  $\mathcal{H}(-\mathbf{k}) = \mathcal{H}(\mathbf{k})$ . Nevertheless, Eq. (A6) shows that the spherical average of  $\mathcal{J}\{\mathbf{R}\}$  vanishes; therefore, helicity makes no contribution to either  $\mathbf{H}^{(e)}$  or  $\mathbf{H}^{(c)}$ . The analysis of anisotropic helicity would therefore begin with consideration of the weighted averages  $\int k_m \mathcal{J}\{R_{ij}(\mathbf{k})\} d^3 \mathbf{k}$ , which are nonzero because of the odd parity of  $\mathcal{J}\{R_{ij}(\mathbf{k})\}$  under inversions.

<sup>1</sup>G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, 1953).

<sup>2</sup>J. R. Herring, "Approach of axisymmetric turbulence to isotropy," *Phys. Fluids* **17**, 859 (1974).

<sup>3</sup>Here and throughout, index and index-free notation will both be used as convenient:  $\mathbf{R}$  and  $R_{ij}$  denote the same tensor, and  $\mathbf{k}$  and  $k_i$  the same vector.

<sup>4</sup>C. Cambon, D. Jeandel, and J. Mathieu, "Spectral modelling of homogeneous non isotropic turbulence," *J. Fluid Mech.* **104**, 247 (1981).

<sup>5</sup>T-H. Shih, W. C. Reynolds, and N. N. Mansour, "A spectrum model for weakly anisotropic turbulence," *Phys. Fluids A* **2**, 1500 (1990).

<sup>6</sup>T. Ishihara, K. Yoshida, and Y. Kaneda, "Anisotropic velocity correlation spectrum at small scales in a homogeneous turbulent shear flow," *Phys. Rev. Lett.* **88**, 154501 (2002).

<sup>7</sup>K. Yoshida, T. Ishihara, and Y. Kaneda, "Anisotropic spectrum of homogeneous turbulent shear flow in a Lagrangian renormalized approximation," *Phys. Fluids* **15**, 2385 (2003).

<sup>8</sup>C. Cambon and L. Jacquin, "Spectral approach to non-isotropic turbulence subjected to rotation," *J. Fluid Mech.* **202**, 295 (1989).

<sup>9</sup>C. Cambon, L. Jacquin, and J-L. Lubrano, "Towards a new Reynolds

stress model for rotating turbulent flows," *Phys. Fluids A* **4**, 812 (1992).

<sup>10</sup>C. Cambon, N. N. Mansour, and F. S. Godeferd, "Energy transfer in rotating turbulence," *J. Fluid Mech.* **337**, 303 (1997).

<sup>11</sup>I. Arad, V. S. L'vov, and I. Procaccia, "Correlation functions in isotropic and anisotropic turbulence: The role of the symmetry group," *Phys. Rev. E* **59**, 6753 (1999).

<sup>12</sup>S. Kurien and K. R. Sreenivasan, "Anisotropic scaling contributions to high-order structure functions in high-Reynolds-number turbulence," *Phys. Rev. E* **62**, 2206 (2000).

<sup>13</sup>F. Bellet, F. S. Godeferd, J. F. Scott, and C. Cambon, "Wave turbulence in rapidly rotating flows," *J. Fluid Mech.* **562**, 83 (2006).

<sup>14</sup>L. Liechtenstein, F. S. Godeferd, and C. Cambon, "Nonlinear formation of structures in rotating stratified turbulence," *J. Turbul.* **6**, 1 (2005).

<sup>15</sup>J. C. H. Fung, J. C. R. Hunt, N. A. Malik, and R. J. Perkins, "Kinematic simulation of homogeneous turbulence by unsteady random Fourier modes," *J. Fluid Mech.* **236**, 281 (1992).

<sup>16</sup>C. Cambon, F. S. Godeferd, F. Nicolleau, and J. C. Vassilicos, "Turbulent diffusion in rapidly rotating turbulence with and without stable stratification," *J. Fluid Mech.* **499**, 231 (2004).

<sup>17</sup>F. Waleffe, "Inertial transfer in the helical decomposition," *Phys. Fluids A* **5**, 677 (1993).

<sup>18</sup>D. Besnard, F. Harlow, F. Rauen Zahn, and C. Zemach, "Spectral transport model for turbulence," *Theor. Comput. Fluid Dyn.* **8**, 1 (1996).

<sup>19</sup>C. Cambon and J. F. Scott, "Linear and nonlinear models of anisotropic turbulence," *Annu. Rev. Fluid Mech.* **31**, 1 (1999).

<sup>20</sup>J.-P. Bertoglio, L. Shao, and S. Parpais, "A simplified spectral model for inhomogeneous and nonequilibrium turbulence," *AIAA Paper 94-2385* (1994).

<sup>21</sup>Our definition of  $U$  is somewhat nonstandard; it is customary to define it so that  $E = 2\pi k^2 U$ .

<sup>22</sup>D. C. Leslie, *Developments in the Theory of Turbulence* (Oxford University Press, Oxford, 1972).

<sup>23</sup>A. Yoshizawa, *Hydrodynamic and Magnetohydrodynamic Turbulent Flows* (Kluwer Academic, Dordrecht, 1998).

<sup>24</sup>R. Rubinstein and J. M. Barton, "Nonlinear Reynolds stress models and the renormalization group," *Phys. Fluids A* **2**, 1472 (1990).

<sup>25</sup>J. L. Lumley, "Similarity and the turbulent energy spectrum," *Phys. Fluids* **10**, 855 (1967).

<sup>26</sup>S. Kassinos, W. C. Reynolds, and M. Rogers, "One-point turbulence structure tensors," *J. Fluid Mech.* **428**, 213 (2000).

<sup>27</sup>A. Salhi and C. Cambon, "Advances in rapid distortion theory: From rotating shear flows to the baroclinic instability," *J. Appl. Mech.* **73**, 449 (2006).

<sup>28</sup>B. E. Launder, G. J. Reece, and W. Rodi, "Progress in the development of a Reynolds stress turbulence closure," *J. Fluid Mech.* **68**, 537 (1975).

<sup>29</sup>H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, NJ, 1946).

<sup>30</sup>C. Cambon and C. Teissède, "Application des harmoniques sphériques à la représentation et au calcul des grandeurs cinématiques en turbulence homogène anisotrope," *C. R. Acad. Sci. Paris*, **301**, Série II (2), 65 (1985).

<sup>31</sup>K. R. Sreenivasan and R. Narasimha, "Rapid distortion theory of axisymmetric turbulence," *J. Fluid Mech.* **84**, 497 (1978).