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REFINED CONVERGENCE FOR THE BOOLEAN MODEL

PIERRE CALKA,* *MAP 5, Université Paris Descartes*

JULIEN MICHEL,** *UMPA, ENS Lyon*

KATY PAROUX,** *Laboratoire de Mathématiques, Université de Franche Comté and
INRIA Rennes - Bretagne Atlantique*

Abstract

In a previous work [10] two of the authors proposed a new proof of a well known convergence result for the scaled elementary connected vacant component in the high intensity Boolean model towards the Crofton cell of the Poisson hyperplane process (see e.g. [4]). In this paper, we consider the particular case of the two-dimensional Boolean model where the grains are discs with random radii. We investigate the second-order term in this convergence when the Boolean model and the Poisson line process are coupled on the same probability space. A precise coupling between the Boolean model and the Poisson line process is first established, a result of directional convergence in distribution for the difference of the two sets involved is derived as well.

Keywords: Poisson point process; Crofton cell; Convergence; Stochastic geometry

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* Postal address: MAP5, U.F.R. de Mathématiques et Informatique Université Paris Descartes 45, rue des Saints-Pères 75270 Paris Cedex 06 France, research partially supported by the French ANR project "mipomodim" No. ANR-05-BLAN-0017.

** Postal address: Unité de Mathématiques Pures et Appliquées UMR 5669, ENS Lyon, 46 allée d'Italie, F-69364 Lyon Cedex 07, France.

*** Postal address: Laboratoire de Mathématiques de Besançon, UMR 6623, F-25030 Besançon Cedex, France, and INRIA, centre de Rennes - Bretagne Atlantique Campus de Beaulieu, F-35 042 Rennes Cedex, France.

1. Introduction and notations

Since the first result of P. Hall [4, 5] and its generalizations in [13, 15], the scaled vacancy of the Boolean model is known to converge in some sense to its counterpart in the Poisson hyperplane process. In a previous paper [10] two of the authors gave another proof of this convergence result for the local occupations laws of a Boolean shell model in terms of Hausdorff distance. This convergence appears as a first order result, expressed in terms of weak convergence. We aim at giving here a second order weak convergence for the difference as well as a strong convergence result at this first order.

For sake of simplicity we shall work in the plane \mathbb{R}^2 : let us consider a Boolean model based on a Poisson point process \mathbf{X}_λ with intensity measure $\lambda^2 dx$ and generic shape an open disc centred at 0 of random radius R such that $E[R] = 1$ and $E[R^2] < +\infty$. The law of R will be denoted by μ , and we will assume that there exists $R_\star > 0$ such that $\mu(R_\star, +\infty) = 1$.

The occupied phase of the Boolean model is denoted by

$$\mathcal{O}_\lambda = \bigcup_{x \in \mathbf{X}_\lambda} B(x, R_x),$$

where $B(x, r)$ denotes the disc centred at x and of radius r and where the radii R_x , $x \in \mathbf{X}_\lambda$ are independent and identically distributed, independent of \mathbf{X}_λ [14]. This process is supposed to leave the point 0 uncovered, which occurs with positive probability

$$P(0 \notin \mathcal{O}_\lambda) = \exp(-\pi\lambda^2 E[R^2]).$$

From now on the Boolean model shall be conditioned by this event. Let D_0^λ denote the (closed) connected component of $\mathbb{R}^2 \setminus \mathcal{O}_\lambda$ containing 0. The following asymptotic result for this process (see [4, 13, 10]) is a consequence of Steiner's formula [18]:

Theorem 1. *Let D^λ be the following compact set:*

- $D^\lambda = \lambda^2 D_0^\lambda$ whenever this set is bounded,
- $D^\lambda = K_0$ a given fixed compact set otherwise.

When λ tends to infinity, D^λ converges in law towards the Crofton cell \mathcal{C} of a Poisson line process with intensity measure $d\rho d\theta$.

The Poisson line process with intensity measure $d\rho d\theta$ in \mathbb{R}^2 is defined as the set of (random) lines $D_{\tau,\phi}$ with polar equation $\rho \cos(\theta - \phi) = \tau$, where (τ, ϕ) are the points of a Poisson point process Φ with intensity measure $d\rho d\theta$. The Crofton cell \mathcal{C} is defined as the polygon formed by those lines containing the point 0 (see [19] or [15] for a survey on Poisson line tessellations). Numerous distributional results on this model have been obtained notably by R. E. Miles ([11, 12]) and G. Matheron [8]. More recently, central limit theorems have been derived in [15, 16] for the two-dimensional case and in [6] for the general case. Besides, D. G. Kendall's conjecture on the shape of the Crofton cell when it is large has been proved in [7]. Additional distributional and asymptotic results at large inradius have also been obtained in [2] and [1].

It is a natural problem to try to estimate the error in the previous convergence result: one possible answer is to give a geometric description of the difference of those two sets. This description requires to couple the Boolean model with the Poisson line process. This coupling, which asserts as a consequence the almost sure convergence in Theorem 1, will be described in section 2, and its application to the second order convergence will be treated in section 3. Sections 4 and 5 contain the main results, the likes of which might be summarised in the following way: in the coupled models, both the empty region of the Boolean model (at least with high probability for λ large enough), and the Crofton cell of the Poisson line process are star-shaped domains with respect to the origin. Consequently, we introduce for each direction θ the difference $d_\lambda(\theta)$ between the respective lengths of the segments at angle θ that lie in those two regions starting from the point 0. Our results state that for θ fixed, $\lambda^2 d_\lambda(\theta)$ converges almost surely towards an explicit random variable. A more complete result states the convergence in law of the process $\theta \mapsto \lambda^2 d_\lambda(\theta)$.

All the proofs of results stated in sections 2 to 5 are given in an appendix.

2. Couplings

Coupling the Boolean model with the Poisson line process is an easy task: indeed as λ tends to infinity the rescaled Boolean model *looks like* the Poisson line process as one can see from theorem 1 in [10]. The formal way to state this as a coupling result is the

following proposition: let A denote the subset of triples $(\rho, \theta, R) \in \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}_+$ such that $R < \rho$, then, recalling that we assume $E[R] = 1$,

Proposition 1. *Let \mathbf{Y}_λ be a Poisson point process with intensity measure $\mathcal{N}_\lambda = \lambda^2 \rho d\rho d\theta d\mu$ on $A \subset \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}_+$. We will denote by $H_{\lambda, \rho, \theta, R}$ the line with polar equation $r \cos(\phi - \theta) = \lambda^2(\rho - R) + \frac{1}{2R} \lambda^2(\rho - R)^2$ (i.e. the line $D_{(\lambda^2(\rho - R) + \frac{1}{2R} \lambda^2(\rho - R)^2), \theta}$ as defined in the introduction). Then the following sets:*

- $\mathcal{O}_\lambda = \bigcup_{(\rho, \theta, R) \in \mathbf{Y}_\lambda} B((\rho \cos \theta, \rho \sin \theta), R)$,
- and $\{H_{\lambda, \rho, \theta, R}, (\rho, \theta, R) \in \mathbf{Y}_\lambda\}$

are respectively the Boolean model with intensity λ^2 and law of the radius μ conditioned on $0 \notin \mathcal{O}_\lambda$, and the Poisson line process with intensity measure $d\rho d\theta$.

This coupling will be denoted by *coupling $B \rightarrow L$* (balls to lines).

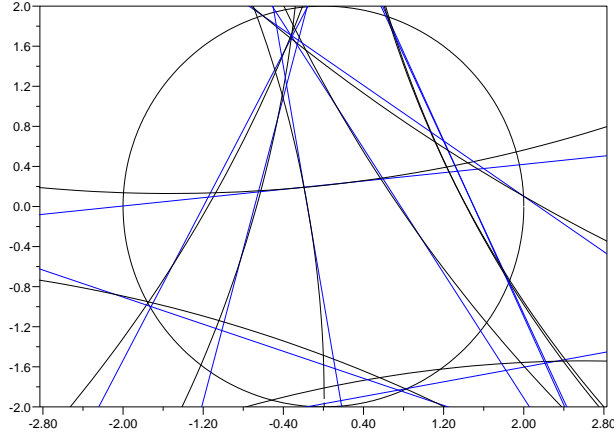


FIGURE 1: Coupling the Boolean model with the line process (the simulation is exact within the circle).

Remark 1. The choice of this coupling is not unique, if we replace the set of lines

$$\left\{ D_{(\lambda^2(\rho - R) + \frac{1}{2R} \lambda^2(\rho - R)^2), \theta}, (\rho, \theta, R) \in \mathbf{Y}_\lambda \right\}$$

with

$$\left\{ D_{(\lambda^2(\rho - R) + \frac{1}{2} \lambda^2(\rho - R)^2), \theta}, (\rho, \theta, R) \in \mathbf{Y}_\lambda \right\}$$

we obtain the same law for the line process. However, our choice relies on the fact that given $R = r$, the mapping $\rho \mapsto \lambda^2(\rho - r) + \frac{1}{2r} \lambda^2(\rho - r)^2$ transforms the initial measure $\lambda^2 \rho d\rho$ into $d\rho$.

From this coupling one states easily that if we denote by $C_0^\lambda(\omega)$ the Crofton cell of this line process and by $D_0^\lambda(\omega)$ the empty connected component containing 0 in the coupled Boolean model, one has:

Proposition 2. *For all $M > 0$ and $\lambda \geq \lambda_0(M) = \sqrt{\frac{M}{R_*}} > 0$ one has*

$$d_H^M(\lambda^2 D_0^\lambda, C_0^\lambda) \leq \frac{M^2}{R_* \lambda^2} \text{ a.s.},$$

where d_H^M denotes the Hausdorff distance in $B(0, M)$ defined by

$$d_H^M(F, G) = \inf \left\{ \alpha > 0 : \begin{aligned} (F \oplus B(0, \alpha)) \cap B(0, M) \supset G \cap B(0, M), \\ (G \oplus B(0, \alpha)) \cap B(0, M) \supset F \cap B(0, M) \end{aligned} \right\},$$

and the set $A \oplus B$ denotes the Minkowski sum of the sets A and B : $A \oplus B = \{x + y, (x, y) \in A \times B\}$.

As can be easily seen from the formula defining the coupling, some discs of the rescaled Boolean model may intersect the disc $B(0, M)$ whereas the associated line does not: the next lemma enables us to assert that the most pertinent points of the point processes that we are dealing with are those such that *at the same time* the disc and the associated line intersect *simultaneously* the given disc of radius $M > 0$:

Lemma 1. *Let $M > 0$, the number of discs of the Boolean model intersecting the disc $B(0, M)$ and such that their associated lines do not intersect this same disc is a Poisson random variable with parameter $\pi M^2 \lambda^{-2}$.*

One may also give the coupling in the other direction, which will be called *coupling $L \rightarrow B$* , as this will prove more easily tractable in the ensuing computations,

Proposition 3. *Let \mathbf{Z} be a Poisson point process with intensity measure $d\rho d\theta d\mu$ on $\mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}_+$, let*

$$\psi_\lambda(\rho, R) = R \sqrt{1 + \frac{2\rho}{\lambda^2 R}},$$

then the following sets

- $\{D_{\rho,\theta}, (\rho, \theta, R) \in \mathbf{Z}\}$, and
- $\mathcal{O}_\lambda = \bigcup_{(\rho,\theta,R) \in \mathbf{Z}} B((\psi_\lambda(\rho, R) \cos \theta, \psi_\lambda(\rho, R) \sin \theta), R)$,

are respectively the Poisson line process with intensity measure $d\rho d\theta$ and the Boolean model with intensity λ^2 and law of the radius μ conditioned on $0 \notin \mathcal{O}_\lambda$.

From now on we shall use this coupling, thus the set D^λ will refer to the rescaled connected component in this Boolean model (when bounded, K otherwise), and \mathcal{C} will be the Crofton cell in this line process.

3. Convergence in law of the difference set

As announced we shall work here with the points of the process \mathbf{Z} . Let us denote by $\theta \mapsto d_\lambda(\theta)$ the defect function:

$$d_\lambda(\theta) = \sup \{r > 0 : (r, \theta) \in D^\lambda\} - \sup \{r > 0 : (r, \theta) \in \mathcal{C}\}.$$

Then we will prove:

Theorem 2. *For all $\theta \in [0, 2\pi)$ one has*

$$\lambda^2 d_\lambda(\theta) \xrightarrow[\lambda \rightarrow +\infty]{\text{law}} Z,$$

where Z is the random variable

$$Z = -\frac{L^2}{2} \frac{\cos 2\Theta}{R \cos \Theta},$$

and the common law of (L, Θ, R) is given by

$$d(L, \Theta, R)(P)(\ell, \alpha, r) = \pi \exp(-2\pi\ell) \cos \alpha \mathbf{1}_{\alpha \in (-\pi/2, \pi/2)} d\ell d\alpha d\mu(r).$$

We remark in this theorem that the limit law does not depend on θ as the process is invariant under rotations. Thus, for sake of simplicity we will take $\theta = 0$, and set Δ_0 the half-line with polar equation $\theta = 0$. The proof, that shall be given in the appendix, proceeds in two steps:

1. The two points of the process \mathbf{Z} realising the suprema in the definition of $d_\lambda(0)$ are distinct with small probability, indeed if we denote by C_λ the event that this is the case (such a case is called a *crossing*), we obtain an upper bound for its probability,

Lemma 2. *Let η be a positive number, then there exists constants B_η and $\lambda_1(\eta)$ such that the probability that two different points of \mathbf{Z} define the defect function is for $\lambda \geq \lambda_1(\eta)$ at most*

$$P(C_\lambda) \leq \frac{B_\eta}{\lambda^{2/3-\eta}},$$

where the constants B_η and $\lambda_1(\eta)$ depend only on η and R_* .

2. Assuming that there is no crossing, the computation of the defect function is reduced to the computation of the difference between the intersection of a line and its associated circle with a fixed half line (here Δ_0), this is given by:

Lemma 3. (a) *Let (Υ, Θ, R) be the point of \mathbf{Z} yielding the first intersecting line with Δ_0 at a distance denoted by $L = \Upsilon / \cos \Theta$, the law of (L, Θ) is given by*

$$d(L, \Theta)(P)(\ell, \alpha) = e^{-2\ell} \cos \alpha \mathbf{1}_{\alpha \in (-\pi/2, \pi/2)} d\ell d\alpha,$$

(b) *if $\lambda \geq \sqrt{(2L \sin^2 \Theta) / (R \cos \Theta)}$ one has*

$$\begin{aligned} \bar{d}_\lambda(0) &= d(0, B(\psi_\lambda(\Upsilon, \Theta, R), R) \cap \Delta_0) - d(0, D_{\Upsilon, \Theta} \cap \Delta_0), \\ &= \lambda^2 R \cos \Theta \sqrt{1 + \frac{2L \cos \Theta}{\lambda^2 R}} - \lambda^2 R \sqrt{1 - \sin^2 \Theta \left(1 + \frac{2L \cos \Theta}{\lambda^2 R}\right)} - L. \end{aligned}$$

From those results one gets the desired asymptotics with a few steps of direct computation.

As a corollary one has

Corollary 1. *The limit expected defect is an integrable random variable with*

$$\lim_{\lambda \rightarrow +\infty} E[\lambda^2 d_\lambda(0)] = 0.$$

However, this limit expected defect is not square-integrable:

$$\lim_{\lambda \rightarrow +\infty} E\left[(\lambda^2 d_\lambda(0))^2\right] = +\infty.$$

Remark 2. In section 1, we gave a second coupling of the Boolean model with the Poisson line process (see Remark 1). In that case, when λ tends to infinity, we obtain

a different limiting distribution:

the quantity $\lambda^2 d_\lambda(\theta)$ converges in distribution to \tilde{Z} defined by

$$\tilde{Z} = \frac{L^2}{2} \left(\frac{\sin^2 \Theta}{R \cos \Theta} - \cos \Theta \right),$$

where the common law of (L, Θ, R) is the same.

4. Convergence of the defect process: finite dimensional distributions

We have so far studied the limit of the defect function at a fixed angle; as this angle varies we obtain for each λ and realisation of the coupling a trajectory of a random process having some remarkable features: indeed it seems from figure 2 that this process is continuous at fixed λ , with rapid variations in regions that are getting smaller as λ increases. The aim of the last two sections is to explore this phenomenon, eventually described by the following theorem whose proof is developed in the appendix:

Theorem 3. *The process $\theta \mapsto \lambda^2 d_\lambda(\theta)$ can be decomposed as*

$$\forall \theta \in [0, 2\pi), \quad \lambda^2 d_\lambda(\theta) = Y_\lambda^c(\theta) + \mu_\lambda^V([0, \theta]),$$

where $\theta \mapsto Y_\lambda^c(\theta)$ is a continuous process, and μ_λ^V is a random signed measure on $[0, 2\pi)$ such that $\mu_\lambda^V(\{0\}) = 0$ almost surely, and

$$Y_\lambda^c \xrightarrow[\lambda \rightarrow +\infty]{\text{weakly}} Y_\infty \quad \text{and} \quad \mu_\lambda^V \xrightarrow[\lambda \rightarrow +\infty]{\text{narrowly}} \mu^V \text{ a.s.},$$

where the common law of Y_∞ and μ^V may be expressed with the law of \mathcal{C} .

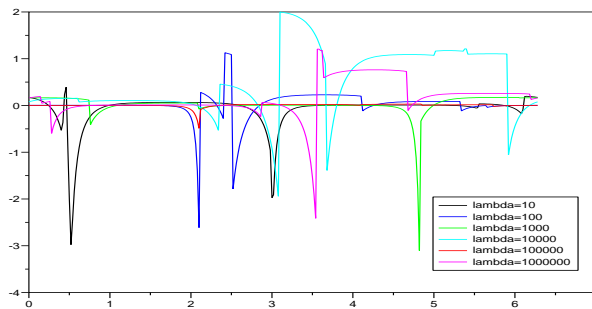


FIGURE 2: Some realisations of the process $\theta \mapsto \lambda^2 d_\lambda(\theta)$

In order to give a proof of this theorem, we shall first study the convergence of the finite dimensional distributions of the process $(\lambda^2 d_\lambda(\theta))_{\theta \in [0, 2\pi]}$: as we will explain below, this convergence does occur, but the limit law remains obscure. We begin by stating the result for two angles: 0 and $\theta \in (0, \pi)$,

Theorem 4. *For all $\theta \in (0, \pi)$, $\lambda^2(d_\lambda(0), d_\lambda(\theta))$ converges in law as λ goes to infinity towards an explicit law.*

A complete statement of this result, as well as a sketch of its proof, is given in the appendix.

Knowing the joint law of those couples gives some knowledge on this process, by simulation we can obtain the covariogram $\theta \mapsto \text{cov}(\lambda^2 d_\lambda(0), \lambda^2 d_\lambda(\theta))$, $\theta \in [0, \pi]$, in figure 3. One clearly observes the divergence as λ tends to infinity of the covariance for $\theta = 0$, as announced in corollary 1.

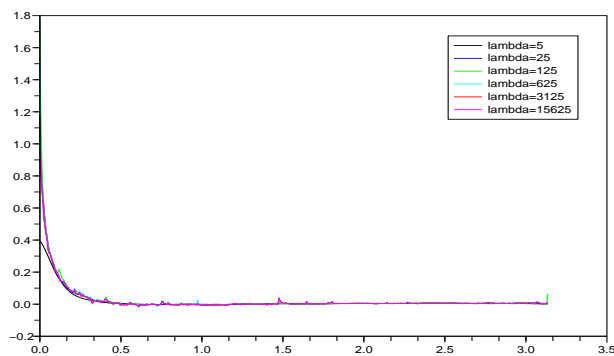


FIGURE 3: Covariograms, sample of size 250000

There is an explicit formula for the limit covariance using the explicit form of the limit law of theorem 4 which is given in the appendix.

For the general finite dimensional distributions one obtains:

Theorem 5. *For all $0 \leq \theta_1 < \dots < \theta_n < 2\pi$, the finite dimensional random vector $\lambda^2(d_\lambda(\theta_1), \dots, d_\lambda(\theta_n))$ converges in law towards $(Z_{\theta_1}, \dots, Z_{\theta_n})$, where the law of this random vector may be explicitated.*

5. Tightness and uniform bounds

The results of the previous section concerned convergence for finite dimensional distributions of the defect process: in order to give results about the global convergence of this process, we need to prove some tightness criterion. In this section we shall study the tail probability for the supremum of the process, and the possibility of a tightness criterion.

Proposition 4. *One has the following estimate: for all $\beta \in (0, 1)$*

$$\lim_{s \rightarrow +\infty} \left\{ s^\beta \limsup_{\lambda \rightarrow +\infty} P \left(\sup_{\theta \in [0, 2\pi)} \lambda^2 |d_\lambda(\theta)| \geq s \right) \right\} = 0.$$

The proof of this result comes from some estimates on the growth of both \mathcal{C} and the empty component of the Boolean model around the origin. Let us recall that for λ large enough, this connected component is almost surely bounded [9], so that if one denotes by R_m the radius of the largest disc centred at 0 contained in \mathcal{C} , by $R_M(\lambda)$ the radius of the smallest disc centred at 0 containing this connected component, and by R_M the radius of the smallest disc centred at 0 containing \mathcal{C} , we have the following estimate:

$$P \left(\sup_{\theta \in [0, 2\pi)} \lambda^2 |d_\lambda(\theta)| \geq s \right) \leq P \left(\sup_{\theta \in [0, 2\pi)} \lambda^2 |d_\lambda(\theta)| \geq s, R_m \geq r, R_M \vee R_M(\lambda) \leq R \right) \\ + P(R_m < r) + P(R_M > R) + P(R_M(\lambda) > R),$$

where r and R will be chosen later such that:

- $P(R_m < r)$ tends to 0,
- $P(R_M > R)$ and $P(R_M(\lambda))$ tend to 0,
- and the first term will be treated directly, indeed from proposition 2 we may state that the Hausdorff distance between the Crofton cell and the empty component of the rescaled Boolean model in the ball $B(0, R)$ is lesser than $R^2/(\lambda^2 R_\star)$, as soon as $\lambda \geq \sqrt{R/R_\star}$. Hence, under this condition, both boundaries of the \mathcal{C} and D^λ are located in the set

$$B(0, R) \cap (\partial \mathcal{C} \oplus B(0, R^2/(\lambda^2 R_\star))),$$

furthermore the edges in $\partial\mathcal{C}$ intersect any half line Δ_θ in $B(0, R)$ at an angle with cosine at most r/R , so that we have

$$\lambda^2 |d_\lambda(\theta)| \leq \frac{R^3}{rR_\star}$$

for every $\theta \in [0, 2\pi)$.

Hence if r and R are chosen such that $R^3/(rR_\star) < s$ we have obtained that if $\lambda \geq \sqrt{R/R_\star}$ then

$$P\left(\sup_{\theta \in [0, 2\pi)} \lambda^2 |d_\lambda(\theta)| \geq s\right) \leq P(R_m < r) + P(R_M > R) + P(R_M(\lambda) > R).$$

In order to check the first two points and conclude the proof of proposition 4, let us recall that the law of R_m is exponential with mean $1/(2\pi)$, and that we know from [1] that the size of \mathcal{C} has a rapidly decreasing tail probability

Theorem 6. (Theorem 8 in [1].) *We have for all $r > 0$*

$$\begin{aligned} & 2\pi r e^{-2r} \left(\cos 1 + \frac{e^{-(2\pi \cos 1 - 1)r}}{2\pi r} \right) \leq P(R_M \geq r) \\ & \leq 2\pi r e^{-2r} \left(1 - (\pi - 2)r e^{-2r} + \frac{2}{3}(\pi - 3)^2 r^2 e^{-4r} + \frac{e^{-2(\pi-1)r}}{2\pi r} \right). \end{aligned}$$

Similarly, we have the following result for the Boolean model:

Theorem 7. *There exist two constants K and C depending only on R_\star such that for all $r \geq 1$ and $\lambda \geq 1$ we have:*

$$P(R_M(\lambda) > r) \leq K \exp(-Cr).$$

From all the results above we deduce the announced result of proposition 4 by choosing $r = s^{-\alpha}$, $\alpha \in (0, 1)$ and $R = s^\gamma$ with $3\gamma + \alpha < 1$. ■

The matter of convergence of the whole process, as may be illustrated from figure 4, will be dealt with the help of a decomposition of this limit process in a pure jump

part and a continuous part.

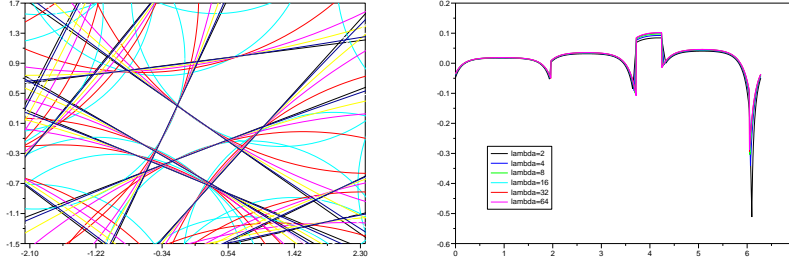


FIGURE 4: The Crofton cell and the related processes, one checks that there are four edges, and four singularities

We want to decompose the process $\theta \mapsto \lambda^2 d_\lambda(\theta)$ as the sum of a smooth process, $\theta \mapsto \lambda^2 d_\lambda^c(\theta)$, and another process $\theta \mapsto \lambda^2 d_\lambda^j(\theta)$ containing the jump phenomena illustrated on figure 4. The convergence of those processes will depend on a tightness criterion for the continuous process, and the precise study of the second one.

Let us define those two processes more precisely: for all $\omega \in \Omega \setminus N$ where $P(N) = 0$, the Crofton cell \mathcal{C} is bounded, and is a polygon with a finite number v of vertices, so that one can define $0 \leq \zeta_1(\omega) < \dots < \zeta_v(\omega) = \zeta_0(\omega) < 2\pi$ ($v \geq 3$) the arguments of those vertices. Let us define $\delta\zeta(\omega)$ as the maximum of the angular width of the zone depicted in figure 5: if we set $\widetilde{R}_M = \max(R_M, R_M(\lambda))$ (depending on ω), then the boundaries of the two cells, Crofton's and the empty connected component of the rescaled Boolean model, are entirely included in the disc $B(0, \widetilde{R}_M)$. From proposition 2 we know that the Hausdorff distance between those two boundaries in the disc $B(0, \widetilde{R}_M)$ is at most $\lambda^{-2} \widetilde{R}_M^2 / R_\star$. In figure 5 the yellow regions are this neighbourhood of the boundary of Crofton's cell containing the associated parts of circles. Parts of those circles may constitute the boundary of D^λ , but other ones might also contribute: this is taken care of thanks to the following procedure. Let us denote by $e_\lambda(\omega)$ the minimum distance between \mathcal{C} and the other lines of the process intersecting $B(0, \widetilde{R}_M)$, with probability one this quantity is positive, and bounded from above as λ tends to infinity, so that for λ large enough (depending on ω) we have

$$2\lambda^{-2} \widetilde{R}_M^2 / R_\star < e_\lambda(\omega),$$

and thus crossings may only occur in the lozenges described on figure 5, so that for $\min_{i \in \{1, \dots, v\}} |\theta - \zeta_i(\omega)| > \delta\zeta(\omega)$ and λ large enough (depending on ω), the defect function $d_\lambda(\theta)$ is equal to $\bar{d}_\lambda(\theta)$.

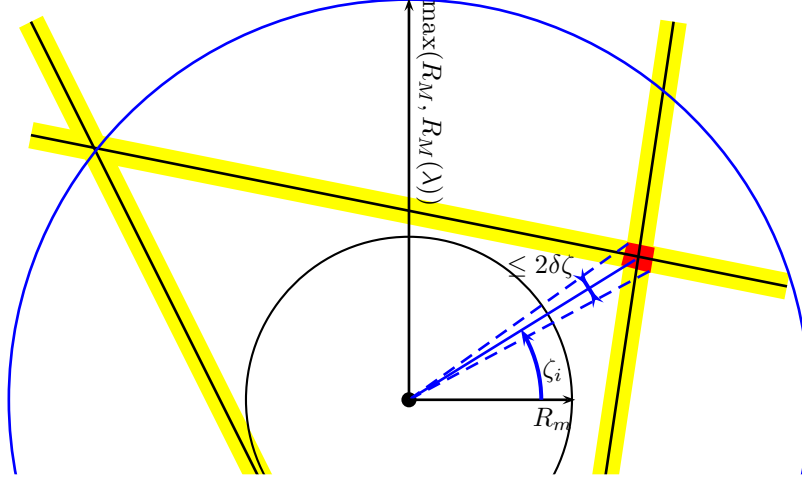


FIGURE 5: The lozenge around an edge where a crossing may occur

The exact computation of $\delta\zeta$ is achieved with a little amount of trigonometry, yielding the following estimate: let $\alpha = \min_{i \neq j} |\theta_i - \theta_j|$ where $\theta_1, \dots, \theta_N$ are the polar angles of the lines intersecting $B(0, R_M)$, then one has

$$\delta\zeta \leq 2 \operatorname{atan} \frac{\widetilde{R}_M}{\lambda^2 R_m \sin \alpha}.$$

Then we set

$$\begin{aligned} d_\lambda^c(\theta) &= \sum_{i=0}^{v-1} \mathbf{1}_{\theta \in (\zeta_i + \delta\zeta, \zeta_{i+1} - \delta\zeta)} \left(d_\lambda(\theta) - \sum_{j=1}^i (d_\lambda(\zeta_j + \delta\zeta) - d_\lambda(\zeta_j - \delta\zeta)) \right) \\ &\quad + \sum_{i=1}^v \mathbf{1}_{\theta \in [\zeta_i - \delta\zeta, \zeta_i + \delta\zeta]} \left(d_\lambda(\zeta_i - \delta\zeta) - \sum_{j=1}^{i-1} (d_\lambda(\zeta_j + \delta\zeta) - d_\lambda(\zeta_j - \delta\zeta)) \right). \end{aligned}$$

This process is clearly continuous, is constant on parts that decrease in size as λ tends to infinity, and for λ large enough for ‘most’ of the θ ’s corresponding to a given edge of the Crofton cell, this process is equal (up to a constant) to $\bar{d}_\lambda(\theta)$, whereas it is constant for θ ’s close to the vertices.

As almost surely there is no vertex for $\theta = 0$, the value of this process at $\theta = 0$ converges in law, and thus one has the boundedness condition

$$\lim_{s \rightarrow +\infty} \left\{ \sup_{\lambda \geq \lambda_0} P(\lambda^2 |d_\lambda^c(0)| \geq s) \right\} = 0.$$

For sake of simplicity, thanks to the invariance under rotations, we may assume that the direction $\theta = 0$ corresponds to the closest line of the line process to the origin.

For the smooth processes to form a tight sequence, as one has already the boundedness condition on one point, one only needs to prove a criterion for the modulus of continuity of the process (see for instance [17] or theorem 11.6.3 in [20]):

$$\forall \eta, \epsilon > 0, \exists \lambda_1, \delta > 0 \text{ such that } \forall \lambda \geq \lambda_1, P \left(\sup_{|t-s| < \delta} \lambda^2 |d_\lambda^c(t) - d_\lambda^c(s)| > \epsilon \right) \leq \eta.$$

One sees easily from the formula defining \bar{d}_λ in lemma 3 that $\lambda^2 d_\lambda^c(\cdot)$ is Lipschitz with constant $R_M^4 / (R_m^2 R_\star)$ as soon as $\lambda \geq \sqrt{2R_M^2 / (R_\star R_m)}$, so that one has

$$\begin{aligned} P \left(\sup_{|t-s| < \delta} \lambda^2 |d_\lambda^c(t) - d_\lambda^c(s)| > \epsilon \right) &\leq P(R_m \leq r) + P(R_M^4 \geq \epsilon \delta^{-1} r^2 R_\star) \\ &\quad + P \left(\lambda < \sqrt{2R_M^2 / (R_\star R_m)}, R_m > r \right), \\ &\leq 2\pi r + P(R_M^4 \geq \epsilon \delta^{-1} r^2 R_\star) \\ &\quad + P(R_M^2 > \lambda^2 R_\star r). \end{aligned}$$

If we take $r = \eta / (6\pi)$, the first term is lesser than $\eta/3$, there remains to bound the last two terms, this is easily achieved by taking adequate values for δ such that the second term becomes lesser than $\eta/3$, and finally the last term tends to 0 thanks to the estimate of theorem 6:

$$P(R_M^2 > \lambda^2 R_\star \eta / (6\pi)) \leq \eta/3$$

as soon as $\lambda \geq \lambda_2(\eta)$, so that

$$\sup_{\lambda \geq \lambda_1(\eta)} P \left(\sup_{|t-s| < \epsilon^2 \eta} \lambda^2 |d_\lambda^c(t) - d_\lambda^c(s)| > \epsilon \right) \leq \eta.$$

To conclude the proof of the convergence theorem for the continuous process, we still have to prove the convergence of the finite dimensional distributions of this process,

this looks a lot like the proof of theorem 5, except that between points θ_i there may be jumps of the original process so that the continuous process is shifted: this does however not make many difficulties, as those shifts behave well with respect to the convergence in law. One may compute the law of the jumps of the limit process, indeed one needs to know the joint law of the couple

$$((\Upsilon_1, \Theta_1, R_1), (\Upsilon_2, \Theta_2, R_2)),$$

such that there is a vertex of \mathcal{C} at the angle α (where $\Theta_1 < \Theta_2$), then the limit jump is given by

$$J_{1 \rightarrow 2} = -\frac{1}{2} \left\{ \frac{\Upsilon_1^2}{R_1 \cos^2(\Theta_1 - \alpha)} \frac{\cos 2(\Theta_1 - \alpha)}{\cos(\Theta_1 - \alpha)} - \frac{\Upsilon_2^2}{R_2 \cos^2(\Theta_2 - \alpha)} \frac{\cos 2(\Theta_2 - \alpha)}{\cos(\Theta_2 - \alpha)} \right\}.$$

From this we may for instance compute the limit of the two-dimensional distribution at angles 0 and θ of the continuous process: if there is no edge of the Poisson polygon between those two angles, then the computation is the same as for the original process, if there is one angle set at α , then, using the notations above one has the convergence of $\lambda^2(d_\lambda^c(0), d_\lambda^c(\theta))$ towards

$$\left(-\frac{\Upsilon_1^2}{2R_1 \cos^2 \Theta_1} \frac{\cos 2\Theta_1}{\cos \Theta_1}, -\frac{\Upsilon_2^2}{2R_2 \cos^2(\Theta_2 - \theta)} \frac{\cos 2(\Theta_2 - \theta)}{\cos(\Theta_2 - \theta)} - J_{1 \rightarrow 2} \right),$$

and so on if there are more than one angle between those two points. All those limit laws may be expressed (though implicitly) with the help of the law of \mathcal{C} .

We have thus obtained:

Proposition 5. *The processes $\theta \mapsto \lambda^2 d_\lambda^c(\theta)$ converge in law as λ tends to infinity.*

For the jump part of the process, one remarks that the locations of the (almost) jumps of d_λ^j converge almost surely with speed of order $\lambda^{-2} \widetilde{R}_M / R_m$ towards the locations of the vertices of \mathcal{C} .

Let us introduce the following signed random measure (with the notation introduced above):

$$\mu^V = \sum_{i=1}^v J_i \delta_{\zeta_i},$$

where J_i is the value of the jump at the angle ζ_i . The law of this process depends only on the law of \mathcal{C} . One may also define a signed random measure associated to $\lambda^2 d_\lambda^j$:

$$\mu_\lambda^V = (\lambda^2 d_\lambda^j)'.$$

One has:

Proposition 6. *Almost surely μ_λ^V converges narrowly towards μ^V .*

There is no doubt that most of those results could be stated and proved for regular random shapes instead of discs, however the leap to higher dimensions, except for the directional convergence, seems to be more tricky.

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Appendix: proofs of the main results

In this section we recall and prove the main propositions and theorems stated without proof in previous sections, the proofs here are mainly geometrical arguments in the euclidean plane.

▷ **Coupling between the Boolean model and the Poisson line process: proof of proposition 1**

This proposition is straightforward, the first point is the definition of the Boolean model, the second one comes from the computation of the intensity measure μ of the (Poisson) point process formed with the points $(\lambda^2(\rho - R) + \frac{1}{2R} \lambda^2(\rho - R)^2, \theta)$ in $\mathbb{R}_+ \times [0, 2\pi)$, indeed one checks easily that its intensity measure \mathcal{M} is rotation-invariant, so that one has to compute the image measure of a disc centred at 0 with radius r :

$$\begin{aligned} \mathcal{M}(B(0, r)) &= \int_A \mathbf{1}_{\lambda^2(\rho - R) + \frac{1}{2R}(\rho - R)^2 \leq r} \lambda^2 \rho \, d\rho \, d\theta \, d\mu(R), \\ &= 2\pi r. \end{aligned}$$

■

▷ **Estimate on the Hausdorff distance: proof of proposition 2**

Let us denote by

$$\phi_\lambda(\rho, R) = \lambda^2(\rho - R) + \frac{1}{2R} \lambda^2(\rho - R)^2, \quad \forall \rho > R > 0$$

the function defining the coupling $B \rightarrow L$, let N_M denote the number of points (ρ_i, θ_i, R_i) of \mathbf{Y}_λ such that their associated disc $B_i = B(\lambda^2 \rho_i(\cos \theta_i, \sin \theta_i), \lambda^2 R_i)$ in the rescaled Boolean model intersects $B(0, M)$. The associated lines $H_{\lambda, \rho_i, \theta_i, R_i}$ may not intersect $B(0, M)$, but no other line of the process may intersect this disc, as $\phi_\lambda(\rho, R) \geq \lambda^2(\rho - R)$.

The centre of those discs are assumed to be located outside the disc $B(0, M)$, which is provided by the condition $\lambda > \sqrt{\frac{M}{R_*}}$ of the hypothesis. As is shown in figure 6, the Hausdorff distance between the intersection of the circle ∂B_i with $B(0, M)$ and the intersection of tangent line T_i with $B(0, M)$ is bounded by the distance between the points A and C , defined as respectively the intersection of T_i and $\partial B(0, M)$, and the

point on ∂B_i aligned with A and the center of B_i . A straightforward computation gives

$$\begin{aligned} AC &= \sqrt{\lambda^4 R_i^2 + M^2 - \lambda^4 (\rho_i - R_i)^2} - \lambda^2 R_i, \\ &= \frac{M^2 - \lambda^4 (\rho_i - R_i)^2}{\sqrt{\lambda^4 R_i^2 + M^2 - \lambda^4 (\rho_i - R_i)^2} + \lambda^2 R_i}, \\ &\leq \frac{M^2}{2\lambda^2 R_i}, \end{aligned}$$

as, since the circle intersects the disc $B(0, M)$, one has $\lambda^2(\rho - R) \leq M$.

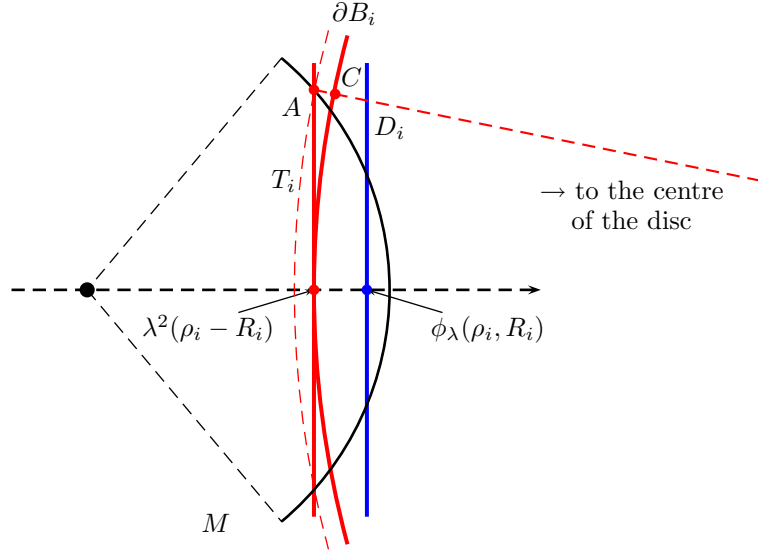


FIGURE 6: The circle ∂B_i and the tangent line T_i .

Consequently, for any disc B of radius $\lambda^2 R > \lambda^2 R_*$ such that $0 \notin B$, and T the tangent to this disc at its closest point to 0 one has

$$d_H^M(\partial B, T) \leq \frac{M^2}{2\lambda^2 R_*}. \quad (1)$$

On the other hand, as $\lambda^2(\rho_i - R_i) \leq M$, one has

$$|\lambda^2(\rho_i - R_i) - \phi_\lambda(\rho_i, R_i)| \leq \frac{M^2}{2\lambda^2 R_*}, \quad (2)$$

so that one obtains the result of proposition 2 by combining the two inequalities (1,2), if we denote by H_i (resp. H'_i) the half plane with boundary D_i (resp. T_i) not containing

the origin:

$$\begin{aligned}
d_H^M(\lambda^2 D_0^\lambda, C_0^\lambda) &\leq \max_{i \in \{1, \dots, N_M\}} d_H^M(\mathbb{C} B_i, \mathbb{C} H_i), \\
&\leq \max_{i \in \{1, \dots, N_M\}} d_H^M(\mathbb{C} B_i, \mathbb{C} H'_i) \\
&\quad + \max_{i \in \{1, \dots, N_M\}} d_H^M(\mathbb{C} H_i, \mathbb{C} H'_i), \\
&\leq \frac{M^2}{\lambda^2 R_\star},
\end{aligned}$$

and one concludes the proof of proposition 2. ■

▷ Rare events: proof of lemma 1

Let us introduce the set $\mathcal{F}(\lambda, M)$ of points (ρ, θ, R) such that

$$\begin{aligned}
\lambda^2(\rho - R) \leq M & \quad : \quad \text{the disc intersects,} \\
\lambda^2(\rho - R) + \frac{\lambda^2}{2R}(\rho - R)^2 > M & \quad : \quad \text{the line does not intersect.}
\end{aligned}$$

If we compute its \mathcal{N}_λ -measure we obtain

$$\begin{aligned}
\mathcal{N}_\lambda(\mathcal{F}(\lambda, M)) &= 2\pi\lambda^2 \int_{R_\star}^{+\infty} \int_{\frac{M\lambda^{-2}+R}{\sqrt{R^2+2MR\lambda^{-2}}}}^{M\lambda^{-2}+R} \rho \, d\rho \, d\mu(R), \\
&= \pi \frac{M^2}{\lambda^2},
\end{aligned}$$

hence the announced result. ■

▷ Convergence in law of the defect function: proof of theorem 2

As said earlier, we will take $\theta = 0$, and set Δ_0 the half-line with polar equation $\theta = 0$.

We begin with some computational lemmas.

Step 0: some technical lemmas

1. One has the following classical result for the law of the first intersecting line:

Lemma 4. *Let L denote the distance from 0 to the first intersection on Δ_0 of the line process, and Θ the polar angle of this intersecting line, then the law of (L, Θ) is given by*

$$d(L, \Theta)(P)(\ell, \theta) = e^{-2\ell} \cos \theta \mathbf{1}_{\theta \in (-\pi/2, \pi/2)} \, d\ell \, d\theta.$$

Proof. Let us denote by Φ the underlying Poisson point process (of intensity $d\rho d\theta$) of the line process. This process induces a marked Point process \mathbf{T} on Δ_0 consisting of the intersections of Δ_0 with its lines, marked with the corresponding polar angles, namely

$$(\rho, \theta) \in \Phi_+ = \Phi \cap (\mathbb{R}_+ \times (-\pi/2, \pi/2)) \mapsto \left(\frac{\rho}{\cos \theta}, \theta \right) \in \Delta_0 \times (-\pi/2, \pi/2).$$

Let us then consider $\ell > 0$ and $\alpha \in (-\pi/2, \pi/2)$, the probability that $L \leq \ell$ and $\Theta \leq \alpha$ may be expressed as

$$\begin{aligned} P(L \leq \ell, \Theta \leq \alpha) &= E \left[\mathbf{1}_{\text{the nearest to 0 point } (r, \theta) \text{ of } \mathbf{T} \text{ satisfies } r \leq \ell \text{ and } \theta \leq \alpha} \right], \\ &= E \left[\sum_{(\rho, \theta) \in \Phi_+} \mathbf{1}_{\frac{\rho}{\cos \theta} \leq \ell} \mathbf{1}_{\theta \leq \alpha} \mathbf{1}_{[0, \rho/\cos \theta] \cap \bigcup_{(\sigma, \phi) \in \Phi \setminus \{(\rho, \theta)\}} D_{\sigma, \phi} = \emptyset} \right]. \end{aligned}$$

Using Slivnyak's well known formula [19], this expectation becomes

$$\begin{aligned} P(L \leq \ell, \Theta \leq \alpha) &= \int_{\mathbb{R}_+ \times (-\pi/2, \pi/2)} \mathbf{1}_{\frac{\rho}{\cos \theta} \leq \ell} \mathbf{1}_{\theta \leq \alpha} \\ &\quad \times P(\text{no line intersects } [0, \rho/\cos \theta]) \, d\rho d\theta, \\ &= \int_{(-\pi/2, \alpha)} \int_{(0, \ell \cos \theta)} \exp\left(-2 \frac{\rho}{\cos \theta}\right) \, d\rho d\theta, \\ &= \int_{(-\pi/2, \alpha)} \int_{(0, \ell)} \exp(-2\rho) \cos \theta \, d\rho d\theta, \end{aligned}$$

which completes the proof of Lemma 4. ■

2. Concerning the intersections of lines and circles with Δ_0 , we get

Lemma 5. *Let $(\rho, \theta, R) \in \mathbf{Z}$, then if $\tan^2 \theta \leq \frac{\lambda^2 R}{2\rho}$ and $\theta \in (-\pi/2, \pi/2)$ one has*

- *the polar line at (ρ, θ) intersects Δ_0 at $\rho/(\cos \theta)$,*
- *the first intersection point of the circle centred at $(\lambda^2 \psi_\lambda(\rho, R), \theta)$ and radius $\lambda^2 R$ with Δ_0 is situated at*

$$\lambda^2 R \cos \theta \left(\sqrt{1 + \frac{2\rho}{\lambda^2 R}} - \sqrt{1 - \tan^2 \theta \frac{2\rho}{\lambda^2 R}} \right).$$

If furthermore one has the following set of conditions on (ρ, θ) :

$$(A) \quad \rho \geq r, \quad \frac{\rho}{\cos \theta} \leq \ell, \quad \text{and } \exists C, \alpha > 0 \text{ such that } \ell \leq C\lambda^{1-\alpha}, \quad \frac{\ell^2}{r} \leq C\lambda^{1-\alpha},$$

then the distance between those two intersection points is lesser than $\frac{2C^2}{R_} \lambda^{-2\alpha}$.*

The proof of this lemma is straightforward computation.

Step 1: almost no crossing

For the proof of this point we will consider the event C_λ that the defect function at $\theta = 0$ is obtained for λ with two different points of the process \mathbf{Z} : (ρ_b, θ_b, R_b) such that the rescaled disc B_b (with obvious notation) intersects Δ_0 before any other, and (ρ_l, θ_l, R_l) such that the line H_l intersects Δ_0 before any other,

$$d_\lambda(0) = d(0, \Delta_0 \cap \lambda^2 B((\rho_b \cos \theta_b, \rho_b \sin \theta_b), R_b)) - d(0, \Delta_0 \cap D_{\rho_l, \theta_l}).$$

Let us denote by L the distance from 0 to the first intersection point on Δ_0 , and by R_m the inner radius of \mathcal{C} , that is the radius of the largest disc centred at 0 included in \mathcal{C} , one has thanks to lemma 4 and to [11]

$$\begin{aligned} P(L > \ell) &= \exp(-2\ell), \\ P(R_m > r) &= \exp(-2\pi r). \end{aligned}$$

Hence, if we compute $P(C_\lambda)$ with the help of R_m and L , we obtain

$$\begin{aligned} P(C_\lambda) &= P(C_\lambda, L > \ell) + P(C_\lambda, L \leq \ell, R_m \leq r) + P(C_\lambda, L \leq \ell, R_m > r), \\ &\leq \exp(-2\ell) + (1 - \exp(-2\pi r)) + P(C_\lambda, L \leq \ell, R_m > r). \end{aligned}$$

The first two terms will tend to 0 as soon as r and ℓ are chosen with care, the last term can be estimated from above by the following method: as $L \leq \ell$, we know from lemma 5 that if condition (A) is satisfied then the associated circle intersects Δ_0 at a distance lesser than $\frac{2C^2}{R_\star} \lambda^{-2\alpha}$ from the intersection point with its associated line. Hence we are reduced to examine the different circles that intersect Δ_0 at a distance lesser than

$$\ell' = \ell + \frac{2C^2}{R_\star} \lambda^{-2\alpha}.$$

We introduce the set $\mathcal{F}_0(\lambda, \ell')$ of those points (ρ, θ, R) of \mathbf{Z} such that the disc of the rescaled Boolean model $B((\lambda^2 \psi_\lambda(\rho, R), \theta), \lambda^2 R)$ intersects $\Delta_0 \cap B(0, \ell')$ and the associated line does not, then one has easily, as for lemma 1:

Lemma 6. *The cardinal of $\mathcal{F}_0(\lambda, \ell')$ is a Poisson random variable with parameter*

$$f_0(\lambda, \ell') \leq \frac{\pi \ell'^2}{2\lambda^2} + \frac{\ell'^3}{3\lambda^4 R_\star}.$$

Proof. Once again we shall use the \mathcal{N}_λ measure, after a short amount of geometry we get that the challenging set has the following shape:

- as the line does not intersect $[0, \ell']$ we have $\rho > R\sqrt{1 + \frac{2\ell' \cos \theta}{\lambda^2}}$,
- as the disc must intersect $[0, \ell']$ we have

$$\begin{aligned} \text{if } |\theta| \leq \arctan \frac{\lambda^2 R}{\ell'} \quad \text{then} \quad \rho &\leq \frac{\ell' \cos \theta}{\lambda^2} + \sqrt{R^2 - \left(\frac{\ell' \sin \theta}{\lambda^2}\right)^2}, \\ \text{else} \quad \rho &\leq \frac{R}{\sin \theta}. \end{aligned}$$

So that

$$\begin{aligned} f_0(\lambda, \ell') &= \lambda^2 E \left[\int_0^{\arctan \frac{\lambda^2 R}{\ell'}} \left(\frac{\ell'^2 \cos 2\theta}{\lambda^4} - \frac{2\ell' \cos \theta R}{\lambda^2} \left(1 - \sqrt{1 - \frac{\sin^2 \theta \ell'^2}{\lambda^4 R^2}} \right) \right) d\theta \right. \\ &\quad \left. + \int_{\arctan \frac{\lambda^2 R}{\ell'}}^{\pi/2} \left(\frac{R^2}{\tan^2 \theta} - \frac{2\ell' \cos \theta R}{\lambda^2} \right) d\theta \right], \\ &\leq \frac{\pi \ell'^2}{4\lambda^2} + E_\mu \left[R^2 \int_{\arctan \frac{\lambda^2 R}{\ell'}}^{\pi/2} \frac{1}{\tan^2 \theta} d\theta \right]. \end{aligned}$$

Thanks to this lemma the probability of a crossing is now lesser than

$$\begin{aligned} P(C_\lambda) &\leq \exp(-2\ell) + (1 - \exp(-2\pi r)) + P(C_\lambda, L \leq \ell, R_m > r), \\ &\leq \exp(-2\ell) + (1 - \exp(-2\pi r)) + P(\mathcal{F}_0(\lambda, \ell') \neq \emptyset) \\ &\quad + P(C_\lambda, L \leq \ell, R_m > r, \mathcal{F}_0(\lambda, \ell') = \emptyset), \\ &\leq \exp(-2\ell) + (1 - \exp(-2\pi r)) + (1 - \exp(-f_0(\lambda, \ell'))) \\ &\quad + P(C_\lambda, L \leq \ell, R_m > r, \mathcal{F}_0(\lambda, \ell') = \emptyset). \end{aligned}$$

There remains to estimate the last term in this inequality: obviously for $\lambda \geq \lambda_0$ the length ℓ' satisfies condition (A) with

$$C' = C + \frac{2C^2}{R_\star \lambda_0^{1+\alpha}},$$

so that still using lemma 5 we know that those lines and associated circles that are of importance for this event intersect Δ_0 at points that are at most $\frac{2C'^2}{R_\star} \lambda^{-2\alpha}$ -apart. Hence if there must be a crossing, there should be two intersection points of the line process with $\Delta_0 \cap B(0, \ell')$ at a distance lesser than twice this quantity. Now,

Lemma 7. *For a Poisson point process with intensity 2 on $[0, m]$ and $\varepsilon > 0$,*

$$P(\exists \text{ two } \varepsilon\text{-close points}) \leq 4\varepsilon m.$$

(The proof is elementary.)

If we gather all those results we get

$$\begin{aligned} P(C_\lambda) &\leq \exp(-2\ell) + (1 - \exp(-2\pi r)) + (1 - \exp(-f_0(\lambda, \ell'))) \\ &\quad + 4 \frac{4C'^2}{R_\star} \lambda^{-2\alpha} \ell', \\ &\leq \exp(-2\ell) + (1 - \exp(-2\pi r)) + \left(1 - \exp\left(-\left(\frac{\pi \ell'^2}{4\lambda^2} + \frac{\ell'^3}{3\lambda^4 R_\star}\right)\right)\right) \\ &\quad + 16 \frac{\left(C + \frac{2C^2}{R_\star \lambda_0^{1+\alpha}}\right)^2}{R_\star} \lambda^{-2\alpha} \left(\ell + \frac{2C^2}{R_\star} \lambda^{-2\alpha}\right), \end{aligned}$$

thus if we take ℓ going to infinity like $(1/3) \log \lambda$ and r going to 0 like $\lambda^{-2/3} (\log \lambda)^2 / 9$ (so that $\alpha = 1/3$), we have just proved lemma 2:

Lemma 2. *Let η be a positive number, then there exists a constant B_η and λ_0 such that the probability that two different points of \mathbf{Z} define the defect function is for $\lambda \geq \lambda_0$ at most*

$$P(C_\lambda) \leq \frac{B_\eta}{\lambda^{2/3-\eta}},$$

where the constant B_η depends only on λ_0 , η and R_\star .

5.1. Step 2: computation of the uncrossed defect

We can now define the uncrossed defect as the difference between the intersection of the rescaled disc associated to this first line and the intersection of this same line with Δ_0 : let (Υ, Θ, R) be the associated point of \mathbf{Z} , and

$$\begin{aligned} \bar{d}_\lambda(0) &:= d(0, B(\psi_\lambda(\Upsilon, \Theta, R), R) \cap \Delta_0) - \frac{\Upsilon}{\cos \Theta}, \\ &= d(0, B(\psi_\lambda(\Upsilon, \Theta, R), R) \cap \Delta_0) - L. \end{aligned}$$

Using lemma 5 this rewrites

$$\bar{d}_\lambda(0) = \lambda^2 R \cos \Theta \sqrt{1 + \frac{2L \cos \Theta}{\lambda^2 R}} - \lambda^2 R \sqrt{1 - \sin^2 \Theta \left(1 + \frac{2L \cos \Theta}{\lambda^2 R}\right)} - L$$

if the inner term of the square root is non negative (i.e. $\lambda \geq \sqrt{(2L \sin^2 \Theta)/(R \cos \Theta)}$), $+\infty$ otherwise. The asymptotic expansion of those square roots gives easily

$$\bar{d}_\lambda(0) = -\frac{L^2 \cos 2\Theta}{2\lambda^2 R \cos \Theta} + O(\lambda^{-4}).$$

From the common law of R, L, Θ one concludes theorem 2.

► Convergence of the two-dimensional distribution: proof of theorem 4

Here is the complete statement of theorem 4:

Theorem 8. *For all $\theta \in (0, \pi)$, $\lambda^2(d_\lambda(0), d_\lambda(\theta))$ converges in law as λ goes to infinity towards*

$$B(Z_0(\Upsilon, \Theta, R), Z_\theta(\Upsilon, \Theta, R)) + (1 - B)(W_0(\Upsilon_1, \Theta_1, R_1)W_\theta(\Upsilon_2, \Theta_2, R_2)),$$

where

- B is Bernoulli random variable stating that the same line determines the intersections in directions 0 and θ : this occurs with probability p ,

$$\begin{aligned} p &= E \left[\sum_{(\rho, \alpha, R) \in \mathbf{Z}} \mathbf{1}_{\text{The lines from } \mathbf{Z} \setminus \{\rho, \alpha, R\} \text{ do not intersect } \Delta_0 \text{ or } \Delta_\theta \text{ before } D_{(\rho, \alpha)}} \right], \\ &= \int_{\mathbb{R}_+ \times [0, 2\pi) \times [R_*, +\infty)} \exp(-\mathfrak{p}(B_{0, \theta}(\rho, \alpha))) \, d\rho \, d\alpha \, d\mu(r), \end{aligned}$$

where $B_{0, \theta}(\rho, \alpha)$ is described by figure 7, \mathfrak{p} denotes the perimeter function. This Bernoulli random variable is independent from the following ones,

- (Υ, Θ, R) has the following distribution:

$$\begin{aligned} d(\Upsilon, \Theta)(P)(\rho, \alpha, r) &= \mathbf{1}_{\alpha \notin (\pi/2, 3\pi/2)} \\ &\quad \mathbf{1}_{\alpha - \theta \notin (\pi/2, 3\pi/2)} \exp(-\mathfrak{p}(B_{0, \theta}(\rho, \alpha))) \, d\rho \, d\alpha \, d\mu(r), \end{aligned}$$

- $Z_0(\Upsilon, \Theta, R) = -\frac{\Upsilon^2 \cos 2\Theta}{2R \cos^2 \Theta \cos \Theta}$,
- $Z_\theta(\Upsilon, \Theta, R) = -\frac{\Upsilon^2 \cos 2(\Theta - \theta)}{2R \cos^2(\Theta - \theta) \cos(\Theta - \theta)}$,
- $(\Upsilon_1, \Theta_1, R_1, \Upsilon_2, \Theta_2, R_2)$ has the following distribution,

$$\begin{aligned} &d(\Upsilon_1, \Theta_1, R_1, \Upsilon_2, \Theta_2, R_2)(P)(\rho_1, \alpha_1, r_1, \rho_2, \alpha_2, r_2) \\ &= \mathbf{1}_{\alpha_1 \notin (\pi/2, 3\pi/2)} \mathbf{1}_{\alpha_2 - \theta \notin (\pi/2, 3\pi/2)} \mathbf{1}_{(\rho_1, \alpha_1) \notin B_\theta(\rho_2, \alpha_2)} \mathbf{1}_{(\rho_2, \alpha_2) \notin B_0(\rho_1, \alpha_1)} \\ &\quad \exp(-\mathfrak{p}(B'_{0, \theta}(\rho_1, \alpha_1, \rho_2, \alpha_2))) \, d\rho_1 \, d\alpha_1 \, d\mu(r_1) \, d\rho_2 \, d\alpha_2 \, d\mu(r_2), \end{aligned}$$

where the sets B_0 , B_θ and $B'_{0,\theta}$ are described by figure 7.

- $W_0(\Upsilon_1, \Theta_1, R_1) = -\frac{\Upsilon_1^2}{2R_1 \cos^2 \Theta_1} \frac{\cos 2\Theta_1}{\cos \Theta_1}$,
- $W_\theta(\Upsilon_2, \Theta_2, R_2) = -\frac{\Upsilon_2^2}{2R_2 \cos^2(\Theta_2 - \theta)} \frac{\cos 2(\Theta_2 - \theta)}{\cos(\Theta_2 - \theta)}$,

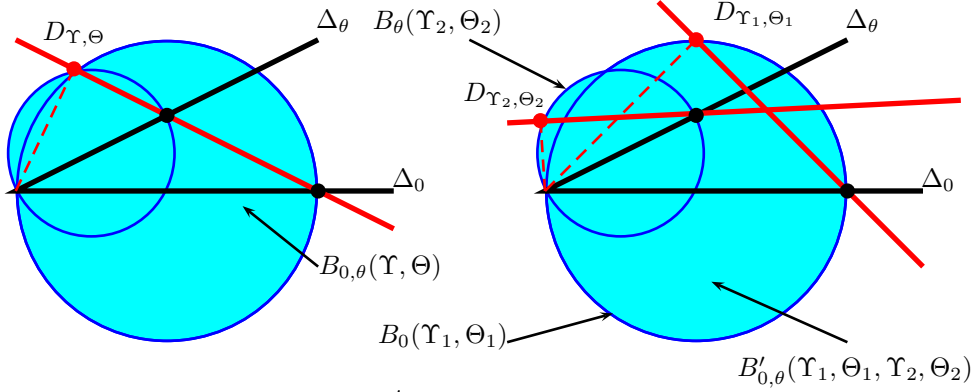


FIGURE 7: The sets $B_{0,\theta}$, $B'_{0,\theta}$, B_0 and B_θ , and the corresponding lines.

Proof. The proof follows the same line as for the case of one direction, the first step becomes a bit more difficult because lines may intersect both Δ_0 and Δ_θ , but the idea is exactly the same. The second step is identical. ■

Remark 3. From the computation of the law of (Υ, Θ) and the joint law of (Υ_1, Θ_1) and (Υ_2, Θ_2) one could also derive the joint law of the distances from the origin to the first line in directions 0 and θ , as given in [3].

From this theorem and the expression of the limit law, we can compute the theoretical form of the limiting covariance:

$$E[\lambda^4 d_\lambda(0) d_\lambda(\theta)] \xrightarrow{\lambda \rightarrow +\infty} p E[Z_0(\Upsilon, \Theta, R) Z_\theta(\Upsilon, \Theta, R)] \\ + (1-p) E[W_0(\Upsilon_1, \Theta_1, R_1) W_\theta(\Upsilon_2, \Theta_2, R_2)].$$

▷ Convergence of the finite-dimensional distributions: proof of theorem 5

The proof proceeds in the same way as for one or two angles, indeed one first approximates the following probability

$$P(\lambda^2(d_\lambda(\theta_1), \dots, d_\lambda(\theta_n)) \leq (x_1, \dots, x_n))$$

up to a factor at most $O(\lambda^{-2/3+n})$ by the same probability where d_λ is replaced by \bar{d}_λ (as defined in lemma 3). The second step is then to introduce the k lines ($k \leq n$) of the line process such that those $\bar{d}_\lambda(\theta_i)$ depend only on those lines, obviously one checks that the set $\{0, \dots, n-1\}$ is equal to $\bigcup_{i=1}^k (\{j_i, \dots, j_{i+1}-1\} \bmod n)$, where $j_1 < j_2 < \dots < j_k < j_{k+1} = j_1 + n$, each of those subsets corresponding to the indices j in $\{0, \dots, n-1\}$ such that $\bar{d}_\lambda(\theta_{j+1})$ is expressed thanks to the line of index $i = i(j)$. The final step is then to perform the computation of the probability

$$\begin{aligned} & P(\lambda^2(\bar{d}_\lambda(\theta_1), \dots, \bar{d}_\lambda(\theta_n)) \leq (x_1, \dots, x_n)) \\ &= \sum_{k=1}^n P(\lambda^2(\bar{d}_\lambda(\theta_1), \dots, \bar{d}_\lambda(\theta_n)) \leq (x_1, \dots, x_n), k \text{ lines}), \\ &= \sum_{k=1}^n P(\lambda^2(\bar{d}_\lambda^{i(1)}(\theta_1), \dots, \bar{d}_\lambda^{i(n)}(\theta_n)) \leq (x_1, \dots, x_n), k \text{ lines}), \end{aligned}$$

the inner term of the sum may be expressed with the help of a random vector

$$(\Upsilon_1, \Theta_1, R_1, \dots, \Upsilon_k, \Theta_k, R_k)$$

whose law may be explicitated in the same way as for theorem 4 (rather tediously), using the same type of computations as for theorem 2 one obtains the convergence of each of those terms, hence the result. ■

▷ **Tail probability for the supremum of the process: estimate on the outer radius of the Boolean model (theorem 7)**

In section 4 we proved under some conditions on the growth of the vacancy around the origin that the supremum of the process had some good properties in proposition 4, the result needed was:

Theorem 7. *There exists constants K and C depending only on R_\star such that for all $r \geq 1$ and $\lambda \geq 1$ we have*

$$P(R_M(\lambda) > r) \leq K \exp(-Cr).$$

Proof. The proof of theorem 7 is the consequence of the following (non-optimal) reasoning : let N be a positive integer greater than 12, and consider angular sectors

$$S_{i,N} = \left\{ (\rho, \theta) : \rho > 0, \theta + \frac{\pi}{N} \in \left[\frac{2i\pi}{N}, \frac{2(i+1)\pi}{N} \right) \right\},$$

for $i \in \{0, \dots, N-1\}$. Then it is clear that if $\lambda^2 D_0^\lambda$ is not included in $B(0, r)$, then at least for one of those sectors there is no circle of the rescaled Boolean model that intersects both sides of this sector inside $B(0, r)$.

Let us denote by $A_{\lambda, r, R}$ the set of the centres (ρ, θ) of discs of radius $\lambda^2 R$ such that this occurs for the sector $S_{0, N}$, we have by invariance under rotations

$$P(R_M(\lambda) \geq r) \leq N \exp \left(-\lambda^{-2} \int_{\mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}_+} \mathbf{1}_{A_{\lambda, r, R}}(\rho, \theta) \rho \, d\rho \, d\theta \, d\mu(R) \right),$$

and the aim of the computations is to bound the Lebesgue measure $a_{\lambda, r, R}$ of the set $A_{\lambda, r, R}$ from below. This set is the intersection of the following three sets: $[0, re^{-i\pi/N}] \oplus B(0, \lambda^2 R)$, $[0, re^{i\pi/N}] \oplus B(0, \lambda^2 R)$ and $\mathbb{R}^2 \setminus B(0, \lambda^2 R)$, according to the relative values of r and $\lambda^2 R$ this set takes three different shapes that we shall denote by

In the following estimations, we assume that $N \rightarrow +\infty$ when $r \rightarrow +\infty$. The choice of N in function of r will be specified at the end of the proof.

We obtain a lower-bound for the area of $A_{\lambda, r, R}$ by considering instead the area $\mathcal{T}_{\lambda, r, R}$ of a triangle which is tangent to $B(0, \lambda^2 R)$ at $\lambda^2 R$ and such that the farthest point from the origin in $A_{\lambda, r, R}$ is one of its vertices.

Three cases have to be considered:

- **The *arrow* case:** the domain of validity of this case is fixed by the inequality

$$r \geq \frac{\lambda^2 R}{\tan(\pi/N)}.$$

It corresponds to the situation where the circular part of $[0, re^{-i\pi/N}] \oplus B(0, \lambda^2 R)$ does not participate in the intersection $A_{\lambda, r, R}$. The area $a_{\lambda, r, R}$ is bounded by $\mathcal{T}_{\lambda, r, R}$ with

$$\mathcal{T}_{\lambda, r, R} = 2\lambda^4 R^2 \frac{(1 - \sin \frac{\pi}{N})^2}{\sin \frac{2\pi}{N}},$$

and thus as $N \geq 6$ we obtain the following bound:

$$a_{\lambda, r, 1} \geq \lambda^2 R \frac{N}{4\pi} \tag{3}$$

- **The *blunt arrow* case:** The domain of validity of that case is

$$\lambda^2 R \frac{1 - \sin(\pi/N)}{\cos(\pi/N)} \leq r \leq \frac{\lambda^2 R}{\tan(\pi/N)}.$$

Here the height of the triangle cuts the cylinder part of $[0, re^{-i\pi/N}] \oplus B(0, \lambda^2 R)$ but the farthest point from the origin is on its circular part. We obtain

$$\mathcal{T}_{\lambda, r, R} = \lambda^2 R \frac{1 - \sin(\pi/N)}{\cos(\pi/N)} \left[r \cos(\pi/N) + \lambda^2 R \sqrt{1 - \frac{r^2}{\lambda^4 R^2} \sin^2(\pi/N)} - \lambda^2 R \right].$$

In particular, in this domain of validity, we have

$$\begin{aligned} & \left[r \cos(\pi/N) + \lambda^2 R \sqrt{1 - \frac{r^2}{\lambda^4 R^2} \sin^2(\pi/N)} - \lambda^2 R \right] \\ & \geq r \cos(\pi/N) - r \left(\frac{r}{\lambda^2 R} \sin^2(\pi/N) \right). \end{aligned}$$

This term is globally of order r since its first subterm is of order r and its second subterm is negligible in front of r . Consequently, there exists a positive constant $C > 0$ such that for r large enough, we have

$$a_{\lambda, r, R} \geq C \lambda^2 R r. \quad (4)$$

- **The eye case:** The domain of validity is

$$r \leq \lambda^2 R \frac{1 - \sin(\pi/N)}{\cos(\pi/N)}.$$

In that case, the height of the triangle cuts the circular part of $[0, re^{-i\pi/N}] \oplus B(0, \lambda^2 R)$. We obtain that

$$\begin{aligned} \mathcal{T}_{\lambda, r, R} &= \left[-r \sin(\pi/N) + \lambda^2 R \sqrt{1 - \left(1 - \frac{r}{\lambda^2 R} \cos(\pi/N)\right)^2} \right] \\ &\quad \times \left[r \cos(\pi/N) + \lambda^2 R \sqrt{1 - \frac{r^2}{\lambda^4 R^2} \sin^2(\pi/N)} - \lambda^2 R \right]. \quad (5) \end{aligned}$$

We remark that in such a domain of validity, we have

$$\begin{aligned} & \left[-r \sin(\pi/N) + \lambda^2 R \sqrt{1 - \left(1 - \frac{r}{\lambda^2 R} \cos(\pi/N)\right)^2} \right] \\ & \geq \frac{\lambda \sqrt{R}}{\sqrt{2}} \sqrt{2r \cos(\pi/N)} - r \sin(\pi/N). \end{aligned}$$

The first term (of order $\lambda \sqrt{R} \sqrt{r}$ with $\lambda \sqrt{R}$ at least of order \sqrt{r}) dominates the second one (which is negligible in front of r). In the same way,

$$\begin{aligned} & \left[r \cos(\pi/N) + \lambda^2 R \sqrt{1 - \frac{r^2}{\lambda^4 R^2} \sin^2(\pi/N)} - \lambda^2 R \right] \\ & \geq r \cos(\pi/N) - \tan^2(\pi/N) (1 - \sin(\pi/N))^2, \end{aligned}$$

so this term is also of order r . Consequently, there exists a positive constant $C > 0$ such that for r sufficiently large, we have

$$\mathcal{T}_{\lambda,r,R} \geq C\lambda\sqrt{R}r^{3/2}. \quad (6)$$

We have then to consider two subcases:

- When $\lambda^2 R \leq 2r$, we deduce from the preceding inequality that

$$a_{\lambda,r,R} \geq \frac{C\sqrt{R_*}}{\sqrt{2}}\lambda^2 Rr; \quad (7)$$

- When $\lambda^2 R \geq 2r$, the area $\mathcal{T}_{\lambda,r,R}$ is too small and we have to estimate more directly the area $a_{\lambda,r,R}$ (there is no obvious geometric description of the following estimation). We define the angle $\theta_0 = \arctan(\lambda^2 R/r) - \pi/N$. As $N \geq 6$, we have $\theta_0 \geq \pi/6$, let us also introduce $\theta_1 = \pi/12$, this angle satisfies $\theta_1 < \theta_0$. The computation of $a_{\lambda,r,R}$ provides

$$\begin{aligned} a_{\lambda,r,R} &= 2 \int_0^{\theta_0} \left[\int_{\lambda^2 R}^{r \cos(\theta+\pi/N) + \sqrt{\lambda^4 R^2 - r^2 \sin^2(\theta+\pi/N)}} \rho d\rho \right] d\theta \\ &\quad + 2 \int_{\theta_0}^{\pi/2 - \pi/N} \left[\int_{\lambda^2 R}^{r \cos(\theta+\pi/N)} \rho d\rho \right] d\theta \\ &\geq \int_0^{\theta_1} \left[r^2 \cos(2\theta_1 + 2\pi/N) + \right. \\ &\quad \left. 2r \cos(\theta_1 + \frac{\pi}{N}) \sqrt{\lambda^4 R^2 - r^2 \sin^2(\theta_1 + \pi/N)} \right] d\theta \\ &\geq 2\theta_1 r \cos(\theta_1 + \pi/N) \frac{\sqrt{3}}{2} \lambda^2 R \\ &\geq \sqrt{\frac{3}{2}} \frac{\pi}{24} \lambda^2 Rr. \end{aligned}$$

Consequently for $N \geq 6$, we obtain that there exists a constant $C > 0$ such that

$$a_{\lambda,r,R} \geq C\lambda^2 Rr. \quad (8)$$

Conclusion. From the results of this tetratomy (3,4,7,8), it follows that the choice $N = \lceil r \rceil$ provides the following result in all four cases: there exists a positive constant C (depending only on R_*) such that for r large enough,

$$a_{\lambda,r,R} \geq C\lambda^2 Rr.$$

This completes the proof of theorem 7. ■

▷ **Convergence of the jump process: proof of proposition 6**

We give only a sketch of the proof of this result: let f be a continuous 2π -periodic function on \mathbb{R} , then

$$\begin{aligned} \langle \mu_\lambda^V - \mu^V, f \rangle &= \sum_{i=1}^v \langle \mu_\lambda^V - \mu^V, f \mathbf{1}_{[\zeta_i - \delta\zeta, \zeta_i + \delta\zeta]} \rangle, \\ &= \sum_{i=1}^v f(\zeta_i) \langle \mu_\lambda^V - \mu^V, \mathbf{1}_{[\zeta_i - \delta\zeta, \zeta_i + \delta\zeta]} \rangle + t_\lambda, \end{aligned}$$

where t_λ is in absolute value of order $2v\delta\zeta \sup_\theta |\lambda^2 d_\lambda(\theta)| \varpi_f(\delta\zeta)$, where ϖ_f is the continuity modulus of f . Obviously we need to give a precise statement here, as $\delta\zeta$ is random, but by conditioning on R_M and α this may be achieved with a few tedious calculations.

The first term can be rewritten as

$$\sum_{i=1}^v (\lambda^2 d_\lambda(\zeta_i + \delta\zeta) - \lambda^2 d_\lambda(\zeta_i - \delta\zeta) - J_i) f(\zeta_i),$$

and using the regularity of $\lambda^2 d_\lambda$ from the right and the left at the points ζ_i one obtains the almost sure convergence towards 0 of this term.

The second term also tends to 0 almost surely, hence proposition 6. ■

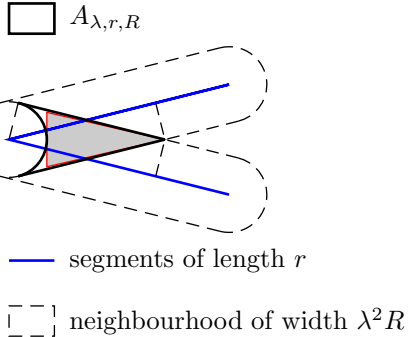
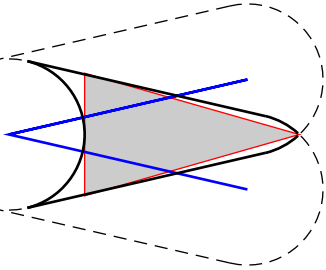
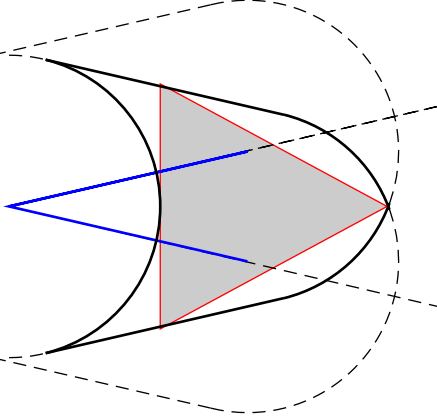
<p><i>arrow</i></p> <p>(3)</p>	$1 \leq \lambda^2 R \leq r \tan \frac{\pi}{N}$	 <p>\square $A_{\lambda,r,R}$</p> <p>— segments of length r</p> <p>- - - neighbourhood of width $\lambda^2 R$</p>
<p><i>blunt arrow</i></p> <p>(4)</p>	$r \tan \frac{\pi}{N} < \lambda^2 R \leq r \sin \frac{2\pi}{N}$	
<p><i>eye</i></p> <p>(first subcase)</p> <p>(7)</p>	$r \frac{\cos \frac{\pi}{N}}{1 - \sin \frac{\pi}{N}} < \lambda^2 R \leq 2r$	

FIGURE 8: The different shapes of the set $A_{\lambda,r,R}$, the grey regions are those chosen to bound its area.