Bipolarization of posets and natural interpolation

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Abstract

The Choquet integral w.r.t. a capacity can be seen in the finite case as a parsimonious linear interpolator between vertices of $[0,1]^n$. We take this basic fact as a starting point to define the Choquet integral in a very general way, using the geometric realization of lattices and their natural triangulation, as in the work of Koshevoy.

A second aim of the paper is to define a general mechanism for the bipolarization of ordered structures. Bisets (or signed sets), as well as bisubmodular functions, bicapacities, bicooperative games, as well as the Choquet integral defined for them can be seen as particular instances of this scheme.

Lastly, an application to multicriteria aggregation with multiple reference levels illustrates all the results presented in the paper.

Keywords: interpolation, Choquet integral, lattice, bipolar structure

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1 Introduction

Capacities and the Choquet integral [6] have become fundamental concepts in decision making (see, e.g., the works of Schmeidler [22], Murofushi and Sugeno [17], and Koshevoy [14]).

An interesting but not so well known fact is that in the finite case, the Choquet integral can be obtained as a parsimonious linear interpolation, supposing that values on the vertices of the hypercube $[0, 1]^n$ are known. The interpolation formula was discovered by Lovász [15], considering the problem of extending the domain of pseudo-Boolean functions to $\mathbb{R}^n$. Later, Marichal [16] remarked that this formula was precisely the Choquet integral (see also Grabisch [8]).

The idea of considering the Choquet integral as a parsimonious linear interpolator could serve as a basic principle for extending the notion of Choquet integral to more general frameworks. An example of this has been done by the authors in [10], considering multiple reference levels in a context of multicriteria aggregation.

Another remarkable example of generalization of the Choquet integral is the one for bicapacities, proposed by the authors [11, 12]. As this paper will make it clear, bicapacities are an example of concept based on the of *bipolarization* of a partially ordered set, in this case Boolean lattices. Specifically, take a finite set N and the set 2^N of all its subsets ordered by inclusion: we obtain a Boolean lattice, and a capacity is an isotone real-valued mapping on 2^N. Introducing $\mathcal{Q}(N) := \{(A, B) \in 2^N \times 2^N \mid A \cap B = \varnothing\}$, a bicapacity is a real-valued mapping on $\mathcal{Q}(N)$, satisfying some monotonicity condition. Observe that $\mathcal{Q}(N)$ could be denoted by 3^N as well: $(A, B) \in \mathcal{Q}(N)$ can be considered as a function ξ of $\{-1, 0, 1\}^N$, where $\xi(i) = 1$ if $i \in A$, $\xi(i) = -1$ if $i \in B$, and 0 otherwise. Then the term "bipolarization" becomes clear, since $2^N \equiv \{0, 1\}^N$ has one "pole" (the value 1, and 0 is the origin or neutral value), and $\{-1, 0, 1\}^N$ has 2 poles, namely -1 and 1, around the neutral value 0.

The set 3^N and functions defined on it are not new in the literature. To the knowledge of the authors, it has been introduced approximately at the same time and independently by Chandrasekaran and Kabadi [5], Bouchet [4], Qi [20], and Nakamura [19] in the field of matroid theory and optimization, and later well developed by Ando and Fujishige [1]. They use the term *biset* or *signed sets* for elements of 3^N, and *bisubmodular functions* for bicapacities (with some more restrictions). In the field of cooperative game theory, Bilbao has introduced *bicooperative games* [2], which corresponds to bicapacities without the monotonicity condition. Other remarkable works on bicapacities and bicooperative games include the one of Fujimoto, who defined the Möbius transform of bicapacities under the name of *bipolar Möbius transform* [7].

The aim of this paper is twofold: First to define the Choquet integral in a very general way, as a parsimonious linear interpolation. This is done through the concept of geometric realization of a distributive lattice and its natural triangulation. Second, to provide a general mechanism for the bipolarization of a poset, and to extend the previous concepts (geometric realization, Choquet integral, etc.) to the bipolarized structure. Then, all concepts around bisets, bicapacities, etc., are recovered as a particular case.

Our work has been essentially inspired and motivated by Koshevoy, who used the geometric realization of a lattice and its natural triangulation [14], and by Fujimoto [7], who first remarked the inadequacy of our original definition of the Möbius transform for bicapacities in [11], and proposed the bipolar Möbius transform.

Section 2 introduces necessary material, in particular geometric realizations, natural triangulations and interpolation. Section 3 is the core section of the paper, which presents the concept of bipolarization, then the bipolar version of the geometric realization, natural triangulations and interpolation. Lastly, Section 4 gives some examples, and develops the particular case of the product of linear lattices, which corresponds to an application in multicriteria aggregation with reference levels. We show that results obtained previously by the authors in [10] are recovered.

2 Preliminaries

In this section, we consider a finite index set $N := \{1, \dots, n\}$.

2.1 Capacities and bicapacities

We recall from Rota [21] that, given a locally finite poset $(X, \leq)$ with bottom element, the *Möbius function* is the function $\mu : X \times X \to \mathbb{R}$ which gives the solution to any equation of the form

$$g(x) = \sum_{y \leq x} f(y), \tag{1}$$

for some real-valued functions f, g on X, by

$$f(x) = \sum_{y \leq x} \mu(y, x) g(y). \tag{2}$$

Function f is called the *Möbius transform* (or *inverse*) of g.

Definition 1 *(i) A function $\nu : 2^N \to \mathbb{R}$ is a* game *if it satisfies $\nu(\varnothing) = 0$.*

(ii) A game which satisfies $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$ (monotonicity) is called a capacity *[6] or* fuzzy measure *[23]. The capacity is* normalized *if in addition $\nu(N) = 1$.*

Unanimity games are capacities of the type

$$u_A(B) := \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{else} \end{cases}$$

for some $A \subseteq N, A \neq \varnothing$. It is well known that the set of unanimity games is a basis for all games, whose coordinates in this basis are exactly the Möbius transform of the game.

Definition 2 *Let us consider $f : N \to \mathbb{R}_+$. The* Choquet integral *of f w.r.t. a capacity ν is given by*

$$\int f \, d\nu := \sum_{i=1}^{n} [f(\pi(i)) - f(\pi(i+1))] \nu(\{\pi(1), \dots, \pi(i)\}),$$

where π is a permutation on N such that $f(\pi(1)) \geq \dots \geq f(\pi(n))$, and $f(\pi(n+1)) := 0$.

The above definition is valid if ν is a game. For any $\{0,1\}$-valued capacity ν on 2^N we have (see, e.g., [18]):

$$\int f\,d\nu = \bigvee_{A \mid \nu(A)=1} \bigwedge_{i\in A} f(i). \tag{3}$$

The expression of the Choquet integral w.r.t. the Möbius transform of ν (denoted by m) is

$$\int f\,d\nu = \sum_{A\subseteq N} m(A) \bigwedge_{i\in A} f(i). \tag{4}$$

We introduce $\mathcal{Q}(N) := \{(A,B) \in 2^N \times 2^N \mid A \cap B = \varnothing\}$.

Definition 3 *(i) A mapping $v : \mathcal{Q}(N) \to \mathbb{R}$ such that $v(\varnothing,\varnothing) = 0$ is called a* bicooperative game *[2].*

(ii) A bicooperative game v such that $v(A,B) \leq v(C,D)$ whenever $(A,B),(C,D) \in \mathcal{Q}(N)$ with $A \subseteq C$ and $B \supseteq D$ (monotonicity) is called a bicapacity*[9, 11]. Moreover, a bicapacity is normalized if in addition $v(N,\varnothing) = 1$ and $v(\varnothing,N) = -1$.*

Definition 4 *Let v be a bicapacity and f be a real-valued function on N. The* (general) Choquet integral *of f w.r.t v is given by*

$$\int f\,dv := \int |f|\,d\nu_{N_f^+}$$

where $\nu_{N_f^+}$ is a game on N defined by

$$\nu_{N_f^+}(C) := v(C \cap N_f^+, C \cap N_f^-), \quad \forall C \subseteq N$$

and $N_f^+ := \{i \in N \mid f(i) \geq 0\}$, $N_f^- = N \setminus N_f^+$.

Note that the definition remains valid if v is a bicooperative game.

Considering on $\mathcal{Q}(N)$ the product order

$$(A,A') \subseteq (B,B') \Leftrightarrow A \subseteq B \text{ and } A' \subseteq B',$$

the Möbius transform b of a bicapacity v is the solution of:

$$\begin{aligned} v(A_1,A_2) &= \sum_{(B_1,B_2)\subseteq(A_1,A_2)} b(B_1,B_2) \\ &= \sum_{\substack{B_1\subseteq A_1 \\ B_2\subseteq A_2}} b(B_1,B_2). \end{aligned}$$

This gives:

$$b(A_1,A_2) = \sum_{\substack{B_1\subseteq A_1 \\ B_2\subseteq A_2}} (-1)^{|A_1\setminus B_1|+|A_2\setminus B_2|} v(B_1,B_2)$$

(see Fujimoto [7]). Unanimity games are then naturally defined by

$$u_{(A_1,A_2)}(B_1,B_2) := \begin{cases} 1, & \text{if } (B_1,B_2) \supseteq (A_1,A_2) \\ 0, & \text{else.} \end{cases}$$

and form a basis of bicooperative games.

The expression of the Choquet integral in terms of b is given by

$$\int f\,dv = \sum_{(A_1,A_2)\in\mathcal{Q}(N)} b(A_1,A_2)\Big[\bigwedge_{i\in A_1} f^+(i) \wedge \bigwedge_{j\in A_2} f^-(j)\Big], \tag{5}$$

with $f^+ := f \vee 0$ and $f^- := (-f)^+$.

2.2 Lattices, geometric realizations, and triangulation

A *lattice* is a set L endowed with a partial order $\leq$ such that for any $x, y \in L$ their least upper bound $x \vee y$ and greatest lower bound $x \wedge y$ always exist. For finite lattices, the greatest element of L (denoted $\top$) and least element $\perp$ always exist. x *covers* y (denoted $x \succ y$) if $x > y$ and there is no z such that $x > z > y$. A sequence of elements $x \leq y \leq \cdots \leq z$ of L is called a *chain* from x to z, while an *antichain* is a sequence of elements such that it contains no pair of comparable elements. A chain from x to z is *maximal* if no element can be added in the chain, i.e., it has the form $x \prec y \prec \cdots \prec z$.

The lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it cannot be expressed as a supremum of other elements. Equivalently, j is join-irreducible if it covers only one element. Join-irreducible elements covering $\perp$ are called *atoms*, and the lattice is *atomistic* if all join-irreducible elements are atoms. The set of all join-irreducible elements of L is denoted $\mathcal{J}(L)$.

For any $x \in L$, we say that x *has a complement in* L if there exists $x' \in L$ such that $x \wedge x' = \perp$ and $x \vee x' = \top$. The complement is unique if the lattice is distributive.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way. We denote by $\eta(x)$ the *(normal) decomposition* of x, defined as the set of join-irreducible elements smaller or equal to x, i.e., $\eta(x) := \{j \in \mathcal{J}(L) \mid j \leq x\}$. Hence

$$x = \bigvee_{j\in\eta(x)} j$$

(throughout the paper, $j, j', \ldots$ will always denote join-irreducible elements). Note that this decomposition may be redundant.

We can rephrase differently the above result in several ways, which will be useful for the sequel. $Q \subseteq L$ is a *downset* of L if $x \in Q$, $y \in L$ and $y \leq x$ imply $y \in Q$. For any subset P of L, we denote by $\mathcal{O}(P)$ the set of all downsets of P. Then the mapping η is an isomorphism of L onto $\mathcal{O}(\mathcal{J}(L))$ (Birkhoff's theorem [3]). Also,

$$\eta(x \vee y) = \eta(x) \cup \eta(y), \quad \eta(x \wedge y) = \eta(x) \cap \eta(y) \tag{6}$$

if L is distributive. Next, downsets of some partially ordered set P correspond bijectively to nonincreasing mappings from P to $\{0,1\}$. Let us denote by $\mathcal{D}(P)$ the set of all nonincreasing mappings from P to $\{0, 1\}$. Then Birkhoff's theorem can be rephrased as follows: *any distributive lattice L is isomorphic to $\mathcal{D}(\mathcal{J}(L))$*. Finally, note that a mapping of $\mathcal{D}(P)$ can be considered as a vertex of $[0,1]^{|P|}$. In summary, we have:

$$x \in L \leftrightarrow \eta(x) \in \mathcal{O}(\mathcal{J}(L)) \leftrightarrow 1_{\eta(x)} \in \mathcal{D}(\mathcal{J}(L)) \leftrightarrow (1_{\eta(x)}, 0_{\eta(x)^c}) \in [0,1]^{|\mathcal{J}(L)|} \qquad (7)$$

where the notation $(1_A, 0_{A^c})$ denotes a vector whose coordinates are 1 if in A, and 0 otherwise. All arrows represent isomorphisms, the leftmost one being an isomorphism if L is distributive.

We introduce now the notion of *geometric realization* of a lattice, and its natural triangulation (see Koshevoy [14] for more details). For any partially ordered set P, we define $\mathcal{C}(P)$ as the set of nonincreasing mappings from P to $[0, 1]$. It is a convex polyhedron, whose set of vertices is $\mathcal{D}(P)$.

Definition 5 *The* geometric realization *of a distributive lattice L is the set $\mathcal{C}(\mathcal{J}(L))$.*

The *natural triangulation* of $\mathcal{C}(\mathcal{J}(L))$ consists in partitioning $\mathcal{C}(\mathcal{J}(L))$ into simplices whose vertices are in $\mathcal{D}(\mathcal{J}(L))$. These simplices correspond to maximal chains of $\mathcal{D}(\mathcal{J}(L))$. The following proposition summarizes all what we need in the sequel.

Proposition 1 *Suppose that L is distributive, with n join-irreducible elements. Consider any maximal chain $C := \{1_\varnothing = 0 \prec 1_{X_1} \prec \cdots \prec 1_{X_{|\mathcal{J}(L)|}} = 1\}$. Then*

(i) The simplex $\sigma(C)$ is n-dimensional, and contains vertices $(0, \dots, 0)$ and $(1, \dots, 1)$ in $[0, 1]^n$.

(ii) The sequence $X_1, \dots, X_n$ induces a permutation $\pi : \{1, \dots, n\} \to \mathcal{J}(L)$ such that $X_i = \{\pi(1), \dots, \pi(i)\}$, $i = 1, \dots, n$, and

$$f(j) = \sum_{i=1}^{n} \alpha_i 1_{X_i}(j) = \sum_{X_i \ni j} \alpha_i = \sum_{i=\pi^{-1}(j)}^{n} \alpha_i, \quad \forall j \in \mathcal{J}(L). \qquad (8)$$

Conversely, a permutation π induces a maximal chain if and only if it fulfills the condition

$$\forall j, j' \in \mathcal{J}(L), j \leq j' \Rightarrow \pi^{-1}(j) \leq \pi^{-1}(j').$$

(iii) The solution of (8) is

$$\alpha_i = f(\pi(i)) - f(\pi(i+1)), \quad i = 1, \dots, n-1, \text{ and } \alpha_n = f(\pi(n)), \qquad (9)$$

and $\alpha_0 = 1 - \sum_{i=1}^{n} \alpha_i = 1 - f(\pi(1))$. In addition, $f(\pi(1)) \geq f(\pi(2)) \geq \cdots \geq f(\pi(n))$.

Definition 6 *For any functional $F : \mathcal{D}(\mathcal{J}(L)) \to \mathbb{R}$ on a distributive lattice L, its* natural extension to the geometric realization of L *is defined by:*

$$\overline{F}(f) := \sum_{i=0}^{p} \alpha_i F(1_{X_i})$$

for all $f \in \mathrm{int}(\sigma(C))$, with C being a chain $\{1_{X_0} < 1_{X_1} < \cdots < 1_{X_p}\}$ in $\mathcal{D}(\mathcal{J}(L))$, and $\sigma(C)$ its convex hull in $\mathcal{C}(\mathcal{J}(L))$, with $f = \sum_{i=0}^{p} \alpha_i 1_{X_i}$.

The following proposition readily follows from Proposition 1 and the above definition.

Proposition 2 *Let L be a distributive lattice, with n join-irreducible elements, and any functional $F : \mathcal{D}(\mathcal{J}(L)) \to \mathbb{R}$. Consider any maximal chain $C := \{1_\varnothing = 0 \prec 1_{X_1} \prec \cdots \prec 1_{X_{|\mathcal{J}(L)|}} = 1\}$.*

(i) For any $f \in \sigma(C)$,

$$\overline{F}(f) = \sum_{i=1}^{n} [f(\pi(i)) - f(\pi(i+1))] F(1_{\{\pi(1),\dots,\pi(i)\}}) \tag{10}$$

with $f(\pi(n+1)) := 0$.

(ii) $\overline{F}$ is linear in each simplex $\sigma(C)$, i.e., $\overline{F}(f+g) = \overline{F}(f) + \overline{F}(g)$ provided that $f, g, f+g$ belongs to the same $\sigma(C)$. Moreover, $\overline{F}$ is linear in F, in the sense that $\overline{F+G}(f) = \overline{F}(f) + \overline{G}(f)$ for any f.

Example 1: If L is the Boolean lattice 2^N, with $N := \{1, \dots, n\}$, then $\mathcal{J}(L) = N$ (atoms). We have $\mathcal{D}(\mathcal{J}(L)) = \{x : N \to \{0,1\}, x \text{ nonincreasing}\}$, but since N is an antichain, there is no restriction on x and $\mathcal{D}(\mathcal{J}(L)) = \{0,1\}^N$, i.e., it is the set of vertices of $[0,1]^n$. Similarly, $\mathcal{C}(\mathcal{J}(L)) = [0,1]^N$, which is the hypercube itself.

Consider now a maximal chain in $\mathcal{D}(\mathcal{J}(L))$, denoted by $C := \{1_{A_0} < 1_{A_1} < \cdots < 1_{A_n}\}$, with $\emptyset =: A_0 \subset A_1 \subset \cdots \subset A_n := N$. It corresponds to a permutation π on N, with $A_i = \{\pi(1), \dots, \pi(i)\}$. Since $\mathcal{J}(L)$ is an antichain, conversely any permutation corresponds to a maximal chain. Using (9), we get

$$\begin{aligned}\overline{F}(f) &= \sum_{i=1}^{n} \alpha_i F(1_{A_i}) \\ &= \sum_{i=1}^{n} [f(\pi(i)) - f(\pi(i+1))] F(1_{\{\pi(1),\dots,\pi(i)\}}),\end{aligned}$$

with the convention $f(\pi(n+1)) := 0$. Putting $\mu(A) := F(1_A)$, we recognize the Choquet integral $\int f\, d\nu$ (see Definition 2). □

This example shows that the Choquet integral is the natural extension of capacities. Hence, by analogy, $\overline{F}(f)$ could be called the *Choquet integral of f w.r.t. F*. Moreover, using Remark 1, we could consider F as a game or capacity defined over a sublattice of the Boolean lattice 2^n.

3 Bipolar structures

3.1 Bipolar extension of L

Definition 7 *Let us consider $(L, \leq)$ an inf-semilattice with bottom element $\perp$. The* bipolar extension *$\widetilde{L}$ of L is defined as follows:*

$$\widetilde{L} := \{(x, y) \mid x, y \in L, x \wedge y = \perp\},$$

which we endow with the product order $\leq$ on L^2.

Remark that $\widetilde{L}$ is a downset of L^2. The following holds.

Proposition 3 *Let $(L, \leq)$ be an inf-semilattice.*

(i) $(\widetilde{L}, \leq)$ *is an inf-semilattice whose bottom element is $(\perp, \perp)$, where $\leq$ is the product order on L^2.*

(ii) The set of join-irreducible elements of $\widetilde{L}$ is

$$\mathcal{J}(\widetilde{L}) = \{(j, \perp) \mid j \in \mathcal{J}(L)\} \cup \{(\perp, j) \mid j \in \mathcal{J}(L)\}.$$

(iii) The normal decomposition writes

$$(x, y) = \bigvee_{j \leq x, j \in \mathcal{J}(L)} (j, \perp) \vee \bigvee_{j \leq y, j \in \mathcal{J}(L)} (\perp, j).$$

Proof: (i) Let us consider $(x, y), (z, t) \in L^2$. Then $(x, y) \wedge (z, t) = (x \wedge z, y \wedge t)$ is the greatest lower bound of (x, y) and (z, t) for the product order. Suppose $x \wedge y = \perp$ and $z \wedge t = \perp$. Then $(x \wedge z) \wedge (y \wedge t) = \perp$ too, which proves that the greatest lower bound always exists in $\widetilde{L}$.

(ii) clear since these are the join-irreducible element of L^2, and they all belong to $\widetilde{L}$.

(iii) clear from (ii). ■

We consider now the Möbius function over $\widetilde{L}$. The aim is to solve

$$f(x, y) = \sum_{(x', y') \leq (x, y), (x', y') \in \widetilde{L}} g(x', y'), \quad \forall (x, y) \in \widetilde{L}, \tag{11}$$

where f, g are real-valued functions on $\widetilde{L}$. The solution is given through the Möbius function on $\widetilde{L}$:

$$g(x, y) = \sum_{\substack{(z,t) \leq (x,y) \\ (z,t) \in \widetilde{L}}} f(z, t) \mu_{\widetilde{L}}((z, t), (x, y)). \tag{12}$$

The following holds.

Proposition 4 *The Möbius function on $\widetilde{L}$ is given by:*

$$\mu_{\widetilde{L}}((z, t), (x, y)) = \mu_L(z, x) \mu_L(t, y).$$

Proof: Let us define $h(x', y) := \sum_{\substack{y' \leq y \\ x' \wedge y' = \perp}} g(x', y')$ for a given $x' \in L$ such that $x' \wedge y = \perp$. Since $y' \leq y$, $x' \wedge y = \perp$ implies $x' \wedge y' = \perp$ too. Hence:

$$h(x', y) = \sum_{y' \leq y} g(x', y'). \tag{13}$$

By a similar argument, note that (11) can be rewritten as

$$f(x, y) = \sum_{x' \leq x} \sum_{y' \leq y} g(x', y'). \tag{14}$$

Putting (13) in (14) gives

$$f(x,y) = \sum_{x' \leq x} h(x',y). \tag{15}$$

Applying Möbius inversion to (13) and (15) gives

$$g(x,y) = \sum_{t \leq y} \mu_L(t,y) h(x,t), \tag{16}$$

for some fixed x, $x \wedge y = \perp$, and

$$h(x,y) = \sum_{z \leq x} \mu_L(z,x) f(z,y) \tag{17}$$

for some fixed y, $x \wedge y = \perp$. Using (17) into (16) leads to, for $(x,y) \in \widetilde{L}$:

$$\begin{aligned} g(x,y) &= \sum_{t \leq y} \mu_L(t,y) \sum_{z \leq x} \mu_L(z,x) f(z,y) \\ &= \sum_{(z,t) \leq (x,y)} \mu_L(z,x)\mu_L(t,y) f(z,y). \end{aligned}$$

Note that in the last equation $(z,t) \in \widetilde{L}$ since $z \leq x$, $t \leq y$ and $x \wedge y = \perp$ imply $z \wedge t = \perp$. Comparing the above last equation with (12) gives the desired result. ∎

Note that as usual, the set of functions $u_{(x,y)}$ defined by

$$u_{(x,y)}(z,t) = \begin{cases} 1, & \text{if } (z,t) \geq (x,y) \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

forms a basis of the functions on $\widetilde{L}$.

Theorem 1 *Let L be a finite distributive lattice, and $c(L)$ be the set of its complemented elements. Then, for any $x \in c(L)$, its complement being denoted by x', the interval $L(x)$ of $\widetilde{L}$ defined by*

$$L(x) := [(\perp,\perp),(x,x')]$$

and endowed with the product order of L^2 is isomorphic to L, by the order isomorphism $\phi_x : L(x) \to L$, $(y,z) \mapsto y \vee z$. The inverse function ϕ_x^{-1} is given by $\phi_x^{-1}(w) = (w \wedge x, w \wedge x')$.

Moreover, the join-irreducible elements of $L(x)$ are the image of those of L by ϕ_x^{-1}, i.e.:

$$\mathcal{J}(L(x)) = \{(j \wedge x, j \wedge x') \mid j \in \mathcal{J}(L)\}.$$

Proof: Take $x \in c(L)$ and show that ϕ_x is an order isomorphism between $L(x)$ and L. First remark that if $y,z \in L$, then $y \vee z \in L$ since L is a lattice. Also for any $(y,z) \in L(x)$, since $y \leq x$, we have $\eta(y) \subseteq \eta(x)$, and similarly $\eta(z) \subseteq \eta(x')$.

Let us show that ϕ_x is a bijection. Observe that since $x \wedge x' = \perp$ and $x \vee x' = \top$, we have $\eta(x) \cap \eta(x') = \varnothing$ and $\eta(x) \cup \eta(x') = \mathcal{J}(L)$ by (6), i.e., x and x' partition the join-irreducible elements of L. It follows that any $w \in L$ can be written uniquely as $w = y \vee z$, with $y, z \in L$ defined by

$$\eta(y) = \eta(w) \cap \eta(x), \quad \eta(z) = \eta(w) \cap \eta(x'). \tag{19}$$

Then $(y, z) \in L(x)$ since $\eta(y) \subseteq \eta(x)$ and $\eta(z) \subseteq \eta(x')$. The expression of the inverse isomorphism $\phi_x^{-1}(w) = (w \wedge x, w \wedge x')$ is clear from (19) and (6).

Take $(y, z) \leq (y', z')$. This means $y \leq y'$ and $z \leq z'$, hence $y \vee z \leq y' \vee z'$. Conversely, take $w \leq w'$. We have $y = w \wedge x \leq w' \wedge x = y'$ and similarly for $z = w \wedge x'$. Hence ϕ_x is an order isomorphism.

Finally, since ϕ_x is an order isomorphism, the two lattices L and $L(x)$ have the same structure, and hence the same join-irreducible elements. ■

Remark that in any finite lattice, $\perp$ and $\top$ are complemented elements, and $L(\top) = L$, $L(\perp) = L^*$, where L^* is the dual of L (i.e., L with the reverse order). An interesting question is whether the union of all $L(x)$, $x \in c(L)$, is equal to $\widetilde{L}$.

Theorem 2 *Let L be a finite distributive lattice. Then the bipolar extension $\widetilde{L}$ can be written as:*

$$\widetilde{L} = \bigcup_{x \in c(L)} L(x)$$

if and only if $\mathcal{J}(L)$ has all its connected components with a single bottom element.

Proof: Take $(y, z) \in \widetilde{L}$, i.e., $y, z \in L$ and $\eta(y) \cap \eta(z) = \varnothing$. To find $x \in c(L)$ such that $(y, z) \in L(x)$ is equivalent to satisfy the conditions

(i) $\mathcal{J}(L) \setminus \eta(x)$ is a downset (x is complemented)

(ii) $\eta(x) \supseteq \eta(y)$, and $\eta(x) \cap \eta(z) = \varnothing$ ((y, z) belongs to $L(x)$).

Consider $\mathcal{J}(L)$. Its Hasse diagram is formed of connected components, say $J_1, \ldots, J_l$. Remark that in a given connected component J_k, it is not possible to partition it into downsets. Indeed, suppose that $J_k = D_1 \cup D_2$, with D_1, D_2 two disjoint nonempty downsets. Since J_k is connected, each $x \in J_k$ is comparable with another $y \in J_k$. Hence, by nonemptiness assumption, there exists $x_1 \in D_1$ which is comparable with some $x_2 \in D_2$, i.e., either $x_1 \leq x_2$ or the converse. But then $x_1 \in D_2$ (or $x_2 \in D_1$), which contradicts the fact they are disjoint. This proves that complemented elements $x \in L$ are such that

$$\eta(x) = \cup_{k \in K(x)} J_k \tag{20}$$

for some index set $K(x) \subseteq \{1, \ldots, l\}$.

Take some $(y, z) \in \widetilde{L}$ and suppose that $\eta(y) \subseteq \cup_{k \in K(y)} J_k$ and $\eta(z) \subseteq \cup_{k \in K(z)} J_k$. Suppose that all J_k's have a single bottom element $\perp_k$. Then necessarily, $K(y) \cap K(z) = \varnothing$, otherwise $\eta(y) \cap \eta(z) = \varnothing$ would not be true. Then it suffices to take $K(x) := K(y)$, $K(x') = \{1, \ldots, l\} \setminus K(x)$ and the conditions (i) and (ii) above are satisfied. Conversely, assume that there exist some connected component J_k with two bottom elements, say $\perp_k$ and $\perp'_k$. Consider y, z such that $\eta(y) = \perp_k$ and $\eta(z) = \perp'_k$. Then $(y, z) \in \widetilde{L}$, but due to (20), no x can satisfy condition (ii) above. ■

Example 1 (ctd): Consider $L = 2^N$. Then $\widetilde{L} = \mathcal{Q}(N)$. Since 2^N is Boolean, any element $A \subseteq N$ is complemented ($A' = A^c$), and $2^N(A) = [(\varnothing,\varnothing),(A,A^c)]$. Obviously the conditions of Theorem 2 are satisfied, thus

$$\mathcal{Q}(N) = \bigcup_{A\subseteq N} [(\varnothing,\varnothing),(A,A^c)].$$

This important result shows that $\widetilde{L}$ is composed by "tiles", all identical to L (note however that the union is not disjoint). This suggests the following definition.

Definition 8 *Let L be a finite distributive lattice, and $\widetilde{L}$ its bipolar extension. $\widetilde{L}$ is said to be a* regular mosaic *if $\mathcal{J}(L)$ has all its connected components with a single bottom element.*

There are two important particular cases of regular mosaics:

(i) L is a product of m linear lattices (totally ordered). Then

$$c(L) = \{(\top_A, \bot_{A^c}) \mid A \subseteq \{1,\dots,m\}\}$$

where $(\top_A, \bot_{A^c})$ has coordinate number i equal to $\top_i$ if $i \in A$, and $\bot_i$ otherwise. Also, $(\top_A, \bot_{A^c})' = (\bot_A, \top_{A^c})$. This case covers Boolean lattices (case of capacities), and lattices of the form k^m, which we will address in Section 4.

(ii) $\mathcal{J}(L)$ has a single connected component with one bottom element. Then $\widetilde{L}$ contains only elements of the form $(y,\bot)$ or $(\bot,z)$, i.e., $\widetilde{L} = L(\bot) \cup L(\top)$.

The following example shows a case where $\tilde{L}$ is not a regular mosaic.

Example 2: we consider L and $\mathcal{J}(L)$ given on Figure 1. Obviously, $\mathcal{J}(L)$ does not satisfy the condition for producing a regular mosaic, and as it can be seen on Fig. 2, the bipolar structure cannot be obtained as a replication of L.

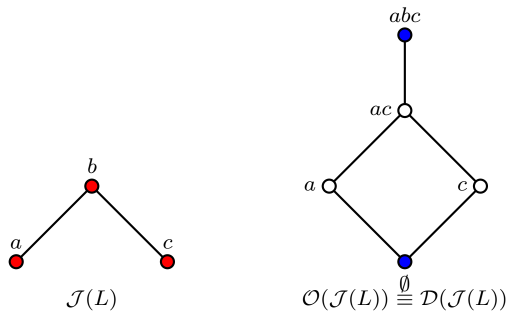

Figure 1: A lattice L and the associated $\mathcal{J}(L)$. In grey, the complemented elements

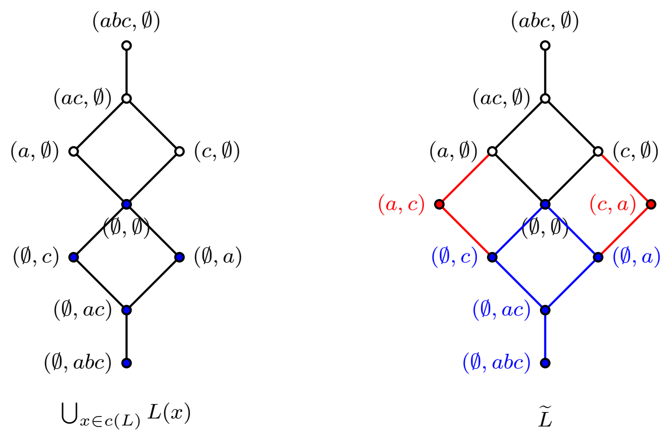

Figure 2: Left: bipolar structure computed as a replication of L. Right: the true bipolar structure

3.2 Bipolar geometric realization

Since $\widetilde{L}$ is not a distributive lattice, it is not possible to define its geometric realization in the sense of Def. 5. Assuming that $\widetilde{L}$ is a regular mosaic, we propose the following definition.

Definition 9 *Let $\widetilde{L}$ be a regular mosaic, and $x \in c(L)$. We consider the mappings $\xi_x : \mathcal{J}(L) \to \{-1, 0, 1\}$ such that*

(i) $|\xi_x|$ is nonincreasing

(ii) $\xi_x(j) \geq 0$ if $j \in \eta(x)$

(iii) $\xi_x(j) \leq 0$ if $j \in \eta(x')$.

The set of such functions is denoted by $\mathcal{D}_x(\mathcal{J}(L))$. Similarly, we introduce

$$\mathcal{C}_x(\mathcal{J}(L)) := \{f_x : \mathcal{J}(L) \to [-1, 1] \text{ such that } |f_x| \text{ is nonincreasing},\\ f_x(j) \geq 0 \text{ if } j \in \eta(x), f_x(j) \leq 0 \text{ if } j \in \eta(x')\}. \quad (21)$$

Then the bipolar geometric realization of L *is*

$$\widetilde{|L|} := \bigcup_{x \in c(L)} \mathcal{C}_x(\mathcal{J}(L)).$$

Proposition 5 *For any $x \in c(L)$, $\mathcal{D}_x(\mathcal{J}(L))$ is the set of vertices of $\mathcal{C}_x(\mathcal{J}(L))$.*

Proof: It is plain that any ξ_x is a vertex of $\mathcal{C}_x(\mathcal{J}(L))$. Conversely, assume f_x is a vertex such that for some $j \in \mathcal{J}(L)$, $f_x(j) = \alpha > 0$ (or < 0). Then we define

$$f^+(j) := f_x(j) + \epsilon, \quad f^-(j) := f_x(j) - \epsilon,$$

and $f^+ = f^- = f_x$ elsewhere, choosing $0 < \epsilon < \alpha$ small enough so that $|f^+|, |f^-|$ remain nonincreasing. Then f^+, f^- belong to $\mathcal{C}_x(\mathcal{J}(L))$, and $f_x = \frac{1}{2}(f^+ + f^-)$, which proves that f_x is not a vertex. ■

Proposition 6 *Let $x \in c(L)$. There is a bijection $\psi_x : \mathcal{D}_x(\mathcal{J}(L)) \to L(x)$ defined by $\psi_x(\xi) := (y_\xi, z_\xi)$ with*

$$\eta(y_\xi) = \{j \in \mathcal{J}(L) \mid \xi(j) = 1\}, \quad \eta(z_\xi) = \{j \in \mathcal{J}(L) \mid \xi(j) = -1\}, \tag{22}$$

and the inverse function is defined by $\psi_x^{-1}(y, z) := \xi_{(y,z)}$ with

$$\xi_{(y,z)}(j) := \begin{cases} 1, & \text{if } j \in \eta(y) \\ -1, & \text{if } j \in \eta(z) \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

for any $j \in \mathcal{J}(L)$, or in more compact form

$$\xi_{(y,z)} = 1_{\eta(y)} - 1_{\eta(z)}.$$

Proof: Since $|\xi|$ is nonincreasing, $\{j \in \mathcal{J}(L) \mid \xi(j) = 1\}$ and $\{j \in \mathcal{J}(L) \mid \xi(j) = -1\}$ are downsets. Hence y_ξ, z_ξ are well-defined, and by construction $(y_\xi, z_\xi) \in L(x)$.

Let us show that $|\xi_{(y,z)}|$ is nonincreasing. Assume $\xi_{(y,z)}(j) = 1$ or -1. Then $j \in \eta(y) \cup \eta(z)$. Since these are downsets, any $j' \leq j$ belongs also to $\eta(y) \cup \eta(z)$. Assume $\xi_{(y,z)}(j) = 0$, i.e., $j \notin \eta(y) \cup \eta(z)$. Then $j' \geq j$ cannot belong to $\eta(y) \cup \eta(z)$ since they are downsets, hence $\xi_{(y,z)}(j') = 0$.

Finally, ψ_x is one-to-one because L is distributive, and so is $L(x)$ (Birkhoff's theorem). ■

Example 1 (ctd): Consider $L = 2^N$, and some $N^+ \subseteq N$, $N^- := N \setminus N^+$. Then

$$\mathcal{D}_{N^+}(N) = \Big\{\xi_{N^+} : N \to \{-1, 0, 1\} \text{ such that } (\xi_{N^+})_{|N^+} \geq 0, \quad (\xi_{N^+})_{|N^-} \leq 0\Big\}.$$

Moreover, $\psi_{N^+}(\xi_{N^+}) = (\{j \in N \mid \xi_{N^+}(j) = 1\}, \{j \in N \mid \xi_{N^+}(j) = -1\})$.

Figure 3 should make things clear for notions introduced till this point. Observe that functions $\xi_x \in \mathcal{D}_x(\mathcal{J}(L))$ corresponds to a subset of points of $[-1, 1]^{|\mathcal{J}(L)|}$ of the form $(1_A, (-1)_B, 0_{(A\cup B)^c})$, with $A \subseteq \eta(x)$ and $B \subseteq \eta(x')$, and that $\mathcal{C}_x(\mathcal{J}(L))$ is the convex hull of these points.

We end this section by addressing the natural triangulation of the bipolar geometric realization. Let us consider some f in $\mathcal{C}(\mathcal{J}(L))$, assuming $f = \sum_{i=0}^p \alpha_i 1_{X_i}$, with $1_{X_0}, \ldots, 1_{X_p}$ forming a chain in $\mathcal{D}(\mathcal{J}(L))$. Given $x \in c(L)$, let us define the corresponding f_x in $\mathcal{C}_x(\mathcal{J}(L))$ as follows:

$$\begin{aligned} f_x &:= \sum_{i=0}^p \alpha_i \psi_x^{-1}(\phi_x^{-1}(\eta^{-1}(X_i))) \\ &= \sum_{i=0}^p \alpha_i (1_{X_i \cap \eta(x)} - 1_{X_i \cap \eta(x')}). \end{aligned}$$

$$
\begin{tikzcd}
y = t \wedge x, z = t \wedge x' & & 1_{\eta(t)\cap\eta(x)} - 1_{\eta(t)\cap\eta(x')} & & f_x = \sum_i \alpha_i(1_{X_i\cap\eta(x)} - 1_{X_i\cap\eta(x')}) \\
(y,z) & & 1_{\eta(y)} - 1_{\eta(z)} & (1_{\eta(y)}, -1_{\eta(z)}, 0_{(\eta(y)\cup\eta(z))^c}) & \\
\Cap & & \Cap & & \\
L(x) \arrow[rr, shift left, leftarrow, "\psi_x"] \arrow[rr, shift right, "\psi_x^{-1}"'] \arrow[d, shift right, "\phi_x"'] & & \mathcal{D}_x(\mathcal{J}(L)) \arrow[r, leftrightarrow] \arrow[rr, bend left, "\text{convex hull}"] \arrow[d, shift right, "|\cdot|"'] & \mathrm{ter}([-1,1]^{|\mathcal{J}(L)|}) \arrow[d, shift right, "|\cdot|"'] & \mathcal{C}_x(\mathcal{J}(L)) \arrow[ll, bend left, "\text{vertices}"] \arrow[d, shift right, "|\cdot|"'] \\
L \arrow[u, shift right, "\phi_x^{-1}"'] \arrow[r, leftrightarrow, "\eta"'] & \mathcal{O}(\mathcal{J}(L)) \arrow[r, leftrightarrow, "1_\eta"'] & \mathcal{D}(\mathcal{J}(L)) \arrow[u, shift right] \arrow[r, leftrightarrow] \arrow[rr, bend left, "\text{convex hull}"] & \mathrm{ext}([0,1]^{|\mathcal{J}(L)|}) \arrow[u, shift right] & \mathcal{C}(\mathcal{J}(L)) \arrow[ll, bend left, "\text{vertices}"] \arrow[u, shift right] \\
\Cup & \Cup & \Cup & & \Cup \\
t & \eta(t) & 1_{\eta(t)} & (1_{\eta(t)}, 0_{\eta(t)^c}) & f = \sum_i \alpha_i 1_{X_i} \\
t = y \vee z & & 1_{\eta(y)\cup\eta(z)} & &
\end{tikzcd}
$$

Figure 3: Relations among various concepts introduced. $|\cdot|$ indicates absolute value, and ter() indicates vectors whose components are -1, 1 or 0.

Explicitely, this gives, for any $j \in \mathcal{J}(L)$:

$$f_x(j) = \begin{cases} \sum_{i|j\in X_i} \alpha_i, & \text{if } j \in \eta(x) \\ -\sum_{i|j\in X_i} \alpha_i, & \text{if } j \in \eta(x'). \end{cases}$$

Hence $|f_x|$ takes value 1 on X_0, $1 - \alpha_0$ on $X_1 \setminus X_0$, etc., and is nonincreasing.

Remark that $|f_x| = f$ if $f \in \mathcal{C}(\mathcal{J}(L))$, and $|f|_x = f$ if $f \in \mathcal{C}_x(\mathcal{J}(L))$.

3.3 Natural interpolation on bipolar structures

Again we suppose that $\widetilde{L}$ is a regular mosaic. Assume $F : \bigcup_{x\in c(L)} \mathcal{D}_x(\mathcal{J}(L)) \to \mathbb{R}$ is given. We want to define the extension $\overline{F}$ of this functional on the bipolar geometric realization $|\widetilde{L}|$.

Let us take $f \in |\widetilde{L}| = \bigcup_{x\in c(L)} \mathcal{C}_x(\mathcal{J}(L))$. First, we must choose $x \in c(L)$ such that f belongs to $\mathcal{C}_x(\mathcal{J}(L))$ (x is not unique in general since in the definition of $|\widetilde{L}|$ the union is not disjoint (see Def. 9)). Defining

$$\mathcal{J}(L)^+ := \{j \in \mathcal{J}(L) \mid f(j) \geq 0\}, \quad \mathcal{J}(L)^- := \mathcal{J}(L) \setminus \mathcal{J}(L)^+,$$

it suffices to take x, x' defined by

$$\eta(x) := \bigcup_{k\in K} J_k, \quad \eta(x') := \mathcal{J}(L) \setminus \eta(x)$$

with K the smallest one such that $\mathcal{J}(L)^+ \subseteq \bigcup_{k\in K} J_k$ (using notations of proof of Theorem 2). Now, consider $|f|$, which belongs to $\mathcal{C}(\mathcal{J}(L))$, and its expression using the natural triangulation:

$$|f| = \sum_{i=0}^{p} \alpha_i 1_{X_i}$$

with $1_{X_0}, \dots, 1_{X_p}$ a chain in $\mathcal{D}(\mathcal{J}(L))$. Then we have $|f|_x = f$, and we propose the following definition.

Definition 10 *Assume $\widetilde{L}$ is a regular mosaic. For any functional $F : \bigcup_{x\in c(L)} \mathcal{D}_x(\mathcal{J}(L)) \rightarrow \mathbb{R}$, its* natural extension to the bipolar geometric realization *of $\widetilde{L}$ is defined by:*

$$\overline{F}(f) := \sum_{i=0}^{p} \alpha_i F_x(1_{X_i})$$

for all $f \in \mathcal{C}_x(\mathcal{J}(L))$, letting $|f| := \sum_{i=0}^{p} \alpha_i 1_{X_i}$ for some chain $\{1_{X_0} < 1_{X_1} < \cdots < 1_{X_p}\}$ in $\mathcal{D}(\mathcal{J}(L))$, and $F_x : \mathcal{D}(\mathcal{J}(L)) \rightarrow \mathbb{R}$ defined by:

$$F_x(1_{X_i}) := F(1_{X_i\cap\eta(x)} - 1_{X_i\cap\eta(x')}).$$

Example (end): Let us take once more $L = 2^N$. For a given f, we define $N^+ := \{j \in N \mid f(j) \geq 0\}$ and $N^- := N \setminus N^+$, we have:

$$\overline{F}(f) = \sum_{i=1}^{n} \alpha_i F_{N^+}(1_{X_i}) = \sum_{i=1}^{n} \big[|f(\pi(i))| - |f(\pi(i+1))|\big] F(1_{X_i\cap N^+} - 1_{X_i\cap N^-}),$$

where we have used (9). Putting $v(A,B) := F(1_A - 1_B)$, we recognize the Choquet integral for bicapacities (see Definition 4). $\square$

Remark 1: Definition 10 can be written equivalently as $\overline{F}(f) = \overline{F_x}(|f|)$, making clear the relation between the functional on L and on $\widetilde{L}$.

Lastly, we address the problem of expressing $\overline{F}$ in terms of the Möbius transform of F, using Prop. 4. For this purpose, it is better to turn a given functional F on $\bigcup_{x\in c(L)} \mathcal{D}_x(\mathcal{J}(L))$ into its equivalent form $\widetilde{F}$ defined on $\widetilde{L}$, thanks to the mappings ψ_x, $x \in c(L)$. Doing so, we can use Prop. 4 and (12), and get the Möbius transform of $\widetilde{F}$, which we denote by $\widetilde{m}$:

$$\widetilde{m}(x,y) = \sum_{\substack{(z,t)\leq(x,y)\\ (z,t)\in\widetilde{L}}} \widetilde{F}(z,t)\mu_L(z,x)\mu_L(t,y), \quad \forall (x,y) \in \widetilde{L}.$$

We need the following result, which is a generalization of (3).

Lemma 1 *Let $f \in \mathcal{C}(\mathcal{J}(L))$ and $F : \mathcal{D}(\mathcal{J}(L)) \rightarrow \{0,1\}$ being nondecreasing and 0-1 valued. Then*

$$\overline{F}(f) = \bigvee_{\substack{T\subseteq\mathcal{J}(L)\\ F(1_T)=1}} \bigwedge_{j\in T} f(j).$$

Proof: (adaptation from [18]) Using notations of Proposition 1, define $i_0 \in \mathcal{J}(L)$ such that

$$f(\pi(i_0)) = \bigvee_{\substack{T \subseteq \mathcal{J}(L) \\ F(1_T)=1}} \bigwedge_{j \in T} f(j).$$

Assume for simplicity that $f(\pi(1)) > f(\pi(2)) > \cdots > f(\pi(n))$, and let us show that

$$F(1_{\{\pi(1),\ldots,\pi(i)\}}) = \begin{cases} 1, & \text{if } i \geq i_0 \\ 0, & \text{else.} \end{cases}$$

Assume $i \geq i_0$. Then for any $T \subseteq \mathcal{J}(L)$ such that $F(1_T) = 1$, we have $f(\pi(i)) \leq \wedge_{j \in T} f(j)$. This inequality implies that $T \subseteq \{\pi(1), \ldots, \pi(i)\}$, and hence by monotonicity of F, we get $F(1_{\{\pi(1),\ldots,\pi(i)\}}) = 1$. Now suppose $i < i_0$. If $F(1_{\pi(1),\ldots,\pi(i)}) = 1$, it follows that $f(\pi(i)) > f(\pi(i_0)) \geq \wedge_{j=1}^{i} f(\pi(j)) = f(\pi(i))$, a contradiction. Hence $F(1_{\{\pi(1),\ldots,\pi(i)\}}) = 0$.

Using this result in (10) gives the desired result. ∎

The following is a generalization of (5).

Proposition 7 *With the above notations, for any $f \in \widetilde{|L|}$ and any F on $\bigcup_{x \in c(L)} \mathcal{D}_x(\mathcal{J}(L))$, the following holds:*

$$\overline{F}(f) = \sum_{(s,t) \in \widetilde{L}} \widetilde{m}(s,t) \Big[\bigwedge_{j \in \eta(s)} f^+(j) \wedge \bigwedge_{j \in \eta(t)} f^-(j) \Big],$$

with $f^+ = f \vee 0$, $f^- = (-f)^+$.

Proof: Taking $\widetilde{F} := u_{(s,t)}$ given by (18), we have by Definition 10

$$\overline{F}(f) = \overline{F_x}(|f|)$$

with $\widetilde{F_x}(y) = u_{(s,t)}(y \wedge x, y \wedge x')$ a nondecreasing 0-1 valued function, with value 1 iff $y \wedge x \geq s$ and $z \wedge x' \geq t$. Since L is distributive, this condition writes $[\eta(y) \cap \eta(x) \supseteq \eta(s)$ and $\eta(z) \cap \eta(x') \supseteq \eta(t)]$, which in turn is equivalent to $[\eta(y) \supseteq \eta(s) \cup \eta(t)$ and $\eta(s) \subseteq \eta(x)$ and $\eta(t) \subseteq \eta(x')]$ since x, x' are complemented. Hence, applying the above lemma, we get:

$$\begin{aligned} \overline{F_x}(|f|) &= \bigvee_{\substack{y \geq s \vee t \\ s \leq x \\ t \leq x'}} \bigwedge_{j \in \eta(y)} |f(j)| \\ &= \bigwedge_{j \in \eta(s)} f^+(j) \wedge \bigwedge_{j \in \eta(t)} f^-(j). \end{aligned}$$

Using linearity of $\overline{F_x}$ versus F_x (see Proposition 2 (ii)) and the decomposition of any $\widetilde{F}$ in the basis of functions $u_{(x,y)}$, the result is proved. ∎

4 Application: k-ary bicapacities and Choquet integral

This section is dedicated to the study of the lattice $L := k^n$. We set $N := \{1, \ldots, n\}$. Elements of L are thus vectors in $\{0, 1, \ldots, k-1\}^n$. For commodity (l_A, l'_{-A}) denotes the element t of L with $t_i = l$ if $i \in A$ and l' otherwise, and we put $M := n(k-1)$.

We begin by giving a motivation of this study rooted in multicriteria decision making.

4.1 Multicriteria aggregation with reference levels

Let us consider N as the set of criteria. In the terminology of multicriteria decision making, an *act* or *option* is a mapping $f : N \to \mathbb{R}$, and $f(i)$ is the *score* of option f on criteria i. We may introduce reference levels for scores, and be interested into the overall score of an option taking values only in the set of reference values (such options are called *pure*, or *prototypical*). Since these options are prototypical, the decision maker is able to assess their overall scores. The question arises then to compute the overall score of an option being not pure. Using our framework, there are basically two ways of answering this question. We put $L := k^n$, where k is the number of reference levels, labelled $\{0, 1, \ldots, k-1\}$. Observe that join-irreducible elements are of the form $(l_i, 0_{-i})$, for any $l \in \{1, \ldots, k-1\}$ and $i \in N$ (see below).

The first way is to say that non-pure options belong to a level only to some degree that can be different from the complete membership and the complete non-membership. Thus, as in Fuzzy Set Theory [24], a membership degree is associated to each level and each criterion, i.e., to each join-irreducible element. From a knowledge of these degrees, it is possible to interpolate between the values known for pure options.

More precisely, an option is an element of $\mathcal{C}(\mathcal{J}(L))$. A degree in $[0, 1]$ is thus associated to all join-irreducible elements. It can be interpreted as a membership degree to the class of levels lower or equal to the join-irreducible element. Hence if an option belongs at a given degree δ to a join-irreducible element, it necessarily belongs to a degree greater or equal to δ to less preferred join-irreducible elements. This explains why options shall be non-increasing functions on $\mathcal{J}(L)$.

The second way is to map the lattice onto a subset of $\mathbb{R}^n$ such that the Pareto order on $\mathbb{R}^n$ corresponds precisely to the order relation on the lattice k^n. For this, we map each reference level on $\mathbb{R}$: $\rho_0 < \cdots < \rho_{k-1}$, which represent the score assigned to each level. The lattice corresponds to the nodes of a rectangular mesh in $\mathbb{R}^n$ composed of the k reference levels for each criterion. The generalized capacity gives the value associated to these nodes (i.e., the pure options). The non-pure options are any point inside the mesh. The problem becomes thus an interpolation problem in $\mathbb{R}^n$.

Consider thus an option $x \in [\rho_0, \rho_{k-1}]^n$ and a generalized capacity $F : \mathcal{D}(\mathcal{J}(L)) \to \mathbb{R}$. Let $I(x) \in \{1, \ldots, k\}^n$ such that for any $i \in N$

$$\rho_{I_i(x)-1} \leq x_i \leq \rho_{I_i(x)}.$$

Define $\Phi : [\rho_0, \rho_{k-1}]^n \to [0, 1]^n$ as

$$\Phi_i(x) := \frac{x_i - \rho_{I_i(x)-1}}{\rho_{I_i(x)} - \rho_{I_i(x)-1}}.$$

Define a capacity v_x on N by

$$v_x(S) := F\Big(1_{\bigcup_{i\in N}\left\{(1_i,0_{-i}),\ldots,((I_i(x)-1)_i,0_{-i})\right\}\cup\bigcup_{i\in S}((I_i(x))_i,0_{-i})}\Big)$$

for all $S \subseteq N$. It corresponds to the value on the 2^n nodes of the mesh just around x. One may have $v_x(\emptyset) \neq 0$. Let $v'_x(S) = v_x(S) - v_x(\emptyset)$ and η a permutation on N such that $\Phi_{\eta(1)}(x) \geq \cdots \geq \Phi_{\eta(n)}(x)$.

$$\begin{aligned} v_x(\emptyset) + C_{v'_x}(\Phi(x)) &= v_x(\emptyset) + \sum_{i=1}^{n} (\Phi_{\eta(i)}(x) - \Phi_{\eta(i+1)}(x))\,(v_x(\{\eta(1),\ldots,\eta(i)\}) - v_x(\emptyset)) \\ &= v_x(\emptyset)\,(1 - \Phi_{\eta(1)}(x)) + \sum_{i=1}^{n} (\Phi_{\eta(i)}(x) - \Phi_{\eta(i+1)}(x))\, v_x(\{\eta(1),\ldots,\eta(i)\}). \end{aligned} \tag{24}$$

4.2 The unipolar case

The set $\mathcal{J}(L)$ of join-irreducible elements is

$$\mathcal{J}(L) = \Big\{(l_i, 0_{-i}) \mid l \in \{1,\ldots,k-1\}, i \in N\Big\}.$$

It is a poset with n connected components, each of them being the linear lattice $\{1,\ldots k-1\}$.

Let us consider f an element of $\mathcal{C}(\mathcal{J}(L))$. We set for commodity $f_i^l := f(l_i, 0_{-i})$. The natural triangulation of $\mathcal{C}(\mathcal{J}(L))$ is done through chains in $\mathcal{D}(\mathcal{J}(L))$, and maximal chains correspond to some permutations on $\mathcal{J}(L)$ (see Proposition 1). For commodity to each permutation $\pi : \{1,\ldots,M\} \to \mathcal{J}(L)$ we assign two functions $\lambda : \{1,\ldots,M\} \to \{1,\ldots,k-1\}$ and $\theta : \{1,\ldots,M\} \to \{1,\ldots,n\}$ such that $\pi(i) = (\lambda(i)_{\theta(i)}, 0_{-\theta(i)})$, for all $i \in \{1,\ldots,M\}$.

Applying Proposition 1 again, we know that for any element f of a simplex of $\mathcal{C}(\mathcal{J}(L))$ corresponding to a permutation π on $\mathcal{J}(L)$, we have

$$f_{\theta(1)}^{\lambda(1)} \geq f_{\theta(2)}^{\lambda(2)} \geq \cdots \geq f_{\theta(M)}^{\lambda(M)}$$

and

$$f(l_p, 0_{-p}) = \sum_{i\in\{1,\ldots,M\}} \alpha_i 1_{X_i}(l_p, 0_{-p})$$

where $X_i := \{(\lambda(1)_{\theta(1)}, 0_{-\theta(1)}),\ldots,(\lambda(i)_{\theta(i)}, 0_{-\theta(i)})\}$, $\alpha_i = f_{\theta(i)}^{\lambda(i)} - f_{\theta(i+1)}^{\lambda(i+1)}$ for $i \in \{1,\ldots,M-1\}$, and $\alpha_M = f_{\theta(M)}^{\lambda(M)}$.

A *k-ary capacity* is a function $F : \mathcal{D}(\mathcal{J}(L)) \to \mathbb{R}$. Applying Proposition 2 the natural extension of $f \in \mathcal{C}(\mathcal{J}(L))$ w.r.t. F is

$$\overline{F}(f) = \sum_{i=1}^{M} \left[f_{\theta(i)}^{\lambda(i)} - f_{\theta(i+1)}^{\lambda(i+1)}\right] \times F\Big(1_{\{(\lambda(1)_{\theta(1)},0_{-\theta(1)}),\ldots,(\lambda(i)_{\theta(i)},0_{-\theta(i)})\}}\Big),$$

with $f_{\theta(M+1)}^{\lambda(M+1)} := 0$. This could be considered as the Choquet integral of f w.r.t F.

To recover the interpolation formula (24) of Section 4.1, we consider a particular class of elements f in $\mathcal{C}(\mathcal{J}(L))$ satisfying for all $i \in N$

$$\begin{aligned} & f_i^1 = \cdots = f_i^{J_i(f)-1} = 1 \\ & f_i^{J_i(f)} = z_i \\ & f_i^{J_i(f)+1} = \cdots = f_i^{k-1} = 0, \end{aligned}$$

for some given integers $J_1(f), \ldots, J_n(f)$ in $\{1, \ldots, k-1\}$, and real numbers $z_1, \ldots, z_n \in [0,1]$.

Let us denote by σ a permutation on N such that

$$z_{\sigma(1)} \geq \cdots \geq z_{\sigma(n)}.$$

Remark that f belongs to all M-dimensional simplices of $\mathcal{C}(\mathcal{J}(L))$ whose corresponding permutation satisfy:

$$\begin{aligned} \forall i \in \{1, \ldots, q^f\}, &\quad f_{\theta(i)}^{\lambda(i)} = 1 \\ \forall i \in \{q^f + 1, \ldots, q^f + n\}, &\quad f_{\theta(i)}^{\lambda(i)} = z_{\sigma(i - q^f)} \\ \forall i \in \{q^f + n + 1, \ldots M\}, &\quad f_{\theta(i)}^{\lambda(i)} = 0 \end{aligned}$$

where $q^f = \sum_{i \in N} (J_i(f) - 1)$. Hence, f belongs to the interior of a n-dimensional simplex corresponding to the chain

$$1_{X_{q^f}} < 1_{X_{q^f} \cup \{((J_{\sigma(1)}(f))_{\sigma(1)}, 0_{-\sigma(1)})\}} < \cdots < 1_{X_{q^f} \cup \{((J_{\sigma(1)}(f))_{\sigma(1)}, 0_{-\sigma(1)}), \ldots, ((J_{\sigma(n)}(f))_{\sigma(n)}, 0_{-\sigma(n)})\}},$$

with $X_{q^f} := \{(l_i, 0_{-i}) \mid 1 \leq l_i \leq J_i(f) - 1\}$. Then

$$\overline{F}(f) = (1 - z_{\sigma(1)}) F(1_{X_{q^f}}) + \sum_{i=1}^{n} (z_{\sigma(i)} - z_{\sigma(i+1)}) F(1_{X_{q^f} \cup \{((J_{\sigma(1)}(f))_{\sigma(1)}, 0_{-\sigma(1)}), \ldots, ((J_{\sigma(i)}(f))_{\sigma(i)}, 0_{-\sigma(i)})\}}) \tag{25}$$

with $z_{\sigma(n+1)} := 0$. Let $x \in [\rho_0, \rho_{k-1}]^n$ defined by

$$x_i := \rho_{J_i(f)-1} + (\rho_{J_i(f)} - \rho_{J_i(f)-1}) \times z_i \ .$$

Then expression (24) and (25) lead to exactly the same value since $J(f) = I(x)$, $\sigma = \eta$, $z = \Phi(x)$ and

$$v_x(S) := F(1_{X_{q^f} \cup \bigcup_{i \in S} ((J_i(f))_i, 0_{-i})}) \ .$$

4.3 The bipolar case

The bipolarization of L is

$$\tilde{L} = \Big\{ (x, y) \in k^n \times k^n \mid \forall i \in N, \quad x_i \neq 0 \Rightarrow y_i = 0, \text{ and } y_i \neq 0 \Rightarrow x_i = 0 \Big\}.$$

Moreover, the set of complemented elements is

$$c(L) = \Big\{ ((k-1)_A, 0_{-A}) \mid A \subseteq N \Big\},$$

and $((k-1)_A, 0_{-A})' = (0_A, (k-1)_{-A})$. Note that $\tilde{L}$ is a regular mosaic, hence Theorem 2 applies and $\tilde{L}$ is the union of all $L(x)$, with $x \in c(L)$, and

$$L((k-1)_A, 0_{-A}) = \Big\{ \big((x_A, 0_{-A}), (0_A, y_{-A})\big) \mid x \in \{0, \dots, k-1\}^{|A|}, y \in \{0, \dots, k-1\}^{n-|A|} \Big\}.$$

Let $f \in \mathcal{C}_{((k-1)_A, 0_{-A})}(\mathcal{J}(L))$, and $f_i^l := f(l_i, 0_{-i})$. We have $f_i^l \geq 0$ if $i \in A$ and $f_i^l \leq 0$ if $i \notin A$.

We consider a simplex of $\mathcal{C}_{((k-1)_A, 0_{-A})}(\mathcal{J}(L))$ containing f, whose corresponding permutation is $\pi : \{1, \dots, M\} \to \mathcal{J}(L)$, and we define as in Section 4.2 the functions $\lambda : \{1, \dots, M\} \to \{1, \dots, k-1\}$, and $\theta : \{1, \dots, M\} \to \{1, \dots, n\}$. Then

$$\left| f_{\theta(1)}^{\lambda(1)} \right| \geq \left| f_{\theta(2)}^{\lambda(2)} \right| \geq \cdots \geq \left| f_{\theta(M)}^{\lambda(M)} \right|$$

and

$$\left| f(l_p, 0_{-p}) \right| = \sum_{i \in \{1, \dots, M\}} \alpha_i \, 1_{X_i}(l_p, 0_{-p})$$

where $X_i := \{(\lambda(1)_{\theta(1)}, 0_{-\theta(1)}), \dots, (\lambda(i)_{\theta(i)}, 0_{-\theta(i)})\}$, $\alpha_i = \left| f_{\theta(i)}^{\lambda(i)} \right| - \left| f_{\theta(i+1)}^{\lambda(i+1)} \right|$ for $i \in \{1, \dots, M-1\}$, and $\alpha_M = \left| f_{\theta(M)}^{\lambda(M)} \right|$.

A *k-ary bicapacity* is a function $F : \cup_{A \subseteq N} \mathcal{D}_{((k-1)_A, 0_{-A})}(\mathcal{J}(L)) \to \mathbb{R}$.

The natural extension $\overline{F}(f)$ is:

$$\begin{aligned} \overline{F}(f) = \sum_{i=1}^{M} & \left(\left| f_{\theta(i)}^{\lambda(i)} \right| - \left| f_{\theta(i+1)}^{\lambda(i+1)} \right| \right) \times \\ & F\Big(1_{\bigcup_{\substack{q \in \{1, \dots, i\} \\ \theta(q) \in A}} (\lambda(q)_{\theta(q)}, 0_{-\theta(q)})} - 1_{\bigcup_{\substack{q \in \{1, \dots, i\} \\ \theta(q) \notin A}} (\lambda(q)_{\theta(q)}, 0_{-\theta(q)})} \Big), \end{aligned}$$

with $f_{\theta(M+1)}^{\lambda(M+1)} := 0$. As before, this could be considered as the Choquet integral of f w.r.t. F.

We consider now a particular class of elements f in $\mathcal{C}(\mathcal{J}(L))$ satisfying for all $i \in N$

$$\begin{aligned} &|f_i^1| = \cdots = |f_i^{J_i(f)-1}| = 1 \\ &|f_i^{J_i(f)}| = z_i \\ &|f_i^{J_i(f)+1}| = \cdots = |f_i^{k-1}| = 0, \end{aligned}$$

for some given integers $J_1(f), \dots, J_n(f)$ in $\{1, \dots, k-1\}$, and real numbers $z_1, \dots, z_n \in [0, 1]$. Let us denote by σ a permutation on N such that

$$z_{\sigma(1)} \geq \cdots \geq z_{\sigma(n)}.$$

Remark that f belongs to all M-dimensional simplices of $\mathcal{C}(\mathcal{J}(L))$ whose corresponding permutation satisfy:

$$\begin{aligned} \forall i \in \{1, \dots, q^f\}, \quad & f_{\theta(i)}^{\lambda(i)} = 1 \text{ if } \theta(i) \in A, \text{ and } -1 \text{ otherwise} \\ \forall i \in \{q^f + 1, \dots, q^f + n\}, \quad & f_{\theta(i)}^{\lambda(i)} = z_{\sigma(i-q^f)} \text{ if } \theta(i) \in A, \text{ and } -z_{\sigma(i-q^f)} \text{ otherwise} \\ \forall i \in \{q^f + n + 1, \dots M\}, \quad & f_{\theta(i)}^{\lambda(i)} = 0 \end{aligned}$$

where $q^f = \sum_{i\in N}(J_i(f)-1)$. Then

$$\overline{F}(f) = (1-z_{\sigma(1)})V(\varnothing) + \sum_{i=1}^{n}(z_{\sigma(i)} - z_{\sigma(i+1)})V(\{\sigma(1),\dots,\sigma(i)\}), \tag{26}$$

with $z_{\sigma(n+1)} := 0$, and

$$V(S) := F\Big(1_{\big(X_{q^f}\cup\bigcup_{i\in S}((J_i(f))_i,0_{-i})\big)\cap\mathcal{J}(L)^+} - 1_{\big(X_{q^f}\cup\bigcup_{i\in S}((J_i(f))_i,0_{-i})\big)\cap\mathcal{J}(L)^-}\Big),$$

with $X_{q^f} := \{(l_i, 0_{-i}) \mid 1 \leq l_i \leq J_i(f)-1\}$.

The positive part of the scale is represented by the positive levels $\rho_0, \dots \rho_{k-1}$. The negative part of the scale is represented by the negative levels $\rho_{-k+1}, \dots, \rho_0$. Hence $\rho_0 = 0$ is the neutral element demarcarting between attractive and repulsive values. Let $x \in [\rho_{-k+1}, \rho_{k-1}]^n$ defined by

$$x_i := \begin{cases} \rho_{J_i(f)-1} + (\rho_{J_i(f)} - \rho_{J_i(f)-1}) \times z_i & \text{if } i \in A \\ \rho_{-J_i(f)+1} + (\rho_{-J_i(f)} - \rho_{-J_i(f)+1}) \times z_i & \text{if } i \notin A \end{cases}$$

Then proceeding as in Section 4.1, it is easy to see that (26) corresponds exactly to the Choquet integral for k-ary bicapacities defined in [10].

4.4 Example

We end this section by illustrating the above results with $k = 3$ and $n = 2$. Clearly, the case $k = 2$ was already well described (capacities and bicapacities).

Elements in $L := 3^2$ are denoted by pairs (l_1, l_2), with $l_i \in \{0, 1, 2\}$, $i = 1, 2$. We have four join-irreducible elements $(1, 0), (2, 0), (0, 1), (0, 2)$. Let us consider the following function f in $\mathcal{C}(\mathcal{J}(L))$:

$$f(1,0) = 0.5, \quad f(2,0) = 0.1, \quad f(0,1) = 0.3, \quad f(0,2) = 0.2.$$

Note that f is indeed nonincreasing on $\mathcal{J}(L)$. The associated permutation is

$$\pi(1) = (1,0), \quad \pi(2) = (0,1), \quad \pi(3) = (0,2), \quad \pi(4) = (2,0),$$

and the corresponding maximal chain is (expressed in L for simplicity)

$$(0,0) < (1,0) < (1,1) < (1,2) < (2,2).$$

(see Fig. 4 for a diagram of L, and the maximal chain in bold) Supposing F being defined on L, the Choquet integral of f w.r.t. F is given by

$$\begin{aligned}\overline{F}(f) = [f(1,0) - f(0,1)]F(1,0) + [f(0,1) - f(0,2)]F(1,1)& \\ + [f(0,2) - f(2,0)]F(1,2) + f(2,0)F(2,2)&.\end{aligned}$$

Let us turn to the bipolar case. To avoid a heavy notation, elements of $\tilde{L}$ are denoted by (ij, kl) instead of $((i,j),(k,l))$. The set of complemented elements together with their complemented elements is

$$\begin{array}{rl} A = \emptyset : & (0,0) \leftrightarrow (2,2) \\ A = \{1\} : & (2,0) \leftrightarrow (0,2) \\ A = \{2\} : & (0,2) \leftrightarrow (2,0) \\ A = \{1,2\} : & (2,2) \leftrightarrow (0,0) \end{array}$$

Then $\tilde{L} = L(0,0) \cup L(2,0) \cup L(0,2) \cup L(2,2)$, with

$$\begin{aligned} L(0,0) &= \{(00, kl) \mid k,l \in \{0,1,2\}\} \\ L(2,0) &= \{(i0, 0l) \mid i,l \in \{0,1,2\}\} \\ L(0,2) &= \{(0j, k0) \mid j,k \in \{0,1,2\}\} \\ L(2,2) &= \{(ij, 00) \mid i,j \in \{0,1,2\}\}. \end{aligned}$$

Consider the function f defined by

$$f(1,0) = 0.5, \quad f(2,0) = 0.1, \quad f(0,1) = -0.3, \quad f(0,2) = -0.2.$$

Then $A = \{1\}$, $f \in \mathcal{C}_{(2,0)}(\mathcal{J}(L))$, and the permutation π is the same as above. Now, assuming F defined on $\tilde{L}$ is given,

$$\begin{aligned} \overline{F}(f) = [|f(1,0)| - |f(0,1)|]F(10,00) + [|f(0,1)| - |f(0,2)|]F(10,01) \\ + [|f(0,2)| - |f(2,0)|]F(10,02) + |f(2,0)|F(20,02). \end{aligned}$$

Fig. 4 shows the bipolar extension $\tilde{L}$, the part $L(2,0)$ used for f is in grey, and the maximal chain is in bold.

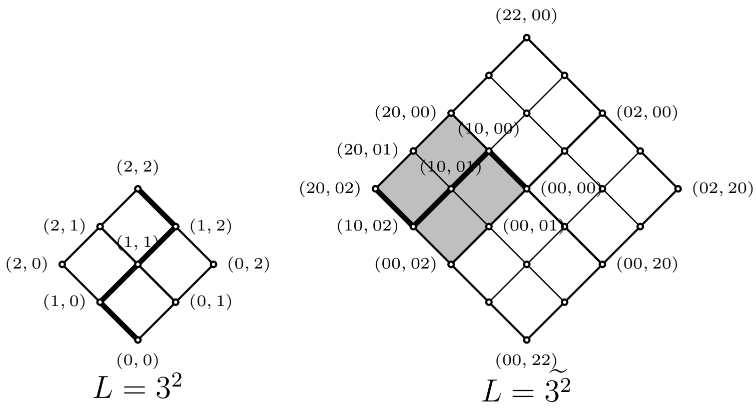

Figure 4: The lattice $L = 3^2$ and its bipolar extension. In bold the maximal chain corresponding to f

5 Concluding remarks

We have provided a general scheme for the bipolarization of a class of posets, precisely of inf-semilattices. The bipolarization is particularly simple in the case of a finite distributive lattice L, where all connected components of the poset of join-irreducible elements have a least element (this is the case, e.g., for Boolean lattices and products of linear lattices). In this case, the bipolarization of L is made from copies of L, and is called for this reason a regular mosaic.

Using the concepts of geometric realization of a distributive lattive and of natural triangulation, we have provided a general interpolation scheme on bipolar structures, which can be considered as a general definition of the Choquet integral.

We have applied our general scheme to multicriteria decision making, where we have shown that our model reduces to the Choquet integral for k-ary capacities, in the special case where the scores assigned to levels for each criterion are either 0 or 1, except on one level. We have also provided an interpretation pertaining to fuzzy set theory when no such restriction exists on the scores.

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