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TOPOLOGICAL PROPERTIES OF SAND AUTOMATA AS CELLULAR AUTOMATA

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Abstract. In this paper, we exhibit a strong relation between the sand automata configuration space and the cellular automata configuration space. This relation induces a compact topology for sand automata, and a new context in which sand automata are homeomorphic to cellular automata acting on a specific subshift. We show that the existing topological results for sand automata, including the Hedlund-like representation theorem, still hold. In this context, we give a characterization of the cellular automata which are sand automata.

1. Introduction

Self-organized criticality (SOC) is a common phenomenon observed in a huge variety of processes in physics, biology and computer science. A SOC system evolves to a "critical state" after some finite transient. Any perturbation, no matter how small, of the critical state generates a deep reorganization of the whole system. Then, after some other finite transient, the system reaches a new critical state and so on. Examples of SOC systems are: sandpiles, snow avalanches, star clusters in the outer space, earthquakes, forest fires, load balance in operating systems [2, 3, 16]. Among them, sandpiles models are a paradigmatic formal model for SOC systems [8, 9].

In [4], the authors introduced sand automata as a generalization of sandpiles models and transposed them in the setting of discrete dynamical systems. A key-point of [4] was

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to introduce a (locally compact) metric topology to study the dynamical behavior of sand automata. A first and important result was a fundamental representation theorem similar to the well-known theorem for cellular automata from Hedlund [10, 4]. In [5, 6], the authors investigate sand automata by dealing with some basic set properties and decidability issues.

In this paper we continue the study of sand automata. First of all, we introduce a different metric on configurations (i.e., spatial distributions of sand grains). This metric is defined by means of the relation between sand automata and cellular automata [6]. With the induced topology, the configuration set turns out to be a compact (and not only locally compact), perfect and totally disconnected space. The "strict" compactness gives a better topological background to study the behavior of sand automata (and in general of discrete dynamical systems) [1, 12]. We show that all the topological results from [4], in particular the Hedlund-like representation theorem, remain valid with the compact topology. Moreover, with this topology, any sand automaton is homeomorphic to a cellular automaton defined on a subset of its usual domain. We prove that it is possible to decide whether a given cellular automaton represents, through that homeomorphism, a sand automaton.

The paper is structured as follows. In Section 2, we recall basic definitions and results about cellular automata and sand automata. Then, in Section 3, we define the topology and prove topological results, in particular the representation theorem.

2. Definitions

For all $a, b \in \mathbb{Z}$ with $a \le b$, let $[a, b] = \{a, a+1, \dots, b\}$ and $\widetilde{[a, b]} = [a, b] \cup \{+\infty, -\infty\}$. For $a \in \mathbb{Z}$, let $[a, +\infty) = \{a, a+1, \dots\} \setminus \{+\infty\}$. Let $\mathbb{N}_+$ be the set of positive integers.

Let A a (possibly infinite) alphabet and $d \in \mathbb{N}^*$. Denote by $\mathcal{M}^d$ the set of all the d-dimensional matrices with values in A. We assume that the entries of any matrix $U \in \mathcal{M}^d$ are all the integer vectors of a suitable d-dimensional hyper-rectangle $[1, h_1] \times \cdots \times [1, h_d] \subset \mathbb{N}_+^d$. For any $h = (h_1, \dots, h_d) \in \mathbb{N}_+^d$, let $\mathcal{M}_h^d \subset \mathcal{M}^d$ be the set of all the matrices with entries in $[1, h_1] \times \cdots \times [1, h_d]$. In the sequel, the vector h will be called the *order* of the matrices belonging to $\mathcal{M}_h^d$. For a given element $x \in A^{\mathbb{Z}^d}$, the *finite portion* of x of reference position $i \in \mathbb{Z}^d$ and order $h \in \mathbb{N}_+^d$ is the matrix $M_h^i(x) \in \mathcal{M}_h^d$ defined as $\forall k \in [1, h_1] \times \cdots \times [1, h_d]$, $M_h^i(x)_k = x_{i+k-\mathbf{1}}$. For any $r \in \mathbb{N}$, let $\mathbf{r}^d$ (or simply $\mathbf{r}$ if the dimension is not ambiguous) be the vector $(r, \dots, r)$.

2.1. Cellular automata and subshifts

Let A be a finite alphabet. A *CA configuration* of dimension d is a function from $\mathbb{Z}^d$ to A. The set $A^{\mathbb{Z}^d}$ of all the CA configurations is called the *CA configuration space*. This space is usually equipped with the Tychonoff metric d_T defined by

$$\forall x, y \in A^{\mathbb{Z}^d}, \quad \mathsf{d}_T(x, y) = 2^{-k} \quad \text{where} \quad k = \min\left\{|j| : j \in \mathbb{Z}^d, x_j \neq y_j\right\} .$$

The topology induced by d_T coincides with the product topology induced by the discrete topology on A. With this topology, the CA configuration space is a Cantor space: it is compact, perfect (i.e., it has no isolated points) and totally disconnected.

For any $k \in \mathbb{Z}^d$ the *shift map* $\sigma^k : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ is defined by $\forall x \in A^{\mathbb{Z}^d}, \forall i \in \mathbb{Z}^d$, $\sigma^k(x)_i = x_{i+k}$. A function $F : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ is said to be *shift-commuting* if $\forall k \in \mathbb{Z}^d$, $F \circ \sigma^k = \sigma^k \circ F$.

A d-dimensional *subshift* S is a closed subset of the CA configuration space $A^{\mathbb{Z}^d}$ which is shift-invariant, i.e., for any $k \in \mathbb{Z}^d$, $\sigma^k(S) \subset S$. Let $\mathcal{F} \subseteq \mathcal{M}^d$ and let $S_{\mathcal{F}}$ be the set of configurations $x \in A^{\mathbb{Z}^d}$ such that all possible finite portions of x do not belong to $\mathcal{F}$, i.e., for any $i, h \in \mathbb{Z}^d$, $M_h^i(x) \notin \mathcal{F}$. The set $S_{\mathcal{F}}$ is a subshift, and $\mathcal{F}$ is called its set of forbidden patterns. Note that for any subshift S, it is possible to find a set of forbidden patterns $\mathcal{F}$ such that $S = S_{\mathcal{F}}$. A subshift S is said to be a *subshift of finite type* (SFT) if $S = S_{\mathcal{F}}$ for some finite set $\mathcal{F}$. The language of a subshift S is $\mathcal{L}(S) = \{U \in \mathcal{M}^d : \exists i \in \mathbb{Z}^d, h \in \mathbb{N}_+^d, x \in S, M_h^i(x) = U\}$ (for more on subshifts, see [13] for instance).

A *cellular automaton* is a quadruple $\langle A, d, r, g\rangle$, where A is the alphabet also called the *state set*, d is the dimension, $r \in \mathbb{N}$ is the *radius* and $g : \mathcal{M}^d_{\mathbf{2r+1}} \to A$ is the *local rule* of the automaton. The local rule g induces a *global rule* $G : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ defined as follows,

$$\forall x \in A^{\mathbb{Z}^d}, \forall i \in \mathbb{Z}^d, \quad G(x)_i = g\big(M^{i-\mathbf{r}}_{\mathbf{2r+1}}(x)\big) \ .$$

Note that CA are exactly the class of all shift-commuting functions which are (uniformly) continuous with respect to the Tychonoff metric (Hedlund's theorem from [10]). For the sake of simplicity, we will make no distinction between a CA and its global rule G.

The local rule g can be extended naturally to all finite matrices in the following way. With a little abuse of notation, for any $h \in [2r+1, +\infty)^d$ and any $U \in \mathcal{M}_h^d$, define $g(U)$ as the matrix obtained by the simultaneous application of g to all the $\mathcal{M}^d_{\mathbf{2r+1}}$ submatrices of U. Formally, $g(U) = M^{\mathbf{r}}_{h-\mathbf{2r}}(G(x))$, where x is any configuration such that $M_h^0(x) = U$.

2.2. SA Configurations

A *SA configuration* (or simply *configuration*) is a set of sand grains organized in piles and distributed all over the d-dimensional lattice $\mathbb{Z}^d$. A *pile* is represented either by an integer from $\mathbb{Z}$ (*number of grains*), or by the value $+\infty$ (*source of grains*), or by the value $-\infty$ (*sink of grains*), i.e., it is an element of $\widetilde{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$. One pile is positioned in each point of the lattice $\mathbb{Z}^d$. Formally, a configuration x is a function from $\mathbb{Z}^d$ to $\widetilde{\mathbb{Z}}$ which associates any vector $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$ with the number $x_i \in \widetilde{\mathbb{Z}}$ of grains in the pile of position i. Denote by $\mathcal{C} = \widetilde{\mathbb{Z}}^{\mathbb{Z}^d}$ the set of all configurations.

When the dimension d is known without ambiguity we note 0 the null vector of $\mathbb{Z}^d$ and $|i|$ the infinite norm of a vector $i \in \mathbb{Z}^d$. A *measuring device* β_r^m of precision $r \in \mathbb{N}$ and reference height $m \in \mathbb{Z}$ is a function from $\widetilde{\mathbb{Z}}$ to $\widetilde{[-r, r]}$ defined as follows

$$\forall n \in \widetilde{\mathbb{Z}}, \quad \beta_r^m(n) = \begin{cases} +\infty & \text{if } n > m + r \ , \\ -\infty & \text{if } n < m - r \ , \\ n - m & \text{otherwise.} \end{cases}$$

A measuring device is used to evaluate the relative height of two piles, with a bounded precision. This is the technical basis of the definition of cylinders, distances and ranges which are used all along this article.

In [4], the authors equipped $\mathcal{C}$ with a metric in such a way that two configurations are at small distance if they have the same number of grains in a finite neighborhood of the pile indexed by the null vector. The neighborhood is individuated by putting the measuring device at the top of the pile, if this latter contains a finite number of grains. Otherwise the measuring device is put at height 0. In order to formalize this distance, the authors introduced the notion of *cylinder*, that we rename *top cylinder*. For any configuration $x \in \mathcal{C}$, for any $r \in \mathbb{N}$, and for any $i \in \mathbb{Z}^d$, the top cylinder of x centered in i and of radius r is the d-dimensional matrix ${C'}^i_r(x) \in \mathcal{M}^d_{\mathbf{2r+1}}$ defined on the infinite alphabet $A = \widetilde{\mathbb{Z}}$ by

$$\forall k \in [1, 2r+1]^d\,,\ \left({C'}^i_r(x)\right)_k = \begin{cases} x_i & \text{if } k = r+1\ , \\ \beta_r^{x_i}(x_{i+k-r-1}) & \text{if } k \neq r+1 \text{ and } x_i \neq \pm\infty\ , \\ \beta_r^0(x_{i+k-r-1}) & \text{otherwise.} \end{cases}$$

In dimension 1 and for a configuration $x \in \mathcal{C}$, we have

$${C'}^i_r(x) = (\beta_r^{x_i}(x_{i-r}), \dots, \beta_r^{x_i}(x_{i-1}), x_i, \beta_r^{x_i}(x_{i+1}), \dots, \beta_r^{x_i}(x_{i+r}))$$

if $x_i \neq \pm\infty$, while

$${C'}^i_r(x) = \left(\beta_r^0(x_{i-r}), \dots, \beta_r^0(x_{i-1}), x_i, \beta_r^0(x_{i+1}), \dots, \beta_r^0(x_{i+r})\right)$$

if $x_i = \pm\infty$.

By means of top cylinders, the distance $\mathsf{d}' : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ has been introduced as follows:

$$\forall x, y \in \mathcal{C}, \quad \mathsf{d}'(x, y) = 2^{-k} \quad \text{where} \quad k = \min\left\{r \in \mathbb{N} : {C'}^0_r(x) \neq {C'}^0_r(y)\right\}\ .$$

Proposition 2.1 ([4, 6])**.** *With the topology induced by* d'*, the configuration space is locally compact, perfect and totally disconnected.* ∎

2.3. Sand automata

For any integer $r \in \mathbb{N}$, for any configuration $x \in \mathcal{C}$ and any index $i \in \mathbb{Z}^d$ with $x_i \neq \pm\infty$, the *range* of center i and radius r is the d-dimensional matrix $R^i_r(x) \in \mathcal{M}^d_{\mathbf{2r+1}}$ on the finite alphabet $A = \widetilde{[-r, r]} \cup \perp$ such that

$$\forall k \in [1, 2r+1]^d\,, \quad \left(R^i_r(x)\right)_k = \begin{cases} \perp & \text{if } k = r+1\ , \\ \beta_r^{x_i}(x_{i+k-r-1}) & \text{otherwise.} \end{cases}$$

The range is used to define a sand automaton. It is a kind of top cylinder, where the observer is always located on the top of the pile x_i (called the *reference*). It represents what the automaton is able to see at position i. Sometimes the central $\perp$ symbol may be omitted for simplicity sake. The set of all possible ranges of radius r, in dimension d, is denoted by $\mathcal{R}^d_r$.

A *sand automaton* (SA) is a deterministic finite automaton working on configurations. Each pile is updated synchronously, according to a local rule which computes the variation of the pile by means of the range. Formally, a SA is a triple $\langle d, r, f\rangle$, where d is the dimension, r is the *radius* and $f : \mathcal{R}^d_r \to [-r, r]$ is the *local rule* of the automaton. By means of the local rule, one can define the *global rule* $F : \mathcal{C} \to \mathcal{C}$ as follows

$$\forall x \in \mathcal{C},\, \forall i \in \mathbb{Z}^d, \quad F(x)_i = \begin{cases} x_i & \text{if } x_i = \pm\infty\ , \\ x_i + f(R^i_r(x)) & \text{otherwise.} \end{cases}$$

Remark that the radius r of the automaton has three different meanings: it represents at the same time the number of measuring devices in every dimension of the range (number of piles in the neighborhood), the precision of the measuring devices in the range, and the highest return value of the local rule (variation of a pile). It guarantees that there are only a finite number of ranges and return values, so that the local rule has finite description.

The following example illustrates a very simple sand automaton. For more examples, we refer to [6].

Example 2.2 (the automaton $\mathcal{N}$)**.** This automaton destroys a configuration by collapsing all piles towards the lowest one. It decreases a pile when there is a lower pile in the neighborhood (see Figure 1). Let $\mathcal{N} = \langle 1, 1, f_{\mathcal{N}} \rangle$ of global rule $F_{\mathcal{N}}$ where

$$\forall a, b \in \widetilde{[-1, 1]}, \quad f_{\mathcal{N}}(a, b) = \begin{cases} -1 & \text{if } a < 0 \text{ or } b < 0 \\ 0 & \text{otherwise.} \end{cases},$$

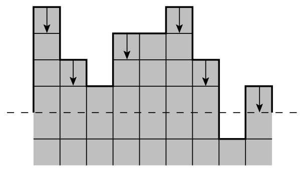


Figure 1: Illustration of the behavior of $\mathcal{N}$.

When no misunderstanding is possible, we identify a SA with its global rule F. For any $k \in \mathbb{Z}^d$, we extend the definition of the *shift map* to $\mathcal{C}$, $\sigma^k : \mathcal{C} \to \mathcal{C}$ is defined by $\forall x \in \mathcal{C}, \forall i \in \mathbb{Z}^d$, $\sigma^k(x)_i = x_{i+k}$. The *raising map* $\rho : \mathcal{C} \to \mathcal{C}$ is defined by $\forall x \in \mathcal{C}, \forall i \in \mathbb{Z}^d$, $\rho(x)_i = x_i + 1$. A function $F : \mathcal{C} \to \mathcal{C}$ is said to be *vertical-commuting* if $F \circ \rho = \rho \circ F$. A function $F : \mathcal{C} \to \mathcal{C}$ is *infinity-preserving* if for any configuration $x \in \mathcal{C}$ and any vector $i \in \mathbb{Z}^d$, $F(x)_i = +\infty$ if and only if $x_i = +\infty$ and $F(x)_i = -\infty$ if and only if $x_i = -\infty$.

Theorem 2.3 ([4, 6])**.** *The class of SA is exactly the class of shift and vertical-commuting, infinity-preserving functions $F : \mathcal{C} \to \mathcal{C}$ which are continuous w.r.t. the metric* d'. ∎

3. Topology and dynamics

In this section we introduce a compact topology on the SA configuration space by means of a relation between SA and CA. With this topology, a Hedlund-like theorem still holds and each SA turns out to be homeomorphic to a CA acting on a specific subshift. We also characterize CA whose action on this subshift represents a SA. Finally, we study some topological properties of SA in this new setting.

3.1. A compact topology for SA configurations

From [6], we know that any SA of dimension d can be simulated by a suitable CA of dimension $d+1$ (and also any CA can be simulated by a SA). In particular, a d-dimensional SA configuration can be seen as a $(d+1)$-dimensional CA configuration on the alphabet $A = \{0, 1\}$. More precisely, consider the function $\zeta : \mathcal{C} \to \{0, 1\}^{\mathbb{Z}^{d+1}}$ defined as follows

$$\forall x \in \mathcal{C}, \quad \forall i \in \mathbb{Z}^d, \forall k \in \mathbb{Z}, \quad \zeta(x)_{(i,k)} = \begin{cases} 1 & \text{if } x_i \geq k \\ 0 & \text{otherwise.} \end{cases},$$

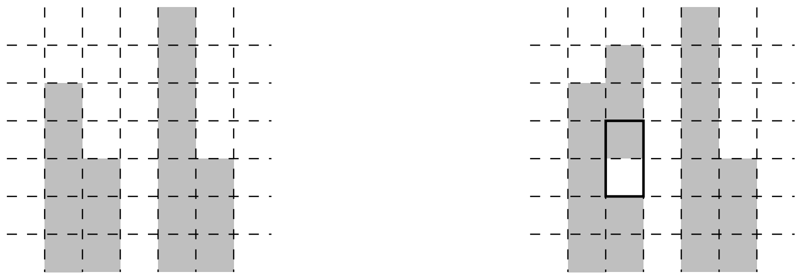
A SA configuration $x \in \mathcal{C}$ is coded by the CA configuration $\zeta(x) \in \{0, 1\}^{\mathbb{Z}^{d+1}}$. Remark that ζ is an injective function.

Consider the $(d+1)$-dimensional matrix $K \in \mathcal{M}^{d+1}_{(}1, \ldots, 1, 2)$ such that $K_{1,\ldots,1,2} = 1$ and $K_{1,\ldots,1,1} = 0$. With a little abuse of notation, denote $S_K = S_{\{K\}}$ the subshift of configurations that do not contain the pattern K.

Proposition 3.1. *The set $\zeta(\mathcal{C})$ is the subshift S_K.*

Proof. Each d-dimensional SA configuration $x \in \mathcal{C}$ is coded by the $(d+1)$-dimensional CA configuration $\zeta(x)$ such that for any $i, h \in \mathbb{Z}^{d+1}, M_h^i(\zeta(x)) \neq K$, then $\zeta(\mathcal{C}) \subseteq S_K$. Conversely, we can define a preimage by ζ for any $y \in S_K$, by $\forall i \in \mathbb{Z}^d, x_i = \sup\{k : y_{(i,k)} = 1\}$. Hence $\zeta(\mathcal{C}) = S_K$. ∎

Figure 2 illustrates the mapping ζ and the matrix $K = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for dimension $d = 1$. The set of SA configurations $\mathcal{C} = \widetilde{\mathbb{Z}}^{\mathbb{Z}}$ can be seen as the subshift $S_K = \zeta(\mathcal{C})$ of the CA configurations set $\{0,1\}^{\mathbb{Z}^2}$.



(a) Valid configuration. (b) Invalid configuration.

Figure 2: The configuration from Figure 2(a) is valid, while the configuration from Figure 2(b) contains the forbidden matrix K: there is a “hole”.

Definition 3.2. The distance $\mathsf{d} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ is defined as follows:

$$\forall x, y \in \mathcal{C}, \quad \mathsf{d}(x, y) = \mathsf{d}_T(\zeta(x), \zeta(y)) \ .$$

In other words, the (well defined) distance d between two configurations $x, y \in C$ is nothing but the Tychonoff distance between the configurations $\zeta(x), \zeta(y)$ in the subshift S_K. The corresponding metric topology is the $\{0,1\}^{\mathbb{Z}^{d+1}}$ product topology induced on S_K.

Remark 3.3. Note that this topology does not coincide with the topology obtained as countable product of the discrete topology on $\widetilde{\mathbb{Z}}$. Nevertheless, if you consider the topology $\mathcal{T}$ on $\widetilde{\mathbb{Z}}$ based on singletons $\{a\}$ where $a \in \mathbb{Z}$ and infinite intervals $[a, \infty]$ or $[-\infty, a]$, where $a \in \mathbb{Z}$, then d corresponds to its product topology. In other words, for any $i \in \mathbb{Z}^d$, the i^{th} projection $\pi_i : \mathcal{C} \to \widetilde{\mathbb{Z}}$ defined by $\pi_i(x) = x_i$ is continuous for $\mathcal{T}$.

By definition of this topology, if one considers ζ as a map from $\mathcal{C}$ onto S_K, ζ turns out to be an isometric homeomorphism between the metric spaces $\mathcal{C}$ (endowed with d) and S_K (endowed with d_T). As an immediate consequence, the following results hold.

Proposition 3.4. *The set $\mathcal{C}$ is a compact and totally disconnected space where the open balls are clopen (i.e., closed and open) sets.* ∎

Proposition 3.5. *The space $\mathcal{C}$ is perfect.*

Proof. Choose an arbitrary configuration $x \in \mathcal{C}$. For any $n \in \mathbb{N}$, let $l \in \mathbb{Z}^d$ such that $|l| = n$. We build a configuration $y \in \mathcal{C}$, equal to x except at site l, defined as follows

$$\forall j \in \mathbb{Z}^d \setminus \{l\}\,,\ y_j = x_j \quad \text{and} \quad y_l = \begin{cases} 1 & \text{if } x_l = 0\ , \\ 0 & \text{otherwise.} \end{cases}$$

By Definition 3.2, $\mathsf{d}(y, x) = 2^{-n}$. ∎

Consider now the following notion.

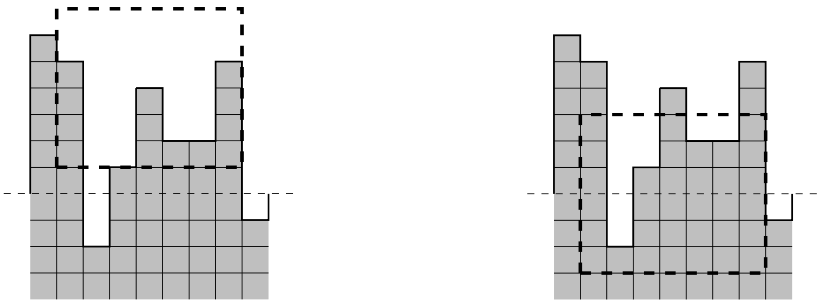
Definition 3.6 (ground cylinder)**.** For any configuration $x \in \mathcal{C}$, for any $r \in \mathbb{N}$, and for any $i \in \mathbb{Z}^d$, the *ground cylinder* of x centered on i and of radius r is the d-dimensional matrix $C^i_r(x) \in \mathcal{M}^d_{\mathbf{2r+1}}$ defined by

$$\forall k \in [1, 2r+1]^d\,, \quad \left(C^i_r(x)\right)_k = \beta^0_r(x_{i+k-r-1})\ .$$

For example in dimension 1,

$$C^i_r(x) = \left(\beta^0_r(x_{i-r}), \ldots, \beta^0_r(x_i), \ldots, \beta^0_r(x_{i+r})\right)\ .$$

Figure 3 illustrates top cylinders and ground cylinders in dimension 1. Remark that the contents of the two kinds of cylinders is totally different.



(a) Top cylinder centered on $x_i = 4$: ${C'}^i_r(x) = (+1, -\infty, -3, \mathbf{4}, -2, -2, +1)$.

(b) Ground cylinder, at height 0: $C^i_r(x) = (+\infty, -2, +1, +\infty, +2, +2, +\infty)$.

Figure 3: Illustration of the two notions of cylinders on the same configuration, with radius 3.

From Definition 3.2, we obtain the following expression of distance d by means of ground cylinders.

Remark 3.7. For any pair of configurations $x, y \in \mathcal{C}$, we have

$$\mathsf{d}(x, y) = 2^{-k} \quad \text{where} \quad k = \min\left\{r \in \mathbb{N} : C^0_r(x) \neq C^0_r(y)\right\}\ .$$

As a consequence, two configurations x, y are compared by putting boxes (the ground cylinders) at height 0 around the corresponding piles indexed by 0. The integer k is the size of the smallest cylinders in which a difference appears between x and y. This way of calculating the distance d is similar to the one used for the distance d', with the difference that the measuring devices and the cylinders are now located at height 0. This is slightly less intuitive than the distance d', since it does not correspond to the definition of the local rule. However, this fact is not an issue all the more since the configuration space is compact and the representation theorem still holds with the new topology (Theorem 3.11).

Finally, for a cylinder U, denote by $[U]_r = \left\{x \in \mathcal{C}, C_r^0(x) = U\right\}$ the *open ball* of radius 2^{-r} centered on U. We may write $[U]$ when the radius of the ball can be omitted.

3.2. SA as CA on a subshift

Let (X, m_1) and (Y, m_2) be two metric spaces. Two functions $H_1 : X \to X$, $H_2 : Y \to Y$ are (topologically) *conjugated* if there exists a homeomorphism $\eta : X \to Y$ such that $H_2 \circ \eta = \eta \circ H_1$.

We are going to show that any SA is conjugated to some restriction of a CA. Let F a d-dimensional SA of radius r and local rule f. Let us define the $(d+1)$-dimensional CA G on the alphabet $\{0, 1\}$, with radius $2r$ and local rule g defined as follows (see [6] for more details). Let $M \in \mathcal{M}^{d+1}_{\mathbf{4r+1}}$ be a matrix on the finite alphabet $\{0, 1\}$ which does not contain the pattern K. If there is a $j \in [r+1, 3r]$ such that $M_{(2r+1,\dots,2r+1,j)} = 1$ and $M_{(2r+1,\dots,2r+1,j+1)} = 0$, then let $R \in \mathcal{R}^d_r$ be the range taken from M of radius r centered on $(2r+1, \dots, 2r+1, j)$. See figure 4 for an illustration of this construction in dimension $d = 1$.

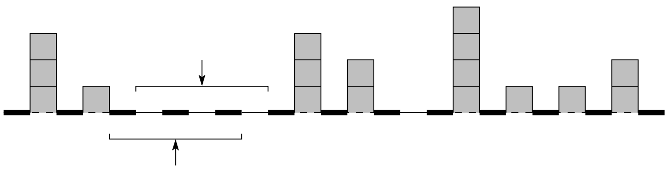


Figure 4: Construction of the local rule g of the CA from the local rule f of the SA, in dimension 1. A range R of radius r is associated to the matrix M of order $\mathbf{4r+1}$.

The new central value depends on the height j of the central column plus its variation. Therefore, define $g(M) = 1$ if $j + f(R) \geq 0$, $g(M) = 0$ if $j + f(R) < 0$, or $g(M) = M_{\mathbf{2r+1}}$ (central value unchanged) if there is no such j.

The following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C} \\ \zeta\big\downarrow & & \big\downarrow\zeta \\ S_K & \xrightarrow[G]{} & S_K \end{array}, \tag{3.1}$$

i.e., $G \circ \zeta = \zeta \circ F$. As an immediate consequence, we have the following result.

Proposition 3.8. *Any d-dimensional SA F is topologically conjugated to a suitable $(d+1)$-dimensional CA G acting on S_K.* ∎

Being a dynamical submodel, SA share properties with CA, some of which are proved below. However, many results which are true for CA are no longer true for SA; for instance, injectivity and bijectivity are no more equivalent, as proved in [5]. Thus, SA deserve to be considered as a new model.

Corollary 3.9. *The global rule $F : \mathcal{C} \to \mathcal{C}$ of a SA is uniformly continuous w.r.t distance* d.

Proof. Let G be the global rule of the CA which simulates the given SA. Since the diagram (3.1) commutes and ζ is a homeomorphism, $F = \zeta^{-1} \circ G \circ \zeta$. The map G is continuous and, by Proposition 3.4, $\mathcal{C}$ is compact, which proves the corollary. ∎

For every $a \in \mathbb{Z}$, let $\mathcal{C}_a = \pi_0^{-1}(\{a\})$ be the clopen (and compact) set of all configurations $x \in \mathcal{C}$ such that $x_0 = a$.

Lemma 3.10. *Let $F : \mathcal{C} \to \mathcal{C}$ be a continuous and infinity-preserving map. There exists an integer $l \in \mathbb{N}$ such that for any configuration $x \in \mathcal{C}_0$ we have $|F(x)_0| \leq l$.*

Proof. Since F is continuous and infinity-preserving, the set $F(\mathcal{C}_0)$ is compact and included in $\pi_0^{-1}(\mathbb{Z})$. From Remark 3.3, π_0 is continuous on the set $\pi_0^{-1}(\mathbb{Z})$ and in particular it is continuous on the compact $F(\mathcal{C}_0)$. Hence $\pi_0(F(\mathcal{C}_0))$ is a compact subset of $\widetilde{\mathbb{Z}}$ containing no infinity, and therefore it is included in some interval $[-l, l]$, where $l \in \mathbb{N}$. ■

Theorem 3.11. *A mapping $F : \mathcal{C} \to \mathcal{C}$ is the global transition rule of a sand automaton if and only if all the following statements hold*

(*i*) *F is (uniformly) continuous w.r.t the distance* d*;*
(*ii*) *F is shift-commuting;*
(*iii*) *F is vertical-commuting;*
(*iv*) *F is infinity-preserving.*

Proof. Let F be the global rule of a SA. By definition of SA, F is shift-commuting, vertical-commuting and infinity-preserving. From Corollary 3.9, F is also uniformly continuous.

Conversely, let F be a continuous map which is shift-commuting, vertical-commuting, and infinity-preserving. By compactness of the space $\mathcal{C}$, F is also uniformly continuous. Let $l \in \mathbb{N}$ be the integer given by Lemma 3.10. Since F is uniformly continuous, there exists an integer $r \in \mathbb{N}$ such that

$$\forall x, y \in \mathcal{C} \quad C_r^0(x) = C_r^0(y) \Rightarrow C_l^0(F(x)) = C_l^0(F(y)) \ .$$

We now construct the local rule $f : \mathcal{R}_r^d \to [-r, r]$ of the automaton. For any input range $R \in \mathcal{R}_r^d$, set $f(R) = F(x)_0$, where x is an arbitrary configuration of $\mathcal{C}_0$ such that $\forall k \in [1, 2r+1]$, $k \neq r+1$, $\beta_r^0(x_{k-r-1}) = R_k$. Note that the value of $f(R)$ does not depend on the particular choice of the configuration $x \in \mathcal{C}_0$ such that $\forall k \neq r+1$, $\beta_r^0(x_{k-r-1}) = R_k$. Indeed, Lemma 3.10 and uniform continuity together ensure that for any other configuration $y \in \mathcal{C}_0$ such that $\forall k \neq r+1$, $\beta_r^0(y_{k-r-1}) = R_k$, we have $F(y)_0 = F(x)_0$, since $\beta_l^0(F(x)_0) = \beta_l^0(F(y)_0)$ and $|F(y)_0| \leq l$. Thus the rule f is well defined.

We now show that F is the global mapping of the sand automaton of radius r and local rule f. Thanks to (*iv*), it is sufficient to prove that for any $x \in \mathcal{C}$ and for any $i \in \mathbb{Z}^d$ with $|x_i| \neq \infty$, we have $F(x)_i = x_i + f\left(R_r^i(x)\right)$. By (*ii*) and (*iii*), for any $i \in \mathbb{Z}^d$ such that $|x_i| \neq \infty$, it holds that

$$\begin{aligned} F(x)_i &= \left[\rho^{x_i} \circ \sigma^{-i}\left(F(\sigma^i \circ \rho^{-x_i}(x))\right)\right]_i \\ &= x_i + \left[\sigma^{-i}\left(F(\sigma^i \circ \rho^{-x_i}(x))\right)\right]_i \\ &= x_i + \left[F(\sigma^i \circ \rho^{-x_i}(x))\right]_0 \ . \end{aligned}$$

Since $\sigma^i \circ \rho^{-x_i}(x) \in \mathcal{C}_0$, we have by definition of f

$$F(x)_i = x_i + f\left(R_r^0(\sigma^i \circ \rho^{-x_i}(x))\right) \ .$$

Moreover, by definition of the range, for all $k \in [1, 2r+1]^d$,

$$R_r^0(\sigma^i \circ \rho^{-x_i}(x))_k = \beta_r^{[\sigma^i \circ \rho^{-x_i}(x)]_0}(\sigma^i \circ \rho^{-x_i}(x)_k) = \beta_r^0(x_{i+k} - x_i) = \beta_r^{x_i}(x_{i+k}) \ ,$$

hence $R_r^0(\sigma^i \circ \rho^{-x_i}(x)) = R_r^i(x)$, which leads to $F(x)_i = x_i + f\left(R_r^i(x)\right)$. ■

We now deal with the following question: given a $(d+1)$-dimensional CA, does it represent a d-dimensional SA, in the sense of the conjugacy expressed by diagram 3.1? In order to answer to this question we start to express the condition under which the action of a CA G can be restricted to a subshift $S_{\mathcal{F}}$, i.e., $G(S_{\mathcal{F}}) \subseteq S_{\mathcal{F}}$ (if this fact holds, the subshift $S_{\mathcal{F}}$ is said to be G-invariant).

Lemma 3.12. *Let G and $S_{\mathcal{F}}$ be a CA and a subshift of finite type, respectively. The condition $G(S_{\mathcal{F}}) \subseteq S_{\mathcal{F}}$ is satisfied iff for any $U \in \mathcal{L}(S_{\mathcal{F}})$ and any $H \in \mathcal{F}$ of the same order than $g(U)$, it holds that $g(U) \neq H$.*

Proof. Suppose that $G(S_{\mathcal{F}}) \subseteq S_{\mathcal{F}}$. Choose arbitrarily $H \in \mathcal{F}$ and $U \in \mathcal{L}(S_{\mathcal{F}})$, with $g(U)$ and H of the same order. Let $x \in S_{\mathcal{F}}$ containing the matrix U. Since $G(x) \in S_{\mathcal{F}}$, then $g(U) \in \mathcal{L}(S_{\mathcal{F}})$, and so $g(U) \neq H$. Conversely, if $x \in S_{\mathcal{F}}$ and $G(x) \notin S_{\mathcal{F}}$, then there exist $U \in \mathcal{L}(S_{\mathcal{F}})$ and $H \in \mathcal{F}$ with $g(U) = H$. ∎

The following proposition gives a sufficient and necessary condition under which the action of a CA G on configurations of the G-invariant subshift $S_K = \mathcal{C}$ preserves any column whose cells have the same value.

Lemma 3.13. *Let G be a $(d+1)$-dimensional CA with state set $\{0,1\}$ and S_K be the subshift representing SA configurations. The following two statements are equivalent:*

- (i) *for any $x \in S_K$ with $x_{(0,\dots,0,i)} = 1$ (resp., $x_{(0,\dots,0,i)} = 0$) for all $i \in \mathbb{Z}$, it holds that $G(x)_{(0,\dots,0,i)} = 1$ (resp., $G(x)_{(0,\dots,0,i)} = 0$) for all $i \in \mathbb{Z}$.*
- (ii) *for any $U \in \mathcal{M}^d_{\mathbf{2r+1}} \cap \mathcal{L}(S_K)$ with $U_{(r+1,\dots,r+1,k)} = 1$ (resp., $U_{(r+1,\dots,r+1,k)} = 0$) and any $k \in [1, 2r+1]$, it holds that $g(U) = 1$ (resp., $g(U) = 0$).*

Proof. Suppose that (1) is true. Let $U \in \mathcal{M}^d_{\mathbf{2r+1}} \cap \mathcal{L}(S_K)$ be a matrix with $U_{(r+1,\dots,r+1,k)} = 1$ and let $x \in S_K$ be a configuration such that $x_{(0,\dots,0,i)} = 1$ for all $i \in \mathbb{Z}$ and $M^{-\mathbf{r}}_{\mathbf{2r+1}}(x) = U$. Since $G(x)_{(0,\dots,0,i)} = 1$ for all $i \in \mathbb{Z}$, and $M^{0}_{\mathbf{2r+1}}(x) = U$, then $g(U) = 1$. Conversely, let $x \in S_K$ with $x_{(0,\dots,0,i)} = 1$ for all $i \in \mathbb{Z}$. By shift-invariance, we obtain $G(x)_{(0,\dots,0,i)} = 1$ for all $i \in \mathbb{Z}$. ∎

Lemmas 3.12 and 3.13 immediately lead to the following conclusion.

Proposition 3.14. *It is decidable to check whether a given $(d+1)$-dimensional CA corresponds to a d-dimensional SA.* ∎

3.3. Some dynamical behaviors

SA are very interesting models, whose complexity lies between that of d-dimensional and $d+1$-dimensional CA. Indeed, we have seen in the previous section that the latter can simulate SA, and it was shown in [6] that SA could simulate the former. A classification of one-dimensional cellular automata in terms of their dynamical behavior was given in [11]. Things appear to be very different as soon as we get into the second dimension, as noted in [15, 14]. This classification is based on the following notions.

Let (X, m) be a metric space and let $H : X \to X$ be a continuous application. An element $x \in X$ is an *equicontinuity* point for H if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in X$, $m(x, y) < \delta$ implies that $\forall n \in \mathbb{N}$, $m(H^n(x), H^n(y)) < \varepsilon$. The map H is *equicontinuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, $m(x, y) < \delta$ implies that $\forall n \in \mathbb{N}$, $m(H^n(x), H^n(y)) < \varepsilon$. An element $x \in X$ is *ultimately periodic* for H if there exist two integers $n \geq 0$ (the preperiod) and $p > 0$ (the period) such that $H^{n+p}(x) = H^n(x)$. H is *ultimately periodic* if there exist $n \geq 0$ and $p > 0$ such that $H^{n+p} = H^n$. H is *sensitive* (to the initial conditions) if there is a constant $\varepsilon > 0$ such that for all points $x \in X$ and all $\delta > 0$, there is a point $y \in X$ and an integer $n \in \mathbb{N}$ such that $m(x, y) < \delta$ but $m(F^n(x), F^n(y)) > \varepsilon$. H is *positively expansive* if there is a constant $\varepsilon > 0$ such that for all distinct points $x, y \in X$, there exists $n \in \mathbb{N}$ such that $m(H^n(x), H^n(y)) > \varepsilon$.

We consider these notions in the setting of sand automata with the metric topology induced by d. First we complete the definitions of equicontinuity and ultimate periodicity.

Proposition 3.15. *A SA F is equicontinuous if and only if all configurations of $\mathcal{C}$ are equicontinuity points.*

Proof. Suppose that all configurations are equicontinuity points. Let $\varepsilon > 0$. For all $x \in \mathcal{C}$, there is some δ_x such that for every configuration y such that $\mathsf{d}(x, y) < \delta_x$, we have $\forall n \in \mathbb{N}, \mathsf{d}(F^n(x), F^n(y)) < \frac{\varepsilon}{2}$. From the open covering $\mathcal{C} = \bigcup_{x \in \mathcal{C}} \left\{y | \mathsf{d}(x, y) < \frac{\delta_x}{2}\right\}$, we can extract a finite covering $\mathcal{C} = \bigcup_{x \in \mathcal{D}} \left\{y | \mathsf{d}(x, y) < \frac{\delta_x}{2}\right\}$, where $\mathcal{D} \subset \mathcal{C}$ is finite. Let $\delta = \min_{x \in \mathcal{D}} \frac{\delta_x}{2}$. Then for every $x, y \in \mathcal{C}$, such that $\mathsf{d}(x, y) < \delta$, there is some $z \in \mathcal{D}$ such that $\mathsf{d}(x, z) < \frac{\delta_z}{2}$. We also have $\mathsf{d}(y, z) < \delta + \frac{\delta_z}{2} \leq \delta_z$. Hence, for any $n \in \mathbb{N}$, $\mathsf{d}(F^n(x), F^n(y)) < \mathsf{d}(F^n(x), F^n(z)) + \mathsf{d}(F^n(y), F^n(z)) < \varepsilon$. Since this is true for any $\varepsilon > 0$, F is equicontinuous. The converse is trivial. ∎

We introduce a helpful lemma, used to refine the notion of ultimate periodicity.

Lemma 3.16. *Any covering $\mathcal{C} = \bigcup_{k \in \mathbb{N}} \Sigma_k$ by closed shift-invariant subsets Σ_k contains $\mathcal{C} = \Sigma_k$ for some $k \in \mathbb{N}$.*

Proof. If $\mathcal{C} = \bigcup_{k \in \mathbb{N}} \Sigma_k$ where the Σ_k are closed, then by the Baire Theorem, some Σ_k has nonempty interior. Hence, it contains some ball $[U]$ where U is a cylinder. If it is shift-invariant, then it contains $\overline{\bigcup_{k \in \mathbb{Z}^d} \sigma^k([U])}$, which is the complete space. ∎

Proposition 3.17. *A SA F is ultimately periodic if and only if all configurations of $\mathcal{C}$ are ultimately periodic points for F.*

Proof. Let F be a SA such that all configurations $x \in \mathcal{C}$ are ultimately periodic for F. For any $n \geq 0$ and $p > 0$, let $D_{n,p}$ be the closed shift-invariant subset $\{x : F^{n+p}(x) = F^n(x)\}$. Since $\mathcal{C} = \bigcup_{n,p \in \mathbb{N}} D_{n,p}$, by Lemma 3.16, $\mathcal{C} = D_{n,p}$ for some $n \geq 0$ and some $p > 0$. The converse is obvious. ∎

Using the new compact topological framework, it is possible to prove that equicontinuity and ultimate periodicity are equivalent (proof in [7]).

Proposition 3.18 ([7])**.** *A SA is equicontinuous if and only if it is ultimately periodic.* ∎

Despite these classical results, it appears that the classification from [11] into four classes (equicontinuous CA, non equicontinuous CA admitting an equicontinuity configuration, sensitive but not positively expansive CA, positively expansive CA) becomes irrelevant for one-dimensional SA. In particular, none of them satisfy the last topological concept of the classification (positive expansivity).

Proposition 3.19 ([7])**.** *There are no positively expansive SA.* ∎

It also seems that the trichotomy between the other classes might be false. We conjecture that there exist non-sensitive SA without equicontinuity points, which would lead to another classification into four classes: equicontinuous, admitting an equicontinuity configuration (but not equicontinuous), non-sensitive without equicontinuity configurations, sensitive.

4. Conclusion

In this article we have continued the study of sand automata, by introducing a compact topology on the SA configurations set. In this new context of study, the characterization of SA functions of [4, 6] still holds. Moreover, a topological conjugacy of any SA with a suitable CA acting on a particular subshift might facilitate future studies about dynamical and topological properties of SA.

In particular, injectivity and surjectivity and their corresponding dimension-dependent decidability problems could help to understand if SA look more like CA of the same dimension or of the following one. Still in that idea is the open problem of the dichotomy between sensitive SA and those with equicontinuous configurations. A potential counter-example would give a more precise idea of the specificities of the dynamical behaviors represented by SA.

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