Amalgamation of cellular automata

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ABSTRACT. In this paper, we study amalgamations of cellular automata (CA), i.e. ways of combining two CA by disjoint union. We show that for several families of CA obtained by simple amalgamation operations (including the well-known families of majority and minority CA), the density of a large class of properties follows a zero-one law. Besides, we establish that intrinsic universality in those families is always non-trivial and undecidable for some of them. Therefore we obtain various syntactical means to produce CA which are almost all intrinsically universal. These results extend properties already obtained for captive cellular automata. We additionally prove that there exists some reversible captive CA which are (intrinsically) universal for reversible CA.

1. Introduction

Cellular automata (CA) are discrete dynamical systems made of an infinite lattice of cells evolving synchronously and uniformly according to a common local rule. The model of cellular automata offers a minimal formal setting to tackle the broad questioning from the field of complex systems: how repetitions of a simple local rule can lead to complex global behaviour?

It is well known through numerous undecidability results [8, 9] that determining global behaviour of CA from their local rule is very challenging. However, the inverse problem of constructing a local rule to achieve a given global behaviour has received a lot of attention in the literature with some success. Hence, one of the most celebrated result of CA theory is the existence of small universal CA (see [2, 3, 15] for smallest known examples in various settings). Historically, the notion of universality used for CA was a more or less formalised adaptation from classical Turing-universality. Later [1, 10, 6, 13], a stronger notion called *intrinsic universality* was proposed: a CA is intrinsically universal if it is able to simulate step by step any other CA. This definition relies on a notion of simulation which, in addition to defining intrinsic universality, provides a sharp formal tool (an infinite pre-order) to classify cellular automata.

As said before, the existence of intrinsically universal CA is clearly established and the present paper (in the continuation of previous work of the author [17]) aims at going further by showing that this property, although highly non-trivial, is in fact very common

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and "malleable". More precisely, we will show that one can define various families of CA, by purely syntactical constraints on local rules, which have simultaneously the following properties:

- almost all CA of the family are intrinsically universal,
- intrinsic universality is undecidable in the family.

To achieve this, we study various notions of *amalgamation* of CA. By amalgamation, we mean an operation which combines two CA by disjoint union. More precisely, an amalgamation operation consists in defining the behaviour of the local rule for transitions involving states from both CA (see definition 2.4 below). These operations are interesting on their own and some were already specifically studied from different points of view [4, 5]. We give several natural examples of amalgamation operations in section 3 and consider associated families of CA (obtained as the cloture by amalgamation of a finite set of CA). We then show in section 4 that they all satisfy a zero-one law for a large class of properties (including intrinsic universality): a property has either probability 0 or probability 1 in the family. Finally, in section 5, we focus on the class of intrinsically universal CA and show that its intersection with the families above is complex (notably undecidable) although probabilistically trivial. Besides, we prove the existence of reversible captive CA which are (intrinsically) universal for reversible CA (i.e. able to simulate any reversible CA). This last result gives another indication of the "malleability" of the notion of (intrinsic) universality.

2. Definitions

For clarity of exposition and although some of the following results extends to any dimension, we assume throughout the paper that dimension is 1. Moreover, we consider only centered connected neighbourhoods. In this setting, a CA is triple $F = (Q_F, r, \delta_F)$ where Q_F is a finite set of *states*, r (the neighbourhood's radius) is a positive integer δ_F is a map from Q_F^{2r+1} to Q_F. Configurations are maps from $\mathbb{Z}$ to Q_F. The local transition function δ_F induces a global evolution rule on configurations denoted Q_F and defined as follows: $\forall c \in Q_F^{\mathbb{Z}}, \forall i \in \mathbb{Z}, F(c)(i) = \delta_F\big(c(i-r), c(i-r+1), \ldots, c(i+r)\big)$.

Let F be any CA with state set Q_F. A subset $Q \subseteq Q_F$ is *F-stable* if $F(Q^{\mathbb{Z}}) \subseteq Q^{\mathbb{Z}}$. Then F induces a cellular automaton on $Q^{\mathbb{Z}}$, denoted by $F_{|Q}$. G with state set Q_G is a *sub-automaton* of F, denoted by $G \sqsubseteq F$, if there is a F-stable subset Q such that G is isomorphic to $F_{|Q}$: formally, there is a one-to-one map $\iota : Q_G \to Q$ such that

$$\iota \circ G(x_{-r}, \ldots, x_r) = F_{|Q}(\iota(x_{-r}), \ldots, \iota(x_r))$$

for any $x_{-r}, \ldots, x_r \in Q_G$.

In the sequel, we denote by $F \equiv G$ the fact that F and G are isomorphic (both $F \sqsubseteq G$ and $G \sqsubseteq F$). Moreover, we say that a state q is quiescent for F if the set $\{q\}$ is F-stable.

The relation $\sqsubseteq$ provides a local comparison relation on CA which is very restrictive. We now define a pre-order relation generalizing $\sqsubseteq$ by allowing some rescaling operations in the CA to be compared. Our formalisation below follows[1] that of [14]. Rescaling transformations considered are very simple: they allow grouping several cells in one block and

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[1]To be precise, we don't use the shift parameter present in [14] in order to simplify notations. However all our proofs remains correct when using this parameter in rescaling transformations.

running several steps at a time. Formally, for any finite set A and any $m \in \mathbb{N}$ ($m \neq 0$), let $b_m : A^{\mathbb{Z}} \to (A^m)^{\mathbb{Z}}$ be the map such that

$$\forall c \in A^{\mathbb{Z}}, \forall z \in \mathbb{Z} : \big(b_m(c)\big)(z) = \big(c(mz), c(mz+1), \dots, c(m(z+1)-1)\big).$$

We denote by $F^{<m,t>}$ the CA $b_m \circ F^t \circ b_m^{-1}$. Once rescaling transformation are defined, the simulation relation is simply the relation $\sqsubseteq$ up to rescaling.

Definition 2.1 (simulation). F *simulates* G, denote by $G \preceq F$, if there are parameters m, t and m', t' such that $G^{<m,t>} \sqsubseteq F^{<m',t'>}$.

From a dynamical systems point of view, if $G \preceq F$ then there exists a set of configurations Σ such that the dynamical system $\big(G^t, Q_G^{\mathbb{Z}}\big)$ is isomorphic to $\big(F^{t'}, \Sigma\big)$. Using the pre-order $\preceq$ as a tool to measure complexity of CA, we will particularly focus on properties which are *increasing* for $\preceq$. A property $\mathcal{P}$ is increasing if whenever $F \preceq G$ then $F \in \mathcal{P} \Rightarrow G \in \mathcal{P}$. A property is decreasing if its complement is increasing.

As evoked in the introduction, the notion of universality used throughout this paper is defined as the global maximum class of $\preceq$.

Definition 2.2 (universality).   • F is *universal* if for any G we have $G \preceq F$.
    • F is *reversible-universal* if it is reversible and for any reversible G we have $G \preceq F$.
We denote by $\mathcal{U}$ the set of universal CA.

Throughout this paper we consider several families $\mathcal{F}$ of CA and we are interested in the typical properties of elements of $\mathcal{F}$ when their state set gets larger and larger. This is formalised by the notion of density, that is, the limit probability (of some property) for cellular automata with increasing state set but fixed neighbourhood. Unless it is specified in the context, the neighbourhood radius r is arbitrary and the arity of local rules is denoted by k with $k = 2r + 1$.

Any alphabet considered in this paper is a subset of $\mathbb{N}$. We denote by $\mathtt{CA}$ the set of CA and by $\mathtt{CA}_n$ the set of CA on state set $\{0, \dots, n-1\}$. Given a CA F we denote by $|F|$ the cardinal of its state set. Moreover, we fix once for all a collection of bijections between state sets of same size (which are always subsets of $\mathbb{N}$): we choose the only bijection which is increasing according to the natural order of $\mathbb{N}$. Thus we get a standard renaming function $\mathtt{std}()$ such that for any $F \in \mathtt{CA}$ and any set X with $|X| = |F|$ we have $\mathtt{std}_X(F) \equiv F$ and the state set of $\mathtt{std}_X(F)$ is X.

Definition 2.3 (density).   • Given a family $\mathcal{F}$ and a property $\mathcal{P}$ (both are sets of CA), we define $d_{\mathcal{F}}(\mathcal{P})$, the *density* of $\mathcal{P}$ in $\mathcal{F}$, as the following limit (if it exists):

$$\lim_{\substack{n \in I_{\mathcal{F}} \\ n \to \infty}} \frac{|\mathcal{P} \cap \mathcal{F}_n|}{|\mathcal{F}_n|},$$

   where $\mathcal{F}_n = \mathcal{F} \cap \mathtt{CA}_n$ and $I_{\mathcal{F}} = \{n : \mathcal{F} \cap \mathtt{CA}_n \neq \emptyset\}$.
    • In the sequel, when $\mathcal{F} = \mathtt{CA}$, $d_{\mathcal{F}}(\mathcal{P})$ is abbreviated to $d(\mathcal{P})$.

The present paper is devoted to families of CA obtained by amalgamation. Intuitively, an amalgamation consists in making the (disjoint) union of two CA and completing the transition table in some way (i.e. choosing a value for each transition involving states from both CA). Precisely, given two CA of size n and p, there are $(n+p)^k - n^k - p^k$ transitions to fix in order to completely define an amalgamation of them. An amalgamation operation is a description of allowed completion for any pair of CA.

Definition 2.4 (amalgamation). An *amalgamation operation* Γ is function from pair of CA to sets of CA verifying for all $F, G \in \mathtt{CA}$:

(1) $\Gamma(F, G) \neq \emptyset$,
(2) $|\Gamma(F, G)|$ is a function of $|F|$ and $|G|$ only,
(3) if $H \in \Gamma(F, G)$ then
    (a) $H \in \mathtt{CA}_{p+q}$,
    (b) $H_{|Q} = \mathtt{std}_Q(F)$ and
    (c) $H_{|Q'} = \mathtt{std}_{Q'}(G)$
    where $p = |F|$, $q = |G|$, $Q = \{0, \ldots, p-1\}$ and $Q' = \{p, \ldots, p+q-1\}$.

Γ is said *associative* if for any F, G, H it verifies:

$$\Gamma\big(F, \Gamma(G, H)\big) = \Gamma\big(\Gamma(F, G), H\big),$$

where the notation Γ is naturally extended to sets of CA.

Given an amalgamation operation Γ, and a finite set of CA $\mathbb{G}$, called *generators*, we consider the set $\mathcal{F}_{\mathbb{G}, \Gamma}$ (or simply $\mathcal{F}$ when the context is clear) which is the smallest set containing $\mathbb{G}$ and closed by Γ.

For any $F_1, \ldots, F_n \in \mathbb{G}$, we denote by $\Gamma(F_1, \ldots, F_n)$ the set

$$\Gamma\left(F_1, \Gamma(F_2, \ldots, \Gamma(F_{n-1}, F_n) \cdots)\right).$$

If Γ is associative, any $F \in \mathcal{F}$ belongs to some $\Gamma(F_1, \ldots, F_n)$ for a convenient choice of $F_1, \ldots, F_n$. Moreover, if $F \in \Gamma(G, H) \cap \Gamma(G', H')$ then we have necessarily either $G \sqsubseteq G'$ or $G' \sqsubseteq G$. Therefore, if we suppose that there is no $G, G' \in \mathbb{G}$ with $G \sqsubseteq G'$, then, for any $F \in \mathcal{F}$, there is a unique list of generators $F_1, \ldots, F_n \in \mathbb{G}$ such that $F \in \Gamma(F_1, \ldots, F_n)$ (straightforward by induction on the size of F).

In the sequel, any family $\mathcal{F}_{\mathbb{G}, \Gamma}$ with such properties will be called an *unambiguous amalgamation family*.

3. Examples

We will establish some properties shared by all unambiguous and associative amalgamation families in section 4. The present section gives several examples of amalgamation families interesting in their own to illustrate the previous definitions. Before giving examples of amalgamation operations and considering the associated families, we will define 3 natural families of CA already considered in the literature which turn out to be amalgamation families. First, captive CA (introduced in [16]) are automata where the transition function is constrained to always choose a state already present in the neighbourhood.

Definition 3.1. A *captive* CA of arity k is a CA where each transition verifies the captivity constraint: $\delta_F(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}$. The set of such CA is denoted by $\mathcal{K}$.

A cellular automaton with 2 states known has the majority vote CA has received a lot of attention in the literature (see for instance [11]). Its transition rule simply consists in choosing the state which has the greatest number of occurrences in the neighbourhood. We will consider generalisations to any number of states of the majority vote CA and its symmetric, the minority vote CA.

Definition 3.2.   • A *majority* CA of arity k is a CA where each transition verifies the majority constraint: $\delta_F(x_1, \ldots, x_k) \in \big\{x_i : \forall j, c(i) \geq c(j)\big\}$, where $c(i) = |\{j : x_j = x_i\}|$. The set of majority CA is denoted by $\mathcal{MAJ}$.

- The set $\mathcal{MIN}$ of *minority* CA is defined analogously, the condition on each transition becoming: $\delta_F(x_1, \ldots, x_k) \in \{x_i : \forall j, c(i) \le c(j)\}$.

We now proceed the opposite way and define natural amalgamation operations in order to consider associated amalgamation families in the sequel.

First, we define the general amalgamation operation Γ_g as the one where all possible completion of transition tables are allowed. Γ_g is obviously non-associative since there are some $G \in \Gamma_g\big(F_1, \Gamma_g(F_2, F_3)\big)$ having transitions involving only states corresponding (through $\mathtt{std}()$) to F_1 and F_2 which leads to some state corresponding to F_3: this is impossible in $\Gamma_g\big(\Gamma_g(F_1, F_2), F_3\big)$.

Since associativity plays an important role in determining the density of properties in amalgamation families, we define the amalgamation operation Γ_a obtained by adding the following restriction: any completion of the transition table must ensure when possible that the union of any pair of stable subsets is itself a stable subset.

Definition 3.3. Γ_a is defined as follows: for any $F \in \mathtt{CA}_p$ and $G \in \mathtt{CA}_q$, $\Gamma_a(F, G)$ is the set of all automata $G \in \mathtt{CA}_{p+q}$ such that if $Q = \{0, \ldots, p-1\}$ and $Q' = \{p, \ldots, p+q-1\}$:

(1) $H_{|Q} = \mathtt{std}_Q(F)$ and $H_{|Q'} = \mathtt{std}_{Q'}(G)$,
(2) for any $L \subseteq Q$ and $R \subseteq Q'$, if both L and R are H-stable then $L \cup R$ is also H-stable.

Proposition 3.4. Γ_a *is associative.*

Proof. Let $F_1, F_2, F_3 \in \mathtt{CA}$ and consider any $G \in \Gamma_a(F_1, \Gamma_a(F_2, F_3))$. Denote by Q_1, Q_2 and Q_3 the state sets corresponding to F_1, F_2 and F_3 (respectively) in G. First it is straigthforward to check that $G_{|Q_1 \cup Q_2} \in \Gamma_a(F_1, F_2)$.

Now consider any $L \subseteq Q_1 \cup Q_2$ and $R \subseteq Q_3$ which are both G-stable. Let $L_1 = L \cap Q_1$ and $L_2 = L \cap Q_2$. $L_2 \cup R$ is G-stable because $G_{|Q_2 \cup Q_3} \in \Gamma_a(F_2, F_3)$. Therefore $L \cup R$ is G-stable because it is the union of stable sets $L_1 \subseteq Q_1$ and $L_2 \cup R \subseteq Q_2 \cup Q_3$ and $G \in \Gamma_a(F_1, \Gamma_a(F_2, F_3))$. Hence $G \in \Gamma_a(\Gamma_a(F_1, F_2), F_3)$. ∎

One can notice that the condition on stable subsets satisfied by Γ_a is a necessary condition for associativity (remark on Γ_g above). Thus, Γ_a is the largest associative amalgamation operation.

The 3 families $\mathcal{K}, \mathcal{MIN}, \mathcal{MAJ}$ rely on a constraint applying to each transition individually. It is thus natural to consider for each one the amalgamation operation where any completion in the transition table fulfils the constraint.

Definition 3.5.
- $\Gamma_{\mathcal{K}}$ is the amalgamation operation defined as follows: for any $F \in \mathtt{CA}_p$ and $G \in \mathtt{CA}_q$, $\Gamma_{\mathcal{K}}(F, G)$ is the set of all automata $G \in \Gamma_a(F, G)$ such that any transition involving states from both $\{0, \ldots, p-1\}$ and $\{p, \ldots, p+q-1\}$ satisfy the captivity constraint.
- The amalgamation operations $\Gamma_{\mathcal{MIN}}$ and $\Gamma_{\mathcal{MAJ}}$ are defined analogously using the minority and majority constraint respectively.

Proposition 3.6. *Let $\mathbb{G}_0$ be the set containing only the trivial CA with a single state. Each of the sets $\mathcal{K}$, $\mathcal{MIN}$ and $\mathcal{MAJ}$ forms an unambiguous amalgamation family associated to $\mathbb{G}_0$ and $\Gamma_{\mathcal{K}}$, $\Gamma_{\mathcal{MIN}}$ or $\Gamma_{\mathcal{MAJ}}$ (respectively).*

Proof. We consider the family $\mathcal{K}$ (proofs for $\mathcal{MIN}$ and $\mathcal{MAJ}$ are similar). First, it is straightforward to check that $\Gamma_{\mathcal{K}}$ is associative. Moreover, since any state q of any captive

CA F is quiescent, it follows that $F \in \Gamma_{\mathcal{K}}(\underbrace{\mathbb{G}_0, \ldots, \mathbb{G}_0}_{|F|})$. Conversely, any F in $\Gamma_{\mathcal{K}}(\mathbb{G}_0, \ldots, \mathbb{G}_0)$

is a captive CA since it has only quiescent states and all other transitions fulfils the captivity constraint by definition of $\Gamma_{\mathcal{K}}$. ∎

4. Density of Properties

In this section, we are interested in typical properties of CA from a given amalgamation family. First, one can establish that any amalgamation family is negligible in the set CA. Therefore, the typical behaviour of CA from some amalgamation family is *a priori* not related to the typical behaviour of CA in general. In the next section, we will establish that typical behaviour of several amalgamation families is interesting (theorem 5.4), whereas nothing is known for CA in general.

Proposition 4.1. *If $\mathcal{F}$ is an amalgamation family then $d(\mathcal{F}) = 0$.*

Proof. This is a straightforward corollary of proposition 1 of [17] since by definition any sufficiently large $F \in \mathcal{F}$ possesses a non-trivial sub-automaton. ∎

We will now focus on unambiguous amalgamation families and establish the main result of this section which gives a simple sufficient condition for a property to have density 1 in such a family.

Definition 4.2. Given a set of generators $\mathbb{G}$ and an amalgamation operation Γ, we say that a property $\mathcal{P}$ is *malleable* for $\mathbb{G}$ and Γ if the two following conditions hold:
(1) $\mathcal{P}$ is increasing,
(2) there is i such that, for any $F_1, \ldots, F_i \in \mathbb{G}$, $\Gamma(F_1, \ldots, F_i) \cap \mathcal{P} \neq \emptyset$.

It is straightforward to check that for an associative amlgamation operation and a set of generators which is a singleton, the malleable properties are exactly the increasing (non-void) properties. this is the case for $\mathcal{K}$, $\mathcal{MIN}$ and $\mathcal{MAJ}$. Concerning Γ_a and a non-trivial set of generators, we have the following proposition.

Proposition 4.3. *Let $\mathbb{G}$ be any set of generators. Any increasing property $\mathcal{P}$ containing a captive CA is a malleable property for $\mathbb{G}$ and Γ_a.*

Proof. Consider any $F \in \mathcal{P} \cap \mathcal{K}$ and let $\{a_1, \ldots, a_i\}$ denotes its state set. Let $F_1, \ldots, F_i \in \mathbb{G}$. For any $1 \leq j \leq i$ there is a positive integer t_j and a state q_j of F_j such that q_j is a quiescent state of $F_j^{t_j}$ (because the phase space of F_j restricted to uniform configurations necessarily contains a cycle). Now let $m = \text{lcm}(t_j)$ and denote by e_j the state of any CA from $\Gamma_a(F_1, \ldots, F_i)$ corresponding to q_j (for any $1 \leq j \leq i$). Since F is captive, one can choose $H \in \Gamma_a(F_1, \ldots, F_i)$ such that : for any k-tuple $j_1, \ldots, j_k \in \{1, \ldots, i\}$ with at least 2 distinct elements:

$$\delta_F(a_{j_1}, \ldots, a_{j_k}) = a_\alpha \implies \delta_H(e_{j_1}, \ldots, e_{j_k}) = e_\alpha.$$

Hence, although the set $\{e_1, \ldots, e_i\}$ may not be H-stable, the only potential obstruction to this stability comes from the fact that states e_j are not all quiescent for H. In any case, we have:

$$F^m \sqsubseteq H^m$$

and thus $F \preceq H_i$ and $H \in \mathcal{P}$ which concludes the proof. ∎

We now establish the central result of this section.

Theorem 4.4. *Let* $\mathcal{F} = \mathcal{F}_{\mathbb{G},\Gamma}$ *be any unambiguous amalgamation family. Then, for any malleable property* $\mathcal{P}$*, we have* $d_\mathcal{F}(\mathcal{P}) = 1$*.*

Proof. Let α and β denote the minimum and maximum size (respectively) of elements of $\mathbb{G}$. By unambiguity of $\mathcal{F}$, we can partition $\mathcal{F}_n$ for any $n \in I_\mathcal{F}$ according to:

$$\mathcal{F}_n = \bigcup_{\substack{n/\beta \leq p \leq n/\alpha, \\ F_1,\ldots,F_p \in \mathbb{G}, \\ \sum |F_i| = n}} \Gamma(F_1,\ldots,F_p).$$

Let us fix some p with $n/\beta \leq p \leq n/\alpha$ and some list of generators $F_1,\ldots,F_p \in \mathbb{G}$ with $\sum |F_i| = n$. We will show that there exist some $0 \leq \lambda < 1$ depending only on $\mathcal{P}$, $\mathbb{G}$ and Γ such that we have for sufficiently large p:

$$\frac{|\Gamma(F_1,\ldots,F_p) \setminus \mathcal{P}|}{|\Gamma(F_1,\ldots,F_p)|} \leq \lambda^p.$$

By the above partition, this property implies

$$\frac{|\mathcal{F}_n \setminus \mathcal{P}|}{|\mathcal{F}_n|} \leq \lambda^{n/\beta}$$

which concludes the proof.

By malleability of $\mathcal{P}$ there exists i such that we have $\Gamma(F_{ji+1},\ldots,F_{(j+1)i}) \cap \mathcal{P} \neq \emptyset$ for any $0 \leq j < \lfloor \frac{p}{i} \rfloor$. Moreover, since Γ is associative, one has the following partition

$$\Gamma(F_1,\ldots,F_p) = \bigcup_{\substack{G_j \in \Gamma(F_{ji+1},\ldots,F_{(j+1)i}) \\ \forall j,\ 0 \leq j < \lfloor \frac{p}{i} \rfloor}} \Gamma\big(G_0,\ldots,G_{\lfloor \frac{p}{i} \rfloor - 1}, \Gamma(F_{i\lfloor \frac{p}{i} \rfloor + 1},\ldots,F_p)\big).$$

By definition of amalgamation operations, each set of the above partition has the same cardinal. Therefore, if we denote by ϵ the proportion of $\lfloor \frac{p}{i} \rfloor$-tuples $(G_0,\ldots,G_{\lfloor \frac{p}{i} \rfloor - 1})$ from the above list such that $G_j \notin \mathcal{P}$ for every j, then we have:

$$\frac{|\Gamma(F_1,\ldots,F_p) \setminus \mathcal{P}|}{|\Gamma(F_1,\ldots,F_p)|} \leq \epsilon.$$

Finally, let m depending only on $\mathbb{G}$, Γ and $\mathcal{P}$ be defined by

$$m = \max\{|G| : G \in \Gamma(\underbrace{\mathbb{G},\ldots,\mathbb{G}}_{i})\}.$$

By choice of i (malleability of $\mathcal{P}$), we have for any sufficiently large p:

$$\epsilon \leq \left(\frac{m-1}{m}\right)^{\lfloor \frac{p}{i} \rfloor}.$$

Thus, choosing $\lambda = \left(\frac{m-1}{m}\right)^{\frac{1}{2i}}$, we have the desired property. $\blacksquare$

Given a family $\mathcal{F}$ of CA, we say that it has the *zero-one law for monotone properties* if any property $\mathcal{P}$ which is non-trivial in $\mathcal{F}$ (i.e. both $\mathcal{F} \cap \mathcal{P} \neq \emptyset$ and $\mathcal{F} \setminus \mathcal{P} \neq \emptyset$) verifies:

- if $\mathcal{P}$ is increasing then $d_{\mathcal{F}}(\mathcal{P}) = 1$,
- if $\mathcal{P}$ is decreasing then $d_{\mathcal{F}}(\mathcal{P}) = 0$.

We have shown that malleable properties are of density 1 in unambiguous amalgamation families (theorem 4.4), and that some sets of increasing properties are malleable for some unambiguous amalgamation families (proposition 4.3 and remark above). Therefore we have the following corollary.

Corollary 4.5. *Each of the following families has the zero-one law for monotone properties:*

(1) $\mathcal{K}$,
(2) $\mathcal{MIN}$,
(3) $\mathcal{MAJ}$.

Moreover, any unambiguous family associated to Γ_a verifies the zero-one law for monotone properties which are non-trivial in $\mathcal{K}$.

5. Universality

In this section, we are interested in the intersection of $\mathcal{U}$ with the different families considered until now. In order to show that these intersection are generally non-empty, we will study the general problem of how to encode CA from a family into CA of another family preserving the intersection with $\mathcal{U}$.

Formally, given two families of CA $\mathcal{F}_1$ and $\mathcal{F}_2$, a computable injective function $\Phi : \mathcal{F}_1 \to \mathcal{F}_2$ is an *encoding* form $\mathcal{F}_1$ into $\mathcal{F}_2$ if for all $F_1 \in \mathcal{F}_1$ we have: $F_1 \preceq \Phi(F_1)$. It is *faithful* if we additionally have $F_1 \in \mathcal{U} \iff \Phi(F_1) \in \mathcal{U}$.

Recall that our convention throughout the paper is to consider CA of fixed neighbourhood. Thus, an encoding send CA of a family to CA of another family with the same neighbourhood. In the sequel we will use the following alternative characterisation of universal CA proved in [12].

Proposition 5.1. *$F \in \mathcal{U}$ if and only if for any G, there are parameters m, t such that $G \sqsubseteq F^{<m,t>}$.*

We now give encoding results of captive cellular automata into different families.

Proposition 5.2. *There exists an encoding from $\mathcal{K}$ to $\mathcal{MAJ}$ and an encoding from $\mathcal{K}$ to $\mathcal{MIN}$.*

Proof. Let $F \in \mathcal{K}$ and let Q its state set. Denote by H the CA from $\mathcal{MAJ}$ defined as follows[2]:
$$\delta_H(x_1, \ldots, x_k) = \max\{x_i \ : \ \forall j, c(i) \geq c(j)\},$$
where $c(i) = |\{j : x_j = x_i\}|$. Now let $Q' = Q \times \{1, \ldots, k\}$ and denote by π_1 and π_2 the projection from Q' to Q and from Q' to $\{1, \ldots, k\}$ (respectively). We furthermore denote

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[2]This particular choice of H is not important to establish the proposition. However we conjecture that this particular choice ensure that the encoding is faithful.

by X the set of words on alphabet $\{1, \ldots, k\}$ of the form $i \cdot (i+1) \cdots k \cdot 1 \cdots (i-1)$. We define the CA G (in a computable way) as follows:

$$\delta_G(x_1, \ldots, x_k) = \begin{cases} \left(\delta_F(\pi_1(x_1), \ldots, \pi_1(x_k)), \pi_2(x_{\lfloor \frac{k}{2} \rfloor + 1})\right) & \text{if } \pi_2(x_1) \cdots \pi_2(x_k) \in X, \\ \delta_H(x_1, \ldots, x_k) & \text{else.} \end{cases}$$

Since the first case of the above definition implies that there are no two occurrences of a same state in the neighbourhood, and by choice of H, we have $G \in \mathcal{MAJ}$. Moreover, considering the set of configuration whose second component is periodic of period $1 \cdot 2 \cdots k$, one verifies that definition of G implies $F^{<k,1,0>} \sqsubseteq G^{<k,1,0>}$ and therefore $F \preceq G$.

The construction of an encoding from $\mathcal{K}$ to $\mathcal{MIN}$ is very similar. Indeed, the simulation takes place on a set of configuration where the minority/majority constraints reduce to the captivity constraint. ∎

Concerning unambiguous amalgamation families associated to Γ_a, one can establish a stronger result.

Proposition 5.3. *Let $\mathbb{G}$ be a set generators such that $\mathbb{G} \cap \mathcal{U} = \emptyset$ and let $\mathcal{F}$ be the family associated to $\mathbb{G}$ and Γ_a. There exists a faithful encoding from $\mathcal{K}$ to $\mathcal{F}$.*

Proof. The faithful encoding relies on a modified version of the construction in the proof of proposition 4.3. Let $F \in \mathcal{K}$ with state set $\{a_1, \ldots, a_i\}$ and consider any $F_1, \ldots, F_i \in \mathbb{G}$. Using the same notation let $Q_j = \{n_j, \ldots, n_j + |F_j| - 1\}$ and $E = \{e_1, \ldots, e_i\}\}$. With the same construction method, one can easily prove that there exists $H_i \in \mathcal{F}$ such that:

(1) for any k-tuple $j_1, \ldots, j_k \in \{1, \ldots, i\}$ with at least 2 distinct elements we have

$$\delta_F(a_{j_1}, \ldots, a_{j_k}) = a_\alpha \implies \delta_{H_i}(e_{j_1}, \ldots, e_{j_k}) = e_\alpha,$$

(2) for any k-tuple $x_1, \ldots, x_k$ with $\{x_1, \ldots, x_k\} \not\subseteq Q_j$ (for all j) and $\{x_1, \ldots, x_k\} \not\subseteq E$ we have

$$\delta_{H_i}(x_1, \ldots, x_k) = \max\{x_j : x_j \notin E\},$$

(3) $F^{m_0} \sqsubseteq H_i^{m_0}$.

Now suppose $H_i \in \mathcal{U}$ and consider some $G_u \in \mathcal{U}$ with only 2 states (but with a neighbourhood possibly larger than k). Since by hypothesis $G_u \preceq H_i$, proposition 5.1 implies that there are parameters m, t such that $G_u \sqsubseteq H_i^{<m,t>}$. Denote by Σ the set of configurations of H_i where the simulation occurs. First, we have $\Sigma \not\subseteq Q_j^{\mathbb{Z}}$ (for all j) since otherwise it would imply that $F_j \in \mathcal{U}$. Moreover, if we suppose $\Sigma \not\subseteq E^{\mathbb{Z}}$, then there is some configuration $c \in \Sigma$ and some position $z \in \mathbb{Z}$ such that $\{c_z, \ldots, c_{z+k-1}\} \not\subseteq Q_j$ and $\{c_z, \ldots, c_{z+k-1}\} \not\subseteq E$. Then condition 2 of the definition of H_i applies at position z. Intuitively, starting from position z, a region of states all in Q_j for the same j will grow unless it meets some state in $Q_{j'}$ with $j' > j$ in which case a larger $Q_{j'}$ region appears. Applying this reasoning until the maximal j is reached, it is straightforward to show that there is t_0 such that $\forall t_+ \geq t_0$ the configuration $d = H_i^{t_+}(c)$ has for some $z' \in \mathbb{Z}$ and some j the property:

$$\forall z'', z' \leq z'' \leq z' + 2m \ : \ d(z'') \in Q_j.$$

It means that some state of G_u is simulated by group of states of H_i all in Q_j for the same j. This implies either $\Sigma \subseteq Q_j$ or that G_u has a spreading state (a state x such that each cell with x in its neighbourhood turns into state x). The two cases being contradictory with hypothesis ($F_j \notin \mathcal{U}$ and $G_u \in \mathcal{U}$ and has only two states), we conclude that $\Sigma \subseteq E^{\mathbb{Z}}$.

Therefore, by condition 3 of the definition of H_i, we have $G_u \sqsubseteq F^{<mm_0,t>}$ and thus $F \in \mathcal{U}$ which concludes the proof. ∎

Using results, one can prove the following theorem showing that the intersection with $\mathcal{U}$ of families $\mathcal{K}$, $\mathcal{MAJ}$, $\mathcal{MIN}$ and any unambiguous $\mathcal{F}$ associated to Γ_a are non-trivial.

Theorem 5.4. *There exists r_0 such that for any fixed Von Neumann neighbourhood of radius $r \geq r_0$, and for any family $\mathcal{F}$ among $\mathcal{K}$, $\mathcal{MIN}$, $\mathcal{MAJ}$ or an unambiguous family associated to Γ_a, then we have $d_{\mathcal{F}}(\mathcal{U}) = 1$.*

Moreover if $\mathcal{F}$ is neither $\mathcal{MIN}$ nor $\mathcal{MAJ}$, we have also:

- *$\mathcal{U}$ is undecidable in $\mathcal{F}$,*
- *for any $F \in \mathcal{F} \setminus \mathcal{U}$ there is $G \in \mathcal{F} \setminus \mathcal{U}$ with $F \preceq G$ but $G \not\preceq F$.*

Proof. The first part of the theorem follows from propositions 5.2 and 5.3, from the existence of universal captive CA (proven in [17] for $r \geq 7$) and from corollary 4.5.

The second part is proved in theorem 2 and corollary 3 of [17] for the family $\mathcal{K}$. It generalises to unambiguous families associated to Γ_a by propositions 5.3 above. ∎

Much less is known about the structure of $\preceq$ concerning reversible CA. However, as we will show below, there exists reversible-universal captive CA. This naturally raises the question (unanswered in this paper) of the density of reversible-universal CA among captive reversible CA. The existence of reversible-universal captive CA relies on the existence of reversible-universal CA of a special kind, equipped with an unalterable state.

Proposition 5.5. *There exists a reversible-universal F such that F and its inverse F^{-1} possesses a common wall state, i.e. a state q such that any cell in state q remains in state q under both the action of F and F^{-1}.*

Proof. Consider any reversible-universal CA G (see [7] for an existence proof) of state set Q_G and radius r. Let q be an additional state not in Q_G and let $Q = Q_G \times Q_G \cup \{q\}$ (it is straitghtforward to translate this state set into a subset of $\mathbb{N}$, but we don't for clarity). We will see any configuration $c \in Q^{\mathbb{Z}}$ as the disjoint union of configurations from $Q_G^{\mathbb{Z}}$ corresponding to zones between occurrences of state q. A finite zone between two occurrences of q will be seen as a torus of states from Q_G, semi-infinite zones as bi-infinite configuration in $Q_G^{\mathbb{Z}}$ and bi-infinite zones as pair of configurations in $Q_G^{\mathbb{Z}}$. We formalise this through functions $L^{\uparrow}$, $L^{\downarrow}$, $R^{\uparrow}$, $R^{\downarrow}$ which give the state of the "logical" neighbours of a "logical" cell (according to the previous description): L/R stand for left/right neighbours and $\uparrow/\downarrow$ for up/down layers (in a zone without q, cells contain 2 layers of "logical" cells).

Formally, for any $z \in \mathbb{Z}$ and $c \in Q^{\mathbb{Z}}$ such that $c(z) \neq q$ these functions are defined by:

$$L^\uparrow(z,p) = \begin{cases} \epsilon & \text{if } p = 0 \\ L^\uparrow(z-1, p-1) \cdot a & \text{if } c(z-1) = (a,b) \in Q_G^2 \\ L^\downarrow(z, p-1) \cdot \beta & \text{if } c(z-1) = q \text{ and } c(z) = (\alpha, \beta) \end{cases}$$

$$L^\downarrow(z,p) = \begin{cases} \epsilon & \text{if } p = 0 \\ L^\downarrow(z+1, p-1) \cdot b & \text{if } c(z+1) = (a,b) \in Q_G^2 \\ L^\uparrow(z, p-1) \cdot \alpha & \text{if } c(z+1) = q \text{ and } c(z) = (\alpha, \beta) \end{cases}$$

$$R^\uparrow(z,p) = \begin{cases} \epsilon & \text{if } p = 0 \\ a \cdot R^\uparrow(z+1, p-1) & \text{if } c(z+1) = (a,b) \in Q_G^2 \\ \beta \cdot R^\downarrow(z, p-1) & \text{if } c(z+1) = q \text{ and } c(z) = (\alpha, \beta) \end{cases}$$

$$R^\downarrow(z,p) = \begin{cases} \epsilon & \text{if } p = 0 \\ b \cdot R^\downarrow(z-1, p-1) & \text{if } c(z-1) = (a,b) \in Q_G^2 \\ \alpha \cdot R^\uparrow(z, p-1) & \text{if } c(z-1) = q \text{ and } c(z) = (\alpha, \beta) \end{cases}$$

Now define F on any $c \in Q^{\mathbb{Z}}$ by:

$$\big(F(c)\big)(z) = \begin{cases} q & \text{if } c(z) = q, \\ \big(\delta_G(L^\uparrow(z,r)c(z)R^\uparrow(z,r)), \delta_G(L^\downarrow(z,r)c(z)R^\downarrow(z,r))\big) & \text{else.} \end{cases}$$

F is a well-defined CA since $L^\uparrow(z,r)$, $R^\uparrow(z,r)$, $(L^\downarrow(z,r)$ and $R^\downarrow(z,r)$ depend only on the $2r+1$ cells surrounding z. It is clear from the definition that $G \preceq F$ since F behave like the product of G and its symmetric on configuration without q.

To show that F is reversible, it is sufficient to notice that applying the same construction to G^{-1}, one gets a CA F' such that for any $c \in Q^{\mathbb{Z}}$: $F'\big(F(c)\big) = c$. ∎

Proposition 5.6. *There exists a reversible-universal captive CA.*

Proof. Let F be any CA of radius r, state set $Q_F = \{q_1, \dots, q_n\}$ and having a wall state q. One defines the captive CA F_κ on state set $Q = Q_F \cup \{L, R\}$ with radius $r' = (n+3)(r+1)$ as follows. Let $u = Lq_1q_2 \cdots q_n R$. Any word $w = w_{-r'} \cdots w_0 \cdots w_{r'}$ of length $2r'+1$ over Q such that $w_0 \in Q_F$ can be written in the form:

$$w = \underbrace{v_l u x_i u x_{i-1} u \cdots u}_{r'} x_0 \underbrace{u x_1 u \cdots x_j u v_r}_{r'}$$

where $x_p \in Q_F$ for all p and i and j are maximal for this form. To the word w we associate the word $\pi(w) = q^{r-i} x_i \cdots x_0 \cdots x_j q^{r-j}$. Now we define F_κ by:

$$\delta_{F_\kappa}\big(e_{-r'}, \dots, e_{r'}\big) = \begin{cases} e_0 & \text{if } e_{-(n+2)} \cdots e_{n+2} \notin u Q_F u, \\ \delta_F\big(\pi(e_{-r'} \cdots e_{r'})\big) & \text{else.} \end{cases}$$

For any $c \in Q^{\mathbb{Z}}$, let $A(c) = \{z \in \mathbb{Z} : c(z) \cdots c(z+n+1) = u\}$. It is straightforward from the definition of F_κ to check that $A(c) = A(F_\kappa(c))$. Therefore we also have $B(c) = B\big(F_\kappa(c)\big)$ where $B(c) = \{z \in \mathbb{Z} : c(z-(n+2)) \cdots c(n+2) \in u Q_F u\}$. Now, if we apply this construction to F and F^{-1} obtained by proposition 5.5, we have from the above remark that $F_\kappa \circ F_\kappa^{-1}$ is the identity and therefore F_κ is reversible.

Finally, one can easily check that $F \preceq F_\kappa$ by considering configurations of the form $^\omega(u Q_F)^\omega$ and the proposition follows. ∎

6. Perspectives

This paper leaves many question unanswered. First, although it was not directly addressed in this paper, determining the density of $\mathcal{U}$ in the whole set of CA remains a completely open problem of prior interest.

Concerning the amalgamation operations themselves, it would be interesting to follow [4] and [5] and define new amalgamation operations where the way of completing transition tables is controlled by a two-state CA. For such amalgamation operations (and for Γ_g also) we believe that the zero-one law is still valid (at least when the set of generators is unambiguous). However, for the amalgamation of J. Mazoyer *et al.*, knowing when and how universality can be achieved starting from non-universal generators seems very difficult.

More generally, for any amalgamation operation, the associated family has only a finite set of nilpotent. Therefore nilpotency is a decidable property in the family and many classical undecidable problems are to be reconsidered when restricted to such families. Among them, we think that reversibility and surjectivity in dimension 2 and properties of limit sets are particularly intersting.

Concerning reversible CA in dimension 1, we wonder whether it is possible to define a non-trivial amalgamation family made of reversible CA only and for which density of reversible-universality is 1. To that extent, we remark that no majority CA is reversible and there are reversible-universal captive CA, so amalgamation families lying between $\mathcal{MAJ}$ and $\mathcal{K}$ are worth being considered.

Finally, even if some amalgamation families share many properties concerning the class $\mathcal{U}$, there is no reason why typical CA from two different families should have a similar dynamical behaviour. A possible formalization of this could be to study the μ-limit sets of typical CA from different families. For instance, one can establish that μ-limit sets of majority CA consists in fixed-point configuration only (proposition omitted from the present paper due to lack of space). We believe that non-trivial properties of μ-limit sets of typical CA from $\mathcal{K}$ or other families could be established.

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