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In this paper we propose a generalization of the concept of symmetric fuzzy measure based in a decomposition of the universal set in what we have called subsets of indifference. Some properties of these measures are studied, as well as their Choquet integral. Finally, a degree of interaction between the subsets of indifference is defined.

Keywords: Fuzzy measures, symmetry, OWA operator, Choquet integral.

1. Introduction

Fuzzy measures are a generalization of probability measures for which additivity is removed and monotonicity is imposed instead. These measures have become a powerful tool in Decision Theory (see e.g. [12], the work of Schmeidler [21] and [3]); moreover, the Choquet Expected Utility model generalizes the Expected Utility one, and this model offers a simple theoretical foundation for explaining phenomena that cannot be accounted for in the framework of Expected Utility Theory, as the well known Ellsberg's and Allais' paradoxes (see [3] for a survey about this topic).

However, the richness of fuzzy measures has its counterpart in the complexity. If we deal with a space of n elements, a probability measure only needs $n-1$ coefficients, while a fuzzy measure needs $2^n - 2$. In an attempt to decrease the exponential complexity of fuzzy measures in practical applications, Grabisch has introduced in [11] the concept of k-order additive measures or k-additive measures for short; k-additive measures are defined from the Möbius transform, and they can be represented by a limited set of coefficients, at most $\sum_{i=1}^{k}\binom{n}{i}$. A characterization of k-additive measures based in Choquet Expected Utility Model can be found in [15].

On the other hand, it is a well known fact that an OWA operator [26] is a discrete Choquet integral with respect to a symmetric fuzzy measure. Hence, Choquet integral generalizes OWA operators and, as before, the richness of Choquet integral is paid by the complexity. Our goal is to introduce a concept, similar to k-additive measures, bridging the gap between symmetric fuzzy measures and general fuzzy measures. We propose a definition of p-symmetry based in what we will call *subsets of indifference*, and we study some of their properties.

Of course, Choquet integral with respect to a p-symmetric fuzzy measure generalizes the concept of OWA. Another generalization of OWA operators can be found in [2], in which it is defined the so-called *double aggregation operators* as an aggregation of two other aggregation operators.

The paper is organized as follows: In Section 2, we recall some basic concepts. Next, in Section 3, we give the definition of p-symmetric measures and study some of their properties. In Section 4, we study the expressions of p-symmetric measures for other representations of fuzzy measures. In Section 5 we deal with the Choquet integral of p-symmetric measures; in this section we also define a degree of interaction and study its relationship with the decomposition of Choquet integral. We finish with the conclusions and open problems.

2. Notations and basic concepts

In the sequel, we will consider a finite universal set of n elements, denoted $X = \{x_1, ..., x_n\}$. Subsets of X are denoted with capital letters A, B, and so on, and also by $A_1, ..., A_p$. The set of all subsets of X is denoted $\mathcal{P}(X)$. Finally, $\bigwedge$ (resp. $\bigvee$) denotes the min (resp. max) operation.

In order to be self-contained, let us now give some definitions:

Definition 1 [24] *A (discrete)* **fuzzy measure** *on X is a set function $\mu : \mathcal{P}(X) \mapsto [0,1]$ satisfying*

(i) $\mu(\emptyset) = 0, \quad \mu(X) = 1$ *(boundary conditions).*

(ii) $A \subset B$ *implies* $\mu(A) \leq \mu(B)$ *(monotonicity).*

To any fuzzy measure, we can assign another one, called dual measure whose definition is the following:

Definition 2 *Consider $(X, \mathcal{X})$ a measurable space and let μ be a fuzzy measure over X; we define the* **dual** *or* **conjugate measure of** μ *as the fuzzy measure $\bar{\mu}$ given by $\bar{\mu}(A) = 1 - \mu(A^c)$, where $A^c = X \backslash A$.*

Other alternative representations of fuzzy measures are given by the Möbius transform and the Shapley interaction.

Definition 3 [20] *Let μ be a fuzzy measure on X. The* **Möbius transform** *of μ is defined by*

$$m(A) := \sum_{B \subset A} (-1)^{|A \setminus B|} \mu(B), \forall A \subset X. \tag{1}$$

In the Theory of Cooperative Games, the Möbius transform is interpreted as the importance of each subset by itself, without considering its parts. In this sense, this transformation is called *dividend* [7].

When m is given, it is possible to recover the original μ by the so-called *Zeta transform* (see [4]):

$$\mu(A) = \sum_{B \subset A} m(B). \tag{2}$$

We can define m for any set function, not limited to fuzzy measures. In order to m being the Möbius transform of a fuzzy measure we need to impose some monotonicity constraints. These constraints are given in the following proposition:

Proposition 1 [4] *A set of 2^n coefficients $m(A), A \subset X$ corresponds to the Möbius representation of a fuzzy measure if and only if*

(i) $m(\emptyset) = 0, \displaystyle\sum_{A \subset X} m(A) = 1,$

(ii) $\displaystyle\sum_{x_i \in B \subset A} m(B) \geq 0, \forall A \subset X, \forall x_i \in A.$

¿From Möbius transform, we can derive the definition of belief function, given by Dempster [6] and Shafer [22]:

Definition 4 *A fuzzy measure μ is a* **belief function** *if $m(A) \geq 0, \forall A \subset X$.*

Shapley interaction is another equivalent representation of fuzzy measures.

Definition 5 [10] *Let μ be a fuzzy measure on X. The* **Shapley interaction index** *of $A \subset X$, is defined by:*

$$I_\mu(A) = \sum_{B \subset X \setminus A} \frac{(n-b-a)!b!}{(n-a+1)!} \sum_{C \subset A} (-1)^{a-c} \mu(B \cup C),$$

with $a = |A|$, $b = |B|$, $c = |C|$.

Shapley interaction for singletons is just the Shapley value of a game [23], and it recovers the interaction index of Murofushi and Soneda [16] for pairs.

I and m are related through the following formulas:

$$I(A) = \sum_{B \subset X \setminus A} \frac{1}{|B|+1} m(A \cup B), \forall A \subset X. \tag{3}$$

$$m(A) = \sum_{B \subset X \setminus A} B_{|B|} I(B \cup A), \forall A \subset X, \tag{4}$$

where B_k denotes the Bernoulli numbers defined by recurrence through $B_0 = 1$ and $B_k = -\sum_{l=0}^{k-1} \frac{B_l}{k-l+1} \binom{k}{l}$.

Definition 6 [5] *The* **Choquet integral****of a measurable function* $f : X \mapsto \mathbb{R}^+$ *is defined by*

$$\mathcal{C}_\mu(f) := \int_0^\infty \mu(\{x|f(x) \geq \alpha\})d\alpha.$$

For simple functions the expression reduces to:

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n}(f(x_{(i)}) - f(x_{(i-1)}))\mu(B_i),$$

where parenthesis mean a permutation such that $0 = f(x_{(0)}) \leq f(x_{(1)}) \leq ... \leq f(x_{(n)})$ *and* $B_i = \{x_{(i)}, ..., x_{(n)}\}$. *Another equivalent expression for simple functions is*

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n} f(x_{(i)})(\mu(B_i) - \mu(B_{i+1}))$$

with $B_{n+1} = \emptyset$.

Choquet integral in terms of m is given by:

Theorem 1 [4] *The Choquet integral* $\mathcal{C}_\mu : [0,1]^n \mapsto \mathbb{R}^+$ *can be written as*

$$\mathcal{C}_\mu(f) = \sum_{T \subset X} m(T) \left[\bigwedge_{x_i \in T} f(x_i) \right], \quad f \in [0,1]^n. \tag{5}$$

Definition 7 [26] *An* **ordered weighted averaging operator (OWA)** *is an operator defined by*

$$\mathrm{OWA}_w(f) = \sum_{i=1}^{n} w_i f(x_{(i)}),$$

where w *is the weight vector,* $w = (w_1, ..., w_n) \in [0,1]^n$ *and such that* $\sum_{i=1}^n w_i = 1$.

Definition 8 *A fuzzy measure is said to be* **symmetric** *if it satisfies*

$$|A| = |B| \Rightarrow \mu(A) = \mu(B), \forall A, B \subset X.$$

It can be proved (see [9] and [18]) that:

Proposition 2 *Let* μ *be a fuzzy measure on* X. *Then, the following statements are equivalent:*

1. *There exists* $w \in [0,1]^n$, $\sum_{i=1}^n w_i = 1$, *such that* $\mathcal{C}_\mu = OWA_w$.
2. μ *is a symmetric fuzzy measure.*

Choquet integral model can be regarded as the generalization of a linear model in the sense that $(\mathcal{C}) \int f\, d\mu + (\mathcal{C}) \int g\, d\mu = (\mathcal{C}) \int (f+g)\, d\mu$ for a pair of comonotone functions [13]. The expressive power of Choquet integral is much higher than that of a linear model. However, as it can be seen from Definition 6, Choquet integral

*If X is continuous, the measurability is needed. Let $(X, \mathcal{X})$ be a measurable space. We say a mapping f is a measurable function if $\{x|\, f(x) \geq \alpha\}$ is in the σ-algebra $\mathcal{X}$ for any $\alpha \geq 0$.

model is difficult to handle. These are the reasons for which it has been proposed a *hierarchical Choquet integral model*, that allows to compute Choquet integral from combinations of other Choquet integrals. The underlying idea here is to be able to decompose the integral into a sum of other integrals over smaller referential sets.

Definition 9 [19] *Let $(X, \mathcal{X})$ be a measurable space. An* **interadditive partition** *of X is a finite measurable partition $\mathcal{Q}$ of X such that for every $A \in \mathcal{X}$*

$$\mu(A) = \sum_{P \in \mathcal{Q}} \mu(P \cap A).$$

Then, the following holds.

Proposition 3 [19] *Let $(X, \mathcal{X})$ be a measurable space and $\mathcal{Q}$ be a finite measurable partition of X. Then, $\mathcal{Q}$ is an interadditive partition if and only if for every measurable function f*

$$(\mathcal{C}) \int_X f d\mu = \sum_{P \in \mathcal{Q}} (\mathcal{C}) \int_P f d\mu. \tag{6}$$

A more general hierachical Choquet integral model based in what is called *inclusion-exclusion coverings* appears in [25].

3. p-symmetric measures

Let us consider an OWA operator. If we look at the definition, we can see that only the order in the scores is important, i.e. we are interested in the scores, but we do not care about which criterium each score has been obtained. Mathematically, this means that the fuzzy measure defining the OWA operator only depends on the cardinality of the subsets, and not in the elements of the subset themselves.

Thus, all criteria have the same importance or, in other words, we have a "subset of indifference" (X itself). Then, it makes sense to define 2-symmetric measures as those measures for which we have two subsets of indifference, 3-symmetric measures as those with three subsets of indifference, and so on. Let us now translate this idea.

Definition 10 *Given two elements x_i, x_j of the universal set X, we say that x_i and x_j are* **indifferent elements** *if and only if*

$$\forall A \subset X \backslash \{x_i, x_j\},\ \mu(A \cup x_i) = \mu(A \cup x_j).$$

This definition translates the idea that we do not care about which element, x_i or x_j, is in the coalition; that is, we are indifferent between x_i and x_j. This concept can be generalized for subsets of more than two elements, as shown in the following definition:

Definition 11 *Given a subset A of X, we say that A is a* **set of indifference** *if and only if*

$$\forall B_1, B_2 \subset A,\ |B_1| = |B_2|,\ \forall C \subset X \backslash A,\ \mu(B_1 \cup C) = \mu(B_2 \cup C).$$

It is easy to see the following:

Lemma 1 *Given $A \subset X$, A is a set of indifference if and only if*

$$\forall B_1, B_2 \subset A,\ |B_1| = |B_2|,\ \forall C \subset X \backslash \{B_1 \cup B_2\},\ \mu(B_1 \cup C) = \mu(B_2 \cup C).$$

Proof: For $C \subset X \backslash A$, we have, applying Definition 11, $\mu(C \cup B_1) = \mu(C \cup B_2)$.

Let us consider $C \subset X \backslash (B_1 \cup B_2)$ but $C \not\subset X \backslash A$. Then, $\exists D \subset A \backslash (B_1 \cup B_2)$ such that $C = D \cup C'$, with $C' \subset X \backslash A$. Thus, by Definition 11,

$$\mu(C \cup B_1) = \mu(C' \cup D \cup B_1) = \mu(C' \cup D \cup B_2) = \mu(C \cup B_2),$$

and therefore the result holds. ■

Another property of sets of indifference is:

Lemma 2 *If A is a set of indifference and $A' \subset A$, then A' is itself a set of indifference.*

Example 1 *Consider A, a set of indifference. Then, taking $C = \emptyset$, we obtain*

$$\mu(x_i) = \mu(x_j),\ \forall x_i, x_j \in A.$$

$$\mu(x_i, x_j) = \mu(x_k, x_l),\ \forall x_i, x_j, x_k, x_l \in A,$$

and so on.

An example of sets of indifference are null sets, defined in [1] and [17]:

Definition 12 *A subset $A \subset X$ is called a* **null set** *with respect to μ if*

$$\mu(A \cup B) = \mu(B),\ \forall B \subset X \backslash A.$$

A consequence of Definition 11 is:

Lemma 3 *A null set is a set of indifference.*

Proof: Let A be a null set. Then, $\mu(A \cup B) = \mu(B)$, $\forall B \subset X$. Let us consider now $A_1, A_2 \subset A$, $|A_1| = |A_2|$. For $B \subset X \backslash A$

$$\mu(B) \leq \mu(A_1 \cup B) \leq \mu(A \cup B) = \mu(B).$$

$$\mu(B) \leq \mu(A_2 \cup B) \leq \mu(A \cup B) = \mu(B).$$

Then, $\mu(A_1 \cup B) = \mu(B) = \mu(A_2 \cup B)$ and hence, A is a set of indifference. ■

We are now able to define p-symmetric fuzzy measures (p-symmetric measures for short). We start with 2-symmetric measures.

Definition 13 *Given a fuzzy measure μ, we say that μ is a* **2-symmetric measure** *if and only if there exists a partition of the universal set $\{A, A^c\}$, $A, A^c \neq \emptyset$ such that both A and A^c are sets of indifference and X is not a set of indifference.*

For the general case we have:

Definition 14 *Given two partitions* $\{A_1, ..., A_p\}$, $\{B_1, ..., B_r\}$ *of a referential* X, *we say* $\{A_1, ..., A_p\}$ *is* **coarser** *than* $\{B_1, ..., B_r\}$ *if the following holds:*

$$\forall A_i, \, \exists B_j \text{ such that } B_j \subset A_i.$$

Definition 15 *Given a fuzzy measure* μ, *we say that* μ *is a* **p-symmetric measure** *if and only if the coarsest partition of the universal set in sets of indifference is* $\{A_1, ..., A_p\}, A_i \neq \emptyset, \forall i \in \{1, ..., p\}$.

Note that by Lemma 2 we need to work with the coarsest partition. Otherwise, a p-symmetric measure would be also a p'-symmetric measure for any $p' > p$.

For the 2-symmetric case, we will use both $\{A_1, A_2\}$ and $\{A, A^c\}$ for denoting the partition of X in sets of indifference.

With these definitions, a symmetric measure is just a 1-symmetric measure.

Example 2 *Consider the 2-symmetric case. Consider the partition given by* $A = \{x_1, ..., x_k\}$, $A^c = \{x_{k+1}, ..., x_n\}$, *with* A, A^c *two sets of indifference. Then, in order to define the fuzzy measure we just need to know*

$$\mu(x_1), \mu(x_{k+1}), \text{ for singletons.}$$

$$\mu(x_1, x_2), \mu(x_{k+1}, x_{k+2}), \mu(x_1, x_{k+1}), \text{ for sets of two elements,}$$

and so on.

Then, it suffices to know the cardinality and the number of elements of A *in the subset.*

Remark 1 *It is important to note that, in order to define a* p*-symmetric measure, we need to know which are the sets of indifference partitioning the universal set. For symmetric measures, we have only one set of indifference* (X) *and thus we omit it, but a symmetric measure is a very particular measure and this does not hold for the general* p*-symmetric case.*

Let us now propose a situation in which p-symmetric measures may appear:

Example 3 *Suppose that a jury of four members is evaluating some students. Moreover, suppose that two members of the jury are mathematicians* M_1, M_2 *and the other two are physicists* P_1, P_2. *Suppose also that we do not have information about which one of the two mathematicians is the best, nor for the physicists. However, let us suppose that, for us, the marks in Mathematics are more important than those in Physics. The fuzzy measure could be defined as follows:* $\mu(M_i) = 0.3$, $\mu(P_i) = 0.2$, $i = 1, 2$ *as the marks in Mathematics are more important than the marks in Physics. Now, for pairs, we can define* $\mu(M_1, M_2) = 0.5, \mu(P_1, P_2) = 0.3, \mu(M_i, P_j) = 0.8$. *This is due to the fact that a student should be considered better (in the sense of more complete) if he obtains a good evaluation for both subjects than in the case in which he is very good in just one of them. Finally, we can define* $\mu(M_1, M_2, P_i) = 0.9, \mu(P_1, P_2, M_i) = 0.85, \mu(X) = 1$.

In this example, we have two sets of indifference, one for the mathematicians and another one for the physicists, and μ *is a 2-symmetric measure. These subsets* *model the fact that we are not able to distinguish between the mathematicians nor between the physicists. Then, for example, the coalition between a physicist and a mathematician has always the same importance for us, regardless which is the mathematician and the physicist in it.*

Example 4 *Consider a finite referential set X on which a probability measure has been defined. However, suppose that we only know the probability values on some subsets of X, namely $B_1, ..., B_p$. Then, we have a set of coherent probabilities with this information. The lower bound of this set is given by*

$$\mu(A) = \sum_{B_i \subset A} P(B_i),$$

and similarly, the upper bound is the corresponding dual measure. This concepts has been introduced by De Finetti in [8].

Let us suppose now that sets $B_1, ..., B_p$ determines a partition on X. Then, it is easy to see that the corresponding measure μ is at most a p-symmetric measure, where sets of indifference are $B_1, ..., B_p$. Indeed, the p-symmetric measure is given by

$$\mu(i_1, ..., i_p) = \begin{cases} 0 & \text{if } i_k < |B_k|, \forall k \\ P(B_{i_r}) & \text{if } i_r = |B_r|, \, i_k < |B_k|, \forall k \neq r \\ ... & ... \end{cases}$$

Example 5 *Consider the 2-step Choquet integral defined in* [14]. *Proposition 2 speaks about 2-step Choquet integral with second step based on additive measures which can be represented as a single Choquet integral with respect to a fuzzy measure; now, if the first steps are OWA operators, i.e. Choquet integral with respect to 1-symmetric fuzzy measures $\mu_1, .., \mu_p$ with disjoint supports, we have that the corresponding μ in Choguet integral representation is a p-symmetric measure.*

In the following, as we only need to know the number of elements of each set of indifference that belong to a given subset C of the universal set X, when dealing with a p-symmetric measure defined by the partition $\{A_1, ..., A_p\}$, we use the notation $C \equiv (c_1, ..., c_p)$, where c_i is the number of elements of A_i in C. Then, we can identify the different subsets with p-dimensional vectors whose i-th coordinate is an integer number from 0 to $|A_i|$. Hence, the number of different subsets C is $(|A_1| + 1) \times ... \times (|A_p| + 1)$, and this is the number of necessary values that we need to know to completely determine the p-symmetric fuzzy measure. Moreover, as $\mu(0, ..., 0) = 0, \mu(|A_1|, ..., |A_p|) = 1$, it follows that we only need to determine $(|A_1| + 1) \times ... \times (|A_p| + 1) - 2$ values. This is written in next proposition:

Proposition 4 *Let μ be a p-symmetric measure with respect to the partition $\{A_1, ..., A_p\}$. Then, the number of values that are needed in order to determine μ is*

$$\big[(|A_1| + 1) \times \cdots \times (|A_p| + 1)\big] - 2.$$

Example 6 *Consider the special 2-symmetric case in which $A = \{x_1\}$. Then, in order to define the fuzzy measure, we just need to know*

$$\mu(x_1), \mu(x_2), \mu(x_1, x_2), \mu(x_2, x_3), ..., \mu(x_1, ..., x_{n-1}), \mu(x_2, ..., x_n),$$

or, in the notation proposed before

$$\mu(1,0), \mu(0,1), ..., \mu(1, n-2), \mu(0, n-1),$$

i.e. $2n-2$ *values.*

Remark 2 *Note that the number of different subsets depends not only on the degree of symmetry, but also on the sets of indifference that determine the partition of* X. *In Example 6, we only needed* $2n-2$ *coefficients. However, if we take* $n = 6, |A_1| = 3, |A_2| = 3$, *by Proposition 4, we need 4*4-2=14 coefficients.*

As a consequence of Proposition 4, a p-symmetric fuzzy measure can be represented in a $(|A_1|+1) \times ... \times (|A_p|+1)$ matrix $\mathbf{M}$ such that $\mathbf{M}[c_1, ..., c_p] = \mu(c_1, ..., c_p)$.

Let us see some special cases as examples:

- If we have a 1-symmetric measure, we just need to know a $(n+1)$-dimensional vector $\vec{v}$ such that $\vec{v}(i) = \mu(i)$, where $\mu(0) = 0$ and $\mu(n) = 1$.
- If we have a 2-symmetric measure, we obtain a $(|A|+1) \times (|A^c|+1)$ matrix.

$$\begin{pmatrix} \mu(0,0) & \mu(0,1) & \dots & \mu(0,|A_2|-1) & \mu(0,|A_2|) \\ \mu(1,0) & \mu(1,1) & \dots & \mu(1,|A_2|-1) & \mu(1,|A_2|) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu(|A_1|-1,0) & \mu(|A_1|-1,1) & \dots & \mu(|A_1|-1,|A_2|-1) & \mu(|A_1|-1,|A_2|) \\ \mu(|A_1|,0) & \mu(|A_1|,1) & \dots & \mu(|A_1|-1,|A_2|-1) & \mu(|A_1|,|A_2|) \end{pmatrix}$$

- In the extreme case of a n-symmetric measure, we obtain a $2 \times ... \times 2$ matrix, i.e. we need 2^n coefficients (two of them are $\mu(\emptyset)$ and $\mu(X)$).

We finish this section with the following result related to dual measures.

Lemma 4 *Let* μ *be a* p*-symmetric measure with respect to a partition* $\{A_1, ..., A_p\}$. *Then,* $\bar{\mu}$ *is also a* p*-symmetric measure with respect to the same partition.*

Proof: Let us consider a p-symmetric measure μ with respect to the partition $\{A_1, ..., A_p\}$. To show that $\bar{\mu}$ is another p-symmetric measure, it suffices to note that

$$\bar{\mu}(i_1, ..., i_p) = 1 - \mu(|A_1| - i_1, ..., |A_p| - i_p),$$

whence the result holds. ■

4. Other representations of a p-symmetric measure

In this section, we deal with the problem of obtaining the different representations of a fuzzy measure in the special case of p-symmetric measures. More concretely, we obtain the Möbius transform and the Shapley interaction.

Let us start with the Möbius transform.

Proposition 5 *Let* μ *be a* p*-symmetric measure associated to the partition* $\{A_1, ..., A_p\}$. *Then, for* $B \equiv (b_1, ..., b_p) \subset X$, *we have*

$$m(b_1, ..., b_p) = \sum_{i_1 \leq b_1, ..., i_p \leq b_p} (-1)^{b_1 + ... + b_p - i_1 - ... - i_p} \binom{b_1}{i_1} ... \binom{b_p}{i_p} \mu(i_1, ..., i_p).$$

Proof: Consider $B \equiv (b_1, ..., b_p)$. Then, the number of subsets of B with i_1 elements of A_1, i_2 elements of A_2..., i_p elements of A_p is

$$\binom{b_1}{i_1} ... \binom{b_p}{i_p},$$

and we know that they have all the same measure.

Now, as

$$m(B) = \sum_{C \subset B} (-1)^{|B|-|C|} \mu(C),$$

by (1), we obtain

$$m(b_1, ..., b_p) = \sum_{i_1 \leq b_1, ..., i_p \leq b_p} (-1)^{b_1 + ... + b_p - i_1 - ... - i_p} \binom{b_1}{i_1} ... \binom{b_p}{i_p} \mu(i_1, ..., i_p),$$

whence the result. ∎

Let us now find the expression of the measure in terms of the Möbius transformation.

Proposition 6 *Let μ be a p-symmetric measure associated to the partition $\{A_1, ..., A_p\}$. Now, suppose m denotes its Möbius transform. Then, for $B \equiv (b_1, ..., b_p) \subset X$, it is*

$$\mu(b_1, ..., b_p) = \sum_{c_1 \leq b_1, ..., c_p \leq b_p} \binom{b_1}{c_1} \cdots \binom{b_p}{c_p} m(c_1, ..., c_p).$$

Proof: Consider $C \equiv (c_1, ..., c_p) \subset B$. Then, the number of possibilities for such a C is

$$\binom{b_1}{c_1} \cdots \binom{b_p}{c_p},$$

and thus the expression holds applying (2). ∎

Let us now turn to the Shapley interaction:

Proposition 7 *Let μ be a p-symmetric measure associated to the partition $\{A_1, ..., A_p\}$. Then, for $B \equiv (b_1, ..., b_p) \subset X$, we have*

$$I(b_1, ..., b_p) = \sum_{c_1 \geq b_1, ..., c_p \geq b_p} \frac{1}{c - b + 1} \binom{a_1 - b_1}{c_1 - b_1} \cdots \binom{a_p - b_p}{c_p - b_p} m(c_1, ..., c_p),$$

with $c = \sum_{i=1}^{p} c_i$, $b = \sum_{i=1}^{p} b_i$.

Proof: We know from (3) that for $B \subset X$

$$I(B) = \sum_{C | B \subset C} \frac{1}{|C| - |B| + 1} m(C).$$

Let us consider $C \equiv (c_1, ..., c_p) | \, B \equiv (b_1, ..., b_p) \subset C$. Then, the number of possible C's is

$$\binom{a_1 - b_1}{c_1 - b_1} \cdots \binom{a_p - b_p}{c_p - b_p}.$$

Thus, we obtain

$$I(b_1, ..., b_p) = \sum_{c_1 \geq b_1, ..., c_p \geq b_p} \frac{1}{c - b + 1} \binom{a_1 - b_1}{c_1 - b_1} \cdots \binom{a_p - b_p}{c_p - b_p} m(c_1, ..., c_p),$$

whence the result. ■

The reciprocal result is given by

Proposition 8 *Let μ be a p-symmetric measure associated to the partition $\{A_1, ..., A_p\}$. Then, for $B \equiv (b_1, ..., b_p) \subset X$, we have*

$$m(b_1, ..., b_p) = \sum_{c_i \leq a_i - b_i, i=1,...,p} \binom{a_1 - b_1}{c_1} \cdots \binom{a_p - b_p}{c_p} B_{c_1+...+c_p} I(c_1+b_1, ..., c_p+b_p).$$

Proof: We know from (4) that for $B \subset X$

$$m(B) = \sum_{C \subset X \setminus B} B_{|C|} I(C \cup B).$$

Let us consider $C \equiv (c_1, ..., c_p) \subset X \setminus B \equiv (a_1 - b_1, ..., a_p - b_p)$. Then, the number of possible C's is

$$\binom{a_1 - b_1}{c_1} \cdots \binom{a_p - b_p}{c_p}.$$

Thus, we obtain

$$m(B) = \sum_{c_1 \leq a_1 - b_1, ..., c_p \leq a_p - b_p} B_{c_1+...+c_p} \binom{a_1 - b_1}{c_1} \cdots \binom{a_p - b_p}{c_p} I(c_1 + b_1, ..., c_p + b_p),$$

whence the result. ■

The expression of Shapley interaction in terms of μ is given in next proposition.

Proposition 9 *Let μ be a p-symmetric measure associated to the partition $\{A_1, ..., A_p\}$. Then, for $B \equiv (b_1, ..., b_p) \subset X$, we have*

$$I(B) = \sum_{d_i \leq a_i, \forall i} \sum_{c_i \geq \{b_i, d_i\}, \forall i} \frac{1}{c - b + 1} (-1)^{c-d} \binom{c_1}{d_1} \cdots \binom{c_p}{d_p} \binom{a_1 - b_1}{c_1 - b_1} \cdots \binom{a_p - b_p}{c_p - b_p} \mu(D),$$

with $d = \sum_{i=1}^{p} d_i, c = \sum_{i=1}^{p} c_i, b = \sum_{i=1}^{p} b_i.$

Proof: We will use the expressions in Proposition 5 and Proposition 7. From Proposition 7, we know that

$$I(b_1, ..., b_p) = \sum_{c_1 \geq b_1, ..., c_p \geq b_p} \frac{1}{c-b+1} \binom{a_1 - b_1}{c_1 - b_1} \cdots \binom{a_p - b_p}{c_p - b_p} m(c_1, ..., c_p).$$

But now, from Proposition 5,

$$m(c_1, ..., c_p) = \sum_{d_1 \leq c_1, ..., d_p \leq c_p} (-1)^{c_1 + ... + c_p - d_1 - ... - d_p} \binom{c_1}{d_1} ... \binom{c_p}{d_p} \mu(d_1, ..., d_p).$$

Joining both results, the proposition is proved. ∎

And the reciprocal result is:

Proposition 10 *Let μ be a p-symmetric measure associated to the partition $\{A_1, ..., A_p\}$. Then, for $B \equiv (b_1, ..., b_p) \subset X$ we have*

$$\mu(B) = \sum_{c_i \leq b_i, \forall i} \binom{b_1}{c_1} \cdots \binom{b_p}{c_p} \sum_{d_i \leq a_i - c_i, \forall i} \binom{a_1 - c_1}{d_1} \cdots \binom{a_p - c_p}{d_p} B_d I(c_1 + d_1, ..., c_p + d_p),$$

with $d = \sum_{i=1}^p d_i$.

Proof: We know that

$$\mu(b_1, ..., b_p) = \sum_{c_1 \leq b_1, ..., c_p \leq b_p} \binom{b_1}{c_1} \cdots \binom{b_p}{c_p} m(c_1, ..., c_p),$$

by Proposition 6, and

$$m(C) = \sum_{d_i \leq a_i - c_i, i=1,...,p} \binom{a_1 - c_1}{d_1} \cdots \binom{a_p - c_p}{d_p} B_{d_1 + ... + d_p} I(c_1 + d_1, ..., c_p + d_p),$$

by Proposition 8. Joining both results, the proposition is proved. ∎

Of course, when considering the representation of a p-symmetric measure in terms of the Möbius transform or the Shapley interaction, we can represent it in a p-dimensional matrix, as we have done in the previous section.

5. Choquet integral with respect to a p-symmetric measure

In this section we study the expression of Choquet integral with respect to a p-symmetric measure, as well as some properties of this integral.

Proposition 11 *Let μ be a p-symmetric measure. Given a function f, the Choquet integral is given by*

$$\sum_{i=1}^n f(x_{(i)}) \sum_{c_k \leq b_k^{i-1}, \forall k} m(c_1, ..., c_j + 1, ..., c_p) \prod_{k=1}^p \binom{b_k^{i-1}}{c_k}$$

with $x_{(i)} \in A_j$, and where $(b_1^{i-1}, ..., b_p^{i-1}) \equiv B_{(i-1)} = \{x_{(1)}, ..., x_{(i-1)}\}$.

Proof: We know from Definition 6

$$(C)\int f d\mu = \sum_{i=1}^{n} f(x_{(i)})(\mu(B_{(i)}) - \mu(B_{(i+1)})),$$

where $B_{(i)} = B_{(i+1)} \cup \{x_{(i)}\}$.

Suppose that $B_{(i+1)} \equiv (b_1^{i-1}, ..., b_p^{i-1})$. Then, by Proposition 6

$$\mu(B_{(i+1)}) = \sum_{c_k \leq b_k^{i-1}, \forall k} m(c_1, ..., c_p) \prod_{k=1}^{p} \binom{b_k^{i-1}}{c_k}.$$

Now, if $x_{(i)} \in A_j$,

$$\mu(B_{(i)}) - \mu(B_{(i+1)}) = \sum_{C \subset B_{(i+1)}} m(x_{(i)} \cup C) = \sum_{c_k \leq b_k^{i-1}, \forall k} m(c_1, ..., c_j + 1, ..., c_p) \prod_{k=1}^{p} \binom{b_k^{i-1}}{c_k},$$

whence the result. ■

As a subset $C \subset X$ is determined by the number of elements in $A_i, \forall i$, we can find all possible Choquet integrals finding all possible paths from $(0, ..., 0)$ to $(|A_1|, ..., |A_p|)$ (see Figure 1 for an example with a 2-symmetric measure).

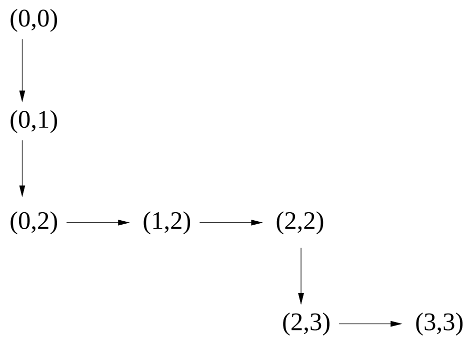

Figure 1: Possible path from (0,0) to (3,3) when $|A_1| = 3$ and $|A_2| = 3$.

The number of such paths is given in next lemma:

Lemma 5 *Let μ be a p-symmetric measure with respect to the partition $\{A_1, ..., A_p\}$. Then, the number of paths from $(0, ..., 0)$ to $(|A_1|, ..., |A_p|)$ is*

$$\binom{n}{|A_1|, ..., |A_p|}.$$

Example 7 *If we are in the 2-symmetric case and $|A| = 1$, then we have just $n+1$ different paths from $(0, ..., 0)$ to $(|A_1|, ..., |A_p|)$ (see Figure 2).*

Let us now see some properties for Choquet integral of a p-symmetric measure.

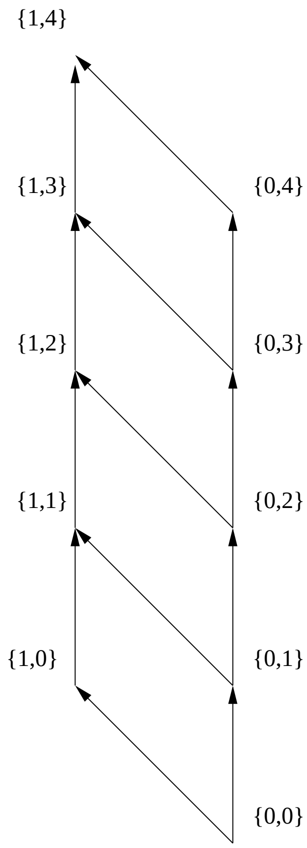

Figure 2: Possible paths when $|A_1| = 1$ and $|A_2| = 4$.

Proposition 12 *Let μ be a p-symmetric measure with respect to the partition $\{A_1, ..., A_p\}$, and suppose $\mu(A_i) > 0, \forall i$. Then, the Choquet integral is given by*

$$\sum_{i=1}^{p} \mu(A_i)(\mathcal{C}) \int f d\mu_{A_i} + \sum_{B \not\subset A_j, \forall j} m(B) \bigwedge_{x_i \in B} f(x_i),$$

where μ_{A_i} is defined by its Möbius transform

$$m_{A_i}(C) = \begin{cases} \frac{m(C)}{\mu(A_i)} & \text{if} C \subset A_i \\ 0 & \text{otherwise} \end{cases}$$

Proof: Suppose that μ is a p-symmetric measure with respect to the partition $\{A_1, ..., A_p\}$. Then, the Choquet integral can be written as

$$\sum_{j=1}^{p} \sum_{B \subset A_j} m(B) \bigwedge_{x_i \in B} f(x_i) + \sum_{B \not\subset A_j, \forall j} m(B) \bigwedge_{x_i \in B} f(x_i),$$

by (5). Now, let us define for $A \subset X$, $\mu(A) > 0$

$$m_A(C) = \begin{cases} \frac{m(C)}{\mu(A)} & \text{if} C \subset A \\ 0 & \text{otherwise} \end{cases}$$

Let us see that m_A is the Möbius transform of a fuzzy measure. To see this, let us show that the conditions of Proposition 1 hold:

First, note that $m_A(\emptyset) = 0$. Now, for $i \in X, C \subset X$, we have

- If $x_i \notin A$, then
$$\sum_{x_i \in B \subset C} m_A(B) = 0.$$
- If $x_i \in A$, then
$$\sum_{x_i \in B \subset C} m_A(B) = \sum_{x_i \in B \subset C \cap A} \frac{m(B)}{\mu(A)} \geq 0,$$
as μ is a fuzzy measure.
- $\sum_{B \subset X} m_A(B) = \frac{\mu(A)}{\mu(A)} = 1$.

Let us denote by μ_A the fuzzy measure associated to m_A. Then, it is trivial to see that

$$\sum_{B \subset A_j} m(B) \bigwedge_{x_i \in B} f(x_i) = \mu(A_j)\,(C) \int f d\mu_{A_j}, \forall j,$$

by (5). This completes the proof. ■

The last summand in Proposition 12 represents the part of the Choquet integral that cannot be assigned to any subset in the partition. When μ is a belief function, the following can be proved:

Proposition 13 *Let μ be a p-symmetric measure with respect to the partition $\{A_1, ..., A_p\}$. Suppose also that μ is a belief function. Then, the Choquet integral can be written as*

$$\sum_{i=1}^{n} \mu(A_i)(\mathcal{C}) \int f d\mu_{A_i} + (C) \int f d\mu^*,$$

where μ_{A_i} and μ^* are defined by

$$m_{A_i}(C) = \begin{cases} \frac{m(C)}{\mu(A_i)} & \text{if} C \subset A_i \\ 0 & \text{otherwise} \end{cases}$$

$$\mu^*(C) = \mu(C) - \mu(C \cap A_1) - ... - \mu(C \cap A_p).$$

Proof: We know from Proposition 12 that the Choquet integral for a p-symmetric fuzzy measure can be written as

$$\sum_i \mu(A_i)(\mathcal{C})\int f d\mu_{A_i} + \sum_{B\not\subset A_j,\forall j} m(B) \bigwedge_{x_i\in B} f(x_i).$$

Now, define

$$\mu^*(C) = \mu(C) - \mu(C\cap A_1) - ... - \mu(C\cap A_p).$$

μ^* is a non-normalized fuzzy measure:

$$\sum_{x_i\in B\subset C} m^*(B) = \sum_{x_i\in B\subset C,\, B\not\subset A_j,\forall j} m(B) \geq 0,$$

as μ is a belief function. Remark that

$$m^*(B) = \begin{cases} m(B) & \text{if} B\not\subset A_j, \forall j \\ 0 & \text{otherwise} \end{cases}$$

Then, it is easy to see that

$$\sum_{B\not\subset A_j,\forall j} m(B) \bigwedge_{x_i\in B} f(x_i) = (C)\int f d\mu^*,$$

and thus, the proposition holds. ■

Note that for belief functions, $\mu^*(X) = \mu(X) - \mu(A_1) - ... - \mu(A_p)$. Then, this value can be seen as a degree of the interaction among the elements of the partition: If $\mu^*(X) = 0$, then necessary $m(B) = 0$ if $B \not\subset A_1, ..., B \not\subset A_p$. We write it in the following definition.

Definition 16 *Consider μ a p-symmetric measure associated to the partition given by $\{A_1, ..., A_p\}$. Suppose also that μ is a belief function. We define the* **degree of interaction among the elements of the partition** *by*

$$\mu(X) - \mu(A_1) - ... - \mu(A_p).$$

Now, we can state the following corollary:

Corollary 1 *Let μ be a p-symmetric measure with respect to the partition given by $\{A_1, ..., A_p\}$ such that $\mu(A_i) > 0$. Suppose also that μ is a belief function. When the interaction degree vanishes, the Choquet integral can be written as*

$$\mu(A_1)(\mathcal{C})\int f d\mu_{A_1} + ... + \mu(A_p)(\mathcal{C})\int f d\mu_{A_p},$$

where μ_{A_i} is defined by

$$m_{A_i}(C) = \begin{cases} \frac{m(C)}{\mu(A_i)} & \text{if} C\subset A_i \\ 0 & \text{otherwise} \end{cases}$$

In this sense, when μ is a belief function and the degree of interaction vanishes, the partition $\{A_1, ..., A_p\}$ in sets of indifference is also an interadditive partition (Equation 6). Moreover, each integral is indeed an OWA operator over a smaller referential set.

Remark 3 *Note that we need μ being a belief function in order to ensure a positive value for the degree of interaction among the elements of the partition. Moreover, if μ is not a belief function, we can find $\mu(X) - \mu(A_1) - ... - \mu(A_p) = 0$ and, on the other hand, there exist interactions among the elements of the partition.*

For Corollary 1, it must be remarked that μ must be a belief function. Otherwise, the result does not necessary hold as next example shows:

Example 8 *Consider $X = \{x_1, x_2, x_3\}$ and let us define the fuzzy measure μ given by the following Möbius transform:*

x_1	x_2	x_3	x_1, x_2	x_1, x_3	x_2, x_3	x_1, x_2, x_3
0.4	*0.3*	*0.3*	*0.1*	*0.1*	*0*	*-0.2*

μ is a 2-symmetric measure, with sets of indifference $A_1 = \{x_1\}$, $A_2 = \{x_2, x_3\}$. On the other hand $\mu(A_1) + \mu(A_2) = 1$, and thus, $\mu(X) - \mu(A_1) - \mu(A_2) = 0$. However, we can not ensure

$$(\mathcal{C}) \int f d\mu = \mu(A_1)\,(\mathcal{C}) \int f d\mu_{A_1} + \mu(A_2)\,(\mathcal{C}) \int f d\mu_{A_2},\ \forall f.$$

To see this, just consider f defined by $f(x_1) = 1$, $f(x_2) = 0.5$, $f(x_3) = 0$. Then, it is straightforward to see

$$(\mathcal{C}) \int f d\mu = 0.6,\ \mu(A_1)\,(\mathcal{C}) \int f d\mu_{A_1} = 0.4,\ \mu(A_2)\,(\mathcal{C}) \int f d\mu_{A_2} = 0.15.$$

6. Conclusions

In this paper, we have proposed a generalization of the concept of symmetry for fuzzy measures. This new concept is based in sets of indifference; these subsets model the fact that some elements are indistinguishable. We have defined p-symmetric fuzzy measures and we have studied some of their properties, as well as other representations. The main property of p-symmetric measures is that they can be represented in a p-dimensional matrix. Once the definition of p-symmetry given, we have obtained an expression for Choquet integral; we have shown that this integral can be easily computed from the matrix representation. Finally, we have derived a value for the interaction among sets of indifference.

We think that p-symmetric measures provide an interesting tool in the field of fuzzy measures, and a graduation between symmetric measures and fuzzy measures.

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