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CONSTRUCTION OF LOCAL CONSERVATION LAWS BY GENERALIZED ISOMETRIC EMBEDDINGS OF VECTOR BUNDLES

NABIL KAHOUADJI

ABSTRACT. This article uses Cartan–Kähler theory to construct local conservation laws from covariant closed vector valued differential forms, objects that can be given, for example, by harmonic maps between two Riemannian manifolds. We apply the article’s main result to construct conservation laws for covariant divergence free energy-momentum tensors. We also generalize the local isometric embedding of surfaces in the analytic case by applying the main result to vector bundles of rank two over surfaces.

1. INTRODUCTION

A conservation law can be seen as a map defined on a space \mathcal{F} (which can be for instance, a function space, a fiber bundle section space, . . . etc) that associates to each element f of \mathcal{F} a vector field X on an m -dimensional Riemannian manifold \mathcal{M} , such that if f is a solution of a given PDE on \mathcal{F} , the vector field X has a vanishing divergence. If we denote by g the Riemannian metric on the manifold \mathcal{M} , we can canonically associate to each vector field $X \in \Gamma(T\mathcal{M})$ a differential 1-form $\alpha_X := g(X, \cdot)$. Since $\operatorname{div}(X) = *d*\alpha_X$ (or $\operatorname{div}(X)\operatorname{vol}_{\mathcal{M}} = d(X \lrcorner \operatorname{vol}_{\mathcal{M}})$), where $*$ is the Hodge operator, $\operatorname{vol}_{\mathcal{M}}$ is the volume form on \mathcal{M} and $X \lrcorner \operatorname{vol}_{\mathcal{M}}$ is the interior product of $\operatorname{vol}_{\mathcal{M}}$ by the vector field X , the requirement $\operatorname{div}(X) = 0$ may be replaced by the requirement $d(X \lrcorner \operatorname{vol}_{\mathcal{M}}) = 0$ and hence, conservation laws may also be seen as maps on \mathcal{F} with values on differential $(m - 1)$ -forms such that solutions of PDEs are mapped to closed differential $(m - 1)$ -forms on \mathcal{M} . More generally, we could extend the notion of conservation laws as mapping to differential p -forms (for instance, Maxwell equations in vacuum can be expressed, as it is well-known, by requiring a system of differential 2-forms to be closed). In this paper, we address the question of finding conservation laws for a class of PDE described as follows:

Question 1. *Let \mathbb{V} be an n -dimensional vector bundle over \mathcal{M} . Let g be a metric bundle and ∇ a connection that is compatible with that metric. We then have a covariant derivative d_{∇} acting on vector valued differential forms. Assume that ϕ is a given closed covariant \mathbb{V} -valued differential p -form on \mathcal{M} , i.e.,*

$$(1.1) \quad d_{\nabla}\phi = 0.$$

does there exist $N \in \mathbb{N}$ and an embedding Ψ of \mathbb{V} into $\mathcal{M} \times \mathbb{R}^N$ given by $\Psi(x, X) = (x, \Psi_x X)$, where Ψ_x is a linear map from \mathbb{V}_x to \mathbb{R}^N such that:

- Ψ is isometric, i.e, for every $x \in \mathcal{M}$, the map Ψ_x is an isometry.
- If $\Psi(\phi)$ is the image of ϕ by Ψ , then

$$(1.2) \quad d\Psi(\phi) = 0.$$

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In this problem, the equation (1.1) represents the given PDE (or a system of PDEs) and the map Ψ plays the role of a conservation law. Note that the problem is trivial when the vector bundle is a line bundle. Indeed, since the connection is compatible with the metric, the 1-form of the connection, which is an $\mathfrak{o}(1)$ -valued differential 1-form, is skew-symmetric and hence all covariant closed \mathbb{V}^1 -valued differential p -forms are actually closed in the standard sense.

A fundamental example is the isometric embedding of Riemannian manifolds in Euclidean spaces and is related to the above problem as follows: \mathcal{M} is an m -dimensional Riemannian manifold, \mathbb{V} is the tangent bundle $T\mathcal{M}$, the connection ∇ is the Levi-Civita connection, $p = 1$ and the $T\mathcal{M}$ -valued differential 1-form ϕ is the identity map on $T\mathcal{M}$. Then (1.1) expresses the torsion-free condition for the connection ∇ and any solution Ψ of (1.2) provides us with an isometric embedding u of the Riemannian manifold \mathcal{M} into an Euclidean space \mathbb{R}^N through $du = d\Psi(\phi)$ and conversely. An answer to the local analytic isometric embeddings of Riemannian manifolds is given by the Cartan–Janet theorem, [Car27] and [Jan26]. Despite the fact that the Cartan–Janet result is local and the analyticity condition of the data regularity may seem to be too restrictive, the Cartan–Janet theorem is important because it actualizes the embedding in an optimal dimension unlike the Nash–Moser isometric embedding which is a smooth and global result. Consequently, if the above problem has a positive answer when the closed covariant vector valued differential p -form on a general vector bundle is a closed covariant vector valued differential 1-form, the notion of isometric embeddings of Riemannian manifolds is extended to a notion of *generalized isometric embeddings* of vector bundles. The general problem, when p is arbitrary, can also be viewed as an *embedding of closed covariant vector valued differential p -forms*.

Another example expounded in [Hél96] of such closed covariant vector valued forms are given by harmonic maps between two Riemannian manifolds. Indeed, let us consider a map u defined on an m -dimensional Riemannian manifold \mathcal{M} which takes values in an n -dimensional Riemannian manifold \mathcal{N} . On the induced bundle¹ by u over \mathcal{M} , the $u^*T\mathcal{N}$ -valued differential $(m - 1)$ -form $*du$ is closed covariant if and only if the map u is harmonic, where the connection on the induced bundle is the pull back by u of the Levi-Civita connection on \mathcal{N} . A positive answer to the above problem in this case will make it possible to construct conservation laws on \mathcal{M} from closed covariant vector valued differential $(m - 1)$ -forms, provided, for example, by harmonic maps. In his book [Hél96], motivated by the question of the compactness of weakly harmonic maps in Sobolev spaces in the weak topology (which is still an open question), Hélein considers harmonic maps between Riemannian manifolds and explains how conservation laws may be obtained explicitly by the Noether’s theorem if the target manifold is symmetric and formulates the problem for non symmetric target manifolds.

In this article, our main result is a positive answer when $p = m - 1$ in the analytic case. We provide, as on the Cartan–Janet theorem, the minimal required dimension that ensures the *generalized isometric embedding* of an arbitrary vector bundle relative to a covariant closed vector valued differential $(m - 1)$ -form.

Theorem 1. *Every real analytic n -dimensional vector bundle \mathbb{V} over a real analytic m -dimensional manifold \mathcal{M} endowed with a metric g and a connection ∇ compatible with g can be, locally, isometrically embedded in the vector bundle $\mathcal{M} \times \mathbb{R}^{n+\kappa_{m,m-1}^n}$ over \mathcal{M} where $\kappa_{m,m-1}^n \geq (m - 1)(n - 1)$ such that the image of a covariant closed \mathbb{V} -valued differential $(m - 1)$ -form is a (local) conservation law.*

¹or the pull back bundle.

On the other hand, obstructions to the existence of a general solution to our question may exist. Indeed, as proved in section 3,

Proposition 1. *Let \mathbb{V} be an n -dimensional vector bundle over an m -dimensional manifold \mathcal{M} . Let g be a metric bundle and ∇ a connection that is compatible with that metric. Let ϕ be a closed covariant \mathbb{V} -valued differential p -form on \mathcal{M} , i.e., $d_{\nabla}\phi = 0$, where $p \leq m - 2$. Then a necessary condition for the existence of an embedding Ψ of \mathbb{V} into $\mathcal{M} \times \mathbb{R}^N$ in general, such that Ψ is isometric and $d(\Psi(\phi)) = 0$ is*

$$(1.3) \quad \frac{n(n-1)}{2} \cdot \frac{m(m-1)}{2} - \frac{n \cdot m!}{(p+2)!(m-p-2)!} > 0.$$

Our existence result theorem 1. can be applied to harmonic maps. We show in the last section of this paper a further application related to energy-momentum tensor which occurs e.g. in general relativity.

The strategy for proving theorem 1. is the following: we reformulate the problem by means of an exterior differential system on a manifold that have to be defined, and since all the datas involved are real analytic, we use the Cartan–Kähler theory to prove the existence of integral manifolds. The problem can be represented by the following diagram which will help later explain the notations

$$\begin{array}{ccc} \mathbb{V}^n & \xrightarrow{T} & \mathcal{M}^m \times \mathbb{R}^N \\ d_{\nabla}\phi=0 \downarrow & & \downarrow dT(\phi)=0 \\ \mathcal{M}^m & & \mathcal{M}^m \end{array}$$

where N is an integer that have to be defined in terms of the problem's datas: n , m and p . Let us then set up a general strategy as an attempt to solve the general problem. We denote by $\kappa_{m,p}^n$ the *embedding codimension*, i.e., the dimension of the fiber extension in order to achieve the desired embedding. Since the Cartan–Kähler theory plays an important role in this paper and since many readers may not be familiar with exterior differential systems (EDS) and the Cartan–Kähler theorem, generalities are made concerning these notions and results in section 2. For details and proofs, the reader may consult Élie Cartan's book [Car71] and the third chapter of [BCG⁺91].

Let us first recast our problem by using moving frames and coframes. For convenience, we adopt the following conventions for the indices: $i, j, k = 1, \dots, n$ are the fiber indices, $\lambda, \mu, \nu = 1, \dots, m$ are the manifold indices and $a, b, c = n+1, \dots, n+\kappa_{m,p}^n$ are the extension indices. We also adopt the Einstein summation convention, i.e., there is a summation when the same index is repeated in high and low positions. However, we will write the sign \sum and make explicit the values of the summation indices when it is necessary. Let $\eta = (\eta^1, \dots, \eta^m)$ be a moving coframe on \mathcal{M} . Let $E = (E_1, \dots, E_n)$ be an orthonormal moving frame of \mathbb{V} . The closed covariant \mathbb{V} -valued differential p -form $\phi \in \Gamma(\wedge^p \mathcal{M} \otimes \mathbb{V}_n)$ can be expressed as follows:

$$(1.4) \quad \phi = E_i \phi^i = E_i \psi_{\lambda_1, \dots, \lambda_p}^i \eta^{\lambda_1, \dots, \lambda_p}$$

where $\psi_{\lambda_1, \dots, \lambda_p}^i$ are functions on \mathcal{M} such that $1 \leq \lambda_1 < \dots < \lambda_p \leq m$ and $\eta^{\lambda_1, \dots, \lambda_p}$ means $\eta^{\lambda_1} \wedge \dots \wedge \eta^{\lambda_p}$.

Definition 1. *Let $\phi \in \Gamma(\wedge^p \mathcal{M} \otimes \mathbb{V})$ be a \mathbb{V} -valued differential p -form on \mathcal{M} . A generalized torsion of a connection relative to ϕ (or for short, a ϕ -torsion) on a vector bundle over \mathcal{M} is a \mathbb{V} -valued differential $(p+1)$ -form $\Theta = (\Theta^i)$ defined as*

follows:

$$(1.5) \quad \Theta = E_i \Theta^i := E_i(d\phi^i + \eta_j^i \wedge \phi^j) \quad \text{for all } i$$

where (η_j^i) is the connection 1-form of ∇ which is an $\mathfrak{o}(n)$ -valued differential 1-form (since ∇ is compatible with the metric bundle).

The condition of closedness covariance $d_\nabla \phi = 0$ is equivalent to the fact that, $d\phi^i + \eta_j^i \wedge \phi^j = 0$ for all $i = 1, \dots, n$. From the above definition, the connection ∇ is ϕ -torsion free. We also notice that the generalized torsion defined above is related to the standard torsion in the tangent bundle case when $\phi = E_i \psi_\lambda^i \eta^\lambda = E_i \eta^i$ (the functions $\psi_\lambda^i = \delta_\lambda^i$ are the Kronecker tensors), and the connection is Levi-Civita.

$$(1.6) \quad d_\nabla \phi = 0 \iff d\phi^i + \eta_j^i \wedge \phi^j = 0 \quad \text{for all } i = 1, \dots, n.$$

Let us assume that the problem has a solution. We consider the flat connection 1-form ω on $\frac{SO(n + \kappa_{m,p}^n)}{SO(\kappa_{m,p}^n)}$, the n -adapted frames of $\mathbb{R}^{(n + \kappa_{m,p}^n)}$, i.e., the set of orthonormal n vectors (e_1, \dots, e_n) of $\mathbb{R}^{(n + \kappa_{m,p}^n)}$ which can be completed by orthonormal $\kappa_{m,p}^n$ vectors $(e_{n+1}, \dots, e_{n + \kappa_{m,p}^n})$ to obtain an orthonormal set of $(n + \kappa_{m,p}^n)$ vectors. The flat standard 1-form of the connection ω is defined as follows: $\omega_j^i = \langle e_i, de_j \rangle$ and $\omega_i^a = \langle e_a, de_i \rangle$, where \langle, \rangle is the standard inner product on $\mathbb{R}^{n + \kappa_{m,p}^n}$. Notice that ω satisfies the Cartan structure equations. Suppose now that such an isometric embedding exists, then, if $e_i = T(E_i)$, the condition $dT(\phi) = 0$ yields to

$$(1.7) \quad e_i(d\phi^i + \omega_j^i \wedge \phi^j) + e_a(\omega_i^a \wedge \phi^i) = 0,$$

a condition which is satisfied if

$$(1.8) \quad \eta_j^i = T^* \omega_j^i \quad \text{and} \quad T^*(\omega_i^a \wedge \phi^i) = 0.$$

The problem then turns to finding moving frames $(e_1, \dots, e_n, e_{n+1}, \dots, e_{n + \kappa_{m,p}^n})$ such that there exist m -dimensional integral manifolds of the exterior ideal generated by the naive exterior differential system $\{\omega_j^i - \eta_j^i, \omega_i^a \wedge \phi^i\}$ on the product manifold

$$(1.9) \quad \Sigma_{m,p}^n = \mathcal{M} \times \frac{SO(n + \kappa_{m,p}^n)}{SO(\kappa_{m,p}^n)}.$$

To find integral manifolds of the naive EDS, we would need to check that the exterior ideal is closed under the differentiation. However, this turns out not to be the case. We then add to the naive EDS the differential of the forms that generate it and therefore, we obtain a closed one.

The objects with which we are dealing, have a meaning in the tangent bundle case with a standard 1-form (the orthonormal moving coframe, as explained above) but not in the arbitrary vector bundle case as we noticed earlier with the notion of torsion of a connection. That leads us to extend them in a generalized sense and recover the standard notions in the tangent bundle case. First of all, the Cartan lemma, which in the isometric embedding problem expresses the symmetry of the second fundamental form, does not hold. Consequently, we can not assure nor assume that the coefficients of the second fundamental form are symmetric as in the isometric embedding problem. In fact, we will show that these conditions should be replaced by *generalized Cartan identities* that express how coefficients of the second fundamental form are related to each other, and of course, we recover the usual symmetry in the tangent bundle case. Another difficulty is the analogue of the Bianchi identity of the curvature tensor. We will define *generalized Bianchi identities* relative to the closed covariant vector valued differential p -form and a *generalized curvature tensor space* which correspond, in the tangent bundle case, to

the usual Bianchi identities and the Riemann curvature tensor space respectively. Finally, from the *generalized Cartan identities* and *generalized curvature tensor space*, we obtain a *generalized Gauss map*.

The key to the proof of theorem 1. is lemma 1. for two main reasons: on one hand, it assures the existence of coefficients of the second fundamental form that satisfy the *generalized Cartan identities* and the *generalized Gauss equation*, properties that simplify the computation of the Cartan characters. On the other hand, the lemma gives the minimal required *embedding codimension* $\kappa_{m,m-1}^n$ that ensures the desired embedding. Using lemma 1. , we give another proof of theorem 1. by an explicit construction of an ordinary integral flag. When the existence of integral manifold is established, we project on $\mathcal{M} \times \mathbb{R}^{n+\kappa_{m,p}^n}$.

2. GENERALITIES

This section is a brief introduction to the Cartan–Kähler theory and is established to state the Cartan test, the proposition 3 and the Cartan–Kähler theorem, results that we use in the proof of theorem 1.

2.1. EDS and exterior ideals. Let us denote by $\mathcal{A}(\mathcal{M})$ the space of smooth differential forms on \mathcal{M} .² An exterior differential system is a finite set of differential forms $I = \{\omega_1, \omega_2, \dots, \omega_k\} \subset \mathcal{A}(\mathcal{M})$ for which we associate the set of equations $\{\omega_i = 0 \mid \omega_i \in I\}$. A subset of differential forms $\mathcal{I} \subset \mathcal{A}(\mathcal{M})$ is an exterior ideal if the exterior product of any differential form of \mathcal{I} by a differential form of $\mathcal{A}(\mathcal{M})$ belong to \mathcal{I} and if the sum of any two differential forms of the same degree belonging to \mathcal{I} , belong also to \mathcal{I} . The exterior ideal generated by I is the smallest exterior ideal containing I . An exterior ideal is said to be an exterior differential ideal if it is closed under the exterior differentiation and hence, the exterior differential ideal generated by an EDS is the smallest exterior differential ideal containing the EDS. Let us notice that an EDS is closed if and only if the exterior differential ideal and the exterior ideal generated by that EDS are equal. In particular, if I is an EDS, $I \cup dI$ is closed.

2.2. Introduction to Cartan–Kähler theory. Let $I \subset \mathcal{A}(\mathcal{M})$ be an EDS on \mathcal{M} and let \mathcal{N} be a submanifold of \mathcal{M} . The submanifold \mathcal{N} is an integral manifold of I if $\iota^*\varphi = 0, \forall \varphi \in I$, where ι is an embedding $\iota : \mathcal{N} \rightarrow \mathcal{M}$. The purpose of this theory is to establish when a given EDS, which represents a PDE, has or has not integral manifolds. We consider in this subsection, an m -dimensional real manifold \mathcal{M} and $\mathcal{I} \subset \mathcal{A}(\mathcal{M})$ an exterior differential ideal on \mathcal{M} .

Definition 2. Let $z \in \mathcal{M}$. A linear subspace E of $T_z\mathcal{M}$ is an integral element of \mathcal{I} if $\varphi_E = 0$ for all $\varphi \in \mathcal{I}$, where φ_E means the evaluation of φ on any basis of E . We denote by $\mathcal{V}_p(\mathcal{I})$ the set of p -dimensional integral elements of \mathcal{I} .

\mathcal{N} is an integral manifold of \mathcal{I} if and only if each tangent space of \mathcal{N} is an integral element of \mathcal{I} . It is not hard to notice from the definition that a subspace of a given integral element is also an integral element. We denote by $\mathcal{I}_p = \mathcal{I} \cap \mathcal{A}^p(\mathcal{M})$ the set of differential p -forms of \mathcal{I} . Thus, $\mathcal{V}_p(\mathcal{I}) = \{E \in G_p(T\mathcal{M}) \mid \varphi_E = 0 \text{ for all } \varphi \in \mathcal{I}_p\}$.

Definition 3. Let E be an integral element of \mathcal{I} . Let $\{e_1, e_2, \dots, e_p\}$ be a basis of $E \subset T_z\mathcal{M}$. The polar space of E , denoted by $H(E)$, is the vector space defined as follows:

$$(2.10) \quad H(E) = \{v \in T_z\mathcal{M} \mid \varphi(v, e_1, e_2, \dots, e_p) = 0 \text{ for all } \varphi \in \mathcal{I}_{p+1}\}.$$

²This is a graded algebra under the wedge product. We do not use the standard notation $\Omega(\mathcal{M})$ to not confuse it with the curvature 2-form of a connection.

Notice that $E \subset H(E)$. The polar space plays an important role in exterior differential system theory as we can notice from the following proposition.

Proposition 2. *Let $E \in \mathcal{V}_p(\mathcal{I})$ be a p -dimensional integral element of \mathcal{I} . A $(p+1)$ -dimensional vector space $E^+ \subset T_z\mathcal{M}$ which contains E is an integral element of \mathcal{I} if and only if $E^+ \subset H(E)$.*

In order to check if a given p -dimensional integral element of an EDS \mathcal{I} is contained in a $(p+1)$ -dimensional integral element of \mathcal{I} , we introduce the following function called the extension rank $r : \mathcal{V}_p(\mathcal{I}) \rightarrow \mathbb{Z}$ that associate to each integral element E , an integer $r(E) = \dim H(E) - (p+1)$. The extension rank $r(E)$ is in fact the dimension of $\mathbb{P}(H(E)/E)$ and is always greater than -1 . If $r(E) = -1$, then E is contained in any $(p+1)$ -dimensional integral element of \mathcal{I} and consequently, there is no hope of extending the integral element. An integral element E is said to be regular if $r(E)$ is constant on a neighborhood of z .

Definition 4. *An integral flag of \mathcal{I} on $z \in \mathcal{M}$ of length n is a sequence $(0)_z \subset E_1 \subset E_2 \subset \dots \subset E_n \subset T_z\mathcal{M}$ of integral elements E_k of \mathcal{I} .*

An integral element E is said to be ordinary if its base point $z \in \mathcal{M}$ is an ordinary zero of $I_0 = I \cap \mathcal{A}^0(\mathcal{M})$ and if there exists an integral flag $(0)_z \subset E_1 \subset E_2 \subset \dots \subset E_n = E \subset T_z\mathcal{M}$ where the $E_k, k = 1, \dots, (n-1)$ are regular. Moreover, if E_n is itself regular, then E is said to be regular. We can now state the following important results of the Cartan–Kähler theory.

Theorem 2. *(Cartan’s test) Let $\mathcal{I} \subset \mathcal{A}^*(\mathcal{M})$ be an exterior ideal which does not contain 0-forms (functions on \mathcal{M}). Let $(0)_z \subset E_1 \subset E_2 \subset \dots \subset E_n \subset T_z\mathcal{M}$ be an integral flag of \mathcal{I} . For any $k < n$, we denote by C_k the codimension of the polar space $H(E_k)$ in $T_z\mathcal{M}$. Then $\mathcal{V}_n(\mathcal{I}) \subset G_n(T\mathcal{M})$ is at least of $C_0 + C_1 + \dots + C_{n-1}$ codimension at E_n . Moreover, E_n is an ordinary integral flag if and only if E_n has a neighborhood U in $G_n(T\mathcal{M})$ such that $\mathcal{V}_n(\mathcal{I}) \cap U$ is a manifold of $C_0 + C_1 + \dots + C_{n-1}$ codimension in U .*

The numbers C_k are called Cartan characters of the k -integral element. The following proposition is useful in the applications. It allows us to compute the Cartan characters of the constructed flag in the proof of the theorem 1.

Proposition 3. *At a point $z \in \mathcal{M}$, let E be an n -dimensional integral element of an exterior ideal $\mathcal{I} \cap \mathcal{A}^*(\mathcal{M})$ which does not contain differential 0-forms. Let $\omega_1, \omega_2, \dots, \omega_n, \pi_1, \pi_2, \dots, \pi_s$ (where $s = \dim \mathcal{M} - n$) be a coframe in a open neighborhood of $z \in \mathcal{M}$ such that $E = \{v \in T_z\mathcal{M} \mid \pi_a(v) = 0 \text{ for all } a = 1, \dots, s\}$. For all $p \leq n$, we define $E_p = \{v \in E \mid \omega_k(v) = 0 \text{ for all } k > p\}$. Let $\{\varphi_1, \varphi_2, \dots, \varphi_r\}$ be the set of differential forms which generate the exterior ideal \mathcal{I} , where φ_ρ is of $(d_\rho + 1)$ degree. For all ρ , there exists an expansion*

$$(2.11) \quad \varphi_\rho = \sum_{|J|=d_\rho} \pi_\rho^J \wedge \omega_J + \tilde{\varphi}_\rho$$

where the 1-forms π_ρ^J are linear combinations of the forms π and the terms $\tilde{\varphi}_\rho$ are, either of degree 2 or more on π , or vanish at z . Moreover, we have

$$(2.12) \quad H(E_p) = \{v \in T_z\mathcal{M} \mid \pi_\rho^J(v) = 0 \text{ for all } \rho \text{ and } \sup J \leq p\}$$

In particular, for the integral flag $(0)_z \subset E_1 \subset E_2 \subset \dots \subset E_n \subset T_z\mathcal{M}$ de \mathcal{I} , the Cartan characters C_p correspond to the number of linear independent forms $\{\pi_\rho^J|_z \text{ such that } \sup J \leq p\}$.

The following theorem is of great importance not only because it is a generalization of the well-known Frobenius theorem but also represents a generalization of the Cauchy–Kovalevskaya theorem.

Theorem 3. (*Cartan–Kähler*)

Let $\mathcal{I} \subset \mathcal{A}^*(\mathcal{M})$ be a real analytic exterior differential ideal. Let $\mathcal{X} \subset \mathcal{M}$ be a p -dimensional connected real analytic Kähler-regular integral manifold of \mathcal{I} . Suppose that $r = r(\mathcal{X}) \geq 0$. Let $\mathcal{Z} \subset \mathcal{M}$ be a real analytic submanifold of \mathcal{M} of codimension r which contains \mathcal{X} and such that $T_x\mathcal{Z}$ and $H(T_x\mathcal{X})$ are transversals in $T_x\mathcal{M}$ for all $x \in \mathcal{X} \subset \mathcal{M}$. There exists then a $(p+1)$ -dimensional connected real analytic integral manifold \mathcal{Y} of \mathcal{I} , such that $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$. Moreover, \mathcal{Y} is unique in the sense that another integral manifold of \mathcal{I} having the stated properties, coincides with \mathcal{Y} on an open neighborhood of \mathcal{X} .

The analyticity condition of the exterior differential ideal is crucial because of the requirements in the Cauchy–Kovalevskaya theorem used in the Cartan–Kähler theorem’s proof. It has an important corollary. Actually, in the application, this corollary is often more used than the theorem and is sometimes called *the Cartan–Kähler theorem* in literature.

Corollary 1. (*Cartan–Kähler*)

Let \mathcal{I} be an analytic exterior differential ideal on a manifold \mathcal{M} . If $E \subset T_z\mathcal{M}$ is an ordinary integral element of \mathcal{I} , there exists an integral manifold of \mathcal{I} passing through z and having E as a tangent space at the point z .

One of the great applications of the Cartan–Kähler theory is the Cartan–Janet theorem concerning local isometric embedding of Riemannian manifolds. We mention this theorem for its historical importance and because the result of this paper generalizes isometric embedding of surfaces.

Theorem 4. (*Cartan [Car27]–Janet [Jan26]*)

Every m -dimensional real analytic Riemannian manifold can be locally embedded isometrically in an $\frac{m(m+1)}{2}$ -dimensional Euclidean space.

3. CONSTRUCTION OF CONSERVATION LAWS

In this section, we continue to explain the general strategy of solving the problem in the general case started in the introduction, and we give a complete proof of theorem 1. In the introduction, we showed that solving the general problem is equivalent to looking for existence of integral manifolds of the naive EDS $\{\omega_j^i - \eta_j^i, \omega_i^a \wedge \phi^i\}$ on the product manifold

$$\Sigma_{m,p}^n = \mathcal{M}_m \times \frac{SO(n + \kappa_{m,p}^n)}{SO(\kappa_{m,p}^n)}$$

This naive EDS is not closed. Indeed, the generalized torsion-free of the connection implies that $d(\omega_i^a \wedge \phi^i) \equiv 0$ modulo the naive EDS, but the second Cartan structure equation yields to $d(\omega_j^i - \eta_j^i) \equiv \sum_a (\omega_i^a \wedge \omega_j^a) - \Omega_j^i$ modulo the naive EDS, where $\Omega = (\Omega_j^i)$ is the curvature 2-form of the connection. Consequently, the exterior ideal that we now consider on the product manifold $\Sigma_{m,p}^n$ is

$$(3.13) \quad \mathcal{I}_{m,p}^n = \{\omega_j^i - \eta_j^i, \sum_a \omega_i^a \wedge \omega_j^a - \Omega_j^i, \omega_i^a \wedge \phi^i\}_{\text{alg}}$$

The curvature 2-form of the connection is an $\mathfrak{o}(n)$ -valued two form and is related to the connection 1-form (η_j^i) by the Cartan structure equation:

$$(3.14) \quad \Omega_j^i = d\eta_j^i + \eta_k^i \wedge \eta_j^k$$

A first covariant derivative of ϕ has led to the generalized torsion. A second covariant derivative of ϕ gives rise to *generalized Bianchi identities*³ as follows:

$$(3.15) \quad d_{\nabla}^2(\phi) = 0 \iff \Omega_j^i \wedge \phi^j = 0 \text{ for all } i = 1, \dots, n.$$

The conditions $\Omega_j^i \wedge \phi^i = 0$ for all $i = 1, \dots, n$ are called *generalized Bianchi identities*. We then define a generalized curvature tensor space $\mathcal{K}_{m,p}^n$ as the space of curvature tensor satisfying the *generalized Bianchi identities*:

$$(3.16) \quad \mathcal{K}_{m,p}^n = \{(\mathcal{R}_{j;\lambda\mu}^i) \in \wedge^2(\mathbb{R}^n) \otimes \wedge^2(\mathbb{R}^m) \mid \Omega_j^i \wedge \phi^i = 0\}$$

where $\Omega_j^i = \frac{1}{2}\mathcal{R}_{j;\lambda\mu}^i \eta^\lambda \wedge \eta^\mu = \mathcal{R}_{j;\lambda\mu}^i \eta^\lambda \otimes \eta^\mu$.

In the tangent bundle case and $\phi = E_i \eta^i$, $\mathcal{K}_{n,1}^n$ is the Riemann curvature tensor space which is of dimension $\frac{1}{12}m^2(m^2 - 1)$.

We are now in position to prove proposition 1. stated in the introduction. When $p = m - 1$, generalized Bianchi identities are trivial, i.e., $\mathcal{K}_{m,m-1}^n = \wedge^2(\mathbb{R}^n) \otimes \wedge^2(\mathbb{R}^m)$. The proposition 1. then holds. When $p \leq m - 2$ and $\frac{n(n-1)}{2} \cdot \frac{m(m-1)}{2} - \frac{n.m!}{(p+2)!(m-p-2)!} \leq 0$, then there are no solutions in general to our question. Indeed, $\Omega_j^i \wedge \phi^j$ is a differential $(p+2)$ -form on \mathcal{M} . When $p \leq m - 2$, the *generalized Bianchi identities* is a system of $\frac{n.m!}{(p+2)!(m-p-2)!}$ equations on the curvature tensor $(\mathcal{R}_{j;\lambda\mu}^i)$. Therefore, the positivity of the necessary condition expresses the fact that there are less constraints than the dimension of the curvature tensors.

All the datas are analytic, we can apply the Cartan–Kähler theory in order to check the involution of the exterior differential system by constructing an m -integral flag. If the exterior ideal $\mathcal{I}_{m,p}^n$ passes the Cartan test, the flag is then ordinary and by the Cartan–Kähler theorem, there exists integral manifolds of $\mathcal{I}_{m,p}^n$. To be able to project the product manifold $\Sigma_{m,p}^n$ on \mathcal{M} , we show the existence of m -dimensional integral manifolds on which the volume form on $\eta^{1,\dots,m}$ on \mathcal{M} does not vanish.

Let us express the 1-forms ω_i^a in the coframe (η^1, \dots, η^m) in order to later make the computation of Cartan characters easier. Let $\mathcal{W}_{m,p}^n$ be an $\kappa_{m,p}^n$ -dimensional Euclidean space. We then write $\omega_i^a = H_{i\lambda}^a \eta^\lambda$ where $H_{i\lambda}^a \in \mathcal{W}_{m,m-1}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^m$ and define the forms $\pi_i^a = \omega_i^a - H_{i\lambda}^a \eta^\lambda$. We can also consider $H_{i\lambda} = (H_{i\lambda}^a)$ as a vector of $\mathcal{W}_{m,p}^n$. The forms that generate algebraically $\mathcal{I}_{m,p}^n$ are then expressed as follows:

$$(3.17) \quad \begin{aligned} \sum_a \omega_i^a \wedge \omega_j^a - \Omega_j^i &= \sum_a \pi_i^a \wedge \pi_j^a + \sum_a (H_{j\lambda}^a \pi_i^a - H_{i\lambda}^a \pi_j^a) \wedge \eta^\lambda \\ &+ \sum_a \underbrace{(H_{i\lambda}^a H_{j\mu}^a - H_{i\mu}^a H_{j\lambda}^a - \mathcal{R}_{j;\lambda\mu}^i)}_* \eta^\lambda \otimes \eta^\mu \end{aligned}$$

and

$$(3.18) \quad \omega_i^a \wedge \phi^i = \psi_{\lambda_1 \dots \lambda_p}^i \pi_i^a \wedge \eta^{\lambda_1 \dots \lambda_p} + \sum_{\substack{\lambda=1, \dots, m \\ 1 \leq \mu_1 < \dots < \mu_p \leq m}} \overbrace{H_{i\lambda}^a \psi_{\mu_1, \dots, \mu_p}^i}^{**} \eta^{\lambda \mu_1 \dots \mu_p}$$

³In the tangent bundle case and $\phi = E_i \eta^i$, we recover the standard Bianchi identities of the Riemann curvature tensor, i.e, $\mathcal{R}_{jkl}^i = \mathcal{R}_{lij}^k$ and $\mathcal{R}_{jkl}^i + \mathcal{R}_{ljk}^i + \mathcal{R}_{klj}^i = 0$.

These new expressions of the forms in term of vectors H and the differential 1-form π lead us later to compute the Cartan characters of an m -integral flag. To make these calculations easier, we will choose $H_{i\lambda}^a$, which are the coefficients of the second fundamental form, so that the quantities marked with (*) and (**) in the equation (3.17) and (3.18) vanish and hence:

$$(3.19) \quad \sum_a (H_{i\lambda}^a H_{j\mu}^a - H_{i\mu}^a H_{j\lambda}^a) = \mathcal{R}_{j;\lambda\mu}^i \quad \text{generalized Gauss equation}$$

$$(3.20) \quad \sum_{\substack{\lambda=1,\dots,m \\ 1 \leq \mu_1 < \dots < \mu_p \leq m}} H_{i\lambda}^a \psi_{\mu_1, \dots, \mu_p}^i \eta^{\lambda\mu_1 \dots \mu_p} = 0 \quad \text{generalized Cartan identities}$$

As we mentioned in the introduction, the system of equation (3.20) is said to be *generalized Cartan identities* because it gives us relations between the coefficients of the second fundamental form which are not necessarily the usual symmetry given by the Cartan lemma. These properties of the coefficients and the fact that the curvature tensor $(\mathcal{R}_{j;\lambda\mu}^i)$ satisfies generalized Bianchi identities yield us to name the equation (3.19) as the *generalized Gauss equation*.

We define now a *generalized Gauss map* $\mathcal{G}_{m,p}^n : \mathcal{W}_{m,p}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^m \longrightarrow \mathcal{K}_{m,p}^n$ defined for $H_{i\lambda}^a \in \mathcal{W}_{m,p}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^m$ by

$$(3.21) \quad \left(\mathcal{G}_{m,p}^n(H) \right)_{j;\lambda\mu}^i = \sum_a (H_{i\lambda}^a H_{j\mu}^a - H_{i\mu}^a H_{j\lambda}^a)$$

Let us specialize in the conservation laws case, i.e., when $p = m - 1$. We adopt these following notations: $\Lambda = (1, 2, \dots, m)$ and $\Lambda \setminus k = (1, \dots, k-1, k+1, \dots, m)$. We have thus $\eta^\Lambda = \eta^1 \wedge \dots \wedge \eta^m$ and $\eta^{\Lambda \setminus k} = \eta^1 \wedge \dots \wedge \eta^{k-1} \wedge \eta^{k+1} \wedge \dots \wedge \eta^m$. Let us construct an ordinary m -dimensional integral element of the exterior ideal $\mathcal{I}_{m,m-1}^n$ on $\Sigma_{m,m-1}^n$. Generalized Bianchi identities are trivial in this case and so $\dim \mathcal{K}_{m,m-1}^n = \frac{n(n-1)}{2} \frac{m(m-1)}{2}$.

The generalized Gauss equation is $H_{i\lambda} \cdot H_{j\mu} - H_{i\mu} \cdot H_{j\lambda} = \mathcal{R}_{j;\lambda\mu}^i$, where $H_{i\lambda}$ is viewed as a vector of the $\kappa_{m,m-1}^n$ -Euclidean space $\mathcal{W}_{m,m-1}^n$. Generalized Cartan identities are

$$(3.22) \quad \sum_{\lambda=1,\dots,m} (-1)^{\lambda+1} H_{i\lambda}^a \psi_{\Lambda \setminus \lambda}^i = 0 \quad \text{for all } a$$

The following lemma, for which a proof is later given, represents the key to the proof of theorem 1. not only because it assures the existence of suitable coefficients of the second fundamental form that satisfy generalized Cartan identities and the generalized Gauss equation, but also, as on the Cartan–Janet theorem, it provides us with the minimal embedding codimension.

Lemma 1. *Let $\kappa_{m,m-1}^n \geq (m-1)(n-1)$. Let $\mathcal{H}_{m,m-1}^n \subset \mathcal{W}_{m,m-1}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^m$ be the open set consisting of those elements $H = (H_{i\lambda}^a)$ so that the vectors $\{H_{i\lambda}^a | i = 1, \dots, n-1 \text{ and } \lambda = 1, \dots, m-1\}$ are linearly independents as elements of $\mathcal{W}_{m,m-1}^n$ and satisfy generalized Cartan identities. Then $\mathcal{G}_{m,m-1}^n : \mathcal{H}_{m,m-1}^n \longrightarrow \mathcal{K}_{m,m-1}^n$ is a surjective submersion.*

Let $\mathcal{Z}_{m,m-1}^n = \{(x, \Upsilon, H) \in \Sigma_{m,m-1}^n \times \mathcal{H}_{m,m-1}^n | \mathcal{G}_{m,m-1}^n(H) = \mathcal{R}(x)\}$. We conclude from lemma 1. that $\mathcal{Z}_{m,m-1}^n$ is a submanifold⁴ and hence,

$$(3.23) \quad \dim \mathcal{Z}_{m,m-1}^n = \dim \Sigma_{m,m-1}^n + \dim \mathcal{H}_{m,m-1}^n$$

⁴ $\mathcal{Z}_{m,m-1}^n$ is the fiber of \mathcal{R} by a submersion. The surjectivity of $\mathcal{G}_{m,m-1}^n$ assures the non-emptiness.

where

$$(3.24) \quad \dim \Sigma_{m,m-1}^n = m + \frac{n(n-1)}{2} + n\kappa_{m,m-1}^n$$

$$(3.25) \quad \dim \mathcal{H}_{m,m-1}^n = (nm-1)\kappa_{m,m-1}^n - \frac{n(n-1)m(m-1)}{4}$$

We define the map $\Phi_{m,m-1}^n : \mathcal{Z}_{m,m-1}^n \longrightarrow \mathcal{V}_m(\mathcal{I}_{m,m-1}^n, \eta^\Lambda)$ which associates to $(x, \Upsilon, H) \in \mathcal{Z}_{m,m-1}^n$ the m -plan on which the differential forms that generate algebraically $\mathcal{I}_{m,m-1}^n$ vanish and the volume form η^Λ on \mathcal{M} does not vanish. $\Phi_{m,m-1}^n$ is then an embedding and hence $\dim \Phi(\mathcal{Z}_{m,m-1}^n) = \dim \mathcal{Z}_{m,m-1}^n$. In what follows, we prove that in fact $\Phi(\mathcal{Z}_{m,m-1}^n)$ contains only ordinary m -integral elements of $\mathcal{I}_{m,m-1}^n$. Since the coefficients $H_{i\lambda}^a$ satisfy the generalized Gauss equation and generalized Cartan identities, the differential forms that generate the exterior ideal $\mathcal{I}_{m,m-1}^n$ are as follows:

$$(3.26) \quad \sum_a \omega_i^a \wedge \omega_j^a - \Omega_j^i = \sum_a \pi_i^a \wedge \pi_i^a + \sum_a (H_{j\lambda}^a \pi_i^a - H_{i\lambda}^a \pi_j^a) \eta^\lambda$$

$$(3.27) \quad \omega_i^a \wedge \phi^i = \psi_{\lambda_1 \dots \lambda_p}^i \pi_i^a \wedge \eta^{\lambda_1 \dots \lambda_p}$$

we recall that Cartan characters are the codimension of the polar space of integral elements. Their computation is a straight forward application of proposition 3. and yields

$$(3.28) \quad C_\lambda = \frac{n(n-1)}{2}(\lambda+1) \text{ for } \lambda = 0, \dots, m-2$$

$$(3.29) \quad C_{m-1} = \frac{n(n-1)}{2}m + \kappa_{m,m-1}^n$$

so

$$(3.30) \quad C_0 + \dots + C_{m-1} = m \frac{n(n-1)}{2} + \frac{n(n-1)m(m-1)}{4} + \kappa_{m,m-1}^n$$

Finally, the codimension of the space on m -integral elements of $\mathcal{I}_{m,m-1}^n$ on which η^Λ does not vanish is:

$$(3.31) \quad \begin{aligned} \text{codim } \mathcal{V}_m(\mathcal{I}_{m,m-1}^n, \eta^\Lambda) &= \dim G_m \left(T_{(x, \Upsilon) \Sigma_{m,m-1}^n} \right) - \Phi(\mathcal{Z}_{m,m-1}^n) \\ &= m \frac{n(n-1)}{2} + \frac{n(n-1)m(m-1)}{4} + \kappa_{m,m-1}^n \end{aligned}$$

By the Cartan test, we conclude that $\Phi(\mathcal{Z}_{m,m-1}^n)$ contains only ordinary m -integral flags. The Cartan-Kähler theorem then assures the existence of an m -integral manifold on which η^Λ does not vanish since the exterior ideal is in involution. We finally project the integral manifold on $\mathcal{M} \times \mathbb{R}^{n+\kappa}$. Let us notice that the requirement of the non vanishing of the volume form η^Λ on the integral manifold yields to project the integral manifold on \mathcal{M} and also to view it as a graph of a function f defined on \mathcal{M} with values in the space of n -adapted orthonormal frames of $\mathbb{R}^{n+\kappa}$. In the isometric embedding problem, the composition of f with the projection of the frames on the Euclidean space is by construction the isometric embedding map.

3.1. Another proof of theorem 1. This proof is based on constructing explicitly an ordinary m -integral element, and Cartan characters are computed by expliciting the polar space of an integral flag. As defined above, let consider $\mathcal{I}_{m,m-1}^n$ an exterior ideal on $\Sigma_{m,m-1}^n$. Let us denote by $\left(\frac{\partial}{\partial x^\lambda}\right)$ the dual basis of (η^λ) and by $\left(\frac{\partial}{\partial y^A}\right)$ the dual basis of $(\varpi^A) = (\varpi^{\sigma^i_j}, \varpi^{\sigma^i_j}) = (\omega_j^i - \eta_j^i, \omega_i^a)$ where $A = 1, \dots, \dim \Sigma_{m,m-1}^n -$

m and $\sigma|_j^i = (j-i) + \frac{n(n-1)}{2} - \frac{(n-i)(n-i+1)}{2}$ for $1 \leq i < j \leq n$ and $\sigma|_i^a = \frac{n(n-1)}{2} + (a-n-1)n + i$ for $i = 1, \dots, n$ and $a = n+1, \dots, n + \kappa_{m,m-1}^n$. Let us consider on the Grassmanian manifold $G_m(\Sigma_{m,m-1}^n, \eta^\Lambda)$ a basis X_λ defined as follows:

$$(3.32) \quad X_\lambda(E) = \frac{\partial}{\partial x^\lambda} + P_\lambda^A(E) \frac{\partial}{\partial y^A} \quad A = 1, \dots, \dim \Sigma_{m,m-1}^n - m$$

In order to compute the codimension in the Grassmanian $G_m(T\Sigma_{m,m-1}^n, \eta^\Lambda)$ of m -integral elements of $\mathcal{I}_{m,m-1}^n$, we pull back the forms that generate the exterior ideal. To do so, we evaluate the forms on the basis $X_\lambda(E)$ and hence the expression of the forms on the Grassmanian are:

$$(3.33) \quad (\varpi^{\sigma|_j^i})_E = P_\lambda^{\sigma|_j^i} \eta^\lambda$$

$$(3.34) \quad \left(\sum_a \varpi^{\sigma|_i^a} \wedge \varpi^{\sigma|_j^a} - \Omega_j^i \right)_E = \left(\sum_a P_\lambda^{\sigma|_i^a} P_\mu^{\sigma|_j^a} - P_\mu^{\sigma|_i^a} P_\lambda^{\sigma|_j^a} - \mathcal{R}_{j;\lambda\mu}^i \right) \eta^{\lambda\mu}$$

$$(3.35) \quad (\varpi^{\sigma_j^a} \wedge \phi^i)_E = \left(\sum_\lambda (-1)^{\lambda+1} \psi_{\Lambda \setminus \lambda}^i P_\lambda^{\sigma|_i^a} \right) \eta^\Lambda$$

The number of functions that have linearly independent differentials represents the desired codimension and hence with lemma 1

$$(3.36) \quad \text{codim } \mathcal{V}_m(\mathcal{I}_{m,m-1}^n, \eta^\Lambda) = m \frac{n(n-1)}{2} + \frac{n(n-1)}{2} \frac{m(m-1)}{2} + \kappa_{m,m-1}^n$$

Let us now construct an explicit m -integral element of $\mathcal{I}_{m,m-1}^n$. Since the exterior ideal does not contain functions, every point of $\Sigma_{m,m-1}^n$ is a 0-integral element. Let $(E_0)_z = z \in \Sigma_{m,m-1}^n$. A vector ξ in the tangent space of $\Sigma_{m,m-1}^n$ is of the form

$$(3.37) \quad \xi = \xi^\lambda \frac{\partial}{\partial x^\lambda} + \xi^A \frac{\partial}{\partial y^A}$$

By considering the polar space, we obtain Cartan characters as previously. We choose then the integral element in the following way:

$$(3.38) \quad e_\lambda = \frac{\partial}{\partial x^\lambda} + H_{i\lambda}^a \frac{\partial}{\partial y^{\sigma|_i^a}}$$

where the coefficients $H_{i\lambda}^a$ are provided by the lemma 1, which assures the existence of solutions of the successive polar systems during the construction of the integral flag. The coefficients $\xi^{\sigma|_j^i}$ for all $1 \leq i < j \leq n$ are zero for all the vecteurs e because of $\varpi^{\sigma|_j^i}$. Let us denote $E_\lambda = \text{span}\{e_1, \dots, e_\lambda\}$. The integral flag is then $F = E_0 \subset E_1 \subset \dots \subset E_{m-1} \subset E_m$. Cartan characters are the same as computed previously and the Cartan test assures that the flag is ordinary. By construction, the flag does not annihilate the volume form η^Λ .

3.2. Proof of lemma 1. The generalized Gauss map $\mathcal{G}_{m,p}^n$ defined on $\mathcal{W}_{m,p}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^m$ with values in $\mathcal{K}_{m,p}^n$ is a submersion if and only if the differential $d\mathcal{G}_{m,p}^n \in \mathcal{L}(\mathcal{W}_{m,p}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^m; \mathcal{K}_{m,p}^n)$ is of maximal rank. Let us define the following subspaces of $\mathcal{K}_{m,m-1}^n$ as follows:

$$(3.39) \quad \mathcal{E}_\nu^k|_{m,p}^n = \{ \mathcal{R} \in \mathcal{K}_{m,p}^n \mid \mathcal{R}_{j;\lambda\mu}^i = 0 \text{ for } 1 \leq i < j \leq k \text{ and } 1 \leq \lambda < \mu \leq \nu \}$$

where $2 \leq k \leq n$ and $2 \leq \nu \leq m$. By convention,

$$(3.40) \quad \mathcal{E}_\nu^1|_{m,p}^n = \{ \mathcal{R} \in \mathcal{K}_{m,p}^n \mid \mathcal{R}_{j;\lambda\mu}^i = 0 \text{ for } 1 \leq \lambda < \mu \leq \nu \text{ and } 2 \leq \nu \leq m \}$$

$$(3.41) \quad \mathcal{E}_1^k|_{m,p}^n = \{ \mathcal{R} \in \mathcal{K}_{m,p}^n \mid \mathcal{R}_{j;\lambda\mu}^i = 0 \text{ for } 1 \leq i < j \leq k \text{ and } 2 \leq \nu \leq m \}$$

$$(3.42) \quad \mathcal{E}_1^1|_{m,p}^n = \mathcal{K}_{m,p}^n$$

In what follows, we make the assumption that neither $\psi_{\Lambda \setminus m}^1$ nor $\psi_{\Lambda \setminus 1}^n$ vanishes on the neighborhood of the considered point where n and m are not equal to 2. This assumption yields to express either H_{1m}^a or H_{n1}^a for all a in terms of the linearly independent vectors of $\mathcal{H}_{m,m-1}^n$ which satisfy generalized Cartan identities. When $n = 2$ and $m > 2$, we assume that $\psi_{\Lambda \setminus m}^1 \neq 0$. When $m = 2$ and $n > 2$, we assume that $\psi_2^n \neq 0$. When both n and m are equal to 2, we assume that the determinant $\det \psi = (\psi_1^1 \psi_2^2 - \psi_2^1 \psi_1^2) \neq 0$. In order to understand the proof of the submersivity of $\mathcal{G}_{m,m-1}^n$, we explain and show the proof for two special cases: when the vector bundle is of rank 2 ($n = 2$) and when the manifold is a surface ($m = 2$). The surjectivity of the generalized Gauss map will follow from these considerations. Calculations in the case $\mathcal{G}_{2,1}^n$ lead to the following expression:

$$(3.43) \quad \mathcal{E}_{1|2,1}^{k|n} \ni d\mathcal{G}_{2,1}^n(H_{k2}^a) = \sum_{i=1}^{k-1} H_{i1}^a \mathcal{R}_{k;12}^i - \sum_{j=k+1}^n H_{j1}^a \mathcal{R}_{j;12}^{j-1}$$

where $2 \leq k \leq n$, with this kind of flag:

$$(3.44) \quad 0 = \mathcal{E}_1^{n|2,1} \subset \mathcal{E}_1^{n-1|2,1} \subset \mathcal{E}_1^{n-2|2,1} \subset \dots \subset \mathcal{E}_1^3|2,1 \subset \mathcal{E}_1^2|2,1 \subset \mathcal{K}_{2,1}^n$$

and in the case $\mathcal{G}_{m,m-1}^2$ to

$$(3.45) \quad \mathcal{E}_{\nu|m,m-1}^1 \ni d\mathcal{G}_{m,m-1}^2(H_{2\nu}^a) = \sum_{\lambda=1}^{\nu-1} H_{1\lambda}^a \mathcal{R}_{2;\lambda\nu}^1 - \sum_{\mu=\nu+1}^m H_{1\mu}^a \mathcal{R}_{2;(\mu-1)\mu}^1$$

where $2 \leq \nu \leq m$, with the flag

$$(3.46) \quad 0 = \mathcal{E}_m^1|2_{m,m-1} \subset \mathcal{E}_{m-1}^1|2_{m,m-1} \subset \mathcal{E}_{m-2}^1|2_{m,m-1} \subset \dots \subset \mathcal{E}_2^1|2_{m,m-1} \subset \mathcal{K}_{m,m-1}^2$$

From the linear map $d\mathcal{G}_{2,1}^n$, we want to extract a submatrix of maximal rank. The equation (3.43) shows that the submatrix $(d\mathcal{G}_{2,1}^n(H_{22}^a), \dots, d\mathcal{G}_{2,1}^n(H_{n2}^a))$ is of maximal rank if the vectors $H_{11}, H_{21}, \dots, H_{(n-1)1}$ are linearly independent vectors of $\mathcal{W}_{2,1}^n$ and $\kappa_{2,1}^n \geq (n-1)$ where the minimal embedding codimension is given by the dimension of the first non-zero subspace of the flag (3.44): $\mathcal{E}_1^{n-1}|2,1$. Similarly, from the linear map $d\mathcal{G}_{m,m-1}^2$, we want to extract a submatrix of maximal rank. The equation (3.45) shows that the submatrix $(d\mathcal{G}_{m,m-1}^2(H_{22}^a), \dots, d\mathcal{G}_{m,m-1}^2(H_{2m}^a))$ is of maximal rank if the vectors $H_{11}, H_{12}, \dots, H_{1(m-1)}$ are linearly independent vectors of $\mathcal{W}_{m,m-1}^2$ and $\kappa_{m,m-1}^2 \geq (m-1)$ where the minimal embedding codimension is given by the dimension of the first non-zero subspace of the flag (3.46): $\mathcal{E}_{m-1}^1|2_{m,m-1}$.

We proceed in the same way to prove lemma 1. We have

$$(3.47) \quad \mathcal{E}_{\nu|m,m-1}^{k|n} \ni d\mathcal{G}_{m,m-1}^n(H_{k\nu}^a) = \sum_{\substack{i=1, \dots, k-1 \\ \lambda=1, \dots, \nu-1}} H_{i\lambda}^a \mathcal{R}_{k;\lambda\nu}^i - \sum_{\substack{j=k+1, \dots, n \\ \mu=\nu+1, \dots, m}} H_{j\mu}^a \mathcal{R}_{j;(\mu-1)\mu}^{j-1}$$

where $2 \leq k \leq n$ and $2 \leq \nu \leq m$ and we have the following multi-flag

$$\begin{array}{cccccc}
 & 0 & & 0 & \cdots & 0 & & 0 & & 0 \\
 & \cap & & \cap & & \cap & & \cap & & \cap \\
 0 & \subset \mathcal{E}_{m-1}^{n-1} |_{m,m-1}^n & \subset \mathcal{E}_{m-1}^{n-2} |_{m,m-1}^n & \subset \cdots & \subset \mathcal{E}_{m-1}^2 |_{m,m-1}^n & \subset \mathcal{E}_{m-1}^1 |_{m,m-1}^n & & & & \\
 & \cap & & \cap & & \cap & & \cap & & \cap \\
 0 & \subset \mathcal{E}_{m-2}^{n-1} |_{m,m-1}^n & \subset \mathcal{E}_{m-2}^{n-2} |_{m,m-1}^n & \subset \cdots & \subset \mathcal{E}_{m-2}^2 |_{m,m-1}^n & \subset \mathcal{E}_{m-2}^1 |_{m,m-1}^n & & & & \\
 & \cap & & \cap & & \cap & & \cap & & \cap \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & \cap & & \cap & & \cap & & \cap & & \cap \\
 0 & \subset \mathcal{E}_2^{n-1} |_{m,m-1}^n & \subset \mathcal{E}_2^{n-2} |_{m,m-1}^n & \subset \cdots & \subset \mathcal{E}_2^2 |_{m,m-1}^n & \subset \mathcal{E}_2^1 |_{m,m-1}^n & & & & \\
 & \cap & & \cap & & \cap & & \cap & & \cap \\
 0 & \subset \mathcal{E}_1^{n-1} |_{m,m-1}^n & \subset \mathcal{E}_1^{n-2} |_{m,m-1}^n & \subset \cdots & \subset \mathcal{E}_1^2 |_{m,m-1}^n & \subset \mathcal{K}_{m,m-1}^n & & & &
 \end{array}$$

As we explained previously, from the linear map $d\mathcal{G}_{m,m-1}^n$, we want to extract a submatrix of maximal rank. The equation (3.47) shows that the submatrix $(d\mathcal{G}_{m,m-1}^n(H_{k\nu}^a))$ is of maximal rank if the vectors $H_{i\lambda}$ are linearly independent vectors of $\mathcal{W}_{m,m-1}^n$ where $i = 1, \dots, n-1$ and $\lambda = 1, \dots, m-1$ and $\kappa_{m,m-1}^n \geq (n-1)(m-1)$. The minimal embedding codimension is given by the dimension of the first non-zero subspace of the multi-flag $\mathcal{E}_{m-1}^{n-1} |_{m,m-1}^n$ because, as previously, the submatrix is triangular by different sized blocks and the terms above the block-diagonal are zero.

4. CONSERVATION LAWS FOR COVARIANT DIVERGENCE FREE ENERGY-MOMENTUM TENSORS

We present here an application for our main result to covariant divergence free energy-momentum tensors. Let us consider a bivector $T \in \Gamma(TM \otimes TM)$ which is expressed in a chart by $T = T^{ij} \partial_i \otimes \partial_j$, where $(\partial_1, \dots, \partial_n)$ is the dual basis of an orthonormal moving coframe (η^1, \dots, η^n) . The volume form is denoted by $\eta^I = \eta^1 \wedge \cdots \wedge \eta^n$. Using the interior product, we can associate to any bivector T , an TM -valued m -differential form τ defined as follows:

$$\begin{aligned}
 \Gamma(TM \otimes TM) &\longrightarrow \Gamma(TM \otimes \wedge^{(m-1)} T^* \mathcal{M}). \\
 T = T^{ij} \partial_i \otimes \partial_j &\longmapsto \tau = \partial_i \otimes \tau^i
 \end{aligned}$$

where $\tau^i = T^{ij} (\partial_j \lrcorner \eta^I)$

The tangent space TM is endowed with the Levi-Civita connection ∇ . Let us compute the covariant derivative of τ .

$$(4.48) \quad d_{\nabla} \tau = \partial_i \otimes (d\tau^i + \eta_j^i \wedge \tau^j)$$

On one hand, using the first Cartan equation⁵ that expresses the vanishing of the torsion of the Levi-Civita connection and the expression of the Christoffel symbols⁶ in term of the connection 1-form, we obtain

$$(4.49) \quad d\tau^i = d(T^{ij} (\partial_j \lrcorner \eta^I)) = d(T^{ij}) \wedge (\partial_j \lrcorner \eta^I) + T^{ij} d(\partial_j \lrcorner \eta^I) = (\partial_j (T^{ij}) + T^{ij} \Gamma_{kj}^k) \eta^I$$

and

$$(4.50) \quad \eta_j^i \wedge \tau^j = \eta_j^i \wedge T_{jk} (\partial_k \lrcorner \eta^I) = (T^{jk} \Gamma_{kj}^i) \eta^I$$

⁵The first Cartan structure equation is: $d\eta^i + \eta_j^i \wedge \eta^j = 0$ for all i

⁶The Christoffel symbols are the functions Γ defined by: $\eta_j^i = \Gamma_{kj}^i \eta^k$.

consequently

$$(4.51) \quad d_{\nabla}\tau = \partial_i \otimes \left[\left(\partial_j(T^{ij}) + T^{ij}\Gamma_{kj}^k + T^{jk}\Gamma_{kj}^i \right) \eta^I \right]$$

On the other hand, a straightforward computation of the divergence of the bivector leads us to

$$(4.52) \quad \nabla_j T^{ij} = \partial_j(T^{ij}) + T^{ij}\Gamma_{kj}^k + T^{jk}\Gamma_{kj}^i$$

We conclude then that

$$(4.53) \quad d_{\nabla}\tau = 0 \Leftrightarrow \nabla_j T^{ij} = 0$$

Hence, for an m -dimensional Riemannian manifold \mathcal{M} , the main result of this article assures the existence of an isometric embedding $\Psi : T\mathcal{M} \longrightarrow \mathcal{M} \times \mathbb{R}^{m+(m-1)^2}$ such that $d(\Psi(\tau)) = 0$ is a conservation law for a covariant divergence free energy-momentum tensor. For instance, if $\dim\mathcal{M} = 4$, then $\Psi(\tau)$ is a closed differential 3-form on \mathcal{M} with values in \mathbb{R}^{13} .

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