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TWO-ORBIT MANIFOLDS WITH BOUNDARY

BENOÎT KLOECKNER

1. INTRODUCTION

In this article we are interested in the classification of some Lie group actions on differentiable manifolds. In full generality, classifying all actions of Lie groups is of course an unachievable task. Let us consider the case when there are very few orbits. The case of transitive actions is easily dealt with: the manifold is then a homogeneous space, and it is sufficient to give the stabilizer of a point to completely describe the action.

As soon as there are two orbits, the question does not remain so simple. Not only does one need a stabilizer for each orbit, but several non differentially conjugate actions can have the same couple of stabilizers.

For example, in [7] we described the case of the “differentiable compactifications” of the real hyperbolic spaces \mathbb{RH}^n . They are defined as the compactifications of \mathbb{RH}^n into a manifold with boundary so that the isometric action of $\mathrm{SO}_0(1, n)$ extends differentiably. Since \mathbb{RH}^n is an isotropic Riemannian manifold, such an action has two orbits: one is the interior \mathbb{RH}^n , the other is its geodesic boundary. All differentiable compactifications of \mathbb{RH}^n are topologically conjugate (in particular, all have the same couple of stabilizers), but there exist infinitely many differentiably nonconjugate actions. This statement remains true for real-analytic actions.

One of our goals is to generalize this result to the other nonpositively curved symmetric spaces of rank one: \mathbb{CH}^m , \mathbb{HH}^m and \mathbb{OH}^2 . The case of higher rank symmetric spaces has been considered in [6], where we proved that their geodesic compactification carries no differentiable structure that is invariant under the action of the isometry group.

The tools we use enable us to consider more generally actions on manifold with boundary with as much transitivity than possible, what we call “two-orbit manifolds with boundary”. Let us introduce some notations we shall use in the sequel.

1.1. Notations. Let ρ_0 be an action of a connected Lie group G on a manifold with boundary M of dimension n . Denote by $\mathrm{Int}(M)$ the interior of M and by ∂M its boundary. When ρ_0 is transitive on both $\mathrm{Int}(M)$ and ∂M , we say that M (together with the action) is a *two-orbit manifold with boundary*. This property is assumed to hold from now on.

Fix a regularity index r that is either ∞ or ω (where \mathcal{C}^ω means “real-analytic”) and assume that ρ_0 is \mathcal{C}^r .

We define the following usual notion of conjugacy between \mathcal{C}^r actions on M .

DEFINITION 1 — *Two \mathcal{C}^r actions ρ_1, ρ_2 of the group G on the manifold M are said to be \mathcal{C}^r conjugate if there is a \mathcal{C}^r diffeomorphism $\Phi : M \rightarrow M$ that is (ρ_1, ρ_2) equivariant, that is:*

$$\Phi(\rho_1(g)x) = \rho_2(g)\Phi(x) \quad \forall x \in M \forall g \in G.$$

These two actions are said to be topologically conjugate if Φ is only a homeomorphism.

When Φ is both a \mathcal{C}^r map and a homeomorphism (but not necessarily a diffeomorphism), we say that ρ_1 is \mathcal{C}^r semi-conjugate to ρ_2 .

Denote by $\mathcal{C}^r(G; M)$ the set of all \mathcal{C}^r actions of G on M , and by \sim the relation of \mathcal{C}^r conjugacy. The conjugacy class of an action ρ is denoted under brackets: $[\rho]$.

Denote by $\mathcal{C}^r(\rho_0)$ the subset of $\mathcal{C}^r(G; M)$ consisting of all actions that are topologically conjugate to ρ_0 .

We are interested in the quotient set $\mathcal{M}^r(\rho_0) = \mathcal{C}^r(\rho_0) / \sim$. More precisely, we aim to answer the following questions.

- (1) What is the size of $\mathcal{M}^r(\rho_0)$? Is it reduced to $[\rho_0]$ and if not, is it finite ?
- (2) What kind of structure can we endow $\mathcal{M}^r(\rho_0)$ with, in a natural fashion ?
- (3) Can we entirely describe $\mathcal{M}^r(\rho_0)$?

1.2. Summary of the results. The answer to the previous questions are of course expected to depend heavily on the action ρ_0 . A positive answer to the first question would be a rigidity result, and we already noticed that some actions, like the usual action of $\mathrm{SO}_0(1, n)$ on the closed ball $\overline{\mathbb{R}\mathbb{H}^n}$ of $\mathbb{R}\mathbb{P}^n$, are flexible. One could expect other actions, like for example the action of $\mathrm{PU}(1, m)$ on the closed ball $\overline{\mathbb{C}\mathbb{H}^m}$ of $\mathbb{C}\mathbb{P}^m$, to be more rigid. This guess, motivated by the presence of totally real and complex subspaces, is however wrong.

In this paper we define a “stretching” operation: for some \mathcal{C}^r functions $f : [0, 1] \rightarrow [0, 1]$ (the typical case being $f(y) = y^p$ with $p \in \mathbb{N}$) we define the stretch of ρ_0 by f , a new action of G on M (proposition 3 and definition 6).

We then obtain the following results.

- The stretches of ρ_0 are \mathcal{C}^r actions that are topologically conjugate to ρ_0 , and thus give elements of $\mathcal{M}^r(\rho_0)$ (Theorem 1).
- At least under an algebraic condition, satisfied in the case of symmetric spaces, the element of $\mathcal{M}^r(\rho_0)$ obtained by stretching ρ_0 by a function f is trivial only if f is a \mathcal{C}^r diffeomorphism (Theorem 4). In particular $\mathcal{M}^r(\rho_0)$ is infinite (Corollary 5).
- Under the same algebraic condition, the relation of semi-conjugacy induces an ordering of $\mathcal{M}^r(\rho_0)$ (Proposition 10).
- Differentiable compactifications of $\mathbb{C}\mathbb{H}^m$, $\mathbb{H}\mathbb{H}^m$ and $\mathbb{O}\mathbb{H}^2$ are stretches of the projective compactification to be defined below, thus we get a complete classification (Theorem 6).

In each of the next three sections we state more precisely and prove these results. The last section gives some possible extensions of this work.

2. STRETCHING AN ACTION

A stretching of ρ_0 consists in gluing together its two orbits in a new way. To achieve that, we need a function that indicates “at what speed” we glue them.

DEFINITION 2 — *By a stretching function we mean a function $f : [0, 1] \rightarrow [0, 1]$ such that*

- (1) f is an orientation-preserving homeomorphism,
- (2) f is a \mathcal{C}^r function and the restriction of f to $]0, 1[$ is a \mathcal{C}^r diffeomorphism

- (3) the function f/f' , which is well defined on $]0, 1]$, can be extended to a \mathcal{C}^r function at 0.

We say that a stretching map f is trivial when it is a diffeomorphism (that is, when $f'(0) \neq 0$), and that two stretching maps f, g are equivalent if $g^{-1}f$ is a diffeomorphism.

The reason why we need the last condition will become clear in the proof of theorem 1. This condition is satisfied by any non-flat (that is: having non-trivial Taylor series) function. In the real-analytic case, any stretching function is equivalent to $y \mapsto y^p$ for some $p \in \mathbb{N}$.

Given a stretching function, we define a map $M \rightarrow M$ that will relate ρ_0 and its stretching. Note that to simplify the construction of manifolds, we consider charts with value not only in \mathbb{R}^n , but in any manifold.

PROPOSITION 3 — *Let f be a stretching function. There is a homeomorphism $\Phi_f : M \rightarrow M$ that is a \mathcal{C}^r map, whose restriction $\text{Int}(M) \rightarrow \text{Int}(M)$ is a \mathcal{C}^r diffeomorphism, and such that for every point $p \in \partial M$ there exist two systems of coordinates (x_1, \dots, x_{n-1}, y) around p and $\Phi_f(p)$, where y is a locally defining function for ∂M and such that in those coordinates*

$$\Phi_f(x_1, \dots, x_{n-1}, y) = (x_1, \dots, x_{n-1}, f(y)).$$

Since the latter formula is valid in two different systems of coordinates on the domain and the range of Φ_f , this map should be considered as defined up to composition with diffeomorphisms on the left and right, and not up to conjugacy. This is natural since we shall use it to pull back ρ_0 , so the domain and range should be considered as two different copies of M .

Proof. Let f be a stretching map. Let C be collar neighborhood of ∂M in M , parametrized by coordinates $(x, y) \in \partial M \times]0, 1[$. Note that we ask that a collar neighborhood has its complement diffeomorphic to M .

Let $c : \partial M \times]0, 1[\rightarrow \text{Int}(M)$ be the inclusion map (where $\partial M \times]0, 1[$ is identified with $C \cap \text{Int}(M)$ via the coordinates). Define a \mathcal{C}^r manifold with boundary M' by two charts that are copies of $\text{Int}(M)$ and C , with change of coordinates given by

$$\begin{aligned} \partial M \times]0, 1[&\rightarrow \text{Int}(M) \\ (x, y) &\mapsto c(x, f(y)) \end{aligned}$$

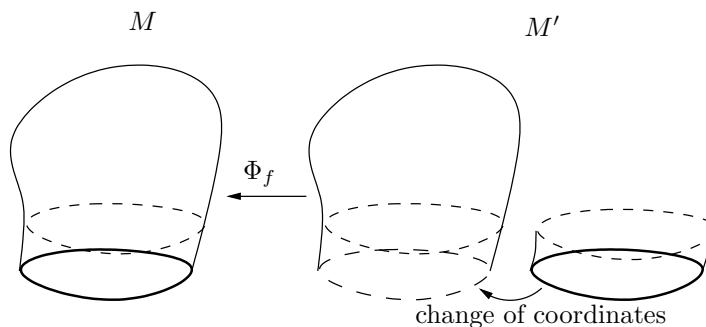


FIGURE 1. Construction of a stretching map

LEMMA 4 — *If $r = \infty$, the resulting manifold M' only depends (up to \mathcal{C}^∞ diffeomorphism) upon the equivalence class of the stretching map.*

Proof. First, the choice of the neighborhood does not matter.

Now let g be a stretching map equivalent to f and denote by M'' the manifold obtained from the two charts $\text{Int}(M)$ and C by the change of coordinates

$$\begin{aligned} \partial M \times]0, 1[&\rightarrow \text{Int}(M) \\ (x, y) &\mapsto c(x, g(y)) \end{aligned}$$

Then the map $(x, y) \mapsto (x, g^{-1}f(y))$ is a diffeomorphism from a collar neighborhood of $\partial M'$ to a collar neighborhood of $\partial M''$. Moreover, this map extends to the image of the extensions $\tilde{c}(\cdot, 1)$ of c that maps $\partial M \times 1$ into the interiors $\text{Int}(M')$, $\text{Int}(M'') \simeq \text{Int}(M)$. Using partition of unity, one can extend further this map to a \mathcal{C}^∞ diffeomorphism between M' and M'' . \square

LEMMA 5 — *In both cases $r = \infty, \omega$, the previous construction leads to a manifold with boundary M' that is \mathcal{C}^r diffeomorphic to M , regardless the choice of the stretching function f .*

Proof. First note that the result is obvious for the manifold with boundary $U \times]0, +\infty[$, where U is any open set of \mathbb{R}^{n-1} .

If $r = \infty$, the previous lemma enables one to assume that f is the identity on a neighborhood $]1 - 2\varepsilon, 1]$ of 1. Take then a covering of ∂M by charts (U_i) : the manifolds M and M' are covered by the same charts $(U_i \times]0, 1 - \varepsilon])$ and $\text{Int}(M) \setminus \partial M \times]0, 1 - 2\varepsilon[$, with the same changes of coordinates. They are therefore \mathcal{C}^∞ diffeomorphic.

If $r = \omega$, the previous discussion shows that M and M' are \mathcal{C}^∞ diffeomorphic. But a given differentiable manifold admits only one real-analytic structure (this is discussed in more details at the end of the present Section, see 2.1), thus M and M' are \mathcal{C}^ω diffeomorphic. \square

We have now two presentations of M by the same pair of charts $C, \text{Int}(M)$ but with different changes of coordinates. The identity on $\text{Int}(M)$ extends continuously into a \mathcal{C}^r map $\Phi_f : M' \rightarrow M$. Up to composition by a diffeomorphism $M \rightarrow M'$, Φ_f has the desired properties. \square

DEFINITION 6 — *A map Φ_f satisfying the conclusion of proposition 3 is said to be a stretching map associated to the stretching function f . A stretch of ρ_0 is an action $\rho_f = \Phi_f^*(\rho_0)$ where Φ_f is a stretching map.*

Let us prove that the conditions we put on f ensure that a stretch ρ_f is as regular as ρ_0 ; recall that ρ_0 is an action of a *connected* Lie group G .

THEOREM 1 — *Any stretch of ρ_0 is a \mathcal{C}^r action on M .*

Proof. We first go to the Lie algebra level.

Let \mathfrak{g} be the Lie algebra of G . For any $X \in \mathfrak{g}$, we denote by $\rho_0(X)$ the corresponding vector field on M . Then $\phi_f^*(\rho_0(X))$ is well defined and \mathcal{C}^r on $\text{Int}(M)$ since by construction Φ_f is a diffeomorphism when restricted to $\text{Int}(M)$. Let us prove that it extends to a \mathcal{C}^r vector field on M . This will be done by considering local charts near the boundary.

We start with the analytic case. Since any analytic stretching map is equivalent to the lower order term of its Taylor series, we can assume that f writes $y \mapsto y^p$ for some integer p . In suitable charts (x_i, y) near the boundary (locally defined by $y = 0$), we can write Φ_f under the form $(x_i, y) \mapsto (x_i, y^p)$ and $\rho_0(X)$ under the form

$$\rho_0(X) = \sum_{i=1}^{n-1} \sum_{a,b} \alpha_{a,b}^i x^a y^b \frac{\partial}{\partial x_i} + \sum_{a,b} \beta_{a,b} x^a y^b \frac{\partial}{\partial y}$$

where the sums are taken over all non-negative integers b and all $(n-1)$ -tuples of non-negative integers a ; x^a means $x_1^{a_1} x_2^{a_2} \dots x_{n-1}^{a_{n-1}}$. By a direct computation, we see that

$$\Phi_f^*(\rho_0(X)) = \sum_{i=1}^{n-1} \sum_{a,b} \alpha_{a,b}^i x^a y^{pb} \frac{\partial}{\partial x_i} + \sum_{a,b} \beta_{a,b} x^a y^{p(b+1)-p} \frac{\partial}{\partial y}$$

hence its analyticity since, $\rho_0(X)$ being tangent to the boundary, $\beta_{a,0} = 0$ for all a .

In the smooth case, we write

$$\rho_0(X) = \sum_{i=1}^{n-1} \sum_{a,b} \alpha^i(x, y) \frac{\partial}{\partial x_i} + \beta(x, y) \frac{\partial}{\partial y}$$

where α_i and β are \mathcal{C}^∞ functions, and $\beta(x, 0) = 0$ for all x . According to the Hadamard lemma (see for example [2]), there is a \mathcal{C}^∞ function β_1 such that $\beta(x, y) = \beta_1(x, y)y$. Then we get

$$\Phi_f^*(\rho_0(X)) = \sum_{i=1}^{n-1} \alpha^i(x, f(y)) \frac{\partial}{\partial x_i} + \beta_1(x, f(y)) \frac{f(y)}{f'(y)} \frac{\partial}{\partial y}$$

and, since f/f' extends differentiably, so does $\Phi_f^*(\rho_0(X))$.

We get that $\Phi_f^*(\rho_0)$ defines a \mathcal{C}^r action of \mathfrak{g} on M . Thus, it defines a \mathcal{C}^r action of the universal covering \tilde{G} of G . But an element that projects on $1 \in G$ must act trivially on $\text{Int}(M)$, and by continuity it acts trivially on M . We thus get a \mathcal{C}^r action of G on M that coincides with $\Phi_f^*(\rho_0)$ on $\text{Int}(M)$. This last action is therefore \mathcal{C}^r . \square

The main reason to restrict ourselves to smooth and analytic actions is the loss of regularity in the Hadamard lemma for \mathcal{C}^r functions, $r < \infty$.

2.1. Uniqueness of the analytic structure of a smooth manifold with boundary.

We used above the following result.

THEOREM 2 — *Let M' and M be two compact real-analytic manifolds with (analytic) boundary. If there is a smooth diffeomorphism $F : M' \rightarrow M$, then there exists also an analytic one.*

The without boundary version of this theorem is well known: Grauert proved in [5] that the set of analytic diffeomorphisms between two analytic manifold without boundary is dense into the set of smooth ones.

However, even if the “with boundary” version is unsurprising and cannot be expected to be new, it is very difficult to find in the litterature. Luckily, it can be deduced from the following result and the Morrey-Grauert theorem that states that any analytic manifold can be analytically embedded in \mathbb{R}^N for some N .

THEOREM 3 (Tognoli [12]) — *Let U be a relatively compact open set in \mathbb{R}^N , V a coherent analytic subset of U and ℓ a smooth function on U whose restriction to V is analytic. Then for all k and all $\varepsilon > 0$ there is an analytic function h defined on U such that*

- (1) $\|\ell - h\|_k < \varepsilon$ where $\|\cdot\|_k$ is the \mathcal{C}^k norm,
- (2) h and ℓ coincide on V

Note that an analytic submanifold is a special case of a coherent analytic subset.

Proof of Theorem 2. Up to doubling, one has to prove that if M and M' are compact analytic manifolds without boundary, N and N' are analytic submanifolds of M and M' , and $F : M \rightarrow M'$ is a smooth diffeomorphism that restricts to a smooth diffeomorphism $N \rightarrow N'$, then there is an analytic diffeomorphism $M \rightarrow M'$ that maps N onto N' .

Thanks to the Morrey-Grauert Theorem we can embed M and M' into R^N , and using the Grauert Theorem we construct an analytic diffeomorphism $L : N \rightarrow N'$. It extends into a smooth diffeomorphism $L : M \rightarrow M'$. Moreover, we can smoothly extend L on a relatively compact open set $U \subset R^N$ containing M . Since M' is analytic, it admits a neighborhood U' that retracts analytically on M' (see [8] Theorem 2.7.10). Using the approximation theorem of Tognoli on the coordinates of L , with $k = 1$, we construct a map $H : M \rightarrow U'$ that is an analytic diffeomorphism onto its image, coincide with L on N , and is \mathcal{C}^1 close to L . Composing H with the retraction $U' \rightarrow M'$, we get the desired analytic diffeomorphism $(M, N) \rightarrow (M', N')$ (the \mathcal{C}^1 closeness to L ensures that this actually is a diffeomorphism). \square

3. NON-CONJUGACY OF STRETCHES

Now, we would like to distinguish between the various stretches of ρ_0 , in order to ensure that $\mathcal{M}^r(\rho_0)$ is large. Unfortunately, we are able to do so only under an algebraic assumption.

Note that for many actions, explicit computations like in the proof or theorem 1 will be sufficient to prove that $\Phi_f^*(\rho_0)$ is not conjugated to ρ_0 . For example, the valuation of the Taylor expansion of $\rho_0(X)$, where X is any given element of \mathfrak{g} , is a conjugacy invariant. Problems are however expected when all elements of \mathfrak{g} act very flatly near the boundary.

3.1. The algebraic condition. Let x be a point of the interior of M and $H \subset G$ be its stabilizer for the action ρ_0 . Denote by $\text{Stab}(H)$ the subgroup of elements $g \in G$ such that $gHg^{-1} = H$, and by $Z(G)$ the center of G . The inclusion $\text{Stab}(H) \supset H \cdot Z(G)$ always holds; we consider the converse inclusion.

$$(A) \quad \text{Stab}(H) = H \cdot Z(G).$$

Note that (A) does not depend upon the choice of x .

Let us stress two particular cases. First, when no two points of $\text{Int}(M)$ have the same stabilizers, $\text{Stab}(H) = H$ and (A) holds. For example, it is the case of symmetric spaces. Second, when G is abelian $\text{Stab}(H) = G = Z(G)$.

Moreover (A) is stable by direct product in the following sense.

PROPOSITION 7 — *Consider two actions ρ_i of groups G_i on manifolds M_i ($i = 1, 2$), one being a homogenous manifold and the second a two-orbit manifold with*

boundary. If (A) holds for both actions, then (A) holds for the action of $G_1 \times G_2$ on $M_1 \times M_2$ defined by

$$(g_1, g_2) \cdot (x_1, x_2) = (\rho_1(g_1)x_1, \rho_2(g_2)x_2).$$

Proof. With obvious notations we have $H = H_1 \times H_2$, $Z(G) = Z(G_1) \times Z(G_2)$ and $\text{Stab}(H) = \text{Stab}(H_1) \times \text{Stab}(H_2)$ and the result follows. \square

3.2. Common regularity of semi-conjugacies. The core result of this section is the following simple lemma.

LEMMA 8 (common regularity) — *Assume that (A) holds, and let ρ_1 and ρ_2 be two \mathcal{C}^r actions of G on M , both topologically conjugate to ρ_0 . Let Φ_0 and Φ be two homeomorphisms of M that are (ρ_1, ρ_2) equivariant. Then Φ is \mathcal{C}^r if and only if Φ_0 is.*

Proof. Let x be any point of $\text{Int}(M)$ and H its stabilizer for ρ_1 .

Since ρ_0 is transitive on $\text{Int}(M)$, so is ρ_2 and there is $g_0 \in G$ such that $\Phi(x) = \rho_2(g_0)\Phi_0(x)$. But by equivariance, the stabilizer for ρ_2 of both $\Phi(x)$ and $\Phi_0(x)$ must be H . Therefore, $g_0 \in \text{Stab}(H)$ and from (A) follows $g_0 = h_0 z_0$ where $h_0 \in H$ and $z_0 \in Z(G)$.

We get for all $g \in G$:

$$\begin{aligned} \Phi(\rho_1(g)x) &= \rho_2(g)\Phi(x) \\ &= \rho_2(g)\rho_2(z_0)\Phi_0(\rho_1(h_0)x) \\ &= \rho_2(z_0)\rho_2(g)\Phi_0(x) \\ &= \rho_2(z_0)\Phi_0(\rho_1(g)x). \end{aligned}$$

that is, $\Phi = \rho_2(z_0)\Phi_0$ on $\text{Int}(M)$. By continuity, this equality holds on the whole of M and since $\rho_2(z_0)$ is a \mathcal{C}^r diffeomorphism, Φ is \mathcal{C}^r if and only if Φ_0 is. \square

The first consequence of the common regularity Lemma is that two stretching functions that are not equivalent lead to non-conjugate stretchings.

THEOREM 4 — *If condition (A) holds, then the two stretches of ρ_0 associated to functions f_1 and f_2 are \mathcal{C}^r conjugate only if f_1 and f_2 are equivalent as stretching functions.*

Proof. The map $\Phi_0 = \Phi_{f_2^{-1}f_1}$ topologically conjugates the two stretches, denoted by ρ_1 and ρ_2 . Moreover, it is not a conjugacy unless f_1 and f_2 are equivalent (read in charts near the boundary).

Let $\Phi : M \rightarrow M$ be any (ρ_1, ρ_2) equivariant homeomorphism. The common regularity Lemma, applied to Φ and Φ_0 or to Φ^{-1} and Φ_0^{-1} , implies that Φ is not a \mathcal{C}^r diffeomorphism unless Φ_0 is one. Therefore, either f_1 and f_2 are equivalent or there exists no \mathcal{C}^r conjugacy between ρ_1 and ρ_2 . \square

As a striking consequence, we get the following.

COROLLARY 5 — *If condition (A) holds, then $\mathcal{M}^\omega(\rho_0)$ is at least countably infinite.*

We will see in the next section some examples where $\mathcal{M}^\omega(\rho_0)$ is countable. It would be interesting to determine if there exist actions ρ_0 for which it is uncountable.

3.3. Semi-conjugacy as an ordering. The common regularity Lemma can be used to define a naturel partial order on $\mathcal{M}^r(\rho_0)$.

DEFINITION 9 — *Let $[\rho_1], [\rho_2]$ be two elements of $\mathcal{M}^r(\rho_0)$. We say that $[\rho_2]$ is tighter than $[\rho_1]$ and we write $[\rho_1] \succ^r [\rho_2]$ (sometimes forgetting the brackets or the r) if ρ_1 is \mathcal{C}^r semi-conjugate to ρ_2 .*

Of course, this definition is consistent: the relation holds or not regardless the choice of a representative in each conjugacy class.

PROPOSITION 10 — *If (A) holds, the relation \succ^r defines a partial order on $\mathcal{M}^r(\rho_0)$.*

Proof. The reflexivity and transitivity of \succ are obvious. Let us show that \succ is antisymmetric.

Let $[\rho_1]$ and $[\rho_2]$ be elements of $\mathcal{M}^r(\rho_0)$ that is, ρ_i are \mathcal{C}^r actions of G on M that are topologically conjugate to ρ_0 . Assume that $[\rho_1] \succ [\rho_2]$ and $[\rho_2] \succ [\rho_1]$. Then there are two homeomorphisms Φ_a and Φ_b of M that are respectively (ρ_1, ρ_2) and (ρ_2, ρ_1) equivariant and are both \mathcal{C}^r maps. But Φ_b^{-1} is a (ρ_1, ρ_2) equivariant homeomorphism of M , and the common regularity lemma implies that it is \mathcal{C}^r . It therefore defines a \mathcal{C}^r conjugacy between ρ_1 and ρ_2 . \square

Given the action ρ_0 , an interesting question is wether there is a tightest element in $\mathcal{M}^r(\rho_0)$. It would be the “fundamental action” to which everyone is semi-conjugate. In the case of the differentiable compactifications of the hyperbolic spaces, we proved in [7] the existence of such a tightest compactification. Moreover, every other compactification is not only semiconjugate to it, but is a stretch of it. In the next section, we generalize this result to the other non-positively curved symmetric spaces of rank one.

4. NON-POSITIVELY CURVED SYMMETRIC SPACES OF RANK ONE

4.1. Differentiable compactifications. In this subsection we consider a homogeneous \mathcal{C}^r Riemannian manifold X . We consider the neutral component G of its isometry group and denote by ρ the corresponding transitive action.

DEFINITION 11 — *A \mathcal{C}^r differentiable compactification of X is the data of a manifold with boundary M and a \mathcal{C}^r embedding $\psi : X \rightarrow M$ such that:*

- (1) $\psi(X) = \text{Int } M$,
- (2) *the action $\psi_*\rho$ of G on $\text{Int } M$ extends to a \mathcal{C}^r action on M , which is also denoted by $\psi_*\rho$ when no confusion is possible.*

DEFINITION 12 — *Let (ψ_1, M_1) and (ψ_2, M_2) be two differentiable compactifications of X and denote ρ_i ($i = 1, 2$) the extension of $\psi_{i*}\rho$ to M_i . Then (ψ_1, M_1) and (ψ_2, M_2) are said to be equivalent if there are \mathcal{C}^r diffeomorphisms $\alpha : X \rightarrow X$ and $\beta : M_1 \rightarrow M_2$ that are respectively ρ and (ρ_1, ρ_2) -equivariant and so that $\psi_2 \alpha = \beta \psi_1$.*

The introduction of α is natural: a mere change of coordinates on X must not change the equivalence class of a differentiable compactification. But the condition $\psi_2 \alpha = \beta \psi_1$ entirely defines α and the equivariance of β implies that of α . As a consequence, two differentiable compactifications of X are equivalent if and only if the actions ρ_1 and ρ_2 are equivalent in the sense of definition 1.

4.2. Recalls and notations. The classical reference on negatively curved symmetric spaces is [9]. See also [3, 4] for the complex case, and [1, 11] for the octonionic case.

Let \mathbb{K} be either \mathbb{C} , \mathbb{H} or the nonassociative field \mathbb{O} and $k = 2, 4$ or 8 the dimension of \mathbb{K} as a real algebra. Denote by \mathbb{KH}^m (where $m = 2$ if $\mathbb{K} = \mathbb{O}$) the \mathbb{K} hyperbolic space, G the neutral component of its isometry group and ρ the isometric action of G on \mathbb{KH}^m . Denote by K a maximal compact subgroup of G ; it is the stabilizer of some point x_0 of \mathbb{KH}^m . Denote by $n = km$ the real dimension of \mathbb{KH}^m .

The space \mathbb{KH}^m is symmetric in the sense that the geodesic symmetry around any point x is a global isometry. The action ρ is transitive and, \mathbb{KH}^n being of rank one it is isotropic (that is: K acts transitively on the unit tangent sphere at x_0). No two points of \mathbb{KH}^m have the same stabilizer: an isometry that fixes two points must fix the geodesic joining them, and the isotropy implies that it cannot be the case for all elements of K . Thus ρ satisfies condition (A).

As a Riemannian space, \mathbb{KH}^m is negatively curved. As a homogeneous space, it can be identified with an open ball in the projective space \mathbb{KP}^m in such a way that ρ extends to a real-analytic projective action. The boundary of \mathbb{KH}^m in \mathbb{KP}^m can be canonically identified with its geodesic boundary $\partial\mathbb{KH}^m$, defined as the space of asymptote classes of geodesic rays. The group K , and thus G as well, act transitively on this boundary since it can also be identified with the unit tangent sphere at any point.

As a consequence, this embedding into \mathbb{KP}^m gives a \mathcal{C}^ω differentiable compactification $\mathbb{KH}^m \rightarrow \overline{\mathbb{KH}^m}$; the corresponding action of G will be denoted by $\bar{\rho}$ and is called the projective compactification.

The intersection of any \mathbb{K} -projective lines with \mathbb{KH}^m is a totally geodesic submanifold of \mathbb{KH}^n isometric to \mathbb{RH}^k (up to a constant that makes it of curvature -4). These are called the \mathbb{K} -lines of \mathbb{KH}^m . Two \mathbb{K} -lines intersect in at most one point, and any vector of the tangent space at a point x is tangent to exactly one \mathbb{K} -line. Therefore, G acts transitively on \mathbb{K} -lines. The geodesic boundary of a \mathbb{K} -line is a properly embedded $(k-1)$ -dimensional sphere in $\partial\mathbb{KH}^m$.

Let ℓ be a \mathbb{K} -line. Then the geodesic symmetry around ℓ is a direct isometry: the subgroup of G that fixes every point of ℓ contains an element whose differential has eigenvalues 1 with multiplicity k and -1 with multiplicity $km - k$. Moreover, any direct isometry of ℓ extends to a direct isometry of the whole of \mathbb{KH}^m , thus the group of direct isometry of ℓ can be considered a subgroup G_ℓ of G . The restriction of $\bar{\rho}$ to $\bar{\ell}$ and G_ℓ is analytically conjugate to the conformal compactification (see [7] and below) of \mathbb{RH}^k .

Each geodesic is also contained in a totally real \mathbb{RH}^m , which is a totally geodesic m -dimensional submanifold isometric to \mathbb{RH}^m (with curvature -1), whose tangent space at each point is totally real. A totally real \mathbb{RH}^m is obtained by moving the canonical embedding $\mathbb{RP}^m \rightarrow \mathbb{KP}^m$ by an element of G . Let λ be a totally real \mathbb{RH}^n . Writing isometries by means of matrices, any direct isometry of λ is given by a real matrix that acts on \mathbb{KP}^m as well. It follows that any direct isometry of λ extends to an element of G . The group of direct isometries of λ can be considered as a subgroup G_λ of G . The restriction of $\bar{\rho}$ to $\bar{\lambda}$ and G_λ is analytically conjugate to the projective compactification (see [7] and below) of \mathbb{RH}^k .

4.3. The projective and conformal compactifications of the real hyperbolic space. The projective compactification of \mathbb{RH}^n is defined like that of \mathbb{KH}^n :

it is the restriction to the closure of Klein's ball $\overline{\mathbb{R}\mathbb{H}^n}$ of the projective action of $\mathrm{SO}_0(1, n)$ on $\mathbb{R}\mathbb{P}^n$. In this model geodesics are affine lines, and it plays a central rôle in the proof that every \mathcal{C}^r differentiable compactification of $\mathbb{R}\mathbb{H}^n$ is a stretch of the projective one, provided $n > 2$ or $r = \omega$.

The conformal compactification is the continuous extension to the closed ball of the action of $\mathrm{SO}_0(1, n)$ on Poincaré's ball. In this model, geodesics are circle arcs orthogonal to the boundary, this orthogonality making sense since the Euclidean conformal structure on the closed ball is preserved by the action. As any other one, the conformal compactification is a stretch of the projective one. The stretching function can be chosen to be $y \mapsto y^2$, and this can be seen by constructing Poincaré's ball from Klein's one: one projects the latter vertically on a hemisphere (here lies the order 2 term), which is in turn stereographically projected on the former.

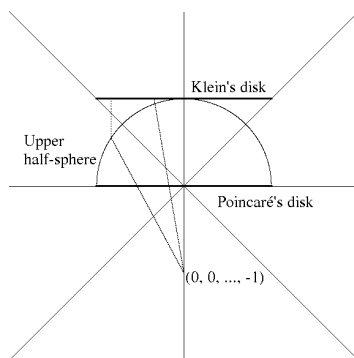


FIGURE 2. Construction of Poincaré's ball from Klein's one.

4.4. Classification of differentiable compactifications. First note that the following proposition can be proved in the same way than the real case (see [7]).

PROPOSITION 13 — *Any differentiable compactification of $\mathbb{K}\mathbb{H}^m$ is topologically conjugate to $\overline{\mathbb{K}\mathbb{H}^m}$.*

As a consequence, we can identify a differentiable compactification and the homeomorphism that conjugates it to $\overline{\mathcal{p}}$. The following definition is thus equivalent to definition 11 (\overline{B}^n denotes the closed ball of dimension n).

DEFINITION 14 — *By a \mathcal{C}^r differentiable compactification of $\mathbb{K}\mathbb{H}^m$ we now mean a homeomorphism $\Phi : \overline{B}^n \rightarrow \overline{\mathbb{K}\mathbb{H}^m}$ whose restriction to the interiors is a \mathcal{C}^r diffeomorphism and such that the action $\Phi^*(\overline{\mathcal{p}})$ is \mathcal{C}^r up to the boundary.*

THEOREM 6 — *If $r = \omega$ or $\mathbb{K} = \mathbb{H}$ or \mathbb{O} , then any \mathcal{C}^r compactification of $\mathbb{K}\mathbb{H}^m$ is a stretch of $\overline{\mathcal{p}}$. In particular, $(\mathcal{M}^\omega(\overline{\mathcal{p}}), \prec)$ is isomorphic to \mathbb{N} endowed with the divisibility ordering.*

We thus get the very same result than in the real case: the projective compactification is in each case the tightest differentiable compactification. The same result may hold in lower regularity, and could be checked by a computation of the action

of G in Klein coordinates. Concerning the $\mathbb{K} = \mathbb{C}$, $r = \infty$ case, the result holds if we ask that the whole group of isometries (and not only its neutral component) acts smoothly. Otherwise it is open: the following proof relies heavily on the classification of differentiable compactification of $\mathbb{R}\mathbb{H}^k$, and the $k = 2$, $r = \infty$ case is open.

Proof. Let Φ be a \mathcal{C}^r compactification of $\mathbb{K}\mathbb{H}^m$ and $\rho_1 = \Phi^*(\bar{\rho})$.

We fix a point $p \in \mathbb{K}\mathbb{H}^m$ and put coordinates $(x, y) \in S_p\mathbb{K}\mathbb{H}^m \times [0, \infty)$ on $\overline{\mathbb{K}\mathbb{H}^m} \setminus \{p\}$.

Let ℓ be any \mathbb{K} -line of $\mathbb{K}\mathbb{H}^m$ through p . Let g be the geodesic symmetry around ℓ . Then $\rho_1(g)$ has eigenvalues 1 and -1 with multiplicities k and $mk - k$ at any point of $\Phi^*\ell$, thus also at points of $\overline{\Phi^*\ell}$. By the implicit function theorem, $\overline{\Phi^*\ell}$ is thus a \mathcal{C}^r submanifold of \overline{B}^n that is transverse to the boundary. Then, the restriction of ρ_1 to $\overline{\Phi^*\ell}$ and G_ℓ is a \mathcal{C}^r compactification of $\mathbb{R}\mathbb{H}^k$. Due to the classification of those and since $\bar{\ell}$ is the conformal compactification of $\mathbb{R}\mathbb{H}^k$, the restriction of Φ to $\overline{\Phi^*\ell}$ is almost a stretch: in coordinates $(x, y) \in S_p\ell \times [0, 1)$ it can be written $(x, y) \mapsto (x, \sqrt{f(y)})$ where f is a stretching map.

But by equivariance Φ can be written under this form on a neighborhood of the whole boundary (just let act the stabilizer K of p). If we look at the restriction of Φ to the closure of a totally real line $\bar{\lambda}$, it writes $(x, y) \mapsto (x, \sqrt{f(y)})$ in coordinates $(x, y) \in S_p\lambda \times [0, 1)$ and defines a \mathcal{C}^r compactification of $\mathbb{R}\mathbb{H}^m$. Due to the classification in the real case, \sqrt{f} itself must be a stretching function, and Φ is a stretching map.

Note that we do not need $m \geq 3$ since $\bar{\lambda}$ is necessarily transverse to the boundary, and so are its geodesics. \square

One could have thought that the presence of different type of embedded real hyperbolic spaces would give more rigidity to the differentiable compactifications of, for example, $\mathbb{C}\mathbb{H}^m$ than those of $\mathbb{R}\mathbb{H}^m$. We saw that it is not the case, but this flexibility is a consequence of theorem 4. Theorem 6 says, unsurprisingly, that the non real cases are as rigid as the real one. We expect the projective compactification to be the only one to preserve a complex structure, since such a condition should introduce a great deal of rigidity.

5. PROSPECTS

Let us stress some limitations of this work that lead to interesting questions.

First, we would like to get rid of condition (A). Without it, can two non-equivalent stretching maps lead to equivalent actions? The existence of a dense open orbit will of course be of primary importance.

Second, the notion of stretch could be used in little more generality when we have a 1-codimensional orbit in the closure of an open orbit, but does not extend as it is to greater codimension. Could one modify it so that the dimension of the closed orbit does not matter?

Third, in most cases we are only able to construct new actions from a given one. Given two subgroups P, K in G , we would like to determine the (possibly empty) set of differentiable action of G that have two orbits, with respective stabilizers P and K . A first step would be to describe by some simple data the way two orbits are glued together, without reference to a given action.

The answers to these question could culminate in a description of differentiable actions of G that have a finite number of orbits in term of a labelled directed graph. The vertices would represent the orbits and be labelled by the (conjugacy class of) their stabilizer. There would be an edge from an orbit A to an orbit B exactly when $B \subset \overline{A}$ and each edge would be labelled with the data describing the way A and B are glued together. To construct this data, we would need some sort of non-proper slice theorem (*cf* [10]).

It might be necessary to use a hypergraph instead of a graph, with an hyperedge containing all orbits contained in \overline{A} .

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