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On Lipschitz compactifications of trees

Benoît Kloeckner

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Abstract

We study the Lipschitz structures on the geodesic compactification of a regular tree, that are preserved by the automorphism group. They are shown to be similar to the compactifications introduced by William Floyd, and a complete description is given.

In [4], we described all possible differentiable structures on the geodesic compactification of the hyperbolic space, for which the action of its isometries is differentiable. We consider here the similar problem for regular trees and obtain a description of "differentiable" compactifications, based on an idea of William Floyd [3]. A tree has a geodesic compactification, but it is obviously not a manifold and we shall in fact replace the differentiability condition by a Lipschitz one.

Note that we only consider regular trees so that we have a large group of automorphisms, hence the greatest possible rigidity in our problem. A close case is that of the universal covering of a finite graph (that is, when the automorphism group is cocompact). Our study does not extend as it is to this case, in particular one can convince oneself by looking at the barycentric division of a regular tree that condition (1) in theorem 2.1 should be modified. However, similar results should hold, up to considering the translates of a fundamental domain instead of the edges at some point.

This note is made of two sections. The first one recalls some facts about regular trees and their automorphisms, Floyd compactifications, and gives the definition of a Lipschitz compactification. The second one contains the result and its proof.

1 Preliminaries

1.1 Regular trees and their automorphisms

We denote by $\mathcal{T}_n$ the regular tree of valency $n \geq 3$ and by T_n is topological realization, obtained by replacing each abstract edge by a segment. All considered metrics on T_n shall be length metrics, since general metrics could have no relation at all with the combinatorial structure of $\mathcal{T}_n$. Up to isometry, two length metrics on T_n that are compatible with the topology differ only by the length of the edges. We shall therefore identify T_n equipped with such a metric and $\mathcal{T}_n$ equipped with a labelling of the edges by positive real numbers (the label corresponding to the length of the edge). When all edges are chosen of length 1, we call the resulting metric space the "standard metric realization" of $\mathcal{T}_n$, denoted by $T_n(1)$. Its metric shall be denoted by d; it coincides on vertices with the usual combinatorial distance.

There is a natural one-to-one correspondence between automorphism of $\mathcal{T}_n$ and isometries of $T_n(1)$. We denote both groups by $\mathrm{Aut}(\mathcal{T}_n)$ and endow them with the compact-open topology, so that a basis of neighborhoods of identity is given by the sets $B_K(\mathrm{Id}) = \{\phi \in \mathrm{Aut}(\mathcal{T}_n); \phi(x) = x \quad \forall x \in K\}$ where K runs over all finite sets of vertices.

Given an automorphism ϕ, one defines the *translation length* of ϕ as the integer $T(\phi) = \min_x\{d(x, \phi(x))\}$ where the minimum is taken over all points (not only vertices) of $T_n(1)$. The following alternative is classical:

1. if $T(\phi) > 0$ then there is a unique invariant bi-infinite path $(x_i)_{i\in\mathbb{Z}}$ and $\phi(x_i) = x_{i+T(\phi)}$ for all i,

2. if $T(\phi) = 0$ then either ϕ fixes some vertex, or ϕ has a unique fixed point in $T_n(1)$, which is the midpoint of an edge.

In the first case, ϕ is said to be a *translation* (a *unitary* translation if $T(\phi) = 1$). Any translation is a power of a unitary translation.

1.2 Compactification of trees

The standard metric tree $T_n(1)$ is a CAT(0) complete length space, thus is a Hadamard space (see for example [2]). Therefore, it has a geodesic compactification we now briefly describe.

A boundary point p is a class of asymptotic geodesic rays, where two geodesic rays $\gamma_1 = x_0, x_1, \ldots, x_i, \ldots$ and $\gamma_2 = y_0, y_1, \ldots, y_j, \ldots$ are said to be *asymptotic* if they are eventually identical: there are indices i_0 and j_0 so that for all $k \in \mathbb{N}$, on has $x_{i_0+k} = y_{j_0+k}$. The point p is said to be the endpoint of any geodesic ray of the given asymptoty class.

The union $\overline{T}_n = T_n \cup \partial T_n$ is given the following topology: for a point that is not on the boundary, a basis of neighborhoods is given by its neighborhoods in T_n; for a boundary point p, a basis of neighborhoods is given by the connected components of $T_n \setminus \{x\}$ containing a geodesic ray asymptotic to p, where x runs over the vertices

It is a general property of Hadamard spaces that $\mathrm{Aut}(\mathcal{T}_n)$ acts on $\overline{T}_n$ by homeomorphisms for this topology. Our goal will be to see which additional structure can be added to this topology, that is preserved by $\mathrm{Aut}(\mathcal{T}_n)$.

We have no differentiable structure on $\overline{T}_n$, but due to the Rademacher theorem it is natural to look at Lipschitz structures instead.

Definition 1.1 *Let X be a metrizable topological space. A Lipschitz structure $[\delta]$ on X is the data of a metric δ that is compatible with the topology of X, up to local Lipschitz equivalence (two metrics δ_1, δ_2 are said to be locally Lipschitz equivalent if the identity map $(X, \delta_1) \to (X, \delta_2)$ is locally bilipschitz).*

The natural isomorphisms of a space X endowed with a Lipschitz structure are the locally bilipschitz maps. Usually, for an action of a Lie group on a manifold to be differentiable, one asks the map $G \times M \to M$ to be differentiable. Similarly, we say that an action of a topological group Γ on a metrizable topological space X is Lipschitz if it is a continuous action by locally bilipschitz maps, and if moreover the Lipschitz factor is locally uniform.

We can now define our main object of study.

Definition 1.2 *A Lipschitz compactification of $\mathcal{T}_n$ is a Lipschitz structure $[\delta]$ on $\overline{T}_n$, where δ is a length metric, and such that the action of $\mathrm{Aut}(\mathcal{T}_n)$ on $\overline{T}_n$ is Lipschitz.*

In [3], Floyd introduced a method for compactifying a graph. We give definitions that are adapted to the simpler case of trees.

Definition 1.3 *By a Floyd function we mean a function $h : \mathbb{N} \to]0, +\infty[$ such that $\sum_r h(r) < +\infty$. Two Floyd functions h_1, h_2 are said to be* comparable *if there is a $C > 1$ such that for all $r \in \mathbb{N}$ one has $C^{-1} h_2(r) \leq h_1(r) \leq C h_2(r)$.*

Definition 1.4 *A Floyd metric on $\overline{T}_n$ is the length metric obtained from a vertex x_0 and a Floyd function h by assigning to each edge e the length $h(d)$, where $d \in \mathbb{N}$ is the combinatorial distance between e and x_0.*

By a Floyd compactification of $\mathcal{T}_n$ we mean the topological space $\overline{T}_n$ endowed with the Lipschitz structure corresponding to a Floyd metric.

The condition that $\sum h(r)$ converges ensures that we do get a distance on $\overline{T}_n$. For example, the distance between two boundary points p and p' is $2\sum_{r\geq R} h(r)$ where R is the combinatorial distance between x_0 and the only geodesic joining p and p'.

Two Floyd metrics obtained from the same point x_0 and Floyd functions h_1, h_2 are easily seen to define the same Lipschitz structures if and only if h_1 and h_2 are comparable.

2 Description of all Lipschitz compactifications of regular trees

Theorem 2.1 *Any Lipschitz compactification of $\mathcal{T}_n$ is a Floyd compactification.*

The Floyd compactification of $\mathcal{T}_n$ obtained from a Floyd function h and a base point x_0 is a Lipschitz compactification if and only if there is a constant $0 < \eta < 1$ so that for all $r \in \mathbb{N}$

$$h(r+1) \geq \eta\, h(r). \tag{1}$$

Remark 1 Condition (1) *implies that h decreases at most exponentially fast. It is interesting to compare this with the usual conformal compactification of the hyperbolic space, obtained by multiplying the metric by a factor that is exponential in the distance to a fixed point.*

Remark 2 Condition (1) *implies that the considered Lipschitz structure depends only upon h, not x_0. We can therefore denote this compactification by $\overline{T}_n(h)$.*

Proof. We first prove that any Lipschitz compactification of $\mathcal{T}_n$ is a Floyd compactification.

Let δ' be any length metric in the given Lipschitz class, and fix any vertex x_0 of $\mathcal{T}_n$. We define h by $h(r) = \min \delta'(x,y)$ where the minimum is taken over all edges xy that are at combinatorial distance r from x_0. Then h is a Floyd function because x_0 is at finite δ' distance from the boundary. Denote by δ the Floyd metric obtained from x_0 and h, and let us prove that $[\delta] = [\delta']$. It is sufficient to prove that there is a constant C so that for all r, two edges that are at combinatorial distance r from x_0 have their δ' lengths that differ by a factor at most C.

For any $R \in \mathbb{N}$, let $B(R)$ be the closed ball of radius R and center x_0 in $T_n(1)$. It contains a finite number of edges, so that there is a constant C_R that satisfies the above property for all $r \leq R$.

Since the compactification is assumed to be Lipschitz, for all $p \in \partial T_n$ there are a neighborhood V of p, a neighborhood U of the identity and a constant k so that any $\phi \in U$ is k-Lipschitz on V. Since ∂T_n is compact, we can find a finite number of such quadruples (p_i, V_i, U_i, k_i) so that the V_i cover ∂T_n. Moreover we can assume that the V_i are the connected components of $T_n \setminus B(R)$ for some radius R, and that $U = \cap U_i = B_{B(R)}(\mathrm{Id})$. Since for all i and $r > R$, U acts transitively on the set of edges of V_i that are at combinatorial distance r from x_0, those edges have their δ'-length that differ by a factor at most $C' = \sup k_i$. Moreover, there is an automorphism ϕ_0 that fixes x_0 and permutes cyclically the V_i. Since ϕ_0 is locally Lipschitz, there is a R' and a C'' so that for all $r \geq R'$ and all couple (i_1, i_2), there are edges of V_{i_1} and V_{i_2} that are at combinatorial distance r from x_0 and whose δ' lengths differ by a factor at most C''. The supremum C of $C_{R'}$ and $C''C'^2$ is the needed constant.

Consider now the Floyd compactification obtained from x_0 and h and denote by δ the associated Floyd metric. By construction, any automorphism ϕ of $\mathcal{T}_n$ that fixes x_0 is an isometry for δ, thus is locally bilipschitz for the corresponding Lipschitz structure.

Two translations are close to one another when they differ by an element close to identity. An element close enough to identity must fix x_0, thus is an isometry. Therefore, we only need to prove that a given translation is Lipschitz to get that all automorphisms in a neighborhood are equilipschitz. Checking unitary translations is sufficient since any translation is an iterate of one of those.

Let ϕ be a unitary translation, and $\gamma = \ldots, y_{-1}, y_0, y_1, \ldots$ be its translated geodesic, where we assume that y_0 realizes the minimal combinatorial distance d_0 between vertices of γ and x_0. By local finiteness, ϕ is locally bilipschitz around any point of T_n and we need only check the boundary.

Let us start with the attractive endpoint p of γ. Assume that our Floyd compactification is Lipschitz. It implies that ϕ is locally bilipschitz around p, in particular there is a $r_0 > 0$ and a $k > 1$ such that for any $r \geq r_0$,

$$\begin{aligned} k\,\delta(y_{r+1}, y_{r+2}) &\geq \delta(y_r, y_{r+1}) \\ h(r + d_0 + 1) &\geq k^{-1} h(r + d_0) \end{aligned}$$

which gives condition (1).

Conversely, assume that (1) holds.

For any vertex x we have

$$|d(\phi(x), x_0) - d(x, x_0)| \leq 1 + 2d_0$$

since the worst case is when $x = x_0$ or x is in a connected component of $\mathcal{T}_n \setminus \{x_0\}$ other than that of γ. Therefore, the length of an edge and of its

image by ϕ differ by a factor bounded by $\eta^{-(1+2d_0)}$. Therefore, ϕ is Lipschitz. Since ϕ^{-1} is also a unitary translation, ϕ is bilipschitz. $\square$

It would be interesting to consider more general spaces, for example euclidean buildings or CAT(-1) buildings like the I_{pq} described by Bourdon in [1]. It is not obvious how to define the Floyd compactification: for example, a mere scaling of the distance in each cell by a factor depending on the combinatorial distance to a fixed cell would create gluing problems (an edge shared by two faces having two different length). This spaces could therefore be less flexible than trees.

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