Random changes of flow topology in two dimensional and geophysical turbulence

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We study the two dimensional (2D) stochastic Navier Stokes (SNS) equations in the inertial limit of weak forcing and dissipation. The stationary measure is concentrated close to steady solutions of the 2D Euler equation. For such inertial flows, we prove that bifurcations in the flow topology occur either by changing the domain shape, the nonlinearity of the vorticity-stream function relation, or the energy. Associated to this, we observe in SNS bistable behavior with random changes from dipoles to unidirectional flows. The theoretical explanation being very general, we infer the existence of similar phenomena in experiments and in models of geophysical flows.

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The largest scales of turbulent flows are at the heart of a number of geophysical processes : climate, meteorology, ocean dynamics, the Earth magnetic field. The Earth is affected on a very large range of time scales, up to millennia, by the structure and variability of these flows. Many of these undergo extreme and abrupt qualitative changes, seemingly randomly, after very long period of apparent stability. This occurs for instance for magnetic field reversal for the Earth or in MHD experiments [1], for 3D flows [2], for multiple equilibria of atmospheric flows [3], for 2D turbulence experiments [4, 5] and for the paths of the Kuroshio and Gulf Stream currents [6].

Understanding these phenomena requires a statistical description of the largest scales of turbulent flows. Very few theoretical approaches exists due to the prohibitively huge number of degrees of freedom involved. Fruitful hints may be drawn from qualitative analogies with bistability in system with few degrees of freedom perturbed by noise [7]. However the range of validity of this approach remains a tough scientific issue, because of the complexity of turbulent flows. What is the good theoretical framework for such phenomena ? In the following, we argue that 2D turbulence, because of its relative theoretical simplicity, is a very interesting framework in order to address such an issue.

In this letter we predict and prove the existence of random switches from dipoles to unidirectional flows (see Fig 1), in the 2D Navier Stokes Eq. with random force (SNS). Similar random changes have already been observed in rotating tank experiments for quasi-geostrophic dynamics [3]. Following analogous theoretical considerations as for SNS Eq., we infer that such changes will generically occur within a large class of models like quasi-geostrophic (QG) or shallow-water (SW) models that describe atmospheric [3], and oceanic [6] large scales. The recipe we propose is to exhibit bifurcation lines representing abrupt change in steady solutions in the inertial limit and then look for the corresponding transitions in real flows.

Geophysical and 2D inviscid flows are characterized by the conservation of energy and an infinite number of quantities (Casimirs), such as enstrophy. This property prevents direct energy cascade towards the small scales,

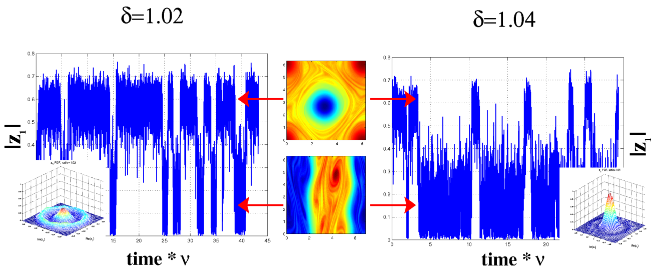

Figure 1: Time series and probability density functions (PDFs) for the order parameter z_1 (see page 4) illustrating random changes between dipoles and unidirectional flows.

by contrast with 3D turbulence. Then, the first phenomenon is an inverse energy cascade towards the large scales and a direct enstrophy cascade. Kraichnan classical theory [8] studies the self-similar processes associated with these two cascades (see the recent spectacular discovery of conformal invariance consequences for the inverse cascade [9]). The second phenomenon, the self organization of the flow into jets and vortices, occurs if energy is not dissipated before reaching the largest scale. Then coherent structures break the self-similarity so that their study cannot be properly addressed using Kraichnan theory [8]. A second classical theory, the so-called Robert-Sommeria-Miller (RSM) equilibrium statistical mechanics [10], predicts the self-organized structures for inviscid decaying turbulence. However, this inviscid theory does not take into account the long-term effects of forcing and dissipation as well as the slow dynamics of the flow. Therefore, random changes of flow topologies cannot be explained by these two classical theories.

As an alternative theoretical approach, we study statistically stationary states of SNS Eq. Note that a self-similar growth of a dipole has been studied in [11] emphasizing transient growths: both approaches complement each other. SNS Eq. on a doubly-periodic domain $\mathcal{D} = (0; 2\pi\delta) \times (0; 2\pi)$ reads

$$\frac{\partial \omega}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\omega = -\alpha\omega + \nu\Delta\omega + \sqrt{\sigma}\eta,\ \mathbf{v} = \mathbf{e}_z \times \boldsymbol{\nabla}\psi,\ \omega = \Delta\psi, \quad (1)$$

where ω, $\mathbf{v}$ and ψ are respectively the vorticity, velocity and stream function ; α is the Rayleigh friction coefficient and ν the viscosity. The force curl is $\eta = \sum_{\mathbf{k}} f_{\mathbf{k}} \eta_{\mathbf{k}}(t) \exp(i\mathbf{k}.\mathbf{x})/(2\pi)$, with $\{\eta_{\mathbf{k}}\}$ independent Gaussian white noises : $\langle \eta_{\mathbf{k}}(t) \eta_{\mathbf{k}'}(t') \rangle = \delta_{\mathbf{k}\mathbf{k}'} \delta(t-t')$. We impose $B_0 \equiv \sum_{\mathbf{k}} |f_{\mathbf{k}}|^2 / |\mathbf{k}|^2 = 1$ so that σ is the average energy injection rate.

Euler Eq. ($\alpha = \nu = \sigma = 0$) conserve the kinetic energy E and vorticity moments Ω_n (Ω_2 is the enstrophy)

$$E = \frac{1}{2}\int_{\mathcal{D}} d^2x \, \mathbf{v}^2 \text{ and } \Omega_n = \int_{\mathcal{D}} d^2x \, \omega^n. \tag{2}$$

Application of Ito formula to the energy, and averaging over the noise, leads to $d\langle E\rangle/dt = -2\alpha \langle E\rangle + \sigma - \nu\langle \Omega_2\rangle$. If $\langle . \rangle_S$ denotes averages over the stationary measure, we have $2\alpha\langle E\rangle_S + \nu \langle \Omega_2\rangle_S = \sigma$. It expresses the balance between energy injection and energy dissipation. Clearly, for flows with energetic large scales, Rayleigh friction dominates dissipation $2\alpha\langle E\rangle_S \gg \nu\langle\Omega_2\rangle_S$ (mathematically we consider the limit $\nu \to 0$ for fixed α and assume $\nu\langle\Omega_2\rangle_S \to 0$). It is natural to fix the average energy to be of order 1 by using a typical turnover time as a new time unit. Putting $t' = \sqrt{\sigma/(2\alpha)}t$, $\omega' = \sqrt{2\alpha/\sigma}\omega$, $\alpha' = (2\alpha)^{3/2}/(2\sigma^{1/2})$ and $\nu' = \nu (2\alpha/\sigma)^{1/2}$ and dropping the primes, the dimensionless Eq. are

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\, \omega = -\alpha\omega + \nu\Delta\omega + \sqrt{2\alpha}\eta. \tag{3}$$

The energy balance now reads $\langle E\rangle_S + (\nu/2\alpha)\langle\Omega_2\rangle_S = 1$. In these dimensionless unit the Reynolds number is $1/\nu$ and the Rayleigh number is $R_\alpha = \mathcal{O}(\mathbf{v}\cdot\boldsymbol{\nabla}\,\omega/\alpha\omega) = 1/\alpha$ (arresting the inverse cascade before energy reaches the largest scale would requires $\alpha > 1$). For most geophysical flows and experiments the case of weak forcing and dissipation is the most relevant one. We thus study the inertial limit $\alpha \ll 1$ (more precisely the limit $\lim_{\alpha\to 0}\lim_{\nu\to 0}$).

Without Rayleigh friction ($\alpha = 0$), the previous discussion is meaningless and the balance relation becomes $2\nu\langle\Omega_2\rangle_S = \sigma$. By a natural time unit change, we can fix $\langle\Omega_2\rangle_S = 1$. The nondimensional equation is then

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\, \omega = \nu\Delta\omega + \sqrt{2\nu}\eta. \tag{4}$$

From a physical point of view, this last model is less relevant than (3) but is still very interesting from an academic point of view. A series of recent works has proved the existence of invariant measures, validity of the law of large numbers, central limit theorems, ergodicity and some properties of stationary measures in the inertial limit $\nu \to 0$, balance relations (see [12] and references therein). All following considerations are relevant for both models (3,4), in their respective inertial limits.

We know since decades from real [13] or numerical [5, 14] experiments, that for times large compared to the turnover time but small compared to the dissipation time, the largest scales of 2D Navier-Stokes turbulent flows converge towards steady solutions of Euler Eq. :

$$\mathbf{v} \cdot \boldsymbol{\nabla}\, \omega = 0 \text{ or equivalently } \omega = f(\psi). \tag{5}$$

It appears to be true as well for the Euler Eq. For instance, RSM theory predicts f from given initial conditions. Given this empirical evidence, it is thus extremely natural to expect that in the inertial limit, measures for SNS are concentrated near steady Euler flows. We show numerical evidences of this fact in the following.

The ensemble of steady Euler flows is huge, as it is parametrized by the function f. It will be proven that when either f or the domain shape is changed, bifurcations may occur. Such abrupt transitions lead to strong qualitative changes in the flow topology. In this critical regime and under the action of a small random force in SNS, the system switches randomly from one type of topology to another. In the following, we show that this scenario is valid.

We study a bifurcation diagram for stable steady Euler solutions, by considering

$$S(E) = \sup_{\omega}\{\mathcal{S}[\omega] = \int_{\mathcal{D}} d^2x \; s(\omega) \mid \mathcal{E}(\omega) = E\}, \tag{6}$$

where $S(E)$ is the equilibrium entropy, $\mathcal{S}$ the entropy of ω ; the specific entropy $s(\omega) = -\omega^2/2 + \sum_{n\geq 2} a_{2n}/2^n \omega^{2n}$ is concave assuming s even for simplicity. Critical points of (6) verify $\omega = f(\psi) = (s')^{-1}(-\beta\psi)$, where β is the Lagrange multiplier associated with energy conservation. They are thus steady Euler flows, satisfying (5), and the knowledge of f or s are equivalent. From Arnold's theorems [15] or its generalization, maxima of (6) are dynamically stable. One can also prove that any solutions for (6) are RSM equilibria [16]. Even if it seems appealing, there are no clear theoretical arguments for giving a thermodynamical interpretation to (6) in the SNS out-of-equilibrium context. We thus consider (6) only as a practical way to describe ensembles of *stable* steady Euler solutions.

Dipoles and unidirectional flows ("bars") have been obtained numerically [17] as entropy maxima for 2D Euler Eq. with periodic boundary conditions, assuming sinh, tanh and 3-level Poisson $\omega - \psi$ (5) relations. According to [17] "which has the greater entropy (between dipoles and bars) depends on seemingly arbitrary choices".

The fact that both unidirectional flows and dipole may be equilibria can be understood from the small energy limit of (6). Let us call $\{e_i\}_{i\geq 1}$ the orthonormal family of eigenfunctions of the Laplacian $-\Delta e_i = \lambda_i e_i$, $\langle e_i e_j\rangle_{\mathcal{D}} = \delta_{ij}$ ($< . >_{\mathcal{D}} \equiv \int_{\mathcal{D}} dx$ and λ_i are arranged in increasing order). We decompose the vorticity as $\omega = \sum_{i\geq 1}\omega_i e_i$. The energy is then $2\mathcal{E}(\omega) = \sum_{i\geq 1}\lambda_i^{-1}\omega_i^2$. Since $\mathcal{E}(\omega)$ is always positive $\langle \omega^2\rangle_{\mathcal{D}}$ is small in the limit $E \to 0$, and only the quadratic part of $\mathcal{S}[\omega]$ is relevant. Long but straightforward computation of (6) in the limit $E \to 0$ gives

$$\omega \underset{E\to 0}{\sim} (2\lambda_1 E)^{1/2} e_1 \text{ with } S(E) = -\lambda_1 E + \mathcal{O}(a_4 \lambda_1^2 E^2). \tag{7}$$

We thus conclude that the eigenmode with the smallest eigenvalue is selected, corresponding to the heuristic idea that energy condensate to the largest scale.

For instance when the aspect ratio $\delta > 1$, the mode $e_1 = n_1 \sin\left[(x+\phi_1)/\delta\right]$ is selected. This corresponds to a unidirectional velocity field $\mathbf{v}_1 = n_1 \cos\left[(x+\phi_1)/\delta\right] \mathbf{e}_y$ where ϕ_1 is a phase associated to the translational invariance. A dipole is actually a mixed state $\alpha e_1 + \beta e_2$ with $e_2 = n_2 \sin(y+\phi_2)$. In the weak energy limit, it can be selected only for the degenerate case $\lambda_1 = \lambda_2$. This happens for the square box $\delta = 1$. In such a case we can prove that the degeneracy is removed by the contribution of higher order terms in (7). From (7), we conclude that the domain shape (e_1) selects the equilibria for $\lambda_2 - \lambda_1 \gg a_4 \lambda_1^2 E$ whereas for $\lambda_2 - \lambda_1 \ll a_4 \lambda_1^2 E$ the degeneracy is removed by the nonlinearity of $f\ (a_4)$. In order to study the bifurcation between these two behaviors, we define g by $\lambda_2 - \lambda_1 = gE$ and we study the small energy limit $E \to 0$, with fixed g. Straightforward computations lead to

$$S(E) = -\lambda_1 E + E^2 \max_{0 \le X \le 1} h(X), \tag{8}$$

with $h(X) = \left\langle e_1^4 \right\rangle_{\mathcal{D}} a_4 \lambda_1^2 - gX + 2\gamma \lambda_1^2 a_4 X(1-X)$, where $\gamma = 3\left\langle e_1^2 e_2^2 \right\rangle_{\mathcal{D}} - \left\langle e_1^4 \right\rangle_{\mathcal{D}} > 0$. The vorticity equilibria is then $\omega_{eq} \sim_{E\to 0} \sqrt{2\lambda_1 E(1-X_M)}e_1 + \sqrt{2E\lambda_1 X_M} e_2$ where X_M is the maximizer of h in (8). For $X_M = 0$ or $X_M = 1$, ω_{eq} is an unidirectional flow whereas for $0 < X_M < 1$ it is a dipole (symmetric for $X_M = 1/2$). The selection occurs via maximization of h. When maximizing h, the sign of the parameter a_4 plays a crucial role. We note that a_4 is intimately related to the shape of the relationship $\omega = f(\psi) = (s')^{-1}(-\beta\psi)$. Indeed $(s')^{-1}(-x) = x + a_4 x^3 + o(x^3)$ and when $a_4 > 0$ (resp. $a_4 < 0$), the curve $f(\psi)$ bends upward (resp. downward) for positive ψ similarly to sinh (resp. tanh).

The bifurcation diagram is summarized in Fig. 2 a). In the degenerate case ($g = 0$), the dipole is selected for $a_4 > 0$ (sinh like), whereas unidirectional flows are selected for $a_4 < 0$ (tanh like). The term $-gX$ favors the pure state e_1 ($X = 0$). For $a_4 < 0$ the unidirectional flow e_1 is always selected. More interestingly, for $a_4 > 0$ a bifurcation occurs along the critical line $g^\star = 2\gamma\lambda_1^2 a_4$ between dipole and unidirectional flows.

We have obtained the bifurcation diagram in the limit of small energy using the scaling $\lambda_2 - \lambda_1 = gE$. From a practical point of view, it is more convenient to work for a fixed aspect ratio δ. Using the relation $g = (\lambda_2 - \lambda_1)/E$, we obtain that the critical line in a $E - a_4$ plane is the hyperbola $a_4 E = 8\pi^2(\delta-1)/3 + o(\delta-1)$. We use a continuation algorithm in order to numerically compute solution to (6) corresponding to $f_{a_4}(x) = (1/3 - 2a_4)\tanh x + (2/3 + 2a_4)\sinh x$. The inset of Fig 2 **b)** shows good agreement for transition lines obtained either with the continuation algorithm or the low-energy limit theoretical result, for $\delta = 1.01$. Figure 2 **b)** shows the bifurcation diagram for $\delta = 1.1$; in such a case the transition line is still very close to an hyperbola provided energy is small.

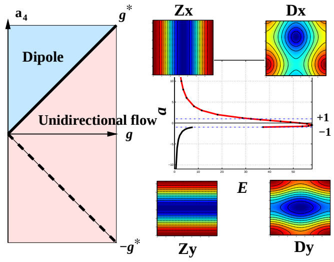

Figure 2: Bifurcation diagrams for steady Euler flows a) in the g-a_4 plane b) obtained numerically in an $E - a_4$ plane with $\delta = 1.1$. The inset illustrates good agreement between numerical and theoretical results in the low energy limit.

Following the same reasoning, small-energy bifurcation diagrams could be computed for any Euler-like model like QG or SW models. Most often, the domain shape selects the flow topology. When domain shape is varied, we meet eigenvalues degeneracy. In all these cases a bifurcation diagram can be computed where the transition line corresponds to the competition between the $\omega - \psi$ (5) nonlinearity and the domain shape.

We expect to observe both dipoles and unidirectional flows in SNS. Numerical simulations in a square domain $\delta = 1$ exhibit statistically stationary ω with a dipole structure (Fig. 3 **a)**), whereas for $\delta \geq 1.1$, nearly unidirectional flows are observed (Fig. 3 **b)**). This result has been confirmed both for $\alpha = 0$ and $\alpha \neq 0$, and for different ν values and forcing spectra. One observes in Fig. 3 a $\omega - \psi$ relation qualitatively similar to a sinh, in the dipole case and to a tanh in the unidirectional case. This confirms that ω remains close to steady Euler flows.

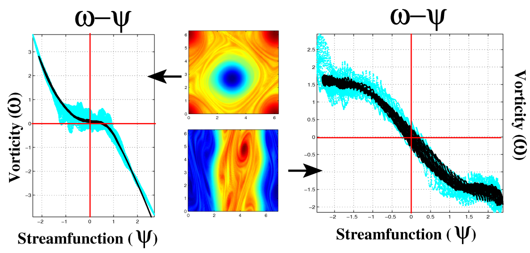

Figure 3: $\omega - \psi$ scatter-plots (cyan). In black the same after time averaging (averaging windows $1 \ll \tau \ll 1/\nu$, the drift due to translational invariance has been removed) a) dipole case with $\delta = 1.03$ b) unidirectional case $\delta = 1.10$.

A very natural order parameter is $|z_1|$, where $z_1 = \frac{1}{(2\pi)^2}\left\langle \omega(x,y)\exp(iy)\right\rangle_{\mathcal{D}}$. Indeed, for unidirectional flow $\omega = \alpha e_1$, $z_1 = 0$, whereas for a dipole $\omega = \alpha(e_1 + e_2)$, $|z_1| = \alpha$. Fig. 1 shows $|z_1|$ time series for $\delta = 1.02$ and $\delta = 1.04$. The remarkable observation is the bimodal behavior in this transition range. The switches from $|z_1|$ values close to zero to values of order of 0.6 correspond to genuine transitions between unidirectional and dipole flows. The PDF of the complex variable z_1 (Fig. 1) exhibits a circle corresponding to the dipole state (a slow dipole random translation corresponds into to a phase drift for z_1, explaining the circular symetry). The zonal state corresponds to the central peak. As δ increases, one observes less occurrences of the dipole. For larger (resp. smaller) values of δ only unidirectional (dipole) flows exist. The transition is also visible in other physical variables. For instance $\Omega_4 = \left\langle |\omega|^4 \right\rangle_{\mathcal{D}}$ switches between a state with weak variance and low mean value (unidirectional) to an intermittent state with large variance and larger mean value (dipole). Topology changes are very slow dynamical processes : for the model (4), an average transition time is of order $1/\nu$. For instance Fig. 1 represents 3.10^4 turnover times. For this reason, because of numerical limitations it has not been yet possible to obtain convincing analysis of the switch time statistics.

In the spirit of [7] we look for low-dimensional analogies. When looking how evolves the PDF for the order parameter, while changing the control parameter, the dipole-unidirectional transition has striking similarities with the stochastic differential equation

$$dx = x(\mu + x^2 - x^4)dt + \sigma dW. \qquad (9)$$

The deterministic part of (9) is the normal form for a generalized subcritical pitchfork bifurcation. For $\mu < -1/4$, one has a single stable fixed point $x^\star = 0$. For $\mu > 0$ there are three fixed point, one unstable $x_0 = 0$ and two stable $x_{1,2} = \pm(1+(1+4\mu)^{1/2})^{1/2}$. For $\mu \in]-1/4, 0[$, three stable fixed points coexist (x_0 and $x_{1,2}$) and two unstable ones $x_{3,4} = \pm(1-(1+4\mu)^{1/2})^{1/2}$. With additive noise ($\sigma \neq 0$), when $\mu < -1/4$, the $|x|$ PDF has a single peak centered at $x = 0$. In the interval corresponding to $\mu \in]-1/4, 0[$, an additional peak appears related to $|x_{1,2}|$. Finally, there is a transition for μ larger than 0 and only one peak corresponding to $|x_1|$ remains.

However, we stress that a low-dimensional model like (9), as useful as it may be, lacks part of the phenomena. For instance, it can not explain why Ω_4 is intermittent while $|z_1|$ is not. Moreover, the role of turbulence here is not only to act as noise, but also to build up the large-scale flow by inverse cascade. The inverse cascade properties are strongly affected by the existing large-scale flow, leading to the observed self-organization process. From a theoretical point of view, the main issue, beyond the scope of this letter, is to explain which of the Euler steady states will be selected by turbulence and to predict the relative frequency of such states. We thus need an alternative theoretical approach bridging the gap between the two classical theories : self-similar inverse energy cascade on one hand and RSM equilibrium statistical mechanics on the other hand.

In this letter, we have not addressed the other crucial issue : fluctuations. Some very interesting results and considerations on small-scale fluctuations for turbulence dominated by large-scale flows may be found in [11, 18, 19]. In forthcoming works, the statistical properties of these random change of flow topologies and of fluctuations will be investigated. Finally, it will also be extremely interesting to analyze the connexions with similar transitions observed in other contexts [1, 3, 4, 6].

We infer similar random flow topology changes for other geometry for 2D SNS, QG and SW models. Using simple generalization of our analysis, rotating tanks experiments can be designed in order to observe similar phenomena. This study also suggests that flows like the Kuroshio currents [6] or the Gulf Stream might be close to steady solutions of inertial models.

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