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# COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPS

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## Abstract

In this paper, we establish a common fixed point theorem for two pairs of occasionally weakly compatible single and set-valued maps satisfying a strict contractive condition in a metric space. Our result unifies and extends many results existing in the literature including those of Aliouche [3], Bouhadjera [4] and Popa [20]-[25]. Also we establish another common fixed point theorem for four owc single and set-valued maps of Greguš type which improves the results of Djoudi and Nisse [6], Pathak et al. [18] and others and we end our work by giving a third theorem which generalizes the results given by Elamrani and Mehdaoui [7], Mbarki [15] and references therein.

**Key words and phrases:** Occasionally weakly compatible maps, weakly compatible maps, compatible and compatible maps of type (A), (B), (C) and (P), implicit relation, common fixed point theorem, Greguš type, strict contractive condition, metric space.

**2000 Mathematics Subject Classification:** 47H10, 54H25.

## 1 Introduction and preliminaries

Throughout this paper,  $(\mathcal{X}, d)$  denotes a metric space and  $\mathcal{P}_{fb}(\mathcal{X})$  the class of all nonempty bounded closed subsets of  $\mathcal{X}$ . We recall these usual notations: for  $x \in \mathcal{X}$  and  $A \subseteq \mathcal{X}$ ,

$$d(x, A) = \inf \{d(x, y) : y \in A\}$$

and let  $H$  be the associated Hausdorff metric on  $\mathcal{P}_{fb}(\mathcal{X})$ : for every  $A, B$  in  $\mathcal{P}_{fb}(\mathcal{X})$ ,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}.$$

In the following, we use small letters:  $f, g, \dots$  to denote maps from  $\mathcal{X}$  to  $\mathcal{X}$  and capital letters:  $F, G, S, T, \dots$  for set-valued maps, that is, maps from  $\mathcal{X}$  to  $\mathcal{P}_{fb}(\mathcal{X})$  and we write  $fx$  for  $f(x)$  and  $Fx$  for  $F(x)$ .

The concepts of weak commutativity, compatibility, noncompatibility and weak compatibility were frequently used to prove existence theorems in fixed and common fixed points for single and set-valued maps satisfying certain conditions in different spaces. The study of common fixed points on occasionally weakly compatible maps is new and also interesting. This notion which is defined by Al-Thagafi and Shahzad [2] and which is published in 2008, has been used by Jungck and Rhoades [12] in 2006 and by Abbas and Rhoades [1] in 2007.

We begin by a short historical evolution of these different notions. Generalizing the concept of commuting mappings, Sessa [26] introduced the concept of weakly commuting mappings.  $f$  and  $g$  are weakly commuting if

$$d(fgx, gfx) \leq d(gx, fx)$$

for all  $x \in \mathcal{X}$ , where  $f$  and  $g$  are two self-maps of  $(\mathcal{X}, d)$ .

In 1986, Jungck [8] made more generalized commuting and weakly commuting maps called compatible maps.  $f$  and  $g$  are said to be compatible if

$$(1) \quad \lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$$

whenever  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{X}$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$  for some  $t \in \mathcal{X}$ . This concept has been useful as a tool for obtaining more comprehensive fixed point theorems. Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible, but neither implication is reversible (see [8]).

Further, the same author with Murthy and Cho [10] gave another generalization of weakly commuting maps by introducing the concept of compatible maps of type (A).  $f$  and  $g$  are said to be compatible of type (A) if in place of (1) we have the two equalities

$$\lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) = 0.$$

Obviously, weakly commuting maps are compatible of type (A). From [10] it follows that the implication is not reversible.

In their paper [17], Pathak and Khan extended type (A) maps by introducing the concept of compatible maps of type (B) and compared these maps with compatible and compatible maps of type (A) in normed spaces. To be compatible of type (B),  $f$  and  $g$  above have to satisfy, in lieu of condition (1), the inequalities

$$\begin{aligned} \lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) &\leq \frac{1}{2} \left[ \lim_{n \rightarrow \infty} d(fgx_n, ft) + \lim_{n \rightarrow \infty} d(ft, f^2x_n) \right] \\ &\text{and} \\ \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) &\leq \frac{1}{2} \left[ \lim_{n \rightarrow \infty} d(gfx_n, gt) + \lim_{n \rightarrow \infty} d(gt, g^2x_n) \right]. \end{aligned}$$

It is clear that compatible maps of type (A) are compatible of type (B). The converse is not true ([17]).

In 1998, Pathak et al. [18] introduced an extension of compatibility of type (A) by giving the notion of compatible maps of type (C).  $f$  and  $g$  are compatible of type (C) if they satisfy the two inequalities

$$\begin{aligned} \lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) &\leq \frac{1}{3} \left[ \lim_{n \rightarrow \infty} d(fgx_n, ft) + \lim_{n \rightarrow \infty} d(ft, f^2x_n) \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} d(ft, g^2x_n) \right] \\ \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) &\leq \frac{1}{3} \left[ \lim_{n \rightarrow \infty} d(gfx_n, gt) + \lim_{n \rightarrow \infty} d(gt, g^2x_n) \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} d(gt, f^2x_n) \right]. \end{aligned}$$

The same authors gave some examples to show that compatible maps of type (C) need not be neither compatible nor compatible of type (A).

In [16] the concept of compatible maps of type (P) was introduced and compared with compatible and compatible maps of type (A) (resp. (B)).  $f$  and  $g$  are compatible of type (P) if in lieu of (1) we have

$$\lim_{n \rightarrow \infty} d(f^2x_n, g^2x_n) = 0.$$

Note that compatibility, compatibility of type (A) (resp. (B), (C) and (P)) are equivalent if  $f$  and  $g$  are continuous.

Afterwards, Jungck [9] generalized the compatibility, the compatibility of type (A), (B), (C) and (P) by introducing the concept of weak compatibility. He defines  $f$  and  $g$  to be weakly compatible if  $ft = gt, t \in \mathcal{X}$  implies  $fgt = gft$ .

It is known that all of the above compatibility notions imply weakly compatible notion, however, there exist weakly compatible maps which are neither compatible nor compatible of type (A), (B), (C) and (P) (see [3]).

Recently in a paper submitted before 2006 but published only in 2008, Al-Thagafi and Shahzad [2] weakened the concept of weakly compatible maps by giving the new concept of occasionally weakly compatible maps. Two self-maps  $f$  and  $g$  of  $\mathcal{X}$  are called occasionally weakly compatible maps (shortly owc) if there is a point  $x$  in  $\mathcal{X}$  such that  $fx = gx$  at which  $f$  and  $g$  commute. This notion is used in 2006 by Jungck and Rhoades [12] to prove some common fixed point theorems in symmetric spaces.

In their paper [13], Kaneko and Sessa extended the compatibility to the setting of single and set-valued maps as follows:  $f : \mathcal{X} \rightarrow \mathcal{X}$  and  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  are said to be compatible if  $fFx \in \mathcal{P}_{fb}(\mathcal{X})$  for all  $x \in \mathcal{X}$  and

$$\lim_{n \rightarrow \infty} H(Ffx_n, fFx_n) = 0$$

whenever  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{X}$  such that  $fx_n \rightarrow t, Fx_n \rightarrow A \in \mathcal{P}_{fb}(\mathcal{X})$  and  $t \in A$ .

After, in [11] Jungck and Rhoades extend the concept of compatible single

and set-valued maps by giving the concept of weak compatibility. Maps  $f$  and  $F$  are weakly compatible if they commute at their coincidence points, i.e., if  $fFx = Ffx$  whenever  $fx \in Fx$ .

More recently, Abbas and Rhoades [1] extended the definition of owc maps to the setting of set-valued maps and they proved some common fixed point theorems satisfying generalized contractive condition of integral type.  $f$  and  $F$  are said to be owc if and only if there exists some point  $x$  in  $\mathcal{X}$  such that  $fx \in Fx$  and  $fFx \subseteq Ffx$ . Clearly, weakly compatible maps are occasionally weakly compatible. However, the converse is not true in general. The example below illustrate this fact.

**1.1 Example** Let  $\mathcal{X} = [1, \infty[$  with the usual metric. Define  $f : \mathcal{X} \rightarrow \mathcal{X}$  and  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  by, for all  $x \in \mathcal{X}$ ,

$$fx = 2x + 1, Fx = [1, 2x + 1].$$

$$fx = 2x + 1 \in Fx \text{ and } fFx = [3, 4x + 3] \subset Ffx = [1, 4x + 3].$$

Hence,  $f$  and  $F$  are occasionally weakly compatible but non weakly compatible.

The following property will be used. It is immediately proved from definition of the Hausdorff metric  $H$ .

**1.2 Property** Let  $A$  and  $B \in \mathcal{P}_{fb}(\mathcal{X})$ , then for any  $a \in A$ , we have

$$d(a, B) \leq H(A, B).$$

## 2 A strict fixed point theorem

In this section, before giving our first main result, we recall this definition.

**2.1 Definition** Let  $F : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a set-valued map on  $\mathcal{X}$ .  $x \in \mathcal{X}$  is a fixed point of  $F$  if  $x \in Fx$  and is a strict fixed point of  $F$  if  $Fx = \{x\}$ .

We recall that a strict fixed point is also denoted by "absolutely fixed point" in [14] and "stationary point" in [5].

**2.2 Theorem** Let  $f, g : \mathcal{X} \rightarrow \mathcal{X}$  be mappings and  $F, G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued mappings such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\varphi : \mathbb{R}^6 \rightarrow \mathbb{R}$  be a real map satisfying the following conditions:

( $\varphi_1$ ) :  $\varphi$  is nonincreasing in variables  $t_2, t_5$  and  $t_6$ ,

( $\varphi_2$ ) :  $\varphi(t, t, 0, 0, t, t) \geq 0 \forall t > 0$ .

If, for all  $x$  and  $y \in \mathcal{X}$  for which  $\max\{d(fx, gy), d(fx, Fx), d(gy, Gy)\} > 0$ ,

(\*)  $\varphi(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) < 0$

then  $f, g, F$  and  $G$  have a unique common fixed point which is a strict fixed point for  $F$  and  $G$ .

**Proof**

*i)* We begin to show existence of a common fixed point.

Since the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc then, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu$ ,  $gv \in Gv$ ,  $fFu \subseteq Ffu$  and  $gGv \subseteq Ggv$ . Also, using the triangle inequality and Property 1.2, we obtain

$$(2) \quad d(fu, gv) \leq H(Fu, Gv)$$

and

$$(3) \quad d(f^2u, gv) \leq H(Ffu, Gv).$$

First, we show that  $gv = fu$ . Suppose that it is not the case, then condition (\*) implies that

$$\begin{aligned} & \varphi(H(Fu, Gv), d(fu, gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu)) \\ &= \varphi(H(Fu, Gv), d(fu, gv), 0, 0, d(fu, Gv), d(gv, Fu)) < 0. \end{aligned}$$

By  $(\varphi_1)$  we have

$$\varphi(H(Fu, Gv), H(Fu, Gv), 0, 0, H(Fu, Gv), H(Fu, Gv)) < 0$$

which from  $(\varphi_2)$  gives  $H(Fu, Gv) = 0$ . So  $Fu = Gv$  and, by (2),  $fu = gv$ . Again we have  $d(f^2u, fu) \leq H(Ffu, Gv)$ . Next, we claim that  $f^2u = fu$ . If not, then condition (\*) implies that

$$\begin{aligned} & \varphi(H(Ffu, Gv), d(f^2u, gv), d(f^2u, Ffu), \\ & \quad d(gv, Gv), d(f^2u, Gv), d(gv, Ffu)) \\ &= \varphi(H(Ffu, Gv), d(f^2u, fu), 0, 0, d(f^2u, Gv), d(fu, Ffu)) < 0. \end{aligned}$$

By  $(\varphi_1)$  we have

$$\varphi(H(Ffu, Gv), H(Ffu, Gv), 0, 0, H(Ffu, Gv), H(Ffu, Gv)) < 0$$

which, from  $(\varphi_2)$ , gives  $H(Ffu, Gv) = 0$ . By (3), we obtain  $f^2u = fu$ . Since  $(f, F)$  and  $(g, G)$  have the same role, we have:  $Fu = Ggv$  and  $gv = g^2v$ . Therefore,  $ffu = fu = gv = ggv = gfu$ ,  $fu = f^2u \in fFu \subset Ffu$ , so  $fu \in Ffu$  and  $fu = gfu \in Gfu$ . Then  $fu$  is a common fixed point of  $f, g, F$  and  $G$ .

*ii)* Now, we show uniqueness of the common fixed point.

Put  $fu = w$  and let  $w'$  be another common fixed point of the four maps such that  $w \neq w'$ , then we have  $d(w, w') = d(fw, gw') \leq H(Fw, Gw')$ ; by (\*), we get

$$\begin{aligned} & \varphi(H(Fw, Gw'), d(fw, gw'), d(fw, Fw), \\ & \quad d(gw', Gw'), d(fw, Gw'), d(gw', Fw)) \\ &= \varphi(H(Fw, Gw'), d(fw, gw'), 0, 0, d(fw, Gw'), d(gw', Fw)) < 0. \end{aligned}$$

By  $(\varphi_1)$ , we get

$$\varphi(H(Fw, Gw'), H(Fw, Gw'), 0, 0, H(Fw, Gw'), H(Fw, Gw')) < 0.$$

So, by  $(\varphi_2)$ ,  $H(Fw, Gw') = 0$  and thus  $d(fw, gw') = d(w, w') = 0$ .  
iii) Let  $w \in Ffu$ . Using the triangle inequality and Property 1.2, we have

$$d(fu, w) \leq d(fu, Ffu) + H(Ffu, Gv) + d(w, Gv).$$

Since  $fu \in Ffu$  and  $H(Ffu, Gv) = 0$ ,  $d(w, fu) \leq d(w, Gv) \leq H(Ffu, Gv) = 0$ .  
So  $w = fu$  and  $Ffu = \{fu\} = \{gv\} = Ggv$ , this last equality is given thanks to the same role of  $F$  and  $G$ . ■

**2.3 Remark** Truly Theorem 2.2 is a generalization of corresponding theorems of [3], [4], [20]-[25] and others since we extended the setting of single-valued maps to the one of single and set-valued maps, also we deleted the compactness in [3],[23], we further add that we not required the continuity, although we used the strict contractive condition  $(*)$  which is substantially more general than the inequalities in the cited papers, and we weakened the concepts of compatibility, compatibility of type  $(A)$ , compatibility of type  $(C)$ , compatibility of type  $(P)$  and weak compatibility to the more general one say occasionally weak compatibility. Finally we deleted some assumptions of functions  $\varphi$  which are superfluous for us but are necessary in the papers [3], [4], [20]-[25].

If we let  $f = g$  and  $F = G$  in Theorem 2.2, we get the following corollary

**2.4 Corollary** Let  $f : \mathcal{X} \rightarrow \mathcal{X}$  and let  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  such that the pair  $\{f, F\}$  is owc. Let  $\varphi : \mathbb{R}^6 \rightarrow \mathbb{R}$  be a real map satisfying conditions  $(\varphi_1)$ ,  $(\varphi_2)$  and

$$\varphi(H(Fx, Fy), d(fx, fy), d(fx, Fx), d(fy, Fy), d(fx, Fy), d(fy, Fx)) < 0$$

for all  $x$  and  $y \in \mathcal{X}$  for which  $\max\{d(fx, fy), d(fx, Fx), d(fy, Fy)\} > 0$ , then  $f$  and  $F$  have a unique common fixed point.

Now, if we let  $f = g$ , we get the next result

**2.5 Corollary** Let  $f$  be a self-mapping of a metric space  $(\mathcal{X}, d)$  and let  $F, G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued mappings. Suppose pairs  $\{f, F\}$  and  $\{f, G\}$  are owc and  $\varphi : \mathbb{R}^6 \rightarrow \mathbb{R}$  satisfies conditions  $(\varphi_1)$ ,  $(\varphi_2)$  and

$$\varphi(H(Fx, Gy), d(fx, fy), d(fx, Fx), d(fy, Gy), d(fx, Gy), d(fy, Fx)) < 0$$

for all  $x$  and  $y \in \mathcal{X}$  for which  $\max\{d(fx, fy), d(fx, Fx), d(fy, Gy)\} > 0$ , then  $f, F$  and  $G$  have a unique common fixed point.

With different choices of the real map  $\varphi$ , we obtain the following corollaries.

**2.6 Corollary** If in the hypotheses of Theorem 2.2, we have instead of  $(*)$  one of the following inequalities, for all  $x$  and  $y \in \mathcal{X}$  whenever the right hand side of each inequality is not zero, then the four maps have a unique common fixed point.

$$(a) \quad H(Fx, Gy) < k \max\{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}$$

where  $0 < k \leq 1$

$$(b) \quad H^2(Fx, Gy) < c_1 \max \{d^2(fx, gy), d^2(fx, Fx), d^2(gy, Gy)\} + \\ + c_2 \max \{d(fx, Fx)d(fx, Gy), d(gy, Gy)d(gy, Fx)\} + c_3 d(fx, Gy)d(gy, Fx)$$

where  $c_1 > 0$ ,  $c_2, c_3 \geq 0$  and  $c_1 + c_3 \leq 1$ ,

$$(c) H(Fx, Gy) < [\alpha H^{p-1}(Fx, Gy)d(fx, gy) + \beta H^{p-2}(Fx, Gy)d(fx, Fx)d(gy, Gy) \\ + \gamma d^{p-1}(fx, Gy)d(gy, Fx) + \delta d(fx, Gy)d^{p-1}(gy, Fx)]^{\frac{1}{p}}$$

where  $\alpha > 0$ ,  $\beta, \gamma, \delta \geq 0$ ,  $\alpha + \gamma + \delta \leq 1$  and  $p \geq 2$ ,

$$(d) \quad H^2(Fx, Gy) < \frac{1}{\alpha} \left[ \beta d^2(fx, gy) + \frac{\gamma d(fx, Gy)d(gy, Fx)}{1 + \delta d^2(fx, Fx) + \epsilon d^2(gy, Gy)} \right]$$

where  $\alpha > 0$ ,  $\beta, \gamma, \delta, \epsilon \geq 0$  and  $\beta + \gamma \leq \alpha$ ,

$$(e) H(Fx, Gy) < [\alpha d^p(fx, gy) + \beta d^p(fx, Fx) + \gamma d^p(gy, Gy)]^{\frac{1}{p}} + \delta [d(fx, Gy)d(gy, Fx)]^{\frac{1}{2}}$$

where  $0 < \alpha \leq (1 - \delta)^p$ ,  $\beta, \gamma, \delta \geq 0$  and  $p \in \mathbb{N}^* = \{1, 2, \dots\}$ .

### Proof

For proof of (a), (b), (c), (d) and (e), we use Theorem 2.2 with the following functions  $\varphi$  which satisfy, for every case, hypothesis  $(\varphi_1)$  and  $(\varphi_2)$

for (a):

$$\varphi(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) = \\ = H(Fx, Gy) - k \max \{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}.$$

This function  $\varphi$  is used by many authors with single maps, for example: [12] in Theorem 1, Example 3.4 in [19].

For (b):

$$\varphi(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ = H^2(Fx, Gy) - c_1 \max \{d^2(fx, gy), d^2(fx, Fx), d^2(gy, Gy)\} \\ - c_2 \max \{d(fx, Fx)d(fx, Gy), d(gy, Gy)d(gy, Fx)\} \\ - c_3 d(fx, Gy)d(gy, Fx).$$

This function  $\varphi$  is Example 2 of [23].

For (c):

$$\varphi(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ = H(Fx, Gy) - [\alpha H^{p-1}(Fx, Gy)d(fx, gy) \\ + \beta H^{p-2}(Fx, Gy)d(fx, Fx)d(gy, Gy) \\ + \gamma d^{p-1}(fx, Gy)d(gy, Fx) + \delta d(fx, Gy)d^{p-1}(gy, Fx)]^{\frac{1}{p}}.$$



For  $p = 3$ , we have Example 3.4 de [4] and Example 3 of [24]. If we take  $p = 2$ ,  $\varphi$  is Example 1 of [21].

For (d):

$$\begin{aligned} & \varphi(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ = & H^2(Fx, Gy) - \frac{1}{\alpha} \left[ \beta d^2(fx, gy) + \frac{\gamma d(fx, Gy) d(gy, Fx)}{1 + \delta d^2(fx, Fx) + \epsilon d^2(gy, Gy)} \right]. \end{aligned}$$

This function  $\varphi$  is that one of Example 6 of [20].

And for (e):

$$\begin{aligned} & \varphi(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \\ = & H(Fx, Gy) - [\alpha d^p(fx, gy) + \beta d^p(fx, Fx) + \gamma d^p(gy, Gy)]^{\frac{1}{p}} \\ & - \delta [d(fx, Gy) d(gy, Fx)]^{\frac{1}{2}} \end{aligned}$$

■

**2.7 Corollary** *Let  $f, g$  be two self-maps of a metric space  $(\mathcal{X}, d)$  and let  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Suppose that, for all  $x, y \in \mathcal{X}$ , we have the inequality*

$$(f) \quad H^p(Fx, Gy) < \alpha d^p(fx, gy) + \beta d^p(fx, Fx) + \gamma d^p(gy, Gy)$$

*such that  $0 < \alpha \leq 1$ ,  $\beta$  and  $\gamma \geq 0$  and  $p \in \mathbb{N}^* = \{1, 2, \dots\}$  whenever the right hand side of the above inequality is positive. Then  $f, g, F$  and  $G$  have a unique common fixed point.*

**Proof**

We give this corollary because it is an interesting particular case of the previous corollary. We obtain the result by using (e) in Corollary 2.6 with  $\delta = 0$ . ■

### 3 A type Gregus fixed point theorem

As we already said, in 1998, Pathak et al. [18] gave an extension of compatibility of type (A) by introducing the concept of compatibility of type (C) and they proved a common fixed point theorem of Greguš type for four compatible maps of type (C) in a Banach space. Further, Djoudi and Nisse [6] extended the result of [18] by weakening compatibility of type (C) to the weak one without continuity.

Our objective here is to establish a common fixed point theorem for four occasionally weakly compatible single and set-valued mappings of Greguš type in a metric space which improves the results of [6], [18] and others.

**3.1 Theorem** *Let  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  be mappings,  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued mappings such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let*

$\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) < t$  and satisfying the following condition:

$$(\star) \quad H^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1-a)d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy)]$$

for all  $x$  and  $y \in \mathcal{X}$ , where  $0 < a \leq 1$  and  $p \geq 1$ .

Then  $f$ ,  $g$ ,  $F$  and  $G$  have a unique common fixed point which is a strict fixed point for  $F$  and  $G$ .

**Proof**

Since  $f, F$  and  $g, G$  are owc, as in proof of Theorem 2.2, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu$ ,  $gv \in Gv$ ,  $fFu \subseteq Ffu$ ,  $gGv \subseteq Ggv$ ,

$$(2) \quad d(fu, gv) \leq H(Fu, Gv)$$

and

$$(3) \quad d(f^2u, gv) \leq H(Ffu, Gv).$$

*i)* As in proof of Theorem 2.2, we begin to show existence of a common fixed point.

We have,

$$H^p(Fu, Gv) \leq \Psi(ad^p(fu, gv) + (1-a)d^{\frac{p}{2}}(gv, Fu)d^{\frac{p}{2}}(fu, Gv))$$

and by (2) and Property 1.2,

$$H^p(Fu, Gv) \leq \Psi(aH^p(Fu, Gv) + (1-a)H^p(Gv, Fu)) = \Psi(H^p(Fu, Gv)).$$

So, if  $H(Fu, Gv) > 0$ ,  $\Psi(t) < t$  for  $t > 0$ , we obtain

$$H^p(Fu, Gv) \leq \Psi(H^p(Fu, Gv)) < H^p(Fu, Gv)$$

which is a contradiction, thus we have  $H(Fu, Gv) = 0$ , hence  $fu = gv$ .

Again, if  $H(Ffu, Gv) > 0$ , then by (3) and  $(\star)$ , we have

$$\begin{aligned} H^p(Ffu, Gv) &\leq \Psi[ad^p(f^2u, gv) + (1-a)d^{\frac{p}{2}}(gv, Ffu)d^{\frac{p}{2}}(f^2u, Gv)] \\ &\leq \Psi[aH^p(Ffu, Gv) + (1-a)H^p(Ffu, Gv)] = \Psi(H^p(Ffu, Gv)). \end{aligned}$$

If  $H(Ffu, Gv) > 0$ , we obtain  $H^p(Ffu, Gv) \leq \Psi(H^p(Ffu, Gv)) < H^p(Ffu, Gv)$  what it is impossible. Then we have  $H(Ffu, Gv) = 0$ ,  $Ffu = Gv$  and  $f^2u = fu$ . Similarly, we can prove that  $g^2v = gv$ , let  $fu = w$  then,  $fw = w = gw$ ,  $w \in Fw$  and  $w \in Gw$ , this completes the proof of the existence.

*ii)* For the uniqueness, let  $w'$  be a second common fixed point of  $f, g, F$  and  $G$  with  $w' \neq w$ . Then,  $d(w, w') = d(fw, gw') \leq H(Fw, Gw')$  and, by assumption  $(\star)$ , we obtain

$$H^p(Fw, Gw') \leq \Psi[ad^p(fw, gw') + (1-a)d^{\frac{p}{2}}(fw, Gw')d^{\frac{p}{2}}(gw', Fw)]$$

$$\leq \Psi(H^p(Fw, Gw')).$$

If  $H(Fw, Gw') > 0$ , we have a contradiction. So,  $Fw = Gw'$ . Since  $w$  and  $w'$  are common fixed points of  $f, g, F$  and  $G$ , we have

$$d(fw, gw') \leq d(fw, Fw) + H(Fw, Gw') + d(gw', Gw') \leq H(Fw, Gw').$$

So,  $w = fw = gw' = w'$  and there exists a unique common fixed point of  $f, g, F$  and  $G$ .

iii) The proof that the fixed point of  $F$  and  $G$  is a strict fixed point is identical of that of Theorem 2.2.  $\blacksquare$

**3.2 Theorem** *Let  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  be mappings,  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued mappings such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) < t$  and satisfying the following condition:*

$$(\star) \quad H^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1 - a) \max\{\alpha d^p(fx, Fx), \beta d^p(gy, Gy),$$

$$d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(gy, Fx), d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy), \frac{1}{2}(d^p(fx, Fx) + d^p(gy, Gy))\}]$$

for all  $x$  and  $y \in \mathcal{X}$ , where  $0 < a \leq 1$ ,  $0 < \alpha, \beta \leq 1$  and  $p \geq 1$ .

Then  $f, g, F$  and  $G$  have a unique common fixed point which is a strict fixed point for  $F$  and  $G$ .

**Proof**

Since  $f, F$  and  $g, G$  are owc, as in proof of Theorem 2.2, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu, gGv \subseteq Ggv$ , (2) and (3). Since  $\Psi$  is a nondecreasing function and since for any real numbers  $c$  and  $d$ ,  $\frac{c+d}{2} \leq \max\{c, d\}$  we have, for all  $x, y \in \mathcal{X}$ ,

$$H^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1 - a) \max\{d^p(fx, Fx), d^p(gy, Gy),$$

$$d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(gy, Fx), d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy)\}]$$

and, for  $u$  and  $v$ ,

$$H^p(Fu, Gv) \leq \Psi[ad^p(fu, gv) + (1 - a)d^{\frac{p}{2}}(gv, Fu)d^{\frac{p}{2}}(fu, Gv)].$$

The continuation of the proof is identical of that of Theorem 3.1.  $\blacksquare$

**3.3 Remark** Obviously, Theorems 3.1 and 3.2 extend the results of [6], [18] and others to the class of four single and set-valued maps. In particular, Theorem 3.2 improves the cited results since we not required the closedness of the sets  $F(\mathcal{X})$  and  $G(\mathcal{X})$ , also we deleted the inclusions  $F(\mathcal{X}) \subset f(\mathcal{X})$  and  $G(\mathcal{X}) \subset g(\mathcal{X})$  in [6], we weakened the weakly compatibility in [6] and the compatibility of type (C) in [18] to the wider one cited occasionally weakly compatibility and we deleted the continuity which is indispensable in [18] and the upper semicontinuity imposed on  $\Psi$  in [6].

If we put  $f = g$  in Theorem 3.1, then we get the corollary

**3.4 Corollary** *Let  $f : \mathcal{X} \rightarrow \mathcal{X}$  be a map and let  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued mappings. Let  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) < t$ . Suppose that pairs  $\{f, F\}$  and  $\{f, G\}$  are owc and satisfy the inequality*

$$H^p(Fx, Gy) \leq \Psi[ad^p(fx, fy) + (1-a)d^{\frac{p}{2}}(fy, Fx)d^{\frac{p}{2}}(fx, Gy)]$$

for all  $x, y \in \mathcal{X}$ , where  $0 < a \leq 1$  and  $p \geq 1$ , then  $f, F$  and  $G$  have a unique common fixed point.

If we put  $f = g$  and  $F = G$  in Theorem 3.2, then we obtain the following result

**3.5 Corollary** *Let  $f : \mathcal{X} \rightarrow \mathcal{X}$  be a map and let  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued mapping such that  $f$  and  $F$  are owc . Let  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing map such that, for every  $t > 0$ ,  $\Psi(t) < t$ . If*

$$\begin{aligned} H^p(Fx, Fy) \leq & \Psi[ad^p(fx, fy) + (1-a) \max \{ \alpha d^p(fx, Fx), \beta d^p(fy, Fy), \\ & d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(fy, Fx), d^{\frac{p}{2}}(fy, Fx)d^{\frac{p}{2}}(fx, Fy), \\ & \frac{1}{2}(d^p(fx, Fx) + d^p(fy, Fy)) \}] \end{aligned}$$

for all  $x, y \in \mathcal{X}$ , where  $0 < a \leq 1$ ,  $\{\alpha, \beta\} \subset ]0, 1]$  and  $p \geq 1$ , then  $f$  and  $F$  have a unique common fixed point.

## 4 Another type fixed point theorem

Now, we end our work by establishing another result which improves those given by Elamrani and Mehdaoui [7], Mbarki [15] and others since our version does not impose continuity and we use occasionally weak compatibility which is more general than compatibility and weak compatibility; also we delete, on  $\Phi$ , some strong conditions which are necessary in papers [7] and [15] on a metric space instead of a complete metric space.

**4.1 Theorem** *Let  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  be mappings,  $F$  and  $G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be set-valued maps and  $\Phi$  be a nondecreasing function of  $[0, \infty[$  into itself such that  $\Phi(t) = 0$  iff  $t = 0$  and satisfying inequality*

$$\begin{aligned} (\otimes) \quad \Phi(H(Fx, Gy)) \leq & \alpha(d(fx, gy))\Phi(d(fx, gy)) + \\ & + \gamma(d(fx, gy)) \min\{\Phi(d(fx, Gy)), \Phi(d(gy, Fx))\} \end{aligned}$$

for all  $x, y \in \mathcal{X}$  and  $\alpha, \gamma : [0, \infty[ \rightarrow [0, 1[$  satisfying condition:

$$(4) \quad \alpha(t) + \gamma(t) < 1 \quad \forall t > 0.$$

If the pairs  $f, F$  and  $g, G$  are owc, then  $f, g, F$  and  $G$  have a unique common fixed point in  $\mathcal{X}$  which is a strict fixed point of  $F$  and  $G$ .

**Proof**

Since  $f, F$  and  $g, G$  are owc, as in proof of Theorem 2.2, there exist  $u, v$  in  $\mathcal{X}$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu, gGv \subseteq Ggv$  and inequalities (2) and (3) are satisfied.

i) First we prove that  $fu = gv$ . By  $(\otimes)$ , nondecrease of  $\Phi$ , Property 1.2 and (2), we have

$$\begin{aligned} \Phi(H(Fu, Gv)) &\leq \alpha(d(fu, gv))\Phi(d(fu, gv)) + \\ &\quad + \gamma(d(fu, gv)) \min \{ \Phi(d(fu, Gv)), \Phi(d(gv, Fu)) \} \\ &\leq [\alpha(d(fu, gv)) + \gamma(d(fu, gv))] \Phi(H(Fu, Gv)). \end{aligned}$$

If  $H(Fu, Gv) > 0$ , since  $\Phi(t) = 0$  iff  $t = 0$ , using inequality (4), we get  $\Phi(H(Fu, Gv)) \leq [\alpha(d(fu, gv)) + \gamma(d(fu, gv))] \Phi(H(Fu, Gv)) < \Phi(H(Fu, Gv))$  which is a contradiction. Hence  $H(Fu, Gv) = 0$  and thus  $fu = gv$ .

Now we claim that  $f^2u = fu$ . Suppose not, then, from (3),  $H(Gv, Ffu) > 0$  and since  $\Phi$  is nondecreasing and  $\Phi(t) = 0$  iff  $t = 0$ , the use of  $(\otimes)$  and inequality (4) gives

$$\begin{aligned} \Phi(H(Ffu, Gv)) &\leq \alpha(d(f^2u, gv))\Phi(d(f^2u, gv)) \\ &\quad + \gamma(d(f^2u, gv)) \min \{ \Phi(d(f^2u, Gv)), \Phi(d(gv, Ffu)) \} \\ &\leq [\alpha(d(f^2u, fu)) + \gamma(d(f^2u, fu))] \Phi(H(Ffu, Gv)) \\ &< \Phi(H(Ffu, Gv)) \end{aligned}$$

this contradiction implies that  $H(Ffu, Gv) = 0$  and hence  $f^2u = fu$ . Similarly, we can prove that  $g^2v = gv$ . So, if  $w = fu = gv$  therefore  $fw = w = gw$ ,  $w \in Fw$  and  $w \in Gw$ . Existence of a common fixed point is proved.

ii) Assume that there exists a second common fixed point  $w'$  of  $f, g, F$  and  $G$  such that  $w' \neq w$ . We have  $d(w, w') = d(fw, gw') \leq H(Fw, Gw')$ . If  $H(Fw, Gw') > 0$ , by inequality  $(\otimes)$  and properties of functions  $\Phi, \alpha$  and  $\gamma$ , we obtain

$$\begin{aligned} \Phi(H(Fw, Gw')) &\leq \alpha(d(fw, gw'))\Phi(d(fw, gw')) \\ &\quad + \gamma(d(fw, gw')) \min \{ \Phi(d(fw, Gw')), \Phi(d(gw', Fw)) \} \\ &\leq [\alpha(d(w, w')) + \gamma(d(w, w'))] \Phi(H(Fw, Gw')) \\ &< \Phi(H(Fw, Gw')) \end{aligned}$$

this contradiction implies that  $H(Fw, Gw') = 0$ , hence  $w' = w$ .

iii) This part of the proof is analogous of that of Theorem 2.2. ■

**4.2 Remark** Theorem 4.1 remains valid if we replace  $(\otimes)$  by the following inequality:

$$\Phi(H(Fx, Gy)) \leq \alpha(d(fx, gy))\Phi(d(fx, gy)) + \gamma(d(fx, gy)) \left[ \frac{\Phi(d(fx, Gy)) + \Phi(d(gy, Fx))}{2} \right].$$

If we put  $f = g$  with or not  $F = G$  in Theorem 4.1, then we obtain the following corollaries.

**4.3 Corollary** Let  $f : \mathcal{X} \rightarrow \mathcal{X}$  be a single map of a metric space  $(\mathcal{X}, d)$  and let  $F : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be a set-valued map such that  $f$  and  $F$  are owc and satisfy the inequality

$$\begin{aligned} \Phi(H(Fx, Fy)) &\leq \alpha(d(fx, fy))\Phi(d(fx, fy)) \\ &\quad + \gamma(d(fx, fy)) \min \{\Phi(d(fx, Fy)), \Phi(d(fy, Fx))\} \end{aligned}$$

for all  $x$  and  $y \in \mathcal{X}$ , where  $\Phi$ ,  $\alpha$  and  $\gamma$  are as in Theorem 4.1. Then,  $f$  and  $F$  have a unique common fixed point in  $\mathcal{X}$ .

**4.4 Corollary** Let  $f : \mathcal{X} \rightarrow \mathcal{X}$  be a single map of a metric space  $(\mathcal{X}, d)$  and let  $F, G : \mathcal{X} \rightarrow \mathcal{P}_{fb}(\mathcal{X})$  be two set-valued maps. If the pairs  $\{f, F\}$  and  $\{f, G\}$  are owc and

$$\begin{aligned} \Phi(H(Fx, Gy)) &\leq \alpha(d(fx, fy))\Phi(d(fx, fy)) \\ &\quad + \gamma(d(fx, fy)) \min \{\Phi(d(fx, Gy)), \Phi(d(fy, Fx))\} \end{aligned}$$

for all  $x$  and  $y \in \mathcal{X}$ , where  $\Phi$ ,  $\alpha$  and  $\gamma$  are as in Theorem 4.1. Then,  $f$ ,  $F$  and  $G$  have a unique common fixed point in  $\mathcal{X}$ .

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