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# The General Vector Addition System Reachability Problem by Presburger Inductive Invariants

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## Abstract

*The reachability problem for Vector Addition Systems (VAS) is a central problem of net theory. The general problem is known decidable by algorithms exclusively based on the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney decomposition. This decomposition is used in this paper to prove that Parikh images of languages accepted by VAS are semi-pseudo-linear; a class of sets that can be precisely over-approximated by sets definable in the Presburger arithmetic. We provide an application of this result; we prove that if a final configuration is not reachable from an initial one, there exists a Presburger inductive invariant proving this property. Since we can decide with any decision procedure for the Presburger arithmetic if formulas denote inductive invariants, we deduce that there exist checkable certificates of non-reachability. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by non-deterministically selecting finite sequences of actions and a second one that tries to prove the non-reachability by non-deterministically selecting Presburger formulas.*

## 1 Introduction

Vector Addition Systems (VAS) or equivalently Petri Nets are one of the most popular formal methods for the representation and the analysis of parallel processes [2]. The reachability problem is central since many computational problems (even outside the parallel processes) reduce to the reachability problem. Sacerdote and Tenney provided in [9] a partial proof of the decidability of this problem. The proof was completed in 1981 by Mayr [8] and simplified by Kosaraju [6] from [9, 8]. Ten years later [7], Lambert provided a more simplified version based on [6]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known non-primitive recursive. Nowadays, the exact complexity of

the reachability problem for VAS is still an open-problem. Even an elementary upper-bound complexity is open. In fact, the known general reachability algorithms are exclusively based on the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition.

In this paper, by using the KLMST decomposition we prove that Parikh images of languages accepted by VAS are semi-pseudo-linear, a class of sets that can be precisely over-approximated by Presburger sets, or equivalently by semi-linear sets [3]. We provide an application of this result; we prove that if a final configuration is not reachable from an initial one, there exists a Presburger inductive invariant proving this property. Since we can decide with any decision procedure for the Presburger arithmetic if Presburger formulas denote inductive invariants, we deduce that there exist checkable certificates of non-reachability. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by non-deterministically selecting finite sequences of actions and a second one that tries to prove the non-reachability by non-deterministically selecting Presburger formulas. Note [5] that in general, reachability sets are not definable in the Presburger arithmetic. Presburger inductive invariants are obtained by observing that reachability sets are semi-pseudo-linear.

*Outline of the paper:* In Section 2 we introduce the class of *Vector Addition Systems (VAS)*. In section 3 we introduce a dimension function used in the sequel. In Section 4 we provide properties satisfied by additive sub-monoïds of integer vectors. In section 5 we introduce the class of *semi-pseudo-linear sets*. In section 6 Parikh images of languages accepted by VAS are proved semi-pseudo-linear. Finally in section 7 we deduce that if a final configuration is not reachable from an initial one, there exists a Presburger inductive invariant proving this property.

## 2 Vector Addition Systems

In this section *Vector Addition Systems* are recalled.

*Some notations.* We denote by  $\mathbb{Q}, \mathbb{Q}_+, \mathbb{Z}, \mathbb{N}$  respectively the set of rational values, non-negative rational values, the set of integers and the set of non-negative integers. The *components* of a vector  $\mathbf{x} \in \mathbb{Q}^n$  are denoted by  $(\mathbf{x}[1], \dots, \mathbf{x}[n])$ . The sum  $\mathbf{x} + \mathbf{y}$  of two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^n$  is naturally defined component wise. An *alphabet* is a non-empty finite set  $\Sigma$ . The set of words over  $\Sigma$  are denoted by  $\Sigma^*$ . The empty word is denoted by  $\epsilon$ . The concatenation of two words  $\sigma_1$  and  $\sigma_2$  is simply denoted by  $\sigma_1\sigma_2$ . Moreover the concatenation of  $n \geq 1$  times a word  $\sigma$  is denoted by  $\sigma^n$ . We denote by  $\sigma^0$  the empty word. The number of occurrences of an element  $a \in \Sigma$  in a word  $\sigma \in \Sigma^*$  is denoted by  $|\sigma|_a$ . Functions in  $\Sigma \rightarrow \mathbb{Z}^n$  are called *displacement functions*. These functions are naturally extended to functions  $\Sigma^* \rightarrow \mathbb{Z}^n$  satisfying  $\delta(\epsilon) = \mathbf{0}$  and  $\delta(\sigma) = \sum_{i=1}^k \delta(a_i)$  for any word  $\sigma = a_1 \dots a_k$  of  $k \geq 1$  elements  $a_i \in \Sigma$ . A *Parikh image* of a language  $\mathcal{L} \subseteq \Sigma^*$  is a set  $X = \{(|\sigma|_{a_1}, \dots, |\sigma|_{a_n}) \mid \sigma \in \mathcal{L}\}$  where  $a_1, \dots, a_n$  is a finite sequence in  $\Sigma$ .

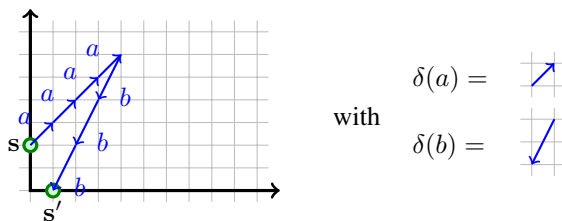


Figure 1. A Vector Addition System.

A *Vector Addition System (VAS)* is a tuple  $\mathcal{V} = (\Sigma, n, \delta)$  where  $\Sigma$  is an alphabet,  $n \in \mathbb{N}$  is the *dimension*, and  $\delta : \Sigma \rightarrow \mathbb{Z}^n$  is the *displacement function*. A *configuration* is a vector in  $\mathbb{N}^n$ . The binary relation  $\xrightarrow{a}_{\mathcal{V}}$  where  $a \in \Sigma$  over the set of configurations is defined by  $\mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'$  if and only if  $\mathbf{s}' = \mathbf{s} + \delta(a)$ . Let  $k \geq 1$ . Given a word  $\sigma = a_1 \dots a_k$  of elements  $a_i \in \Sigma$ , we denote by  $\xrightarrow{\sigma}_{\mathcal{V}}$  the concatenation  $\xrightarrow{a_1}_{\mathcal{V}} \dots \xrightarrow{a_k}_{\mathcal{V}}$ . We also denote by  $\xrightarrow{\epsilon}_{\mathcal{V}}$  the identity binary relation over the set of configurations. We also denote by  $\xrightarrow{*}_{\mathcal{V}}$  the *reachability binary relation* over the set of configurations defined by  $\mathbf{s} \xrightarrow{*}_{\mathcal{V}} \mathbf{s}'$  if and only if there exists  $\sigma \in \Sigma^*$  such that  $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'$ . The *reachability problem* for a tuple  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  where  $(\mathbf{s}, \mathbf{s}')$  are two configurations of a VAS  $\mathcal{V}$  consists to decide if  $\mathbf{s} \xrightarrow{*}_{\mathcal{V}} \mathbf{s}'$ . The *language accepted* by such a tuple  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  is the set  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}') = \{\sigma \in \Sigma^* \mid \mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'\}$ . Given two sets  $S, S'$  of configurations, the set  $\text{post}_{\mathcal{V}}^*(S)$  of *reachable configurations from S* and the set  $\text{pre}_{\mathcal{V}}^*(S')$  of *co-*

*reachable configurations from S'* are formally defined by:

$$\begin{aligned} \text{post}_{\mathcal{V}}^*(S) &= \{\mathbf{s}' \in \mathbb{N}^n \mid \exists \mathbf{s} \in S \quad \mathbf{s} \xrightarrow{*}_{\mathcal{V}} \mathbf{s}'\} \\ \text{pre}_{\mathcal{V}}^*(S') &= \{\mathbf{s} \in \mathbb{N}^n \mid \exists \mathbf{s}' \in S' \quad \mathbf{s} \xrightarrow{*}_{\mathcal{V}} \mathbf{s}'\} \end{aligned}$$

**Example 2.1** A VAS  $\mathcal{V} = (\Sigma, n, \delta)$  with  $\Sigma = \{a, b\}$ ,  $n = 2$ ,  $\delta(a) = (1, 1)$  and  $\delta(b) = (-1, -2)$  is depicted in Figure 1.

Observe that  $\mathbf{s} \xrightarrow{a^4 b^3}_{\mathcal{V}} \mathbf{s}'$  with  $\mathbf{s} = (0, 2)$  and  $\mathbf{s}' = (1, 0)$ . Note that  $\text{post}_{\mathcal{V}}^*(\{\mathbf{s}\}) = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[2] \leq \mathbf{x}[1] + 2\}$  and  $\text{pre}_{\mathcal{V}}^*(\{\mathbf{s}'\}) = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[2] \geq 2(\mathbf{x}[1] - 1)\}$ .

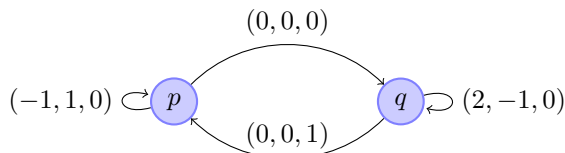


Figure 2. A VASS taken from [5].

**Example 2.2** Recall [5] that sets  $\text{post}_{\mathcal{V}}^*(S)$  and  $\text{pre}_{\mathcal{V}}^*(S')$  are definable in the Presburger arithmetic  $\text{FO}(\mathbb{N}, +, \leq)$  when  $S$  and  $S'$  are definable in this logic and  $n \leq 5$ . Moreover from [5] we deduce an example of 6-dim VAS  $\mathcal{V}$  and a pair of configurations  $(\mathbf{s}, \mathbf{s}')$  such that neither  $\text{post}_{\mathcal{V}}^*(\{\mathbf{s}\})$  nor  $\text{pre}_{\mathcal{V}}^*(\{\mathbf{s}'\})$  are definable in the Presburger arithmetic. This example is obtained by introducing the class of *Vector Addition Systems with States (VASS)*. Let us recall [5] that  $n$ -dim VASS can be simulated by  $(n + 3)$ -dim VAS. Informally, VASS are VAS equipped with control-flow graphs. Let us consider the VASS depicted at Figure 2. This VASS has a loop on state  $p$  and another loop on state  $q$ . Intuitively iterating loop on state  $p$  transfers the content of the first counter to the second counter whereas iterating the loop on state  $q$  transfers and multiply by two the content of the second counter to the first counter. The third counter is incremented each time we come back to state  $p$ . In [5] the set of reachable configurations from  $(p, (1, 0, 0))$  is proved equal to  $(\{p\} \times \{\mathbf{x} \in \mathbb{N}^3 \mid \mathbf{x}[1] + \mathbf{x}[2] \leq 2^{\mathbf{x}[3]}\}) \cup (\{q\} \times \{\mathbf{x} \in \mathbb{N}^3 \mid \mathbf{x}[1] + 2\mathbf{x}[2] \leq 2^{\mathbf{x}[3]+1}\})$ . From this VASS we easily deduce a 6-dim VAS  $\mathcal{V}$  and a pair of configuration  $(\mathbf{s}, \mathbf{s}')$  such that neither  $\text{post}_{\mathcal{V}}^*(\{\mathbf{s}\})$  nor  $\text{pre}_{\mathcal{V}}^*(\{\mathbf{s}'\})$  are definable in the Presburger arithmetic.

## 3 Dimension

In this section, we recall the classical (*mass*) *dimension function*. We associate to any set  $X \subseteq \mathbb{Z}^n$  the sequence  $(r_k)_{k \in \mathbb{N}}$  defined by the following equality (note that  $\ln(0) = -\infty$  by convention):

$$r_k = \frac{\ln(|X \cap \{-k, \dots, k\}^n|)}{\ln(2k + 1)}$$

Observe that  $r_k$  is either  $-\infty$  or a real value such that  $0 \leq r_k \leq n$ . We denote by  $\dim_L(X)$  and  $\dim_U(X)$  respectively the *limit-inf* and the *limit-sup* of  $(r_k)_{r \in \mathbb{N}}$ . In this paper we consider the dimension function  $\dim = \dim_L$ . Note that the other choice is also possible since the sets considered in this paper satisfy  $\dim_L(X) = \dim_U(X)$ .

Let us show some immediate properties satisfied by the dimension function. Observe that  $\dim(X) = -\infty$  if and only if  $X$  is empty. The dimension function is monotonic  $\dim(X_1) \leq \dim(X_2)$  for any  $X_1 \subseteq X_2$ . Moreover it satisfies  $\dim(X_1 \cup X_2) = \max\{\dim(X_1), \dim(X_2)\}$  and  $\dim(X_1 + X_2) \leq \dim(X_1) + \dim(X_2)$ . In particular  $\dim(\mathbf{v} + X) = \dim(X)$  for any  $\mathbf{v} \in \mathbb{Z}^n$ .

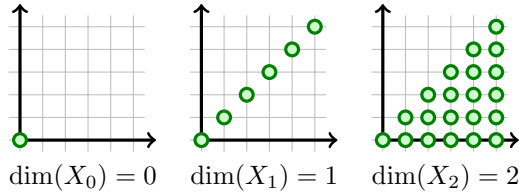


Figure 3. Dimension of some sets.

**Example 3.1** Let  $X_0 = \{(0,0)\}$ ,  $X_1 = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[1] = \mathbf{x}[2]\}$  and  $X_2 = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[2] \leq \mathbf{x}[1]\}$  be the sets depicted in Figure 3. As  $|X_0 \cap \{-k, \dots, k\}^2| = 1$ ,  $|X_1 \cap \{-k, \dots, k\}^2| = k + 1$ , and  $|X_2 \cap \{-k, \dots, k\}^2| = \frac{1}{2}(k + 1)(k + 2)$  we get  $\dim(X_0) = 0$ ,  $\dim(X_1) = 1$  and  $\dim(X_2) = 2$ .

## 4 Monoïds

A sub-monoïd of  $(\mathbb{Z}^n, +)$  is simply called a *monoïd* in the sequel. In this section we provide some properties satisfied by the monoïds.

*Additional notations.* Given a function  $f : E \rightarrow F$  where  $E, F$  are sets, we denote by  $f(X) = \{f(x) \mid x \in X\}$  for any subset  $X \subseteq E$ . This definition naturally defines sets  $X_1 + X_2$  where  $X_1, X_2 \subseteq \mathbb{Q}^n$ . With slight abuse of notation,  $\{\mathbf{x}_1\} + X_2$  and  $X_1 + \{\mathbf{x}_2\}$  are simply denoted by  $\mathbf{x}_1 + X_2$  and  $X_1 + \mathbf{x}_2$ . The total order  $\leq$  over  $\mathbb{Q}$  is extended component-wise to the partial order  $\leq$  over  $\mathbb{Q}^n$  satisfying  $\mathbf{x} \leq \mathbf{x}'$  if and only if  $\mathbf{x}[i] \leq \mathbf{x}'[i]$  for any  $1 \leq i \leq n$ . The set of minimal elements for  $\leq$  of a set  $X \subseteq \mathbb{N}^n$  is denoted by  $\min(X)$ . As  $(\mathbb{N}^n, \leq)$  is a *well partially ordered set*, note that  $\min(X)$  is finite and  $X \subseteq \min(X) + \mathbb{N}^n$  for any  $X \subseteq \mathbb{N}^n$ .

The monoïd generated by a set  $X \subseteq \mathbb{Z}^n$  is denoted by  $X^* = \{\mathbf{0}\} \cup \{\sum_{i=1}^k \mathbf{x}_i \mid k \geq 1 \mathbf{x}_i \in X\}$ . A finite set  $P \subseteq \mathbb{Z}^n$  is called a *set of periods*. A monoïd is said *finitely generated* if it can be generated by a set of periods.

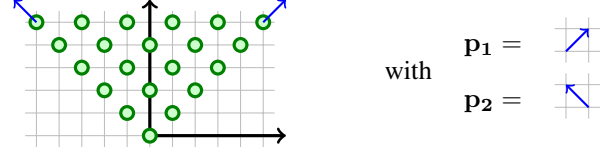


Figure 4. Monoïd  $P^*$  with  $P = \{\mathbf{p}_1, \mathbf{p}_2\}$ .

**Example 4.1** Figure 4 depicts the monoïd  $P^*$  generated by the set of periods  $P = \{\mathbf{p}_1, \mathbf{p}_2\}$  where  $\mathbf{p}_1 = (1,1)$  and  $\mathbf{p}_2 = (-1,1)$ .

### 4.1 Dimension

In this section, dimensions of monoïds are characterized by introducing the class of *vector spaces*.

A *vector space*  $V$  of  $\mathbb{Q}^n$  is a subset  $V \subseteq \mathbb{Q}^n$  that contains the zero vector  $\mathbf{0} \in V$ , that is stable by addition  $V + V \subseteq V$  and that is stable by product  $\lambda \mathbf{v} \in V$  for any  $\lambda \in \mathbb{Q}$  and for any  $\mathbf{v} \in V$ . Observe that for any set  $X \subseteq \mathbb{Q}^n$  the set  $V = \{\mathbf{0}\} \cup \{\sum_{i=1}^k \lambda_i \mathbf{x}_i \mid k \geq 1 \lambda_i \in \mathbb{Q} \mathbf{x}_i \in X\}$  is the unique minimal for the inclusion vector space that contains  $X$ . This vector space is called the vector space generated by  $X$ . Recall that for any vector space  $V$  of  $\mathbb{Q}^n$  there exists a finite set  $B \subseteq V$  that generates  $V$ . The minimal for  $\leq$  integer  $d \in \mathbb{N}$  such that there exists a finite set  $B$  that generated  $V$  is called the *rank* of  $V$  and it is denoted by  $\text{rank}(V)$ . Note that for any set  $X \subseteq \mathbb{Q}^n$  there exists a finite set  $B \subseteq X$  such that the vector space  $V$  generated by  $B$  is equal to the vector space generated by  $X$  and such that  $|B| = \text{rank}(V)$ .

**Proposition 4.2** We have  $\dim(M) = \text{rank}(V)$  where  $V$  is the vector space generated by a monoïd  $M$ .

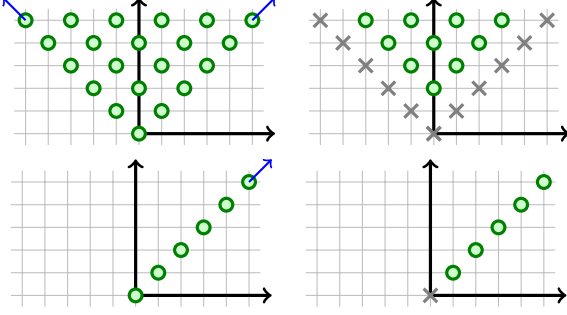
The previous Proposition 4.2 shows that  $\dim(M)$  is an integer for any monoïd  $M$ .

### 4.2 Interiors

We introduce *interiors of monoïds* in this section. This definition is used in the sequel to define the class of *semi-pseudo-linear sets*. The *interior of a monoïd*  $M$  is the set  $\mathcal{I}(M)$  of vectors  $\mathbf{a} \in M$  such that for any  $\mathbf{x} \in M$  there exists an integer  $N \geq 1$  such that  $N\mathbf{a} \in \mathbf{x} + M$ . We denote by  $\mathcal{I}(M)$  the *interior of*  $M$ .

**Example 4.3** Let us consider the vectors  $\mathbf{p}_1 = (1,1)$  and  $\mathbf{p}_2 = (-1,1)$  depicted in Figure 4. The monoïds  $\{\mathbf{p}_1, \mathbf{p}_2\}^*$  and  $\{\mathbf{p}_1\}^*$  and their interiors are depicted in Figure 5.

The following Lemma 4.4 characterizes the set  $\mathcal{I}(P^*)$  where  $P$  is a set of periods.



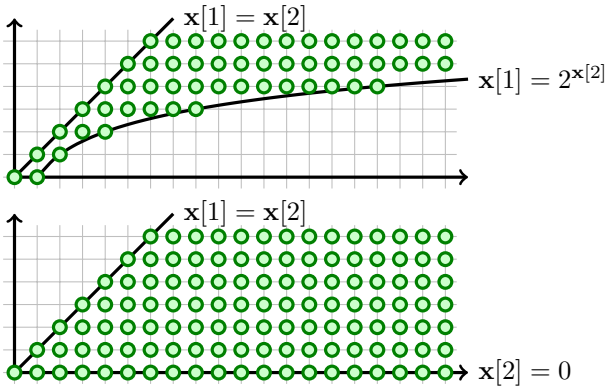
**Figure 5.** On left monoid  $M$ . On right its interior  $\mathcal{I}(M)$ .

**Lemma 4.4** Let  $P = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$  be a set of periods with  $k \in \mathbb{N}$ . We have  $\mathcal{I}(P^*) = \{\mathbf{0}\}$  if  $k = 0$  and  $\mathcal{I}(P^*) = P^* \cap ((\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_k)$  if  $k \geq 1$ .

## 5 Semi-pseudo-linear Sets

A set  $L \subseteq \mathbb{Z}^n$  is said *linear* [3] if there exists a vector  $\mathbf{b} \in \mathbb{Z}^n$  and a finitely generated monoid  $M \subseteq \mathbb{Z}^n$  such that  $L = \mathbf{b} + M$ . A *semi-linear set*  $S \subseteq \mathbb{Z}^n$  is a finite union of linear sets  $L_i \subseteq \mathbb{Z}^n$ . Recall [3] that sets definable in  $\text{FO}(\mathbb{Z}, +, \leq)$  are exactly the semi-linear sets and sets definable in  $\text{FO}(\mathbb{N}, +, \leq)$  also called *Presburger sets* are exactly the non-negative semi-linear sets.

We introduce in this section the class of *pseudo-linear sets* and *semi-pseudo-linear sets*. Intuitively, pseudo-linear sets are sets that can be precisely over-approximated by *linear sets*, and a semi-pseudo-linear sets are finite unions of pseudo-linear sets. We provides properties satisfied by these sets in Sections 5.2 and 5.3.



**Figure 6.** On top a pseudo-linear set, on bottom a linearization.

A set  $X \subseteq \mathbb{Z}^n$  is said *pseudo-linear* if there exists  $\mathbf{b} \in \mathbb{Z}^n$  and a finitely generated monoid  $M \subseteq \mathbb{Z}^n$  such that  $X \subseteq \mathbf{b} + M$  and such that for any finite set  $R$  of interior vectors of  $M$ , there exists  $\mathbf{x} \in X$  such that  $\mathbf{x} + R^* \subseteq X$ . In this case,  $M$  is called a *linearizator* for  $X$  and the linear set  $L = \mathbf{b} + M$  is called a *linearization* of  $X$ . A *semi-pseudo-linear set* is a finite union of *pseudo-linear sets*.

**Example 5.1** The set  $X = \{\mathbf{x} \in \mathbb{Z}^2 \mid 0 \leq \mathbf{x}[2] \leq \mathbf{x}[1] \leq 2^{\mathbf{x}[2]}\}$  is depicted in Figure 6. Observe that  $X$  is included in the linear set  $L = \mathbf{0} + M$  where  $M = \{\mathbf{x} \in \mathbb{Z}^2 \mid 0 \leq \mathbf{x}[2] \leq \mathbf{x}[1]\}$ . Note that  $M$  is the monoid generated by  $P = \{(1, 1), (1, 0)\}$ . The interior of  $M$  is equal to  $\mathcal{I}(M) = \{\mathbf{x} \in \mathbb{Z}^2 \mid 0 < \mathbf{x}[2] < \mathbf{x}[1]\}$ . In particular for any finite set  $R \subseteq \mathcal{I}(M)$  there exists  $\mathbf{x} \in X$  such that  $\mathbf{x} + R^* \subseteq X$ . We deduce that  $X$  is pseudo-linear,  $M$  is a linearizator for  $X$  and  $L$  is a linearization of  $X$ .

**Remark 5.2** Any linear set  $L = \mathbf{b} + M$  is pseudo-linear:  $M$  is a linearizator for  $L$  and  $L$  is a linearization of  $L$ .

**Remark 5.3** A pseudo-linear set  $X$  is non empty. Let  $M$  be a linearizator for  $X$ , Lemma 4.4 shows that  $\mathcal{I}(M)$  is non empty. In particular there exists a vector  $\mathbf{a} \in \mathcal{I}(M)$ . As  $X$  is pseudo-linear there exists  $\mathbf{x} \in X$  such that  $\mathbf{x} + \{\mathbf{a}\}^* \subseteq X$ . Therefore  $X$  is non-empty.

### 5.1 Dimension

From Proposition 4.2 we deduce that dimensions of semi-linear sets are integral values. As expected, the dimension of a pseudo-linear set is equal to the dimension of any linearization.

**Lemma 5.4** We have  $\dim(X) = \dim(L)$  for any linearization  $L$  of a pseudo-linear set  $X \subseteq \mathbb{Z}^n$ .

### 5.2 Pseudo-linear set images

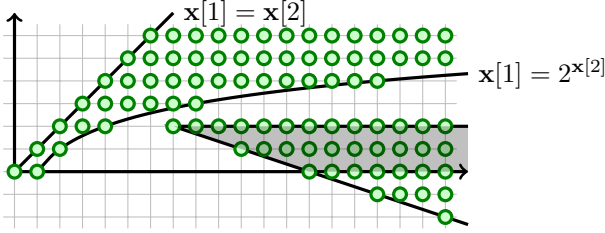
In this section, the class of pseudo-linear sets is proved stable by linear function images. A function  $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n'}$  is said *linear* if there exists a matrix  $A \in \mathbb{Z}^{n' \times n}$  and a vector  $\mathbf{v} \in \mathbb{Z}^{n'}$  such that  $f(\mathbf{x}) = A\mathbf{x} + \mathbf{v}$  for any  $\mathbf{x} \in \mathbb{Z}^n$ .

**Proposition 5.5** Images  $X' = f(X)$  of pseudo-linear sets  $X$  by a linear function  $f$  are pseudo-linear. Moreover the linear set  $L' = f(L)$  is a linearization of  $X'$  for any linearization  $L$  of  $X$ .

### 5.3 Pseudo-linear intersections

In this section we prove that linearizations  $L_1, L_2$  of two pseudo-linear sets  $X_1, X_2$  with an empty intersection

$X_1 \cap X_2 = \emptyset$  satisfy the strict inequality  $\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$ . Note that even if  $X_1 \cap X_2 = \emptyset$ , the intersection  $L_1 \cap L_2$  may be non empty since  $L_1, L_2$  are over-approximation of  $X_1, X_2$ .



**Figure 7. Two pseudo-linear sets with an empty intersection.**

**Example 5.6** Let us consider the pseudo-linear set  $X$  described in Example 5.1 and a linearization  $L = \{(1, 1), (1, 0)\}^*$  of  $X$ . These two sets are depicted in Figure 6. We also consider the linear set  $X' = (8, 2) + \{(1, 0), (3, -1)\}^*$ . Sets  $X$  and  $X'$  are depicted together in Figure 7. Note that  $L' = X'$  is a linearization of the linear set  $X'$ . Observe that  $X \cap X' = \emptyset$  and  $L \cap L' \neq \emptyset$ . The set  $L \cap L'$  is depicted in gray in Figure 7. Observe that  $L \cap L' = \{(8, 2), (11, 1), (14, 0)\} + \{(1, 0)\}^*$ . Therefore  $\dim(L \cap L') = 1$ . Since  $\dim(X) = \dim(X') = 2$  we have  $\dim(L \cap L') < \dim(X \cup X')$ .

We say that two linear sets  $L_1, L_2$  have a *non-degenerate intersection* if  $\dim(L_1) = \dim(L_1 \cap L_2) = \dim(L_2)$ .

**Lemma 5.7** Let  $L_1 = \mathbf{b}_1 + M_1$  and  $L_2 = \mathbf{b}_2 + M_2$  be two linear sets with a non-degenerate intersection. There exists finite sets  $R_1 \subseteq \mathcal{I}(M_1)$  and  $R_2 \subseteq \mathcal{I}(M_2)$  such that  $(\mathbf{x}_1 + R_1^*) \cap (\mathbf{x}_2 + R_2^*) \neq \emptyset$  for any  $(\mathbf{x}_1, \mathbf{x}_2) \in L_1 \times L_2$ .

**Proposition 5.8** Let  $L_1, L_2$  be linearizations of pseudo-linear sets  $X_1, X_2 \subseteq \mathbb{Z}^n$  with an empty intersection  $X_1 \cap X_2 = \emptyset$ . We have:

$$\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$$

*Proof:* Let us consider linearizations  $L_1, L_2$  of two pseudo-linear sets  $X_1, X_2$  such that  $\dim(L_1 \cap L_2) \geq \dim(X_1 \cup X_2)$  and let us prove that  $X_1 \cap X_2 \neq \emptyset$ .

We first show that  $L_1, L_2$  have a non-degenerate intersection. Lemma 5.4 shows that  $\dim(X_1) = \dim(L_1)$  and  $\dim(X_2) = \dim(L_2)$ . Observe that  $\dim(X_1 \cup X_2) \geq \dim(X_1)$  since  $X_1 \subseteq X_1 \cup X_2$ . From  $\dim(X_1) = \dim(L_1)$  and  $\dim(L_1 \cap L_2) \geq \dim(X_1 \cup X_2)$  we deduce that  $\dim(L_1 \cap L_2) \geq \dim(L_1)$ . As  $L_1 \cap L_2 \subseteq L_1$  we also

have  $\dim(L_1 \cap L_2) \leq \dim(L_1)$ . Therefore  $\dim(L_1 \cap L_2) = \dim(L_1)$ . Symmetrically we deduce the equality  $\dim(L_1 \cap L_2) = \dim(L_2)$ .

As  $L_1, L_2$  are two linear sets, there exists  $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{Z}^n$  and two finitely generated monoids  $M_1, M_2$  such that  $L_1 = \mathbf{b}_1 + M_1$  and  $L_2 = \mathbf{b}_2 + M_2$ . As the linear sets  $L_1, L_2$  have a non-degenerate intersection, Lemma 5.7 shows that there exist finite sets  $(R_1, R_2) \subseteq \mathcal{I}(M_1) \times \mathcal{I}(M_2)$  such that  $(\mathbf{x}_1 + R_1^*) \cap (\mathbf{x}_2 + R_2^*) \neq \emptyset$  for any  $(\mathbf{x}_1, \mathbf{x}_2) \in L_1 \times L_2$ . As  $L_1, L_2$  are linearizations of the pseudo-linear sets  $X_1, X_2$  there exists  $(\mathbf{x}_1, \mathbf{x}_2) \in X_1 \times X_2$  such that  $\mathbf{x}_1 + R_1^* \subseteq X_1$  and  $\mathbf{x}_2 + R_2^* \subseteq X_2$ . From  $(X_1, X_2) \subseteq (L_1, L_2)$  we deduce that  $(\mathbf{x}_1, \mathbf{x}_2) \in L_1 \times L_2$ . In particular  $(\mathbf{x}_1 + R_1^*) \cap (\mathbf{x}_2 + R_2^*) \neq \emptyset$ . We have proved that  $X_1 \cap X_2 \neq \emptyset$ .  $\square$

## 6 Parikh Images

Let  $\mathcal{V}$  be a VAS and let  $\mathbf{s}, \mathbf{s}'$  be two configurations. In this section, we prove that Parikh images of  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  are semi-pseudo-linear. In sub-section 6.1 we recall the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition. This decomposition is used in the next sub-section 6.2 to establish the semi-pseudo-linearity of Parikh images of  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ .

### 6.1 The KLMST decomposition

We recall the KLMST decomposition by following notations introduced by Lambert [7].

We first extend the set of integers  $\mathbb{Z}$  with an additional element  $\top$ . The addition function  $+$  :  $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  is extended to the totally-defined function in  $(\mathbb{Z} \cup \{\top\}) \times (\mathbb{Z} \cup \{\top\}) \rightarrow (\mathbb{Z} \cup \{\top\})$  satisfying  $x_1 + x_2 = \top$  if  $x_1 = \top$  or  $x_2 = \top$ . With slight abuse of notation we denote by  $\top - x$  the element  $\top$  when  $x \in \mathbb{Z}$ .

In the sequel, the element  $\top$  is either interpreted as a “very large integer” or a “don’t care integer”. More formally, we denote by  $\mathbb{N}_\top$  the set  $\mathbb{N} \cup \{\top\}$ . The total order  $\leq$  over  $\mathbb{N}$  is extended over  $\mathbb{N}_\top$  by  $x_1 \leq x_2$  if and only if  $x_2 = \top \vee (x_1, x_2 \in \mathbb{N} \wedge x_1 \leq x_2)$ . The equality  $=$  over  $\mathbb{N}$  is also extended to a partial order  $\preceq$  over  $\mathbb{N}_\top$  by  $x_1 \preceq x_2$  if and only if  $x_2 = \top \vee (x_1, x_2 \in \mathbb{N} \wedge x_1 = x_2)$ . Intuitively element  $\top$  denotes a “very large integer” for the total order  $\leq$  whereas it denotes a “don’t care integer” for the partial order  $\preceq$ .

We also extend the semantics of VAS. A vector in  $\mathbb{N}_\top^n$  is called an *extended configuration* of  $\mathcal{V}$ . With slight abuse of notation, the binary relation  $\xrightarrow{a}_{\mathcal{V}}$  where  $a \in \Sigma$  is extended over the set of extended configurations by  $\mathbf{x} \xrightarrow{a}_{\mathcal{V}} \mathbf{x}'$  if and only if  $\mathbf{x}' = \mathbf{x} + \delta(a)$ . Let  $k \geq 1$ . Given a word  $\sigma = a_1 \dots a_k$  of  $k$  elements  $a_i \in \Sigma$ , we denote by  $\xrightarrow{\sigma}_{\mathcal{V}}$

the concatenation  $\xrightarrow{a_1} \mathcal{V} \cdots \xrightarrow{a_k} \mathcal{V}$ . We also denote by  $\xrightarrow{\epsilon} \mathcal{V}$  the identity binary relation over the set of extended configurations. Given an extended configuration  $\mathbf{x}$  we denote by  $\mathbf{x} \xrightarrow{\sigma} \mathcal{V}$  if there exists an extended configuration  $\mathbf{x}'$  such that  $\mathbf{x} \xrightarrow{\sigma} \mathcal{V} \mathbf{x}'$  and symmetrically for any extended configuration  $\mathbf{x}'$  we denote by  $\xrightarrow{\sigma} \mathcal{V} \mathbf{x}'$  if there exists an extended configuration  $\mathbf{x}$  such that  $\mathbf{x} \xrightarrow{\sigma} \mathcal{V} \mathbf{x}'$ .

*Additional notations.* A graph  $G$  is a tuple  $G = (Q, \Sigma, T)$  where  $Q$  is a non-empty finite set of *state*,  $\Sigma$  is an alphabet, and  $T \subseteq Q \times \Sigma \times Q$  is a finite set of *transitions*. Given a state  $q \in Q$  we denote by  $\langle q, T \rangle$  the set of transitions  $T \cap (\{q\} \times \Sigma \times Q)$ . Symmetrically, given a state  $q' \in Q$  we denote by  $\langle T, q' \rangle$  the set of transitions  $T \cap (Q \times \Sigma \times \{q'\})$ . A *path*  $\pi$  is a word  $\pi = t_1 \dots t_k$  of  $k \in \mathbb{N}$  transitions  $t_i \in T$  such that there exists  $q_0, \dots, q_k \in Q$  and  $a_1, \dots, a_k \in \Sigma$  such that  $t_i = (q_{i-1}, a_i, q_i)$  for any  $1 \leq i \leq k$ . In this case we say that  $\pi$  is a *path from  $q_0$  to  $q_k$  labelled by  $\sigma = a_1 \dots a_k$*  and we denote  $\pi$  by  $q_0 \xrightarrow{\sigma} G q_k$  or simply  $q_0 \xrightarrow{*} G q_k$ . When the states  $q_0$  and  $q_k$  are equal, the path  $\pi$  is called a *cycle on this state*. In this following we simply denote by  $|\pi|$  the sequence  $(|\pi|_t)_{t \in T}$  that counts the number of occurrences of transitions  $t \in T$  in  $\pi$ .

A *graph vector*  $G = (Q, \Sigma, T)$  for  $\mathcal{V}$  is a strongly connected graph such that  $Q \subseteq \mathbb{N}^n$  is a non-empty finite set of extended configurations, and  $T \subseteq Q \times \Sigma \times Q$  is a finite set of transitions  $(q, a, q')$  such that  $q \xrightarrow{a} \mathcal{V} q'$ . Note that  $q, q'$  are two extended configurations in the previous relation.

A *marked graph vector*  $\mathcal{M} = (\mathbf{m}, \mathbf{x}, G, \mathbf{x}', \mathbf{m}')$  for  $\mathcal{V}$  is a tuple such that  $G = (Q, \Sigma, T)$  is a graph vector for  $\mathcal{V}$ , such that  $\mathbf{x}, \mathbf{x}'$  are two states in  $Q$  satisfying  $\mathbf{x} = \mathbf{x}'$  (we use both notations  $\mathbf{x}$  and  $\mathbf{x}'$  for the same element in order to keep results symmetrical), and such that  $\mathbf{m}, \mathbf{m}'$  are two extended configurations satisfying  $\mathbf{m} \preceq \mathbf{x}, \mathbf{m}' \preceq \mathbf{x}'$  and both the following symmetrical conditions (i) and (i'):

- (i) there exists a cycle  $\theta = (\mathbf{x} \xrightarrow{w} G \mathbf{x})$  such that  $\mathbf{m} \xrightarrow{w} \mathcal{V}$  and such that  $\mathbf{m} + \delta(w) \geq \mathbf{m}$  and  $\delta(w)[i] > 0$  if  $\mathbf{m}[i] \in \mathbb{N}$  and  $\mathbf{x}[i] = \top$ , and
- (i') there exists a cycle  $\theta' = (\mathbf{x}' \xrightarrow{w'} G \mathbf{x}')$  such that  $\mathbf{m}' \xrightarrow{w'} \mathcal{V}$  and such that  $\mathbf{m}' - \delta(w') \geq \mathbf{m}'$  and  $-\delta(w')[i] > 0$  if  $\mathbf{m}'[i] \in \mathbb{N}$  and  $\mathbf{x}'[i] = \top$ .

A *marked graph vector sequences (MGVS)*  $\mathcal{U}$  for  $(s, \mathcal{V}, s')$  is a sequence  $\mathcal{U} = \mathcal{M}_0 a_1 \mathcal{M}_1 \dots a_k \mathcal{M}_k$  that alternates elements  $a_j \in \Sigma$  and marked graph vectors  $\mathcal{M}_j = (\mathbf{m}_j, \mathbf{x}_j, G_j, \mathbf{x}'_j, \mathbf{m}'_j)$  such that  $\mathbf{m}_0 = s$  and  $\mathbf{m}'_k = s'$ . In the sequel we denote by  $\theta_j = (\mathbf{x}_j \xrightarrow{w_j} G_j \mathbf{x}'_j)$  and  $\theta'_j = (\mathbf{x}'_j \xrightarrow{w'_j} G_j \mathbf{x}_j)$  cycles satisfying conditions (i) and (i') for each marked graph vector  $\mathcal{M}_j$ .

The *language accepted* by a MGVS  $\mathcal{U}$  is the set  $\mathcal{L}(\mathcal{U})$  of words of the form  $\sigma = \sigma_0 a_1 \sigma_1 \dots a_k \sigma_k$  such that there

exists a cycle  $\pi_j = (\mathbf{x}_j \xrightarrow{\sigma_j} G_j \mathbf{x}'_j)$  and such that there exist sequences  $(s_j, s'_j)_j$  of configurations pair such that  $s_j \preceq \mathbf{m}_j$  and  $s'_j \preceq \mathbf{m}'_j$  for any  $0 \leq j \leq k$  and such that :

$$s_0 \xrightarrow{\sigma_0} \mathcal{V} s'_0 \xrightarrow{a_1} \mathcal{V} s_1 \xrightarrow{\sigma_1} \mathcal{V} s'_1 \dots s'_{k-1} \xrightarrow{a_k} \mathcal{V} s_k \xrightarrow{\sigma_k} \mathcal{V} s'_k$$

Observe that  $\mathcal{L}(\mathcal{U}) \subseteq \mathcal{L}(s, \mathcal{V}, s')$  since  $(s_0, s'_k) = (s, s')$ .

Now, we associate integers sequences to accepted words and we show that they are solutions of a linear system. We still use notations introduced in the previous paragraph. We denote by  $(\mu_{j,t})_t$  the sequence  $|\pi_j|$ . The sequence  $\xi = (s_j, (\mu_{j,t})_t, s'_j)_j$  indexed by  $0 \leq j \leq k$  is said *associated* to  $\sigma$ . We observe that  $\xi$  is a non-negative integral solution of the linear system given in Figure 8 where  $\delta(t)$  denotes  $\delta(a)$  for any transition  $t = (q, a, q')$ . This linear system is called the *characteristic system* of  $\mathcal{U}$ . The homogeneous form of the characteristic system, obtained by replacing constant terms by zero is called the *homogeneous characteristic system* of  $\mathcal{U}$ . In the sequel, a solution of the homogeneous characteristic system is denoted by  $\xi_0 = (s_{0,j}, (\mu_{0,j,t})_t, s_{0,j}')_j$ . The *homogeneous characteristic system* of  $\mathcal{U}$  is also given in Figure 8.

$\left\{ \begin{array}{l} \text{for all } 1 \leq j \leq k \\ s'_{j-1} + \delta(a_j) = s_j \\ \\ \text{for all } 0 \leq j \leq k \\ s_j + \sum_{t \in T_j} \mu_{j,t} \delta(t) = s'_j \\ \\ \text{for all } 0 \leq j \leq k, 1 \leq i \leq n \\ s_j[i] = \mathbf{m}_j[i] \text{ if } \mathbf{m}_j[i] \in \mathbb{N} \\ s'_j[i] = \mathbf{m}'_j[i] \text{ if } \mathbf{m}'_j[i] \in \mathbb{N} \\ \\ \text{for all } 0 \leq j \leq k, q_j \in Q_j \\ \sum_{t \in \langle T, q_j \rangle} \mu_{j,t} = \\ \sum_{t \in \langle q_j, T \rangle} \mu_{j,t} \end{array} \right.$	$\left\{ \begin{array}{l} \text{for all } 1 \leq j \leq k \\ s'_{0,j-1} = s_{0,j} \\ \\ \text{for all } 0 \leq j \leq k \\ s_{0,j} + \sum_{t \in T_j} \mu_{0,j,t} \delta(t) = s'_{0,j} \\ \\ \text{for all } 0 \leq j \leq k, 1 \leq i \leq n \\ s_{0,j}[i] = 0 \text{ if } \mathbf{m}_j[i] \in \mathbb{N} \\ s'_{0,j}[i] = 0 \text{ if } \mathbf{m}'_j[i] \in \mathbb{N} \\ \\ \text{for all } 0 \leq j \leq k, q_j \in Q_j \\ \sum_{t \in \langle T, q_j \rangle} \mu_{0,j,t} = \\ \sum_{t \in \langle q_j, T \rangle} \mu_{0,j,t} \end{array} \right.$
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**Figure 8. On left the characteristic system. On right the homogenous characteristic system.**

Naturally there exists non-negative integral solutions  $\xi$  of the characteristic system that are not associated to accepted words. In particular even if there exists non-negative integral solutions of the characteristic linear system we cannot conclude that  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ . However, under the following *perfect condition* [7], we can prove that  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ .

A MGVS  $\mathcal{U}$  is said *perfect* [7] (equivalent to the  $\theta$ -condition of [6]) if its characteristic system has an inte-

gral solution and if there exists a non-negative rational solution  $(\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}_{0,j}')_j$  of its homogeneous characteristic system satisfying the following additional strict inequalities for any  $0 \leq j \leq k$  and  $1 \leq i \leq n$ :

- $\mathbf{s}_{0,j}[i] > 0$  if  $\mathbf{m}_j[i] = \top$ , and
- $\mathbf{s}'_{0,j}[i] > 0$  if  $\mathbf{m}'_j[i] = \top$ , and
- $\mu_{0,j,t} > 0$  for any  $t \in T_j$ .

Let us recall without proof the fundamental decomposition theorem.

**Theorem 6.1 (Fundamental Decomposition[7])** *For any tuple  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ , we can effectively compute a finite sequence of perfect MGVS  $\mathcal{U}_1, \dots, \mathcal{U}_l$  for  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  such that:*

$$\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}') = \mathcal{L}(\mathcal{U}_1) \cup \dots \cup \mathcal{L}(\mathcal{U}_l)$$

In the remaining of this section, we assume that  $\mathcal{U}$  is a perfect MGVS. We show that there exists a non-negative integral solution  $\xi$  of the characteristic system and a non-negative integral solution  $\xi_0$  of the homogeneous characteristic system that explains why  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ . These two solutions  $\xi$  and  $\xi_0$  are respectively defined in Lemma 6.2 and Lemma 6.3.

**Lemma 6.2** *There exist a sequence  $(\pi_j)_j$  of cycles  $\pi_j = (\mathbf{x}_j \xrightarrow{\sigma_j}_{G_j} \mathbf{x}'_j)$  and a sequence  $(\mathbf{s}_j, \mathbf{s}'_j)_j$  of configurations pairs such that  $\xi = (\mathbf{s}_j, |\pi_j|, \mathbf{s}'_j)_j$  is solution of the characteristic system of  $\mathcal{U}$  and such that  $\mathbf{s}_j \xrightarrow{w_j}_{\mathcal{V}}$  and  $\xrightarrow{w'_j}_{\mathcal{V}}$   $\mathbf{s}'_j$  for any  $0 \leq j \leq k$ .*

**Lemma 6.3** *There exist a sequence  $(\pi_{0,j})_j$  of cycles  $\pi_{0,j} = (\mathbf{x}_j \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x}'_j)$  and a sequence  $(\mathbf{s}_{0,j}, \mathbf{s}'_{0,j})_j$  of configurations pairs such that  $\xi_0 = (\mathbf{s}_{0,j}, |\pi_{0,j}| + |\theta_j|, \mathbf{s}'_{0,j})_j$  is a solution of the homogeneous characteristic system, such that  $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{0}$  and  $\mathbf{s}'_{0,j} - \delta(w'_j) \geq \mathbf{0}$  for any  $0 \leq j \leq k$ , and such that for any  $1 \leq i \leq n$  and for any  $0 \leq j \leq k$ :*

- $\mathbf{s}_{0,j}[i] > 0$  if and only if  $\mathbf{m}_j[i] = \top$ .
- $\mathbf{s}'_{0,j}[i] > 0$  if and only if  $\mathbf{m}'_j[i] = \top$ .
- $(\mathbf{s}_{0,j} + \delta(w_j))[i] > 0$  if and only if  $\mathbf{x}_j[i] = \top$ .
- $(\mathbf{s}'_{0,j} - \delta(w'_j))[i] > 0$  if and only if  $\mathbf{x}'_j[i] = \top$ .

Let us fix notations satisfying both Lemma 6.2 and Lemma 6.3. We now provide technical lemmas that prove together that  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ . These lemmas are also used in the next sub-section 6.2.

**Lemma 6.4** *For any  $c \geq 0$  we have:*

$$\begin{aligned} \mathbf{s}_j + c\mathbf{s}_{0,j} &\xrightarrow{w_j^c}_{\mathcal{V}} \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) \\ \mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)) &\xrightarrow{(w'_j)^c}_{\mathcal{V}} \mathbf{s}'_j + c\mathbf{s}'_{0,j} \end{aligned}$$

**Lemma 6.5** *There exists  $c_0 \geq 0$  such that for any  $c \geq c_0$ :*

$$\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) \xrightarrow{\sigma_{0,j}^c}_{\mathcal{V}} \mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$$

**Lemma 6.6** *There exists  $c' \geq 0$  such that for any  $c \geq c'$ :*

$$\mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)) \xrightarrow{\sigma_j}_{\mathcal{V}} \mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$$

Now, let us consider an integer  $c \geq 0$  satisfying  $c \geq c_0$  and  $c \geq c'$  where  $c_0$  and  $c'$  are respectively defined by Lemma 6.5 and Lemma 6.6. Note that we have proved the following relation:

$$\mathbf{s}_j + c\mathbf{s}_{0,j} \xrightarrow{w_j^c \sigma_{0,j}^c \sigma_j (w'_j)^c}_{\mathcal{V}} \mathbf{s}'_j + c\mathbf{s}'_{0,j}$$

Therefore there exists a word in  $\mathcal{L}(\mathcal{U})$  associated to  $\xi + c\xi_0$ . In particular we have proved that  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ .

## 6.2 Parikh images of perfect MGVS

Parikh images of  $\mathcal{L}(\mathcal{U})$  are proved pseudo-linear for any perfect MGVS  $\mathcal{U}$  for  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ . From Theorem 6.1 we deduce that Parikh images of  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  are semi-pseudo-linear.

Let us consider a perfect MGVS  $\mathcal{U}$  for  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ :

$$(\mathbf{m}_0, \mathbf{x}_0, G_0, \mathbf{x}'_0, \mathbf{m}'_0), a_1, \dots, a_k, (\mathbf{m}_k, \mathbf{x}_k, G_k, \mathbf{x}'_k, \mathbf{m}'_k)$$

We denote by  $H$  the non-negative integral solutions of the characteristic system of  $\mathcal{U}$  and we denote by  $H'$  the subset of  $H$  corresponding to sequences  $\xi$  associated to words in  $\mathcal{L}(\mathcal{U})$ . Observe that Parikh images of  $\mathcal{L}(\mathcal{U})$  are images by linear functions of  $H'$ . From Proposition 5.5 it is sufficient to prove that  $H'$  is pseudo-linear. We are going to prove that the set  $H_0$  of non-negative integral solutions of the homogeneous characteristic system is a linearizator for  $H'$ . First of all observe that  $H_0$  is a monoid finitely generated since  $H_0 = P_0^*$  where  $P_0 = \min(H_0 \setminus \{\mathbf{0}\})$ .

Let us consider  $\xi \in H$  and  $\xi_0 \in H_0$  satisfying the following Lemma 6.7. Observe that  $H' \subseteq H$  implies that  $H'$  is included in the linear set  $\xi - \xi_0 + H_0$ .

**Lemma 6.7** *There exists  $\xi \in H$  and  $\xi_0 \in H_0$  such that  $\xi_0 + H \subseteq \xi + H_0$ .*



Now, let us consider a set  $R_0 = \{\xi_1, \dots, \xi_d\}$  included in the interior of  $H_0$ . We are going to prove that there exists  $\xi' \in H'$  such that  $\xi' + R_0^* \subseteq H'$ . We first prove the following lemma.

**Lemma 6.8** *For any  $\xi_i = (\mathbf{s}_{i,j}, (\mu_{i,j,t})_t, \mathbf{s}'_{i,j})_j$  interior vector of  $H_0$  there exists a cycle  $\pi_{i,j} = (\mathbf{x}_j \xrightarrow{\sigma_{i,j}}_{G_j} \mathbf{x}'_j)$  such that  $(\mu_{i,j,t})_t = |\pi_{i,j}|$  for any  $0 \leq j \leq k$ .*

Now, let us fix notations satisfying both Lemma 6.2 and Lemma 6.3.

**Lemma 6.9** *There exists  $c_i \geq 0$  such that for any  $0 \leq j \leq k$  and  $c \geq c_i$ :*

$$\begin{aligned} & \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + \mathbf{s}_{i,j} \\ & \xrightarrow{\sigma_{i,j}}_{\mathcal{V}} \\ & \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + \mathbf{s}'_{i,j} \end{aligned}$$

Now, let us consider an integer  $c \geq 0$  such that  $c \geq c_0$ ,  $c \geq c'$  and  $c \geq c_i$  for any  $1 \leq i \leq d$  where  $c_0, c', c_i$  are respectively defined in Lemma 6.5, Lemma 6.6 and Lemma 6.9. From these lemmas and Lemma 6.4 we deduce that for any sequence  $n_1, \dots, n_d \in \mathbb{N}$  we have the following relation:

$$\begin{aligned} & \mathbf{s}_j + c\mathbf{s}_{0,j} + \sum_{i=1}^d n_i \mathbf{s}_{i,j} \\ & \xrightarrow{w_j^c \sigma_{1,j}^{n_1} \dots \sigma_{d,j}^{n_d} \sigma_{0,j}^c \sigma_j(w'_j)^c}_{\mathcal{V}} \\ & \mathbf{s}'_j + c\mathbf{s}'_{0,j} + \sum_{i=1}^d n_i \mathbf{s}'_{i,j} \end{aligned}$$

We have proved that there exists a word in  $\mathcal{L}(\mathcal{U})$  associated with  $\xi + c\xi_0 + \sum_{i=1}^d n_i \xi_i$ . Let  $\xi' = \xi + c\xi_0$ . We deduce that  $\xi' + R_0^* \subseteq H'$ . Thus  $H'$  is pseudo-linear and  $H_0$  is a linearizator for  $H'$ . That means Parikh images of  $\mathcal{L}(\mathcal{U})$  are pseudo-linear for any perfect MGVS  $\mathcal{U}$  for  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ . From Theorem 6.1, we get the following Theorem 6.10.

**Theorem 6.10** *Parikh images of  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  are semi-pseudo-linear.*

## 7 Easy Algorithm With Separators

In this section, we prove that if a final configuration is not reachable from an initial one, there exists a Presburger inductive invariant proving this property. Since we can decide with any decision procedure for the Presburger arithmetic if formulas denote inductive invariants, we deduce that there exist checkable certificates of non-reachability. We deduce

a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by non-deterministically selecting finite sequences of actions and a second one that tries to prove the non-reachability by non-deterministically selecting Presburger formulas.

The reachability problem can be reformulated by introducing the definition of separators. A pair  $(S, S')$  of configuration sets is called a *separator* for  $\mathcal{V}$  if  $(S \times S') \cap \xrightarrow{*}_{\mathcal{V}} = \emptyset$ . Naturally, a pair  $(\mathbf{s}, \mathbf{s}')$  is in the complement of the reachability relation  $\xrightarrow{*}_{\mathcal{V}}$  if and only if the pair  $(\{\mathbf{s}\}, \{\mathbf{s}'\})$  is a separator.

Let us introduce the *inductive separators*. We first define the following sets for any pair  $(S, S')$  of configurations sets and for any action  $a \in \Sigma$ :

$$\begin{aligned} \text{post}_{\mathcal{V}}^a(S) &= \{\mathbf{s}' \in \mathbb{N}^n \mid \exists \mathbf{s} \in S \quad \mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'\} \\ \text{pre}_{\mathcal{V}}^a(S') &= \{\mathbf{s} \in \mathbb{N}^n \mid \exists \mathbf{s}' \in S' \quad \mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'\} \end{aligned}$$

A set  $I \subseteq \mathbb{N}^n$  is called a *forward invariant* if  $\text{post}_{\mathcal{V}}^a(I) \subseteq I$  for any  $a \in \Sigma$ . A set  $I' \subseteq \mathbb{N}^n$  is called a *backward invariant* if  $\text{pre}_{\mathcal{V}}^a(I') \subseteq I'$  for any  $a \in \Sigma$ . Observe that a pair  $(I, I')$  of configurations sets such that  $I \cap I' = \emptyset$ ,  $I$  is a forward invariant and  $I'$  is a backward invariant is a separator. Such a separator  $(I, I')$  is said *inductive*. As  $(\text{post}_{\mathcal{V}}^*(S), \text{pre}_{\mathcal{V}}^*(S'))$  is an inductive separator for any separator  $(S, S')$ , we deduce that separators are included into inductive separators.

We are interested in inductive separators definable in the Presburger arithmetic FO  $(\mathbb{N}, +, \leq)$ . Let us consider a pair  $(\psi(\mathbf{x}), \psi'(\mathbf{x}'))$  of Presburger formulas denoting a pair  $(I, I')$  of configurations sets. Note that  $(I, I')$  is an inductive separator if and only if  $\psi(\mathbf{x}) \wedge \psi'(\mathbf{x}')$  and the following formulas are unsatisfiable for any  $a \in \Sigma$ .

$$\begin{aligned} \psi(\mathbf{x}) \quad \wedge \quad \mathbf{x}' = \mathbf{x} + \delta(a) \quad \wedge \quad \neg\psi(\mathbf{x}') \\ \psi'(\mathbf{x}') \quad \wedge \quad \mathbf{x}' = \mathbf{x} + \delta(a) \quad \wedge \quad \neg\psi'(\mathbf{x}) \end{aligned}$$

In particular we can effectively decide if  $(\psi(\mathbf{x}), \psi'(\mathbf{x}'))$  denotes an inductive separator. That means pairs  $(\psi(\mathbf{x}), \psi'(\mathbf{x}'))$  of Presburger formulas denoting inductive separators provide *checkable certificates of non-reachability* for any pair  $(\mathbf{s}, \mathbf{s}')$  of configurations satisfying  $(\psi(\mathbf{x}), \psi'(\mathbf{x}'))$ .

In the following we prove that Presburger separators are included in Presburger inductive separators. We deduce that the following algorithm  $\text{Reachability}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  decides the reachability problem. The termination is guaranteed by the previous result. Note [5] that in general  $(\text{post}_{\mathcal{V}}^*(S), \text{pre}_{\mathcal{V}}^*(S'))$  is not Presburger (see also Example 2.2). That means, this inductive separator must be

over-approximated by another inductive separator. Intuitively, the approximation is obtained by observing that for any Presburger sets  $S, S'$  of configurations, the sets  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S'$  and  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S')$  are *semi-pseudo-linear*. This property is proved in section 7.1. In the next section 7.2, we provide an induction to compute Presburger inductive separators.

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1 Reachability(  $s, \bar{\mathcal{V}}, s'$  )
2   repeat forever
3     non deterministically select  $\sigma \in \Sigma^*$ 
4     if  $s \xrightarrow{\sigma}_{\bar{\mathcal{V}}} s'$ 
5       return “reachable”
6     non deterministically select  $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ 
7       formulas in FO  $(\mathbb{N}, +, \leq)$ 
8     if  $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ 
9       inductive separator for  $(\{s\}, \{s'\})$ 
10    return “unreachable”

```

---

## 7.1 Reachability sets

We prove that  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S'$  and  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S')$  are semi-pseudo-linear for any semi-linear sets  $S, S' \subseteq \mathbb{N}^n$ .

Since semi-linear sets are finite unions of linear sets we only prove this result for the special case of two linear sets  $S = \mathbf{s} + P^*$  and  $S' = \mathbf{s}' + (P')^*$  where  $\mathbf{s}, \mathbf{s}' \in \mathbb{N}^n$  and where  $P, P'$  are two sets of periods of  $\mathbb{N}^n$ . We consider two alphabets  $\Sigma_P, \Sigma_{P'}$  disjoint of  $\Sigma$  and a displacement function  $\bar{\delta}$  defined over  $\bar{\Sigma} = \Sigma_P \cup \Sigma \cup \Sigma_{P'}$  that extends  $\delta$  such that:

$$P = \{\bar{\delta}(a) \mid a \in \Sigma_P\} \quad P' = \{-\bar{\delta}(a) \mid a \in \Sigma_{P'}\}$$

We consider the VAS  $\bar{\mathcal{V}} = (\bar{\Sigma}, n, \bar{\delta})$ . Intuitively, since  $\bar{\delta}(\Sigma_P) \subseteq \mathbb{N}^n$  and  $\bar{\delta}(\Sigma_{P'}) \subseteq -\mathbb{N}^n$ , words in  $\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$  can be reordered into words in  $(\Sigma_P^* \Sigma^* \Sigma_{P'}^*) \cap \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$ .

Let us consider the displacement functions  $f$  and  $f'$  defined over  $\bar{\Sigma}$  by:

$$f(a) = \begin{cases} \bar{\delta}(a) & \text{if } a \in \Sigma_P \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$$f'(a) = \begin{cases} -\bar{\delta}(a) & \text{if } a \in \Sigma_{P'} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

**Lemma 7.1** *We have  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S' = \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$  and  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = \mathbf{s} + f(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$ .*

Observe that sets  $\mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$  and  $\mathbf{s} + f(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$  are images by linear functions of Parikh images of  $\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$ . Theorem 6.10 shows that Parikh images of  $\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$  are semi-pseudo-linear. From Proposition 5.5 we deduce the following theorem:

**Theorem 7.2** *For any semi-linear sets  $S, S' \subseteq \mathbb{N}^n$ , the sets  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S'$  and  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S')$  are semi-pseudo-linear.*

## 7.2 Induction with domains

Given a pair  $(S, S') \subseteq (\mathbb{N}^n, \mathbb{N}^n)$  of disjoint sets, the set  $D = \mathbb{N}^n \setminus (S \cup S')$  is called the *domain* of  $(S, S')$ . Note that there exists inductive separators with non-empty domains. However, a separator with an empty domain is necessary inductive.

A Presburger inductive separator that over-approximate a Presburger separator  $(S_0, S'_0)$  is obtain inductively. We build a non-decreasing sequence  $(S_j, S'_j)_{j \geq 0}$  of Presburger separators starting from the initial Presburger separator  $(S_0, S'_0)$  such that the dimension of the domain  $D_j = \mathbb{N}^n \setminus (S_j \cup S'_j)$  is strictly decreasing. In order to obtain this sequence, observe that it is sufficient to show that for any Presburger separator  $(S_0, S'_0)$  with a non-empty domain  $D_0$ , there exists a Presburger separator  $(S, S') \supseteq (S_0, S'_0)$  with a domain  $D$  such that  $\dim(D) < \dim(D_0)$ .

**Remark 7.3**  *$(S, S') \subseteq (\mathbb{N}^n, \mathbb{N}^n)$  is a separator if and only if  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = \emptyset$  if and only if  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S' = \emptyset$  if and only if  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = \emptyset$ .*

We first define a set  $S'$  that over-approximates  $S'_0$  and such that  $(S_0, S')$  is a separator. As  $S_0$  and  $D_0$  are Presburger sets, Theorem 7.2 shows that  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap D_0$  is equal to a finite union of pseudo-linear sets  $X_1, \dots, X_k$ . Let us consider some linearizations  $L_1, \dots, L_k$  of these pseudo-linear sets and let us define the following Presburger set  $S'$ .

$$S' = S'_0 \cup (D_0 \setminus (\bigcup_{j=1}^k L_j))$$

We observe that  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap S' = \emptyset$  since  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap S'_0 = \emptyset$  and  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap D_0 \subseteq \bigcup_{j=1}^k L_j$ . We have proved that  $S'$  contains  $S'_0$  and  $(S_0, S')$  is a separator.

Now we define symmetrically a set  $S$  that over-approximates  $S_0$  and such that  $(S, S')$  is a separator. As  $D_0$  and  $S'$  are Presburger sets, Theorem 7.2 shows that  $D_0 \cap \text{pre}_{\bar{\mathcal{V}}}^*(S')$  is equal to a finite union of pseudo-linear sets  $X'_1, \dots, X'_{k'}$ . Let us consider some linearizations  $L'_1, \dots, L'_{k'}$  of these pseudo-linear sets and let us define the following Presburger set  $S$ .

$$S = S_0 \cup (D_0 \setminus (\bigcup_{j'=1}^{k'} L'_{j'}))$$

Once again, note that  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = \emptyset$ . Thus  $S$  contains  $S_0$  and  $(S, S')$  is a separator.

Let  $D$  be the domain of the separator  $(S, S')$ . From  $D_0 = \mathbb{N}^n \setminus (S_0 \cup S'_0)$ , we get the following equality.

$$D = D_0 \cap \left( \bigcup_{\substack{1 \leq j \leq k \\ 1 \leq j' \leq k'}} (L_j \cap L'_{j'}) \right)$$

From  $X_j, X'_{j'} \subseteq D_0$  we get  $\dim(X_j \cup X'_{j'}) \leq \dim(D_0)$ . As  $X_j \subseteq \text{post}_V^*(S_0) \subseteq \text{post}_V^*(S)$  and  $X'_{j'} \subseteq \text{pre}_V^*(S')$  and  $(S, S')$  is a separator, we deduce that  $X_j$  and  $X'_{j'}$  are two pseudo-linear sets with an empty intersection. Proposition 5.8 provides  $\dim(L_j \cap L'_{j'}) < \dim(X_j \cup X'_{j'})$ . We deduce  $\dim(D) < \dim(D_0)$ .

Since the dimension of non-empty Presburger sets are non-negative integers (see Proposition 4.2), we have proved the following theorem.

**Theorem 7.4** *Presburger separators are included in Presburger inductive separators.*

## 8 Conclusion

Thanks to the classical KLMST decomposition we have proved that Parikh Images of languages accepted by VASs are semi-pseudo-linear.

As application, we have proved the termination of a simple algorithm for deciding the reachability problem for VAS. Even though the proof of termination is based on the classical KLMST decomposition, the complexity of the algorithm does not depend on this decomposition. In fact, the complexity depends on the size of the minimal pair of Presburger formulas denoting an inductive separator when  $(\{s\}, \{s'\})$  is separable and the size of a minimal  $\sigma \in \Sigma^*$  such that  $s \xrightarrow{\sigma}_V s'$  otherwise. This algorithm is the *very first one* that does not require the KLMST decomposition for its implementation.

We left as an open question the problem of computing a lower bound and an upper bound of the size of a pair of Presburger formulas denoting an inductive separator. Note that the VAS exhibiting a large (Ackermann size) but finite reachability set given in [4] does not directly provide a lower-bound for this size since inductive separators can over-approximate reachability sets.

We also left as an open question the problem of adapting *Counter Example Guided Abstract Refinement* approaches [1] to obtain an algorithm for the VAS reachability problem with termination guaranty. In practice, such an algorithm should be more efficient than the enumeration-based algorithm provided in this paper.

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## A Proof of Proposition 4.2

**Proposition 4.2** *We have  $\dim(M) = \text{rank}(V)$  where  $V$  is the vector space generated by a monoid  $M$ .*

*Proof* : Since  $M \subseteq \mathbb{Z}^n \cap V$  it is sufficient to prove that  $\dim(M) \geq \text{rank}(V)$  and  $\dim(\mathbb{Z}^n \cap V) \leq \text{rank}(V)$ . Let us denote by  $\|\mathbf{x}\|_\infty = \max\{|\mathbf{x}[1]|, \dots, |\mathbf{x}[k]|\}$  the usual  $\infty$ -norm of a vector  $\mathbf{x} \in \mathbb{Q}^n$ . As  $M$  generates the vector space  $V$ , there exists a sequence  $\mathbf{m}_1, \dots, \mathbf{m}_d \in M$  with  $d = \text{rank}(V)$  that generates  $V$ . Since the case  $d = 0$  is immediate we assume that  $d \geq 1$ . We denote by  $f : \mathbb{Q}^d \rightarrow V$  the rational linear function  $f(\mathbf{x}) = \sum_{i=1}^d \mathbf{x}[i] \mathbf{m}_i$ .

Let us first prove that  $\dim(M) \geq d$ . By minimality of  $d = \text{rank}(V)$  note that  $f$  is injective. In particular the cardinal of  $f(\{0, \dots, k\}^d)$  is equal to  $(1+k)^d$ . Observe that a vector  $\mathbf{m}$  in this set satisfies  $\|\mathbf{m}\|_\infty \leq k \sum_{i=1}^d \|\mathbf{m}_i\|_\infty$  and  $\mathbf{m} \in M$ . We deduce that  $\dim(M) \geq d$ .

Now, let us prove that  $\dim(\mathbb{Z}^n \cap V) \leq d$ . Since for any matrix, the rank of vector space generated by the column vectors is equal to the rank of the vector space generated by the line vectors, there exists a sequence  $1 \leq j_1 < \dots < j_d \leq n$  such that the linear function  $g : \mathbb{Q}^n \rightarrow \mathbb{Q}^d$  defined by  $g(\mathbf{x}) = (\mathbf{x}[j_1], \dots, \mathbf{x}[j_d])$  satisfies  $h = g \circ f$  is a bijective linear function. In particular we deduce that for any  $\mathbf{v} \in \mathbb{Z}^n \cap V \cap \{-k, \dots, k\}^n$  there exists a vector  $\mathbf{x} = g(\mathbf{v}) \in \{-k, \dots, k\}^d$  such that  $\mathbf{v} = f \circ h^{-1}(\mathbf{x})$ . Therefore  $|\mathbb{Z}^n \cap V \cap \{-k, \dots, k\}^n| \leq (1+2k)^d$  for any  $k \in \mathbb{N}$ . We deduce that  $\dim(\mathbb{Z}^n \cap V) \leq d$ .  $\square$

## B Proof of Lemma 4.4

**Lemma 4.4** *Let  $P = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$  be a set of periods with  $k \in \mathbb{N}$ . We have  $\mathcal{I}(P^*) = \{0\}$  if  $k = 0$  and  $\mathcal{I}(P^*) = P^* \cap ((\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_k)$  if  $k \geq 1$ .*

*Proof* : Since the case  $k = 0$  is immediate, we assume that  $k \geq 1$ . Let us first consider an interior vector  $\mathbf{a} \in \mathcal{I}(P^*)$ . As  $\sum_{j=1}^k \mathbf{p}_j \in P^*$  and  $\mathbf{a} \in \mathcal{I}(P^*)$ , there exists  $N \geq 1$  such that  $N\mathbf{a} \in (\sum_{j=1}^k \mathbf{p}_j) + P^*$ . Let  $\mathbf{p} \in P^*$  such that  $N\mathbf{a} = \sum_{j=1}^k \mathbf{p}_j + \mathbf{p}$ . As  $\mathbf{p} \in P^*$ , there exists a sequence  $(N_j)_{1 \leq j \leq k}$  of elements in  $\mathbb{N}$  such that  $\mathbf{p} = \sum_{j=1}^k N_j \mathbf{p}_j$ . Combining this equality with the previous one provides  $\mathbf{a} = \sum_{j=1}^k \frac{1+N_j}{N} \mathbf{p}_j$ . Thus  $\mathbf{a} \in (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_k$ . Conversely, let us consider  $\mathbf{a} \in P^* \cap ((\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_k)$ . Observe that there exists an integer  $d \geq 1$  large enough such that  $d\mathbf{a} \in (\mathbb{N} \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{N} \setminus \{0\})\mathbf{p}_k$ . In particular for any  $\mathbf{x} \in P^*$  there exists  $N \geq 1$  such that  $Nd\mathbf{a} \in \mathbf{x} + P^*$ .  $\square$

## C Proof of Lemma 5.4

**Lemma 5.4** *We have  $\dim(X) = \dim(L)$  for any linearization  $L$  of a pseudo-linear set  $X \subseteq \mathbb{Z}^n$ .*

*Proof* : There exists  $\mathbf{b} \in \mathbb{Z}^n$  and a linearizer  $M$  for  $X$  such that  $L = \mathbf{b} + M$ . From  $X \subseteq L$  we deduce that  $\dim(X) \leq \dim(L)$ . Let us prove the converse. Let us consider an interior vector  $\mathbf{a} \in \mathcal{I}(M)$ . Since  $M$  is finitely generated, there exists a set of periods  $P$  such that  $M = P^*$ . Observe that  $R = \{\mathbf{a}\} \cup (\mathbf{a} + P)$  is a finite subset of  $\mathcal{I}(M)$ . As  $X$  is pseudo-linear, there exists  $\mathbf{x} \in X$  such that  $\mathbf{x} + R^* \subseteq X$ . Note that the vector space generated by  $R$  is equal to the vector space generated by  $P$ . Thus, from Proposition 4.2 we deduce that  $\dim(R^*) = \dim(P^*)$ . As  $\dim(\mathbf{x} + R^*) = \dim(R^*)$  and  $\dim(\mathbf{b} + P^*) = \dim(P^*)$  we deduce that  $\dim(\mathbf{x} + R^*) = \dim(L)$ . Since  $\mathbf{x} + R^* \subseteq X$  we deduce that  $\dim(L) \leq \dim(X)$ .  $\square$

## D Proof of Proposition 5.5

**Proposition 5.5** *Images  $X' = f(X)$  of pseudo-linear sets  $X$  by a linear function  $f$  are pseudo-linear. Moreover the linear set  $L' = f(L)$  is a linearization of  $X'$  for any linearization  $L$  of  $X$ .*

*Proof* : Let us consider a linear function  $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n'}$  defined by a matrix  $A \in \mathbb{Z}^{n' \times n}$  and a vector  $\mathbf{v} \in \mathbb{Z}^{n'}$ . Let us consider a pseudo-linear set  $X \subseteq \mathbb{Z}^n$ . As  $X$  is pseudo-linear, there exists a linearizer  $M$  of  $X$  and a vector  $\mathbf{b} \in \mathbb{Z}^n$  such that  $X \subseteq \mathbf{b} + M$ . As  $M$  is finitely generated there exists a set of periods  $P$  such that  $M = P^*$ . We are going to prove that  $L' = f(L)$  is a linearization of  $X' = f(X)$ . Let us consider  $\mathbf{b}' = f(\mathbf{b})$  and  $P' = \{A\mathbf{p} \mid \mathbf{p} \in P\}$  and observe that  $L' = \mathbf{b}' + (P')^*$ . In particular  $L'$  is a linear set. Since  $X \subseteq L$  we deduce that  $X' \subseteq L'$ . Let us consider a set  $R' = \{\mathbf{r}'_1, \dots, \mathbf{r}'_d\}$  included in the interior of  $(P')^*$ . As  $\mathbf{r}'_i \in (P')^*$  there exists  $\mathbf{p}_i \in P^*$  such that  $\mathbf{r}'_i = A\mathbf{p}_i$ . Lemma 4.4 shows that  $\mathbf{r}'_i$  is a sum of vectors of the form  $\lambda_{i,\mathbf{p}} A\mathbf{p}$  over all  $\mathbf{p} \in P$  where  $\lambda_{i,\mathbf{p}} > 0$  is a rational value. There exists an integer  $n_i \geq 1$  large enough such that  $n_i \lambda_{i,\mathbf{p}} \in \mathbb{N} \setminus \{0\}$  for any  $\mathbf{p} \in P$ . We deduce that  $\mathbf{r}_i = \sum_{\mathbf{p} \in P} n_i \lambda_{i,\mathbf{p}} \mathbf{p}$  is a vector in  $P^*$ . Moreover, from Lemma 4.4 we deduce that  $\mathbf{r}_i$  is in the interior of  $P^*$ . Let us consider the set  $R$  of vectors  $\mathbf{r}_i + k_i \mathbf{p}_i$  where  $k_i$  is an integer such that  $0 \leq k_i < n_i$ . As  $\mathbf{r}_i$  is in the interior of  $P^*$  and  $\mathbf{p}_i \in P^*$  we deduce that  $\mathbf{r}_i + k_i \mathbf{p}_i$  is also in the interior of  $P^*$ . We have proved that  $R \subseteq \mathcal{I}(P^*)$ . As  $L$  is a linearization of  $X$ , there exists  $\mathbf{x} \in X$  such that  $\mathbf{x} + R^* \subseteq X$ . We deduce that  $f(\mathbf{x}) + AR^* \subseteq X'$ . Let us consider  $\mathbf{x}' = f(\mathbf{x}) + A(\sum_{i=1}^d \mathbf{r}_i)$  and let us prove that  $\mathbf{x}' + (R')^* \subseteq X'$ . Consider  $\mathbf{r}' \in (R')^*$ . There exists a sequence  $(\mu'_i)_{1 \leq i \leq d}$  of integers in  $\mathbb{N}$  such that  $\mathbf{r}' = \sum_{i=1}^d \mu'_i \mathbf{r}'_i$ . The Euclid

division of  $\mu'_i$  by  $n_i$  shows that  $\mu'_i = k_i + n_i\mu_i$  where  $\mu_i \in \mathbb{N}$  and  $0 \leq k_i < n_i$ . From  $n_i\mathbf{r}'_i = A\mathbf{r}_i$  we deduce that  $\mathbf{x}' + \mathbf{r}' = f(\mathbf{x}) + A(\sum_{i=1}^d (\mathbf{r}_i + k_i\mathbf{p}_i) + \sum_{i=1}^d \mu_i\mathbf{r}_i)$ . Observe that  $\mathbf{r}_i + k_i\mathbf{p}_i$  and  $\mathbf{r}_i$  are both in  $R$ . We have proved that  $\mathbf{x}' + \mathbf{r}' \in f(\mathbf{x}) + AR^*$ . Since  $f(\mathbf{x}) + AR^* = f(\mathbf{x} + R^*)$  and  $\mathbf{x} + R^* \subseteq X$  we deduce that  $f(\mathbf{x}) + AR^* \subseteq X'$ . Thus  $\mathbf{x}' + (R')^* \subseteq X'$ . We have proved that  $L'$  is a linearization of  $X'$ .  $\square$

## E Proof of Lemma 5.7

A sub-group of  $(\mathbb{Z}^n, +)$  is simply called a *group* in the sequel. Note that  $G = X^* - X^*$  is the group generated by  $X \subseteq \mathbb{Z}^n$ . Now, let us consider the group  $G = M - M$  generated by a monoid  $M$  and  $\mathbf{a} \in \mathbb{Z}^n$ . Observe that  $\mathbf{a} \in \mathcal{I}(M)$  if and only if for any  $\mathbf{g} \in G$  there exists an integer  $N \geq 1$  such that  $\mathbf{g} + N\mathbf{a} \in M$ .

**Lemma E.1** *For any vector  $\mathbf{v} \in V$  where  $V$  is the vector space generated by a group  $G$ , there exists an integer  $d \geq 1$  such that  $d\mathbf{v} \in G$ .*

*Proof:* A vector  $\mathbf{v} \in V$  can be decomposed into a sum  $\mathbf{v} = \sum_{i=1}^k \lambda_i \mathbf{g}_i$  with  $k \in \mathbb{N}$ ,  $\lambda_i \in \mathbb{Q}$  and  $\mathbf{g}_i \in G$ . Let us consider an integer  $d \geq 1$  such that  $d\lambda_i \in \mathbb{Z}$  and observe that  $d\mathbf{v} \in G$ .  $\square$

**Lemma E.2 ([3])** *For any set of periods  $P_1, P_2$  there exists a set of periods  $P$  such that  $P_1^* \cap P_2^* = P^*$ . Moreover, for any  $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{Z}^n$ , there exists a finite set  $B \subseteq \mathbb{Z}^n$  such that  $(\mathbf{b}_1 + P_1^*) \cap (\mathbf{b}_2 + P_2^*) = B + (P_1^* \cap P_2^*)$ .*

*Proof:* Let us consider an enumeration  $\mathbf{p}_{i,1}, \dots, \mathbf{p}_{i,k_i}$  of the  $k_i \geq 0$  vectors in  $P_i$  where  $i \in \{1, 2\}$ . If  $k_1 = 0$  or if  $k_2 = 0$  then  $P_1^* = \{\mathbf{0}\}$  or  $P_2^* = \{\mathbf{0}\}$  and the lemma is immediate. Thus, we can assume that  $k_1, k_2 \geq 1$ .

Let us consider the set  $X$  of vectors  $(\lambda_1, \lambda_2) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$  such that  $\mathbf{b}_1 + \sum_{j=1}^{k_1} \lambda_1[j]\mathbf{p}_{1,j} = \mathbf{b}_2 + \sum_{j=1}^{k_2} \lambda_2[j]\mathbf{p}_{2,j}$ . Let us also consider the set  $X_0$  of vectors  $(\lambda_1, \lambda_2) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$  such that  $\sum_{j=1}^{k_1} \lambda_1[j]\mathbf{p}_{1,j} = \sum_{j=1}^{k_2} \lambda_2[j]\mathbf{p}_{2,j}$ . Observe that  $X = Z + X_0$  where  $Z$  is the finite set  $Z = \min(X)$  and  $X_0 = Z_0^*$  where  $Z_0$  is the finite set  $Z_0 = \min(X_0 \setminus \{\mathbf{0}\})$ .

Let us denote by  $B$  the finite set of vectors  $\mathbf{b} \in \mathbb{Z}^n$  such that there exists  $(\lambda_1, \lambda_2) \in Z$  satisfying  $\mathbf{b}_1 + \sum_{j=1}^{k_1} \lambda_1[j]\mathbf{p}_{1,j} = \mathbf{b} + \sum_{j=1}^{k_2} \lambda_2[j]\mathbf{p}_{2,j}$ . Let us also denote by  $P$  the finite set of vectors  $\mathbf{p} \in \mathbb{Z}^n$  such that there exists  $(\lambda_1, \lambda_2) \in Z_0$  satisfying  $\sum_{j=1}^{k_1} \lambda_1[j]\mathbf{p}_{1,j} = \mathbf{p} = \sum_{j=1}^{k_2} \lambda_2[j]\mathbf{p}_{2,j}$ . Remark that  $(\mathbf{b}_1 + P_1^*) \cap (\mathbf{b}_2 + P_2^*) = B + P^*$  and  $P_1^* \cap P_2^* = P^*$ .  $\square$

**Lemma 5.7** *Let  $L_1 = \mathbf{b}_1 + M_1$  and  $L_2 = \mathbf{b}_2 + M_2$  be two linear sets with a non-degenerated intersection. There exists finite sets  $R_1 \subseteq \mathcal{I}(M_1)$  and  $R_2 \subseteq \mathcal{I}(M_2)$  such that  $(\mathbf{x}_1 + R_1^*) \cap (\mathbf{x}_2 + R_2^*) \neq \emptyset$  for any  $(\mathbf{x}_1, \mathbf{x}_2) \in L_1 \times L_2$ .*

*Proof:* As  $M_1, M_2$  are finitely generated, there exists some sets of periods  $P_1, P_2$  such that  $M_1 = P_1^*$  and  $M_2 = P_2^*$ . From Lemma E.2 there exists a set of periods  $P$  and a finite set  $B \subseteq \mathbb{Z}^n$  such that  $P_1^* \cap P_2^* = P^*$  and  $L_1 \cap L_2 = B + P^*$ . Note that  $L_1 \cap L_2 = \emptyset$  is not possible since in this case  $\dim(L_1 \cap L_2) = -\infty$ . Thus there exists a vector  $\mathbf{b} \in B$ .

Let us denote by  $V_1, V, V_2$  the vector spaces generated respectively by  $P_1, P, P_2$  and let us prove that  $V_1 = V = V_2$ . Proposition 4.2 shows that  $\dim(L_1) = \text{rank}(V_1)$ ,  $\dim(L_1 \cap L_2) = \text{rank}(V)$  and  $\dim(L_2) = \text{rank}(V_2)$ . From  $\dim(L_1 \cap L_2) = \dim(L_1)$  we deduce that  $\text{rank}(V) = \text{rank}(V_1)$ . Moreover as  $P^* \subseteq P_1^*$  we deduce that  $V \subseteq V_1$ . The inclusion  $V \subseteq V_1$  and the relation  $\text{rank}(V) = \text{rank}(V_1)$  prove together that  $V = V_1$ . Symmetrically we deduce that  $V = V_2$ .

We denote by  $G_1, G, G_2$  the groups generated respectively by  $P_1, P, P_2$ . Note that the vector spaces generated by  $G_1, G, G_2$  are equal to  $V_1, V, V_2$ .

Let  $\mathbf{a}$  be an interior vector of  $P^*$  and let us prove that  $\mathbf{a} \in \mathcal{I}(P_1^*) \cap \mathcal{I}(P_2^*)$ . Let  $j \in \{1, 2\}$ . Note that  $\mathbf{a} \in P^* \subseteq P_j^*$ . Let  $\mathbf{p} \in \mathcal{I}(P_j^*)$ . Since  $-\mathbf{p} \in V$  and  $V$  is the vector space generated by  $G$ , Lemma E.1 shows that there exists an integer  $d \geq 1$  such that  $-d\mathbf{p} \in G$ . From  $\mathbf{a} \in \mathcal{I}(P^*)$  we deduce that there exists  $N \geq 1$  such that  $-d\mathbf{p} + N\mathbf{a} \in P^*$ . From  $P^* \subseteq P_j^*$  we deduce that  $\mathbf{a} \in \frac{1}{N}(d\mathbf{p} + P_j^*)$ . From  $\mathbf{p} \in \mathcal{I}(P_j^*)$  and Lemma 4.4 we get  $\mathbf{a} \in \mathcal{I}(P_j^*)$ .

We define  $R_1$  and  $R_2$  by  $R_j = \{\mathbf{a}\} \cup (\mathbf{a} + P_j)$  for  $j \in \{1, 2\}$ . From  $\mathbf{a} \in \mathcal{I}(P_j^*)$ , Lemma 4.4 shows that  $R_j \subseteq \mathcal{I}(P_j^*)$ . Let us consider  $\mathbf{x}_1 \in L_1$  and  $\mathbf{x}_2 \in L_2$  and let us prove that  $(\mathbf{x}_1 + R_1^*) \cap (\mathbf{x}_2 + R_2^*) \neq \emptyset$ .

From  $\mathbf{b}, \mathbf{x}_j \in \mathbf{b}_j + P_j^*$  we deduce that  $\mathbf{x}_j - \mathbf{b} \in G_j$ . As the group generated by  $R_j$  is equal to  $G_j$ , there exists  $\mathbf{r}_j, \mathbf{r}'_j \in R_j^*$  such that  $\mathbf{x}_j + \mathbf{r}_j = \mathbf{b} + \mathbf{r}'_j$ .

As  $V$  is the vector space generated by  $G_1$  and  $\mathbf{r}'_2 \in R_2^* \subseteq V_2 = V$ , Lemma E.1 shows that there exists an integer  $d_1 \geq 1$  such that  $d_1\mathbf{r}'_2 \in G_1$ . As  $\mathbf{a} \in \mathcal{I}(P_1^*)$ , there exists an integer  $N_1 \geq 1$  such that  $d_1\mathbf{r}'_2 + N_1\mathbf{a} \in P_1^*$ . As  $P_1^* \subseteq R_1^* - N\mathbf{a}$ , we deduce that there exists an integer  $N'_1 \geq 0$  such that  $d_1\mathbf{r}'_2 + (N_1 + N'_1)\mathbf{a} \in R_1^*$ . We denote by  $\mathbf{r}''_1$  this vector. Symmetrically, there exist some integers  $d_2 \geq 1$ ,  $N_2 \geq 1$  and  $N'_2 \geq 0$  such that the vector  $d_2\mathbf{r}'_1 + (N_2 + N'_2)\mathbf{a}$  denoted by  $\mathbf{r}''_2$  is in  $R_2^*$ . We get:

$$\begin{aligned} \mathbf{x}_1 + \mathbf{r}_1 + (d_2 - 1)\mathbf{r}'_1 + \mathbf{r}''_1 + (N_2 + N'_2)\mathbf{a} \\ = \mathbf{b} + d_2\mathbf{r}'_1 + d_1\mathbf{r}'_2 + (N_1 + N'_1 + N_2 + N'_2)\mathbf{a} \\ \mathbf{x}_2 + \mathbf{r}_2 + (d_1 - 1)\mathbf{r}'_2 + \mathbf{r}''_2 + (N_1 + N'_1)\mathbf{a} \\ = \mathbf{b} + d_1\mathbf{r}'_2 + d_2\mathbf{r}'_1 + (N_2 + N'_2 + N_1 + N'_1)\mathbf{a} \end{aligned}$$

We have proved that these vectors are equal. Therefore  $(\mathbf{x}_1 + R_1^*) \cap (\mathbf{x}_2 + R_2^*) \neq \emptyset$ .  $\square$

## F Proofs of Section 6

The following lemma is used in the sequel.

**Lemma F.1 (Euler Cycles)** *Let  $G = (Q, \Sigma, T)$  be a strongly connected graph. For any sequence  $(\mu_t)_{t \in T}$  of integers  $\mu_t > 0$  satisfying the following equality for any state  $q_0 \in Q$ , there exists a cycle  $\pi$  such that  $|\pi|_t = \mu_t$  for any transition  $t \in T$ :*

$$\sum_{t=(q,a,q_0) \in T} \mu_t = \sum_{t'=(q_0,a,q') \in T} \mu_{t'}$$

Given a sequence  $(x_i)_{i \geq 0}$  in  $\mathbb{N}_T$ , we denote by  $x = \lim_{i \rightarrow +\infty} x_i$  the element  $x = \top$  if for any  $r \in \mathbb{N}$  there exists  $i_0 \geq 0$  such that  $x_i \geq r$  for any  $i \geq i_0$  and the element  $x \in \mathbb{N}$  if there exists  $i_0 \geq 0$  such that  $x_i = x$  for any  $i \geq i_0$ . When  $x = \lim_{i \rightarrow +\infty} x_i$  exists we say that  $(x_i)_{i \geq 0}$  converges toward  $x$ . Even if the proof of the following lemma is immediate by induction over the length of  $\sigma$ , it is central in the KLMST decomposition. In fact a path  $\mathbf{x} \xrightarrow{\sigma} \mathbf{x}'$  implies the relation  $\mathbf{x} \xrightarrow{\sigma} \mathbf{x}'$ .

**Lemma F.2 (Graph vector paths)** *For any  $\mathbf{x} \xrightarrow{\sigma} \mathbf{x}'$ , for any sequences  $(\mathbf{x}_c)_{c \in \mathbb{N}}$  and  $(\mathbf{x}'_c)_{c \in \mathbb{N}}$  of extended configurations that converge toward  $\mathbf{x} = \lim_{c \rightarrow +\infty} \mathbf{x}_c$  and  $\mathbf{x}' = \lim_{c \rightarrow +\infty} \mathbf{x}'_c$ , there exists  $c_0 \in \mathbb{N}$  such that  $\mathbf{x}_c \xrightarrow{\sigma} \mathbf{x}'$  and  $\mathbf{x}'_c \xrightarrow{\sigma} \mathbf{x}'$  for any  $c \geq c_0$ .*

**Lemma 6.2** *There exist a sequence  $(\pi_j)_j$  of cycles  $\pi_j = (\mathbf{x}_j \xrightarrow{\sigma_j} \mathbf{x}'_j)$  and a sequence  $(\mathbf{s}_j, \mathbf{s}'_j)_j$  of configurations pairs such that  $\xi = (\mathbf{s}_j, |\pi_j|, \mathbf{s}'_j)_j$  is solution of the characteristic system of  $\mathcal{U}$  and such that  $\mathbf{s}_j \xrightarrow{w_j} \mathbf{s}'_j$  and  $\mathbf{s}'_j \xrightarrow{w'_j} \mathbf{s}_j$  for any  $0 \leq j \leq k$ .*

*Proof :* The definition of perfect MGVS requires that there exists an integral solution  $\xi = (\mathbf{s}_j, (\mu_{j,t})_t, \mathbf{s}'_j)_j$  of its characteristic system. This solution may be negative. However, there exists a non-negative rational solution  $\xi_0 = (\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}'_{0,j})_j$  of the homogeneous characteristic system satisfying the perfect MGVS condition. Naturally, by replacing  $\xi_0$  by a sequence in  $(\mathbb{N} \setminus \{0\})\xi_0$  we can assume that  $\xi_0$  is a non-negative integral solution also satisfying the perfect MGVS condition. Now, just observe that there exists an integer  $c_0 \geq 0$  large enough such that  $\xi + c_0 \xi_0$  is a non-negative integral solution of the characteristic system satisfying  $\mu_{j,t} + c_0 \mu_{0,j,t} > 0$  for any  $t \in T_j$  and for any  $0 \leq j \leq k$ . Moreover, as  $\lim_{c \rightarrow +\infty} (\mathbf{s}_j + c \mathbf{s}_{0,j}) = \mathbf{m}_j$  and  $\lim_{c \rightarrow +\infty} (\mathbf{s}'_j + c \mathbf{s}'_{0,j}) = \mathbf{m}'_j$ , the relations  $\mathbf{m}_j \xrightarrow{\sigma_j} \mathbf{m}'_j$

and  $\mathbf{m}'_j \xrightarrow{\sigma'_j} \mathbf{m}_j$  and Lemma F.2 shows that there exists an integer  $c \geq c_0$  large enough such that  $(\mathbf{s}_j + c \mathbf{s}_{0,j}) \xrightarrow{w_j} \mathbf{m}'_j$  and  $\mathbf{m}'_j \xrightarrow{w'_j} (\mathbf{s}'_j + c \mathbf{s}'_{0,j})$ .

By replacing  $\xi$  by  $\xi + c_0 \xi_0$ , we can assume that  $c_0 = 0$ . As  $\mu_{j,t} > 0$  for any  $t \in T_j$  and  $G_j$  is strongly connected, Lemma F.1 shows that there exists a cycle  $\pi_j = (\mathbf{x}_j \xrightarrow{\sigma_j} \mathbf{x}'_j)$  such that  $\mu_{j,t} = |\pi_j|_t$  for any  $t \in T_j$ .  $\square$

**Lemma 6.3** *There exist a sequence  $(\pi_{0,j})_j$  of cycles  $\pi_{0,j} = (\mathbf{x}_j \xrightarrow{\sigma_{0,j}} \mathbf{x}'_j)$  and a sequence  $(\mathbf{s}_{0,j}, \mathbf{s}'_{0,j})_j$  of configurations pairs such that  $\xi_0 = (\mathbf{s}_{0,j}, |\theta_j| + |\pi_{0,j}| + |\theta'_j|, \mathbf{s}'_{0,j})_j$  is a solution of the homogeneous characteristic system, such that  $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{0}$  and  $\mathbf{s}'_{0,j} - \delta(w'_j) \geq \mathbf{0}$  for any  $0 \leq j \leq k$ , and such that for any  $1 \leq i \leq n$  and for any  $0 \leq j \leq k$ :*

- $\mathbf{s}_{0,j}[i] > 0$  if and only if  $\mathbf{m}_j[i] = \top$ .
- $\mathbf{s}'_{0,j}[i] > 0$  if and only if  $\mathbf{m}'_j[i] = \top$ .
- $(\mathbf{s}_{0,j} + \delta(w_j))[i] > 0$  if and only if  $\mathbf{x}_j[i] = \top$ .
- $(\mathbf{s}'_{0,j} - \delta(w'_j))[i] > 0$  if and only if  $\mathbf{x}'_j[i] = \top$ .

*Proof :* Let  $\xi_0 = (\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}'_{0,j})_j$  be a non-negative rational solution of the homogeneous characteristic system satisfying the perfect MGVS condition. By replacing  $\xi_0$  by  $(\mathbb{N} \setminus \{0\})\xi_0$  we can assume that  $\xi_0$  is a non-negative integral solution satisfying the perfect condition. We are going to prove that there exists an integer  $c \in \mathbb{N}$  large enough such that  $c \xi_0$  satisfies the lemma.

First of all, observe that for any  $c \geq 1$  and for any  $1 \leq i \leq n$ , we have:

- $c \mathbf{s}_{0,j} \geq \mathbf{0}$  and  $c \mathbf{s}_{0,j}[i] > 0$  if and only if  $\mathbf{m}_j[i] = \top$ .
- $c \mathbf{s}'_{0,j} \geq \mathbf{0}$  and  $c \mathbf{s}'_{0,j}[i] > 0$  if and only if  $\mathbf{m}'_j[i] = \top$ .

Let us consider  $1 \leq i \leq n$  and let us prove that there exists an integer  $c_i \geq 0$  such that for any  $c \geq c_i$  we have  $(c \mathbf{s}_{0,j} + \delta(w_j))[i] \geq 0$  and  $(c \mathbf{s}_{0,j} + \delta(w_j))[i] > 0$  if and only if  $\mathbf{x}_j[i] = \top$ . Note that  $\mathbf{m}_j[i] \sqsubseteq \mathbf{x}_j[i]$  thus either  $\mathbf{m}_j[i] = \mathbf{x}_j[i] \in \mathbb{N}$ , or  $(\mathbf{m}_j[i], \mathbf{x}_j[i]) \in \mathbb{N} \times \{\top\}$ , or  $\mathbf{m}_j[i] = \mathbf{x}_j[i] = \top$ . We separate the proof following these three cases. Let us first consider the case  $\mathbf{m}_j[i] = \mathbf{x}_j[i] \in \mathbb{N}$ . As  $\mathbf{m}_j[i] \in \mathbb{N}$  and  $\xi_0$  is solution of the homogeneous characteristic system, we get  $\mathbf{s}_{0,j}[i] = 0$ . Moreover the cycle  $\theta_j = (\mathbf{x}_j \xrightarrow{w_j} \mathbf{x}_j)$  shows that  $\mathbf{x}_j + \delta(w_j) = \mathbf{x}_j$ . From  $\mathbf{x}_j[i] \in \mathbb{N}$  we deduce that  $\delta(w_j)[i] = 0$ . In particular  $(c \mathbf{s}_{0,j} + \delta(w_j))[i] = 0$  and we have proved the case  $\mathbf{m}_j[i] = \mathbf{x}_j[i] \in \mathbb{N}$  by considering  $c_i = 0$ . Let us consider the second case  $(\mathbf{m}_j[i], \mathbf{x}_j[i]) \in \mathbb{N} \times \{\top\}$ . As in the previous case, since  $\mathbf{m}_j[i] \in \mathbb{N}$  we deduce that  $\mathbf{s}_{0,j}[i] = 0$ . Note that the perfect condition shows that

$\delta(w_j)[i] > 0$  in this case. In particular for any  $c \geq 0$  we have  $(cs_{0,j} + \delta(w_j))[i] > 0$  and we have proved the case  $(\mathbf{m}_j[i], \mathbf{x}_j[i]) \in \mathbb{N} \times \{\top\}$  by considering  $c_i = 0$ . Finally, let us consider the case  $\mathbf{m}_j[i] = \mathbf{x}_j[i] = \top$ . As  $\mathbf{m}_j[i] = \top$  we deduce that  $s_{0,j}[i] > 0$  in particular there exists an integer  $c_i \geq 0$  large enough such that  $(cs_{0,j} + \delta(w_j))[i] > 0$  for any  $c \geq c_i$ . We have proved the three cases.

Symmetrically, for any  $1 \leq i \leq n$ , there exists an integer  $c'_i \geq 0$  such that for any  $c \geq c'_i$  we have  $(cs'_{0,j} - \delta(w'_j))[i] \geq 0$  and  $(cs'_{0,j} - \delta(w'_j))[i] > 0$  if and only if  $\mathbf{x}'_j[i] = \top$ .

Finally, as  $\mu_{0,j,t} > 0$  for any  $t \in T_j$  and for any  $0 \leq j \leq k$ , we deduce that there exists an integer  $c \geq 0$  large enough such that  $c\mu_{0,j,t} > |\theta_j|_t + |\theta_{j'}|_t$  for any  $t \in T_j$  for any  $0 \leq j \leq k$ . Naturally, we can also assume that  $c \geq 1$ ,  $c \geq c_i$  and  $c \geq c'_i$  for any  $1 \leq i \leq n$ . Let us replace  $\xi_0$  by  $c\xi_0$ . As  $\mu_{j,t} - |\theta_j|_t + |\theta_{j'}|_t > 0$  for any  $t \in T_j$ , Lemma F.1 shows that there exists a cycle  $\pi_{0,j} = (\mathbf{x}_j \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x}'_j)$  such that  $|\pi_{0,j}|_t = \mu_{0,j,t} - (|\theta_j|_t + |\theta_{j'}|_t)$  for any  $t \in T_j$ .  $\square$

**Lemma 6.4** For any  $c \geq 0$  we have:

$$\begin{array}{ccc} \mathbf{s}_j + c\mathbf{s}_{0,j} & \xrightarrow{w_j^c} & \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) \\ \mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)) & \xrightarrow{(w'_j)^c} & \mathbf{s}'_j + c\mathbf{s}'_{0,j} \end{array}$$

*Proof* : Since the two relations are symmetrical, we just prove the first one. The choice of  $\xi$  satisfying Lemma 6.2 shows that  $\mathbf{s}_j \xrightarrow{w_j} \mathcal{V}$ . The conditions  $\mathbf{s}_{0,j} \geq \mathbf{0}$ ,  $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{0}$  and  $\mathbf{s}_j \xrightarrow{w_j} \mathcal{V}$  with an immediate induction on the integer  $c \geq 0$  provides the required relation.  $\square$

**Lemma 6.5** There exists  $c_0 \geq 0$  such that for any  $c \geq c_0$ :

$$\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) \xrightarrow{\sigma_{0,j}^c} \mathcal{V} \quad \mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$$

*Proof* : Let us recall that  $\mathbf{x}_i = \mathbf{x}'_i$ . We denote by  $\mathbf{u}$  the vector in  $\{0, 1\}^n$  satisfying  $\mathbf{u}[i] = 1$  if  $\mathbf{x}_i[i] = \top = \mathbf{x}'_i[i]$  and satisfying  $\mathbf{u}[i] = 0$  otherwise. From the choice of  $\xi_0$  satisfying Lemma 6.3, we observe that  $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{u}$  and  $\mathbf{s}'_{0,j} - \delta(w'_j) \geq \mathbf{u}$ . Note that  $\lim_{c \rightarrow +\infty} (\mathbf{s}_j + c\mathbf{u}) = \mathbf{x}_j$ . As  $\mathbf{x}_j \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x}'_j$ , Lemma F.2 proves that there exists an integer  $c_0 \geq 0$  such that  $\mathbf{s}_j + c_0\mathbf{u} \xrightarrow{\sigma_{0,j}} \mathcal{V}$ . Now, let us consider integers  $c \geq 1$  and  $c' \geq 0$  such that  $c + c' \geq c_0$  and let us prove the relation:

$$\begin{array}{ccc} \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + c'(\mathbf{s}'_{0,j} - \delta(w'_j)) & & \\ \xrightarrow{\sigma_{0,j}} & & \\ \mathbf{s}_j + (c-1)(\mathbf{s}_{0,j} + \delta(w_j)) + (c'+1)(\mathbf{s}'_{0,j} - \delta(w'_j)) & & \end{array}$$

From  $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{u}$  and  $\mathbf{s}'_{0,j} - \delta(w'_j) \geq \mathbf{u}$  we deduce that  $c(\mathbf{s}_{0,j} + \delta(w_j)) + c'(\mathbf{s}'_{0,j} - \delta(w'_j)) \geq (c+c')\mathbf{u} \geq c_0\mathbf{u}$ . Thus, the previous relation directly comes from  $\mathbf{s}_j + c_0\mathbf{u} \xrightarrow{\sigma_{0,j}} \mathcal{V}$  and  $\mathbf{s}_{0,j} + \delta(w_j) + \delta(\sigma_{0,j}) + \delta(w'_j) = \mathbf{s}'_{0,j}$ . Now, an immediate induction provides  $\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) \xrightarrow{\sigma_{0,j}^c} \mathcal{V}$  and  $\mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$  for any  $c \geq 0$ .  $\square$

**Lemma 6.6** There exists  $c' \geq 0$  such that for any  $c \geq c'$ :

$$\mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)) \xrightarrow{\sigma_j} \mathcal{V} \quad \mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$$

*Proof* : As  $\lim_{c \rightarrow +\infty} (\mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))) = \mathbf{x}'_j$  and  $\mathbf{x}_j \xrightarrow{\sigma_j}_{G_j} \mathbf{x}'_j$ , Lemma F.2 proves that there exists  $c' \geq 0$  such that  $\mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)) \xrightarrow{\sigma_j} \mathcal{V}$  for any  $c \geq c'$ . Since  $\mathbf{s}_j + \delta(\sigma_j) = \mathbf{s}'_j$  we are done.  $\square$

**Lemma 6.7** There exists  $\xi \in H$  and  $\xi_0 \in H_0$  such that  $\xi_0 + H \subseteq \xi + H_0$ .

*Proof* : As the MGVS  $\mathcal{U}$  is perfect the set  $H$  is non empty. Let us consider the set  $I$  of components  $i$  such that  $\xi[i]$  is independent of  $\xi \in H$ . As the MGVS is perfect we deduce that for any integer  $c \geq 0$  there exists  $\xi \in H$  such that  $\xi[i] \geq c$  for any  $i \notin I$ . As  $\min(H)$  is finite, we deduce that there exists  $\xi \in H$  such that  $\xi \geq \xi'$  for any  $\xi' \in \min(H)$ . In particular  $\xi_0 = \sum_{\xi' \in \min(H)} (\xi - \xi')$  is in  $H_0$ . Let us prove that  $\xi_0 + H \subseteq \xi + H_0$ . Consider  $\xi'' \in H$ . By definition of  $\min(H)$ , there exists  $\xi''' \in \min(H)$  such that  $\xi''' \leq \xi''$ . The definition of  $\xi_0$  shows that  $\xi_0 - (\xi - \xi''')$  is equal to a sum of terms  $(\xi - \xi')$  indexed by  $\xi' \in \min(H) \setminus \{\xi'''\}$ . Therefore  $\xi_0 - (\xi - \xi''') \in H_0$ . As  $\xi'' - \xi''' \in H_0$  we have proved that the sum of  $\xi_0 - (\xi - \xi''')$  and  $\xi'' - \xi'''$  is also in  $H_0$ . Note that this sum is equal to  $\xi_0 - \xi + \xi''$ . We have proved that  $\xi_0 + \xi'' \in \xi + H_0$ . Therefore  $\xi_0 + H \subseteq \xi + H_0$ .  $\square$

**Lemma 6.8** For any  $\xi_i = (\mathbf{s}_{i,j}, (\mu_{i,j,t})_t, \mathbf{s}'_{i,j})_j$  interior vector of  $H_0$  there exists a cycle  $\pi_{i,j} = (\mathbf{x}_j \xrightarrow{\sigma_{i,j}}_{G_j} \mathbf{x}'_j)$  such that  $(\mu_{i,j,t})_t = |\pi_{i,j}|$  for any  $0 \leq j \leq k$ .

*Proof* : Since  $\mathcal{U}$  is perfect, for any  $t \in T_j$ , there exists a solution  $\xi_0 = (\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}'_{0,j})_j$  in  $H_0$  such that  $\mu_{0,j,t} > 0$ . As  $H_0 = P_0^*$ , for any  $t \in T_j$  there exists  $\xi_0 \in P_0$  satisfying the same property. Lemma 4.4 shows that  $\xi_i = (\mathbf{s}_{i,j}, (\mu_{i,j,t})_t, \mathbf{s}'_{i,j})_j$  is a sum over all solutions  $\xi_0 \in P_0$  of terms of the form  $\lambda\xi_0$  where  $\lambda > 0$  is a rational value that naturally depends on  $\xi_0$ . In particular we deduce that  $\mu_{i,j,t} > 0$  for any  $t \in T_j$  and for any  $0 \leq j \leq k$ . Lemma F.1 shows that there exists a cycle  $\pi_{i,j} = (\mathbf{x}_j \xrightarrow{\sigma_{i,j}}_{G_j} \mathbf{x}'_j)$  such that  $\mu_{i,j,t} = |\pi_{i,j}|_t$  for any  $t \in T_j$  and any  $1 \leq j \leq k$ .  $\square$

**Lemma 6.9** *There exists  $c_i \geq 0$  such that for any  $1 \leq j \leq k$  and  $c \geq c_i$ :*

$$\begin{aligned} & \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + \mathbf{s}_{i,j} \\ & \xrightarrow{\sigma_{i,j}}_{\mathcal{V}} \\ & \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + \mathbf{s}'_{i,j} \end{aligned}$$

*Proof :* As  $\lim_{c \rightarrow +\infty} (\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j))) = \mathbf{x}_j$  and  $\mathbf{x}_j \xrightarrow{\sigma_{i,j}}_{G_j} \mathbf{x}_j$ , Lemma F.2 proves that there exists an integer  $c_i \geq 0$  such that  $(\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j))) \xrightarrow{\sigma_{i,j}}_{\mathcal{V}}$  for any  $c \geq c_i$  and for any  $0 \leq j \leq k$ . As  $\mathbf{s}_{i,j} \geq \mathbf{0}$  and  $\mathbf{s}_{i,j} + \delta(\sigma_{i,j}) = \mathbf{s}'_{i,j} \geq \mathbf{0}$  we deduce the lemma.  $\square$

## G Proofs of Lemma 7.1

*Additional notations :* A shuffle of a pair  $(u, v)$  of words is a word of the form  $u_1 v_1 \dots u_k v_k$  where  $k \geq 1$  and  $u_1, v_1, \dots, u_k, v_k$  are words satisfying  $u = u_1 \dots u_k$  and  $v = v_1 \dots v_k$ . Given two languages  $\mathcal{L}_1, \mathcal{L}_2 \subseteq \Sigma^*$  we define  $\mathcal{L}_1 || \mathcal{L}_2$  the set of shuffle of pairs  $(u, v) \in \mathcal{L}_1 \times \mathcal{L}_2$ . Observe that the shuffle operator is associative since  $(\mathcal{L}_1 || \mathcal{L}_2) || \mathcal{L}_3 = \mathcal{L}_1 || \mathcal{L}_2 || \mathcal{L}_3$  for any  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \subseteq \Sigma^*$ .

We first prove the following lemma.

**Lemma G.1** *Let  $(\mathbf{s}, \mathbf{s}')$  be a pair of configurations of a VAS  $\mathcal{V} = (\Sigma, n, \delta)$ . Let us consider  $u_1, u_2, v_1, v_2 \in \Sigma^*$  and  $a \in \Sigma$  such that  $\mathbf{s} \xrightarrow{u_1 u_2 a v_1 v_2}_{\mathcal{V}} \mathbf{s}'$ .*

- if  $\delta(a) \geq \mathbf{0}$  then  $\mathbf{s} \xrightarrow{u_1 a u_2 v_1 v_2}_{\mathcal{V}} \mathbf{s}'$ .
- if  $\delta(a) \leq \mathbf{0}$  then  $\mathbf{s} \xrightarrow{u_1 u_2 v_1 a v_2}_{\mathcal{V}} \mathbf{s}'$ .

*Proof :* We only consider the case  $\delta(a) \geq \mathbf{0}$  since the other case is symmetrical by replacing  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  by  $(\mathbf{s}', -\mathcal{V}, \mathbf{s})$  where  $-\mathcal{V} = (\Sigma, n, -\delta)$ .

Let us consider the three configurations  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  such that  $\mathbf{s} \xrightarrow{u_1}_{\mathcal{V}} \mathbf{x} \xrightarrow{u_2}_{\mathcal{V}} \mathbf{y} \xrightarrow{a}_{\mathcal{V}} \mathbf{z} \xrightarrow{v_1 v_2}_{\mathcal{V}} \mathbf{s}'$ . Since  $\delta(a) \geq \mathbf{0}$  we deduce  $\mathbf{x}' = \mathbf{x} + \delta(a)$  is a configuration and it satisfies  $\mathbf{x} \xrightarrow{a}_{\mathcal{V}} \mathbf{x}'$ . Moreover as  $\mathbf{x}' \geq \mathbf{x}$  and  $\mathbf{x} \xrightarrow{u_2}_{\mathcal{V}} \mathbf{y}$  we deduce that there exists a configuration  $\mathbf{y}'$  such that  $\mathbf{x}' \xrightarrow{u_2}_{\mathcal{V}} \mathbf{y}'$ . Observe that  $\mathbf{y}' = \mathbf{x}' + \delta(u_2)$ . Since  $\mathbf{x}' = \mathbf{x} + \delta(a)$  we deduce that  $\mathbf{y}' = \mathbf{x} + \delta(u_2 a)$ . As  $\mathbf{z} = \mathbf{x} + \delta(u_2 a)$  we have proved that  $\mathbf{y}' = \mathbf{z}$ . Therefore  $\mathbf{s} \xrightarrow{u_1 a u_2 v_1 v_2}_{\mathcal{V}} \mathbf{s}'$ .  $\square$

The following lemma formally explains why words in  $\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$  can be reordered into words in  $(\Sigma_P^* \Sigma^* \Sigma_{P'}^*) \cap \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$ . Note that in this lemma  $u||v||u'$  denotes the set of words  $\{u\}||\{v\}||\{u'\}$ .

**Lemma G.2** *For any  $(u, v, u') \in \Sigma_P^* \times \Sigma^* \times \Sigma_{P'}^*$ , we have:*

$$(u||v||u') \cap \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}') \neq \emptyset \iff uvu' \in \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$$

*Proof :* Since  $uvu' \in (u||v||u')$  we deduce that  $uvu' \in \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$  implies  $(u||v||u') \cap \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}') \neq \emptyset$ . The converse is obtained by observing that  $\bar{\delta}(\Sigma_P) \subseteq \mathbb{N}^n$  and  $\bar{\delta}(\Sigma_{P'}) \subseteq -\mathbb{N}^n$  and by applying Lemma G.1  $\square$

**Lemma 7.1** *We have  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S' = \mathbf{s}' + \bar{\delta}_{P'}(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$  and  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = \mathbf{s} + \bar{\delta}_P(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$ .*

*Proof :* Let us consider  $\mathbf{c}' \in \text{post}_{\bar{\mathcal{V}}}^*(S) \cap S'$ . There exists  $\mathbf{c} \in S$  and a word  $v \in \Sigma^*$  such that  $\mathbf{c} \xrightarrow{v}_{\bar{\mathcal{V}}} \mathbf{c}'$ . In particular  $\mathbf{c} \xrightarrow{v}_{\bar{\mathcal{V}}} \mathbf{c}'$ . Since  $S = \mathbf{s} + P^*$  we observe that there exists a word  $u \in \Sigma_P^*$  such that  $\mathbf{s} \xrightarrow{u}_{\bar{\mathcal{V}}} \mathbf{c}$ . Symmetrically since  $S' = \mathbf{s}' + (P')^*$  there exists  $u' \in \Sigma_{P'}^*$  such that  $\mathbf{c}' \xrightarrow{u'}_{\bar{\mathcal{V}}} \mathbf{s}'$ . We have proved that  $uvu' \in \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$ . Note that  $f'(uvu') = -\bar{\delta}(v)$ . From  $\mathbf{s}' = \mathbf{c}' + \bar{\delta}(v)$  we have proved that  $\mathbf{c}' \in \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$ . Conversely, let us consider a vector  $\mathbf{c}' \in \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$ . There exists a word  $\sigma \in \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$  such that  $\mathbf{c}' = \mathbf{s}' + f'(\sigma)$ . Thanks to Lemma G.2 we can assume that  $\sigma = uvu'$  with  $u \in \Sigma_P^*, v \in \Sigma^*$  and  $u' \in \Sigma_{P'}^*$ . Let us consider the two configurations  $\mathbf{c}, \mathbf{c}'$  such that  $\mathbf{s} \xrightarrow{u}_{\bar{\mathcal{V}}} \mathbf{c} \xrightarrow{v}_{\bar{\mathcal{V}}} \mathbf{c}' \xrightarrow{u'}_{\bar{\mathcal{V}}} \mathbf{s}'$ . Since  $u \in \Sigma_P^*$  we deduce that  $\mathbf{c} \in S$  and since  $u' \in \Sigma_{P'}^*$ , we get  $\mathbf{c}' \in S'$ . Moreover from  $v \in \Sigma^*$  we deduce  $\mathbf{c} \xrightarrow{v}_{\bar{\mathcal{V}}} \mathbf{c}'$ . We have proved that  $\mathbf{c}' \in \text{post}_{\bar{\mathcal{V}}}^*(S) \cap S'$ . Thus  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S' = \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$ . Symmetrically we deduce  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = \mathbf{s} + \bar{\delta}_P(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$ .  $\square$