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# The General Vector Addition System Reachability Problem by Presburger Inductive Invariants

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## Abstract

*The reachability problem for Vector Addition Systems (VAS) is a central problem of net theory. The general problem is known decidable by algorithms exclusively based on the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney decomposition. This decomposition is used in this paper to prove that Parikh images of languages accepted by VAS are semi-pseudo-linear; a class of sets that can be precisely over-approximated by sets definable in the Presburger arithmetic. We provide an application of this result; we prove that if a final configuration is not reachable from an initial one, there exists a Presburger inductive invariant proving this property. Since we can decide with any decision procedure for the Presburger arithmetic if formulas denote inductive invariants, we deduce that there exist checkable certificates of non-reachability. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by fairly enumerating finite sequence of actions and a second one that tries to prove the non-reachability by fairly enumerating Presburger formulas.*

## 1 Introduction

Vector Addition Systems (VAS) or equivalently Petri Nets are one of the most popular formal methods for the representation and the analysis of parallel processes [2]. The reachability problem is central since many computational problems (even outside the parallel processes) reduce to the reachability problem. Sacerdote and Tenney provided in [10] a partial proof of the decidability of this problem. The proof was completed in 1981 by Mayr [9] and simplified by Kosaraju [7] from [10, 9]. Ten years later [8], Lambert provided a more simplified version based on [7]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known non-primitive recursive. Nowadays, the exact complexity of

the reachability problem for VAS is still an open-problem. Even an elementary upper-bound complexity is open. In fact, the known general reachability algorithms are exclusively based on the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition.

In this paper, by using the KLMST decomposition we prove that Parikh images of languages accepted by VAS are semi-pseudo-linear, a class of sets that can be precisely over-approximated by Presburger sets, or equivalently by semi-linear sets [3]. We provide an application of this result; we prove that if a final configuration is not reachable from an initial one, there exists a Presburger inductive invariant proving this property. Since we can decide with any decision procedure for the Presburger arithmetic if Presburger formulas denote inductive invariants, we deduce that there exist checkable certificates of non-reachability. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by fairly enumerating finite sequence of actions and a second one that tries to prove the non-reachability by fairly enumerating Presburger formulas. Note [5] that in general, reachability sets are not definable in the Presburger arithmetic. Presburger inductive invariants are obtained by observing that reachability sets are semi-pseudo-linear.

*Outline of the paper:* In Section 2 we introduce the VAS. In Section 3 we define the pseudo-linear sets, a class of sets that can be precisely over-approximated by linear sets called *linearization*. We also define the finite union of pseudo-linear sets called semi-pseudo-linear sets. In Section 4 the class of pseudo-linear sets is proved stable by linear function images. In Section 5 Parikh images of languages accepted by VAS are proved semi-pseudo-linear. In Section 6 we introduce a dimension function  $\dim : P(\mathbb{Z}^n) \rightarrow \{-\infty, 0, \dots, n\}$  and we provide few properties satisfied by this function. In Section 7 we prove that  $\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$  where  $L_1, L_2$  are linearizations of pseudo-linear sets  $X_1, X_2$  such that  $X_1 \cap X_2 = \emptyset$ . Finally in section 8 we show that if a final configuration is

not reachable from an initial one, there exists a Presburger inductive invariant proving this property.

## 2 Vector Addition Systems

In this section, we recall the definition of language accepted by a Vector Addition System.

*Some notations* : As usual we denote by  $\mathbb{Q}, \mathbb{Q}_+, \mathbb{Z}, \mathbb{N}$  respectively the set of rational values, non-negative rational values, the set of integers and the set of non-negative integers. The *cardinal* of a finite set  $X$  is denoted by  $|X|$ . The *components of a vector*  $\mathbf{x} \in \mathbb{Q}^n$  are denoted by  $(\mathbf{x}[1], \dots, \mathbf{x}[n])$ . Given a function  $f : E \rightarrow F$  where  $E, F$  are sets, we denote by  $f(X) = \{f(x) \mid x \in X\}$  for any subset  $X \subseteq E$ . This definition naturally defines sets  $X_1 + X_2$  where  $X_1, X_2 \subseteq \mathbb{Q}^n$ . With slight abuse of notation,  $\{\mathbf{x}_1\} + X_2$  and  $X_1 + \{\mathbf{x}_2\}$  are simply denoted by  $\mathbf{x}_1 + X_2$  and  $X_1 + \mathbf{x}_2$ . The total order  $\leq$  over  $\mathbb{Q}$  is extended component-wise to the partial order  $\leq$  over  $\mathbb{Q}^n$  satisfying  $\mathbf{x} \leq \mathbf{x}'$  if and only if  $\mathbf{x}[i] \leq \mathbf{x}'[i]$  for any  $1 \leq i \leq n$ . The set of minimal elements for  $\leq$  of a set  $X \subseteq \mathbb{N}^n$  is denoted by  $\min(X)$ . As  $(\mathbb{N}^n, \leq)$  is a well partially ordered set, note that  $\min(X)$  is finite and  $X \subseteq \min(X) + \mathbb{N}^n$  for any  $X \subseteq \mathbb{N}^n$ . An *alphabet* is a non-empty finite set  $\Sigma$ . Set of words over  $\Sigma$  are denoted by  $\Sigma^*$ . The number of occurrences  $a \in \Sigma$  in a word  $\sigma \in \Sigma^*$  is denoted by  $|\sigma|_a$ . A *Parikh image* of a language  $\mathcal{L} \subseteq \Sigma^*$  is a set  $X = \{(|\sigma|_{a_1}, \dots, |\sigma|_{a_n}) \mid \sigma \in \mathcal{L}\}$  where  $a_1, \dots, a_n$  is a finite sequence in  $\Sigma$ .

A *Vector Addition System (VAS)* is a tuple  $\mathcal{V} = (\Sigma, n, \delta)$  where  $\Sigma$  is an alphabet,  $n \in \mathbb{N}$  is the *dimension*, and  $\delta \in \Sigma \rightarrow \mathbb{Z}^n$  is the *displacement function*. A *configuration* is a vector in  $\mathbb{N}^n$ . The binary relation  $\xrightarrow{a}_{\mathcal{V}}$  where  $a \in \Sigma$  over the set of configurations is defined by  $\mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'$  if and only if  $\mathbf{s}' = \mathbf{s} + \delta(a)$ . Given a word  $\sigma = a_1 \dots a_k$  of  $k \in \mathbb{N}$  elements  $a_i \in \Sigma$ , we denote by  $\xrightarrow{\sigma}_{\mathcal{V}}$  the binary relation over the set of configurations that is equal to the concatenation  $\xrightarrow{a_1}_{\mathcal{V}} \dots \xrightarrow{a_k}_{\mathcal{V}}$  if  $k \geq 1$  and that is equal to the identity binary relation if  $k = 0$ . We also denote by  $\rightarrow_{\mathcal{V}}$  the *reachability binary relation* over the set of configurations defined by  $\mathbf{s} \rightarrow_{\mathcal{V}} \mathbf{s}'$  if and only if there exists  $\sigma \in \Sigma^*$  such that  $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'$ . The *language accepted* by a tuple  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  where  $(\mathbf{s}, \mathbf{s}')$  are two configurations of a VAS  $\mathcal{V}$  is the set  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}') = \{\sigma \in \Sigma^* \mid \mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'\}$ .

## 3 Semi-pseudo-linear Sets

In this section we recall the definition of semi-linear sets and we introduce the class of *pseudo-linear sets* and *semi-pseudo-linear sets*. Intuitively, a pseudo-linear set  $X$  is a set that can be precisely over-approximated by a *linear set*

$L$  called a *linearization* of  $X$ , and a semi-pseudo-linear set is a finite union of pseudo-linear sets.

Let us first recall the definition of *semi-linear sets*. The *sub-monoid* of  $(\mathbb{Z}^n, +)$  generated by a set  $X \subseteq \mathbb{Z}^n$  is denoted by  $X^* = \{\mathbf{0}\} \cup \{\sum_{i=1}^k \mathbf{x}_i \mid k \geq 1 \mathbf{x}_i \in X\}$ . A finite set  $P \subseteq \mathbb{Z}^n$  is called a *set of periods*. A set  $L \subseteq \mathbb{Z}^n$  is said *linear* [3] if there exists a vector  $\mathbf{b} \in \mathbb{Z}^n$  and a set of periods  $P \subseteq \mathbb{Z}^n$  such that  $L = \mathbf{b} + P^*$ . A *semi-linear set*  $S \subseteq \mathbb{Z}^n$  is a finite union of linear sets  $L_i \subseteq \mathbb{Z}^n$ . Recall [3] that sets definable in  $\text{FO}(\mathbb{Z}, +, \leq)$  are exactly the semi-linear sets and sets definable in  $\text{FO}(\mathbb{N}, +, \leq)$  also called *Presburger sets* are exactly the non-negative semi-linear sets.

The definition of pseudo-linear sets is based on the notion of the *interior of monoids*. The *interior of a monoid*  $M$  is the set  $\mathcal{I}(M)$  of vectors  $\mathbf{a} \in M$  such that for any  $\mathbf{x} \in M$  there exists an integer  $N \in \mathbb{N}$  such that  $N\mathbf{a} \in \mathbf{x} + M$ . We denote by  $\mathcal{I}(M)$  the *interior* of  $M$ . The following Lemma 3.1 characterizes the set  $\mathcal{I}(P^*)$  where  $P$  is a set of periods. In particular, this lemma shows that  $\mathcal{I}(P^*)$  is non empty.

**Lemma 3.1** *Let  $P = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$  be a set of periods with  $k \in \mathbb{N}$ . We have  $\mathcal{I}(P^*) = \{\mathbf{0}\}$  if  $k = 0$  and  $\mathcal{I}(P^*) = P^* \cap ((\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_k)$  if  $k \geq 1$ .*

*Proof* : Since the case  $k = 0$  is immediate, we assume that  $k \geq 1$ . Let us first consider an interior vector  $\mathbf{a} \in \mathcal{I}(P^*)$ . As  $\sum_{j=1}^k \mathbf{p}_j \in P^*$  and  $\mathbf{a} \in \mathcal{I}(P^*)$ , there exists  $N \in \mathbb{N}$  such that  $N\mathbf{a} \in (\sum_{j=1}^k \mathbf{p}_j) + P^*$ . Let  $\mathbf{p} \in P^*$  such that  $N\mathbf{a} = \sum_{j=1}^k \mathbf{p}_j + \mathbf{p}$ . As  $\mathbf{p} + \mathbf{a} \in P^*$ , there exists a sequence  $(N_j)_{1 \leq j \leq k}$  of elements in  $\mathbb{N}$  such that  $\mathbf{p} + \mathbf{a} = \sum_{j=1}^k N_j \mathbf{p}_j$ . Combining this equality with the previous one provides  $\mathbf{a} = \sum_{j=1}^k \frac{1+N_j}{1+N} \mathbf{p}_j$ . Thus  $\mathbf{a} \in (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_k$ . Conversely, let us consider  $\mathbf{a} \in P^* \cap ((\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{Q}_+ \setminus \{0\})\mathbf{p}_k)$ . Observe that there exists an integer  $d \geq 1$  large enough such that  $d\mathbf{a} \in (\mathbb{N} \setminus \{0\})\mathbf{p}_1 + \dots + (\mathbb{N} \setminus \{0\})\mathbf{p}_k$ . In particular for any  $\mathbf{x} \in P^*$  there exists  $N \in \mathbb{N}$  such that  $Nd\mathbf{a} \in \mathbf{x} + P^*$ .  $\square$

**Example 3.2** *Let  $P = \{(1, 1), (1, 0)\}$ . We have  $P^* = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[2] \leq \mathbf{x}[1]\}$  and the interior of  $P^*$  is equal to  $\mathcal{I}(P^*) = \{\mathbf{x} \in \mathbb{N}^2 \mid 0 < \mathbf{x}[2] < \mathbf{x}[1]\}$ .*

A set  $X \subseteq \mathbb{Z}^n$  is said *pseudo-linear* if there exists  $\mathbf{b} \in \mathbb{Z}^n$  and a set of periods  $P \subseteq \mathbb{Z}^n$  such that  $X \subseteq \mathbf{b} + P^*$  and such that for any finite set  $R \subseteq \mathcal{I}(P^*)$  there exists  $\mathbf{x} \in X$  such that  $\mathbf{x} + R^* \subseteq X$ . In this case,  $P$  is called a *linearizator* of  $X$  and the linear set  $L = \mathbf{b} + P^*$  is called a *linearization* of  $X$ . A *semi-pseudo-linear set* is a finite union of *pseudo-linear sets*.

**Example 3.3** The set  $P = \{(1, 1), (1, 0)\}$  is a linearizator of the pseudo-linear set  $X = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[2] \leq \mathbf{x}[1] \leq 2\mathbf{x}[2]\}$ . Moreover  $P^*$  is a linearization of  $X$ .

## 4 Pseudo-Linear Set Images

In this section, the class of pseudo-linear sets is proved stable by linear function images.

A function  $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n'}$  is said *linear* if there exists a matrix  $M \in \mathbb{Z}^{n \times n'}$  and a vector  $\mathbf{v} \in \mathbb{Z}^{n'}$  such that  $f(\mathbf{x}) = M\mathbf{x} + \mathbf{v}$  for any  $\mathbf{x} \in \mathbb{Z}^n$ .

**Proposition 4.1** Images  $X' = f(X)$  of pseudo-linear sets  $X$  by a linear function  $f$  are pseudo-linear. Moreover the linear set  $L' = f(L)$  is a linearization of  $X'$  for any linearization  $L$  of  $X$ .

*Proof* : Let us consider a linear function  $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n'}$  defined by a matrix  $M \in \mathbb{Z}^{n \times n'}$  and a vector  $\mathbf{v} \in \mathbb{Z}^{n'}$ . Let us consider a pseudo-linear set  $X \subseteq \mathbb{Z}^n$ . As  $X$  is pseudo-linear, there exists a linearization  $L$  of  $X$ . We are going to prove that  $L' = f(L)$  is a linearization of  $X' = f(X)$ . There exists a vector  $\mathbf{b} \in \mathbb{Z}^n$  and a set of periods  $P \subseteq \mathbb{Z}^n$  such that  $L = \mathbf{b} + P^*$ . Let us consider  $\mathbf{b}' = f(\mathbf{b})$  and  $P' = \{M\mathbf{p} \mid \mathbf{p} \in P\}$  and observe that  $L' = \mathbf{b}' + (P')^*$ . In particular  $L'$  is a linear set. Since  $X \subseteq L$  we deduce that  $X' \subseteq L'$ . Let us consider a set  $R' = \{\mathbf{r}'_1, \dots, \mathbf{r}'_d\}$  included in the interior of  $(P')^*$ . As  $\mathbf{r}'_i \in (P')^*$  there exists  $\mathbf{p}_i \in P^*$  such that  $\mathbf{r}'_i = M\mathbf{p}_i$ . Lemma 3.1 shows that  $\mathbf{r}'_i$  is a sum of vectors of the form  $\lambda_{i,\mathbf{p}}M\mathbf{p}$  over all  $\mathbf{p} \in P$  where  $\lambda_{i,\mathbf{p}} > 0$  is a rational value. There exists an integer  $n_i \geq 1$  large enough such that  $n_i\lambda_{i,\mathbf{p}} \in \mathbb{N} \setminus \{0\}$  for any  $\mathbf{p} \in P$ . We deduce that  $\mathbf{r}_i = \sum_{\mathbf{p} \in P} n_i\lambda_{i,\mathbf{p}}\mathbf{p}$  is a vector in  $P^*$ . Moreover, from Lemma 3.1 we deduce that  $\mathbf{r}_i$  is in the interior of  $P^*$ . Observe that  $n_i\mathbf{r}'_i = M\mathbf{r}_i$ . Let us consider the set  $R$  of vectors  $\mathbf{r}_i + k_i\mathbf{p}_i$  where  $k_i$  is an integer such that  $0 \leq k_i < n_i$ . As  $\mathbf{r}_i$  is in the interior of  $P^*$  and  $\mathbf{p}_i \in P^*$  we deduce that  $\mathbf{r}_i + k_i\mathbf{p}_i$  is also in the interior of  $P^*$ . We have proved that  $R \subseteq \mathcal{I}(P^*)$ . As  $L$  is a linearization of  $X$ , there exists  $\mathbf{x} \in X$  such that  $\mathbf{x} + R^* \subseteq X$ . We deduce that  $f(\mathbf{x}) + MR^* \subseteq X'$ . Let us consider  $\mathbf{x}' = f(\mathbf{x}) + M(\sum_{i=1}^d \mathbf{r}_i)$  and let us prove that  $\mathbf{x}' + (R')^* \subseteq X'$ . Consider  $\mathbf{r}' \in (R')^*$ . There exists a sequence  $(\mu'_i)_{1 \leq i \leq d}$  of integers in  $\mathbb{N}$  such that  $\mathbf{r}' = \sum_{i=1}^d \mu'_i \mathbf{r}'_i$ . The Euclid division of  $\mu'_i$  by  $n_i$  shows that  $\mu'_i = k_i + n_i\mu_i$  where  $\mu_i \in \mathbb{N}$  and  $0 \leq k_i < n_i$ . From  $n_i\mathbf{r}'_i = M\mathbf{r}_i$  we deduce that  $\mathbf{x}' + \mathbf{r}' = f(\mathbf{x}) + M(\sum_{i=1}^d (\mathbf{r}_i + k_i\mathbf{p}_i) + \sum_{i=1}^d \mu_i \mathbf{r}_i)$ . Observe that  $\mathbf{r}_i + k_i\mathbf{p}_i$  and  $\mathbf{r}_i$  are both in  $R$ . We have proved that  $\mathbf{x}' + \mathbf{r}' \in f(\mathbf{x}) + MR^*$ . Thus  $\mathbf{x}' + (R')^* \subseteq X'$ . We have proved that  $L'$  is a linearization of  $X'$ .  $\square$

## 5 Parikh Images

Let  $\mathcal{V}$  be a VAS and let  $\mathbf{s}, \mathbf{s}'$  be two configurations. In this section, we prove that Parikh images of  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  are semi-pseudo-linear. In sub-section 5.1 we recall the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition. This decomposition is used in the next sub-section 5.2 to establish the semi-pseudo-linearity of Parikh images of  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ .

### 5.1 The KLMST decomposition

We recall the KLMST decomposition by following notations introduced by Lambert [8].

We first extend the set of integers  $\mathbb{Z}$  with an additional element  $\top$ . The addition function  $+$  :  $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  is extended to the totally-defined function in  $(\mathbb{Z} \cup \{\top\}) \times (\mathbb{Z} \cup \{\top\}) \rightarrow (\mathbb{Z} \cup \{\top\})$  satisfying  $x_1 + x_2 = \top$  if  $x_1 = \top$  or  $x_2 = \top$ . With slight abuse of notation we denote by  $\top - x$  the element  $\top$  when  $x \in \mathbb{Z}$ .

In the sequel, the element  $\top$  is either interpreted as a “very large integer” or a “don’t care integer”. More formally, we denote by  $\mathbb{N}_\top$  the set  $\mathbb{N} \cup \{\top\}$ . The total order  $\leq$  over  $\mathbb{N}$  is extended over  $\mathbb{N}_\top$  by  $x_1 \leq x_2$  if and only if  $x_2 = \top \vee (x_1, x_2 \in \mathbb{N} \wedge x_1 \leq x_2)$ . The equality  $=$  over  $\mathbb{N}$  is also extended to a partial order  $\preceq$  over  $\mathbb{N}_\top$  by  $x_1 \preceq x_2$  if and only if  $x_2 = \top \vee (x_1, x_2 \in \mathbb{N} \wedge x_1 = x_2)$ . Intuitively element  $\top$  denotes a “very large integer” for the total order  $\leq$  whereas it denotes a “don’t care integer” for the partial order  $\preceq$ . Given a sequence  $(x_i)_{i \geq 0}$  in  $\mathbb{N}_\top$ , we denote by  $x = \lim_{i \rightarrow +\infty} x_i$  the element  $x = \top$  if for any  $r \in \mathbb{N}$  there exists  $i_0 \geq 0$  such that  $x_i \geq r$  for any  $i \geq i_0$  and the element  $x \in \mathbb{N}$  if there exists  $i_0 \geq 0$  such that  $x_i = x$  for any  $i \geq i_0$ . When  $x = \lim_{i \rightarrow +\infty} x_i$  exists we say that  $(x_i)_{i \geq 0}$  converges toward  $x$ .

We also extend the semantics of VAS. A vector in  $\mathbb{N}_\top^n$  is called an *extended configuration* of  $\mathcal{V}$ . With slight abuse of notation, the binary relation  $\xrightarrow{a}_\mathcal{V}$  where  $a \in \Sigma$  is extended over the set of extended configurations by  $\mathbf{x} \xrightarrow{a}_\mathcal{V} \mathbf{x}'$  if and only if  $\mathbf{x}' = \mathbf{x} + \delta(a)$ . Given a word  $\sigma = a_1 \dots a_k$  of  $k \in \mathbb{N}$  elements  $a_i \in \Sigma$ , we denote by  $\xrightarrow{\sigma}_\mathcal{V}$  the binary relation over the set of extended configurations that is equal to the concatenation  $\xrightarrow{a_1}_\mathcal{V} \dots \xrightarrow{a_k}_\mathcal{V}$  if  $k \geq 1$  and that is equal to the identity binary relation if  $k = 0$ . Given an extended configuration  $\mathbf{x}$  we denote by  $\mathbf{x} \xrightarrow{\sigma}_\mathcal{V}$  if there exists an extended configuration  $\mathbf{x}'$  such that  $\mathbf{x} \xrightarrow{\sigma}_\mathcal{V} \mathbf{x}'$  and symmetrically for any extended configuration  $\mathbf{x}'$  we denote by  $\xrightarrow{\sigma}_\mathcal{V} \mathbf{x}'$  if there exists an extended configuration  $\mathbf{x}$  such that  $\mathbf{x} \xrightarrow{\sigma}_\mathcal{V} \mathbf{x}'$ .

Next we recall some elements of graph theory. A *graph*  $G$  is a tuple  $G = (Q, \Sigma, T)$  where  $Q$  is a non-empty finite set of *states*,  $\Sigma$  is an alphabet, and  $T \subseteq Q \times \Sigma \times Q$  is a finite

set of *transitions*. A *path*  $\pi$  is a word  $\pi = t_1 \dots t_k$  of  $k \in \mathbb{N}$  transitions  $t_i \in T$  such that there exists  $q_0, \dots, q_k \in Q$  and  $a_1, \dots, a_k \in \Sigma$  such that  $t_i = (q_{j-1}, a_j, q_j)$  for any  $1 \leq j \leq k$ . In this case we say that  $\pi$  is a *path from*  $q_0$  *to*  $q_k$  *labelled by*  $\sigma = a_1 \dots a_k$  and we denote  $\pi$  by  $q_0 \xrightarrow{\sigma} q_k$  or simply  $q_0 \rightarrow_G q_k$ . Given a transition  $t \in T$ , we denote by  $|\pi|_t$  the number of occurrences of  $t$  in  $\pi$ . When  $q_0 = q_k$ , the path  $\pi$  is called a *cycle*. Let us recall the following lemma.

**Lemma 5.1 (Euler Cycles)** *Let  $G = (Q, \Sigma, T)$  be a strongly connected graph. For any sequence  $(\mu_t)_{t \in T}$  of integers  $\mu_t > 0$  satisfying the following equality for any state  $q_0 \in Q$ , there exists a cycle  $\pi$  such that  $|\pi|_t = \mu_t$  for any transition  $t \in T$ :*

$$\sum_{t=(q,a,q_0) \in T} \mu_t = \sum_{t'=(q_0,a,q') \in T} \mu_{t'}$$

A *graph vector*  $G = (Q, \Sigma, T)$  for  $\mathcal{V}$  is a graph such that  $Q \subseteq \mathbb{N}^n$  is a non-empty finite set of extended configurations, and  $T \subseteq Q \times \Sigma \times Q$  is a finite set of transitions  $(\mathbf{x}, a, \mathbf{x}')$  such that  $\mathbf{x} \xrightarrow{a} \mathbf{x}'$ . Even if the proof of the following lemma is immediate by induction over the length of  $\sigma$ , it is central in the KLMST decomposition. In fact a path  $\mathbf{x} \xrightarrow{\sigma} \mathbf{x}'$  implies the relation  $\mathbf{x} \xrightarrow{\sigma} \mathbf{x}'$ .

**Lemma 5.2 (Graph vector paths)** *For any  $\mathbf{x} \xrightarrow{\sigma} \mathbf{x}'$ , for any sequences  $(\mathbf{x}_c)_{c \in \mathbb{N}}$  and  $(\mathbf{x}'_c)_{c \in \mathbb{N}}$  of extended configurations that converge toward  $\mathbf{x} = \lim_{c \rightarrow +\infty} \mathbf{x}_c$  and  $\mathbf{x}' = \lim_{c \rightarrow +\infty} \mathbf{x}'_c$ , there exists  $c_0 \in \mathbb{N}$  such that  $\mathbf{x}_c \xrightarrow{\sigma} \mathbf{x}'_c$  for any  $c \geq c_0$ .*

A *marked graph vector* for  $\mathcal{V}$  is a tuple  $(\mathbf{m}, \mathbf{x}, G, \mathbf{x}', \mathbf{m}')$  where  $G$  is a graph vector,  $\mathbf{x}, \mathbf{x}'$  are two states of this graph vector, and  $\mathbf{m} \leq \mathbf{x}$  and  $\mathbf{m}' \leq \mathbf{x}'$  are two extended configurations.

A *marked graph vector sequences (MGVS)* for  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  is an alternating sequence  $\mathcal{U}$  of marked graph vectors for  $\mathcal{V}$  and actions of the following form where  $\mathbf{m}_0 = \mathbf{s}$  and  $\mathbf{m}'_k = \mathbf{s}'$ :

$$(\mathbf{m}_0, \mathbf{x}_0, G_0, \mathbf{x}'_0, \mathbf{m}'_0), a_1, \dots, a_k, (\mathbf{m}_k, \mathbf{x}_k, G_k, \mathbf{x}'_k, \mathbf{m}'_k)$$

The *language accepted* by a MGVS  $\mathcal{U}$  is the set  $\mathcal{L}(\mathcal{U})$  of words of the form  $\sigma_0 a_1 \sigma_1 \dots a_k \sigma_k$  such that for any  $0 \leq j \leq k$  there exists a path  $\mathbf{x}_j \xrightarrow{\sigma_j} \mathbf{x}'_j$  and there exists two configurations  $\mathbf{s}_j \leq \mathbf{m}_j$  and  $\mathbf{s}'_j \leq \mathbf{m}'_j$  such that:

$$\mathbf{s}_0 \xrightarrow{\sigma_0} \mathbf{s}'_0 \xrightarrow{a_1} \mathbf{s}_1 \xrightarrow{\sigma_1} \mathbf{s}'_1 \dots \mathbf{s}'_{k-1} \xrightarrow{a_k} \mathbf{s}_k \xrightarrow{\sigma_k} \mathbf{s}'_k$$

We observe that  $\mathcal{L}(\mathcal{U}) \subseteq \mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  since  $(\mathbf{s}_0, \mathbf{s}'_k) = (\mathbf{s}, \mathbf{s}')$ .

We now associate a characteristic linear system to a MGVS  $\mathcal{U}$ . Denoting by  $\mu_{j,t}$  the number of occurrences of

a transition  $t \in T_j$  in the path  $\mathbf{x}_j \xrightarrow{\sigma_j} \mathbf{x}'_j$  we get a non-negative sequence  $(\mu_{j,t})_t$  indexed by  $t \in T_j$ . We also obtain a sequence  $\xi = (\mathbf{s}_j, (\mu_{j,t})_t, \mathbf{s}'_j)_j$  indexed by  $0 \leq j \leq k$  said *associated* to  $\sigma$ . We observe that  $\xi$  is a non-negative integral solution of the following linear system called the *characteristic system* of the MGVS  $\mathcal{U}$  where  $\chi_{\mathbf{x}}(q) = 1$  if  $q = \mathbf{x}$  and where  $\chi_{\mathbf{x}}(q) = 0$  otherwise:

$$\left\{ \begin{array}{l} \text{for all } 1 \leq j \leq k \\ \mathbf{s}'_{j-1} + \delta(a_j) = \mathbf{s}_j \\ \\ \text{for all } 0 \leq j \leq k \\ \mathbf{s}_j + \sum_{t=(q,a,q') \in T_j} \mu_{j,t} \delta(a) = \mathbf{s}'_j \\ \\ \text{for all } 0 \leq j \leq k \text{ and for all } 1 \leq i \leq n \\ \mathbf{s}_j[i] = \mathbf{m}_j[i] \text{ if } \mathbf{m}_j[i] \in \mathbb{N} \\ \mathbf{s}'_j[i] = \mathbf{m}'_j[i] \text{ if } \mathbf{m}'_j[i] \in \mathbb{N} \\ \\ \text{for all } 0 \leq j \leq k \text{ and for all } q_j \in Q_j \\ \chi_{\mathbf{x}_j}(q_j) + \sum_{t=(q,a,q_j) \in T} \mu_{j,t} = \chi_{\mathbf{x}'_j}(q_j) + \sum_{t'=(q_j,a,q') \in T} \mu_{j,t'} \end{array} \right.$$

Naturally there exists non-negative integral solutions  $\xi$  of the characteristic system that are not associated to accepted words. In particular even if there exists non-negative integral solutions of the characteristic linear system we cannot conclude that  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ . However, under the following *perfect condition*, we can prove that  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ .

The homogeneous form of the characteristic system, obtained by replacing constants by zero is called the *homogeneous characteristic system* of  $\mathcal{U}$ . In the sequel, a solution of the homogeneous characteristic system is denoted by  $\xi_0 = (\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}_{0,j}')_j$ .

A *perfect MGVS*  $\mathcal{U}$  is a MGVS such that the graph  $G_j$  is strongly connected and  $\mathbf{x}_j = \mathbf{x}'_j$  for any  $0 \leq j \leq k$ , the characteristic system has an integral solution, there exists a non-negative rational solution  $\xi_0 = (\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}_{0,j}')_j$  of the homogeneous characteristic system satisfying the following additional inequalities where  $0 \leq j \leq k$  and  $1 \leq i \leq n$ :

- $\mathbf{s}_{0,j}[i] > 0$  if  $\mathbf{m}_j[i] = \top$ , and
- $\mathbf{s}'_{0,j}[i] > 0$  if  $\mathbf{m}'_j[i] = \top$ , and
- $\mu_{0,j,t} > 0$  for any  $t \in T_j$ .

and such that for any  $0 \leq j \leq k$  and  $1 \leq i \leq n$ :

- there exists a cycle  $\theta_j = (\mathbf{x}_j \xrightarrow{w_j} \mathbf{x}_j)$  such that  $\mathbf{m}_j \xrightarrow{w_j}$  and such that  $\mathbf{m}_j + \delta(w_j) \geq \mathbf{m}_j$  and  $\delta(w_j)[i] > 0$  if  $\mathbf{m}_j[i] \in \mathbb{N}$  and  $\mathbf{x}_j[i] = \top$ , and

- there exists a cycle  $\theta'_j = (\mathbf{x}'_j \xrightarrow{w'_j}_{G_j} \mathbf{x}'_j)$  such that  $\xrightarrow{w'_j}_{\mathcal{V}} \mathbf{m}'_j$  and such that  $\mathbf{m}'_j - \delta(w'_j) \geq \mathbf{m}'_j$  and  $-\delta(w'_j)[i] > 0$  if  $\mathbf{m}'_j[i] \in \mathbb{N}$  and  $\mathbf{x}'_j[i] = \top$ .

In the sequel, even if  $\mathbf{x}_j = \mathbf{x}'_j$  for any  $0 \leq j \leq k$ , we still use both notations  $\mathbf{x}_j$  and  $\mathbf{x}'_j$  in order to keep results symmetrical.

Let us recall without proof the fundamental decomposition theorem.

**Theorem 5.3 (Fundamental Decomposition[8])** *For any tuple  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ , we can effectively compute a finite sequence of perfect MGVS  $\mathcal{U}_1, \dots, \mathcal{U}_l$  for this tuple such that:*

$$\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}') = \mathcal{L}(\mathcal{U}_1) \cup \dots \cup \mathcal{L}(\mathcal{U}_l)$$

In the remaining of this section, we associate to a perfect MGVS  $\mathcal{U}$ , a non-negative integral solution  $\xi$  of its characteristic system and a non-negative integral solution  $\xi_0$  of its homogeneous characteristic system that explains why  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ . These two solutions  $\xi$  and  $\xi_0$  are respectively defined in Lemma 5.4 and Lemma 5.5.

**Lemma 5.4** *There exists a non-negative integral solution  $\xi = (\mathbf{s}_j, (\mu_{j,t})_t, \mathbf{s}'_j)_j$  of the characteristic system of a perfect MGVS such that  $\mu_{j,t} > 0$  for any  $0 \leq j \leq k$  and  $t \in T_j$  and such that  $\mathbf{s}_j \xrightarrow{w_j}_{\mathcal{V}}$  and  $\xrightarrow{w'_j}_{\mathcal{V}} \mathbf{s}'_j$  for any  $0 \leq j \leq k$ .*

**Lemma 5.5** *There exists a non-negative integral solution  $\xi_0 = (\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}'_{0,j})_j$  of the homogeneous characteristic system of a perfect MGVS  $\mathcal{U}$  such that  $\mu_{0,j,t} > |\theta_j|_t + |\theta'_j|_t$  for any  $0 \leq j \leq k$  and  $t \in T_j$ , and such that for any  $0 \leq j \leq k$  and  $1 \leq i \leq n$ :*

- $\mathbf{s}_{0,j} \geq \mathbf{0}$  and  $\mathbf{s}_{0,j}[i] > 0$  if and only if  $\mathbf{m}_j[i] = \top$ .
- $\mathbf{s}'_{0,j} \geq \mathbf{0}$  and  $\mathbf{s}'_{0,j}[i] > 0$  if and only if  $\mathbf{m}'_j[i] = \top$ .
- $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{0}$  and  $(\mathbf{s}_{0,j} + \delta(w_j))[i] > 0$  if and only if  $\mathbf{x}_j[i] = \top$ .
- $\mathbf{s}'_{0,j} - \delta(w'_j) \geq \mathbf{0}$  and  $(\mathbf{s}'_{0,j} - \delta(w'_j))[i] > 0$  if and only if  $\mathbf{x}'_j[i] = \top$ .

Now, let us consider a perfect MGVS  $\mathcal{U}$  and let us fix two tuples  $\xi = (\mathbf{s}_j, (\mu_{j,t})_t, \mathbf{s}'_j)_j$  and  $\xi_0 = (\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}'_{0,j})_j$  satisfying respectively Lemma 5.4 and Lemma 5.5. As  $\mu_{j,t} > 0$  for any  $t \in T_j$  and  $G_j$  is strongly connected, Lemma 5.1 shows that there exists a cycle  $\pi_j = (\mathbf{x}_j \xrightarrow{\sigma_j}_{G_j} \mathbf{x}_j)$  such that  $\mu_{j,t} = |\pi_j|_t$  for any  $t \in T_j$ . Note that we have  $\mathbf{s}_j + \delta(\sigma_j) = \mathbf{s}'_j$ . Moreover, as  $\mu_{j,t} - |\theta_j|_t + |\theta'_j|_t > 0$  for any  $t \in T_j$  we also deduce that there exists a cycle  $\pi_{0,j} =$

$(\mathbf{x}_j \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x}'_j)$  such that  $|\pi_{0,j}|_t = \mu_{0,j,t} - (|\theta_j|_t + |\theta'_j|_t)$  for any  $t \in T_j$ . Note that we have  $\mathbf{s}_{0,j} + \delta(w_j) + \delta(\sigma_{0,j}) + \delta(w'_j) = \mathbf{s}'_{0,j}$ .

In the sequel we provide technical lemmas that prove together that  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ . These lemmas are also used in the next sub-section 5.2.

**Lemma 5.6** *For any  $c \geq 0$  we have:*

$$\begin{aligned} \mathbf{s}_j + c\mathbf{s}_{0,j} &\xrightarrow{w_j^c}_{\mathcal{V}} \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) \\ \mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)) &\xrightarrow{(w'_j)^c}_{\mathcal{V}} \mathbf{s}'_j + c\mathbf{s}'_{0,j} \end{aligned}$$

**Lemma 5.7** *There exists  $c_0 \geq 0$  such that for any  $c \geq c_0$ :*

$$\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) \xrightarrow{\sigma_{0,j}^c}_{\mathcal{V}} \mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$$

**Lemma 5.8** *There exists  $c' \geq 0$  such that for any  $c \geq c'$ :*

$$\mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)) \xrightarrow{\sigma_j}_{\mathcal{V}} \mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$$

Now, let us consider an integer  $c \geq 0$  satisfying  $c \geq c_0$  and  $c \geq c'$  where  $c_0$  and  $c'$  are respectively defined by Lemma 5.7 and Lemma 5.8. Note that we have proved the following relation:

$$\mathbf{s}_j + c\mathbf{s}_{0,j} \xrightarrow{w_j^c \sigma_{0,j}^c \sigma_j (w'_j)^c}_{\mathcal{V}} \mathbf{s}'_j + c\mathbf{s}'_{0,j}$$

Therefore there exists a word in  $\mathcal{L}(\mathcal{U})$  associated to  $\xi + c\xi_0$ . In particular we have proved that  $\mathcal{L}(\mathcal{U}) \neq \emptyset$ .

## 5.2 Parikh images of perfect MGVS

Parikh images of  $\mathcal{L}(\mathcal{U})$  are proved pseudo-linear for any perfect MGVS  $\mathcal{U}$  for  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ . From Theorem 5.3 we deduce that Parikh images of  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  are semi-pseudo-linear.

Let us consider a perfect MGVS  $\mathcal{U}$  for  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ :

$$(\mathbf{m}_0, \mathbf{x}_0, G_0, \mathbf{x}'_0, \mathbf{m}'_0), a_1, \dots, a_k, (\mathbf{m}_k, \mathbf{x}_k, G_k, \mathbf{x}'_k, \mathbf{m}'_k)$$

We denote by  $H$  the non-negative integral solutions of the characteristic system of  $\mathcal{U}$  and we denote by  $H'$  the subset of  $H$  corresponding to the sequence  $\xi$  associated to a word in  $\mathcal{L}(\mathcal{U})$ . Observe that Parikh images of  $\mathcal{L}(\mathcal{U})$  are images by linear functions of  $H'$ . From Proposition 4.1 it is sufficient to prove that  $H'$  is pseudo-linear. Intuitively, a linearizator for  $H'$  is obtained by considering the set  $H_0$  of non-negative integral solutions of the homogeneous characteristic system. More formally, we are going to prove that  $P_0 = \min(H_0 \setminus \{\mathbf{0}\})$  is a linearizator for  $H'$ .

Let us consider  $\xi \in H$  and  $\xi_0 \in H_0$  satisfying the following Lemma 5.9. Since  $H_0 = P_0^*$ , we deduce that  $H'$  is included in the linear set  $\xi - \xi_0 + P_0^*$ .

**Lemma 5.9** *There exists  $\xi \in H$  and  $\xi_0 \in H_0$  such that  $\xi_0 + H \subseteq \xi + H_0$ .*

Now, let us consider a set  $R_0 = \{\xi_1, \dots, \xi_d\}$  included in the interior of  $H_0$ . We are going to prove that there exists  $\xi' \in H'$  such that  $\xi' + R_0^* \subseteq H'$ . We first prove the following lemma.

**Lemma 5.10** *For any  $\xi_i = ((s_{i,j})_j, (\mu_{i,j,t})_{j,t}, (s'_{i,j})_j)$  interior vector of  $H_0$  there exists a cycle  $\pi_{i,j} = (\mathbf{x}_j \xrightarrow{\sigma_{i,j}}_{G_j} \mathbf{x}'_j)$  such that  $\mu_{i,j,t} = |\pi_{i,j}|_t$  for any  $t \in T_j$  and any  $0 \leq j \leq k$ .*

Now, let us consider a solution  $\xi$  of the characteristic system and a solution  $\xi_0$  of the homogeneous characteristic system satisfying respectively Lemma 5.4 and Lemma 5.5.

**Lemma 5.11** *There exists  $c_i \geq 0$  such that for any  $1 \leq j \leq k$  and  $c \geq c_i$ :*

$$\begin{aligned} & \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + \mathbf{s}_{i,j} \\ & \xrightarrow{\sigma_{i,j}}_{\mathcal{V}} \\ & \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + \mathbf{s}'_{i,j} \end{aligned}$$

Now, let us consider an integer  $c \geq 0$  such that  $c \geq c_0$ ,  $c \geq c'$  and  $c \geq c_i$  for any  $1 \leq i \leq d$  where  $c_0, c', c_i$  are respectively defined in Lemma 5.7, Lemma 5.8 and Lemma 5.11. From these lemmas and Lemma 5.6 we deduce that for any sequence  $n_1, \dots, n_d \in \mathbb{N}$  we have the following relation:

$$\begin{aligned} & \mathbf{s}_j + c\mathbf{s}_{0,j} + \sum_{i=1}^d n_i \mathbf{s}_{i,j} \\ & \xrightarrow{w_j^c \sigma_{1,j}^{n_1} \dots \sigma_{d,j}^{n_d} \sigma_j \sigma_{0,j}^c (w_j)^c}_{\mathcal{V}} \\ & \mathbf{s}'_j + c\mathbf{s}'_{0,j} + \sum_{i=1}^d n_i \mathbf{s}'_{i,j} \end{aligned}$$

We have proved that there exists a word in  $\mathcal{L}(\mathcal{U})$  associated with  $\xi + c\xi_0 + \sum_{i=1}^d n_i \xi_i$ . Let  $\xi' = \xi + c\xi_0$ . We deduce that  $\xi' + R_0^* \subseteq H'$ . Thus  $H'$  is pseudo-linear. We deduce the following proposition:

**Proposition 5.12** *Parikh images of  $\mathcal{L}(\mathcal{U})$  are pseudo-linear for any perfect MGVS  $\mathcal{U}$  for  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ .*

From Theorem 5.3 and the previous Proposition 5.12, we get the following Theorem 5.13.

**Theorem 5.13** *Parikh images of  $\mathcal{L}(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  are semi-pseudo-linear.*

## 6 Dimension

In this section, we introduce a *dimension function*  $\dim : P(\mathbb{Z}^n) \rightarrow \{-\infty, 0, \dots, n\}$ .

The *dimension*  $\dim(X)$  of a non-empty set  $X \subseteq \mathbb{Z}^n$  is the minimal integer  $d \in \{0, \dots, n\}$  such that:

$$\sup_{k \geq 0} \frac{|X \cap \{-k, \dots, k\}^n|}{|\{-k, \dots, k\}^d|} < +\infty$$

The dimension of the empty-set set is denoted by  $\dim(\emptyset) = -\infty$ . Let us observe some immediate properties satisfied by the dimension function. First of all, we have  $\dim(X) \leq 0$  if and only if  $X$  is finite. The dimension function is monotonic  $\dim(X_1) \leq \dim(X_2)$  for any  $X_1 \subseteq X_2$ . Moreover it satisfies  $\dim(X_1 \cup X_2) = \max\{\dim(X_1), \dim(X_2)\}$  and  $\dim(X_1 + X_2) \leq \dim(X_1) + \dim(X_2)$ . In particular  $\dim(\mathbf{v} + X) = \dim(X)$  for any  $\mathbf{v} \in \mathbb{Z}^n$  and for any  $X \subseteq \mathbb{Z}^n$ .

## 7 Pseudo-linear Intersections

In this section we prove that linearizations  $L_1, L_2$  of two pseudo-linear sets  $X_1, X_2$  with an empty intersection  $X_1 \cap X_2 = \emptyset$  satisfy the strict inequality  $\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$ . In sub-section 7.1 we characterize the dimension of linear sets and pseudo-linear sets. This characterization is used in the next sub-section 7.2 to prove the strict inequality.

In these two sub-sections, *vector spaces* are used. A *vector space*  $V$  of  $\mathbb{Q}^n$  is a subset  $V \subseteq \mathbb{Q}^n$  that contains the zero vector  $\mathbf{0} \in V$ , that is stable by addition  $V + V \subseteq V$  and that is stable by product  $\lambda \mathbf{v} \in V$  for any  $\lambda \in \mathbb{Q}$  and for any  $\mathbf{v} \in V$ . Observe that for any set  $X \subseteq \mathbb{Q}^n$  the set  $V = \{\mathbf{0}\} \cup \{\sum_{i=1}^k \lambda_i \mathbf{x}_i \mid k \geq 1 \lambda_i \in \mathbb{Q} \mathbf{x}_i \in X\}$  is the unique minimal for the inclusion vector space that contains  $X$ . This vector space is called the vector space generated by  $X$ . Recall that for any vector space  $V$  of  $\mathbb{Q}^n$  there exists a finite set  $X \subseteq V$  that generates  $V$ . The minimal for  $\leq$  integer  $d \in \mathbb{N}$  such that there exists a finite set  $X$  that generated  $V$  is called the *rank* of  $V$  and it is denoted by  $\text{rank}(V)$ .

### 7.1 Dimension of (pseudo-)linear sets

In this section, we prove that the dimension of a pseudo-linear set  $X$  is equal to the rank of the vector space  $V$  generated by any linearizator  $P$  of  $X$ .

We first prove the following Lemmas 7.1.

**Lemma 7.1** *We have  $\dim(M) = \text{rank}(V)$  where  $V$  is the vector space generated by a monoid  $M$ .*

*Proof* : Since  $M \subseteq \mathbb{Z}^n \cap V$  it is sufficient to prove that  $\dim(M) \geq \text{rank}(V)$  and  $\dim(\mathbb{Z}^n \cap V) \leq \text{rank}(V)$ . Let us denote by  $\|\mathbf{x}\|_\infty = \max\{|\mathbf{x}[1]|, \dots, |\mathbf{x}[k]|\}$  the usual  $\infty$ -norm of a vector  $\mathbf{x} \in \mathbb{Q}^n$ . As  $M$  generates the vector space  $V$  recall that there exists a sequence  $\mathbf{m}_1, \dots, \mathbf{m}_d \in M$  with  $d = \text{rank}(V)$  that generates  $V$ . Since the case  $d = 0$  is immediate we assume that  $d \geq 1$ . We denote by  $f : \mathbb{Q}^d \rightarrow V$  the rational linear function  $f(\mathbf{x}) = \sum_{i=1}^d \mathbf{x}[i] \mathbf{m}_i$ .

Let us first prove that  $\dim(M) \geq d$ . By minimality of  $d = \text{rank}(V)$  note that  $f$  is injective. In particular the cardinal of  $f(\{0, \dots, k\}^d)$  is equal to  $(1+k)^d$ . Note that a vector  $\mathbf{m}$  in this set satisfies  $\|\mathbf{m}\|_\infty \leq k \sum_{i=1}^d \|\mathbf{m}_i\|_\infty$  and  $\mathbf{m} \in M$ . We deduce that  $\dim(M) \geq d$ .

Now, let us prove that  $\dim(\mathbb{Z}^n \cap V) \leq d$ . Since for any matrix, the rank of the column vectors is equal to the rank of the line vectors, there exists a sequence  $1 \leq j_1 < \dots < j_d \leq n$  such that the rational linear function  $g : \mathbb{Q}^n \rightarrow \mathbb{Q}^d$  defined by  $g(\mathbf{x}) = (\mathbf{x}[j_1], \dots, \mathbf{x}[j_d])$  satisfies  $h = g \circ f$  is a bijective rational linear function. In particular we deduce that for any  $\mathbf{v} \in \mathbb{Z}^n \cap V \cap \{-k, \dots, k\}^n$  there exists a vector  $\mathbf{x} = g(\mathbf{v}) \in \{-k, \dots, k\}^d$  such that  $\mathbf{v} = f \circ h^{-1}(\mathbf{x})$ . Therefore  $|\mathbb{Z}^n \cap V \cap \{-k, \dots, k\}^n| \leq (1+2k)^d$  for any  $k \in \mathbb{N}$ . We deduce that  $\dim(\mathbb{Z}^n \cap V) \leq d$ .  $\square$

**Lemma 7.2** *For any pseudo-linear set  $X \subseteq \mathbb{Z}^n$ , we have  $\dim(X) = \text{rank}(V)$  where  $V$  is the vector space generated by any linearizator  $P$  of  $X$ .*

*Proof* : Let  $P$  be a linearizator of a pseudo-linear set  $X$  and let  $V$  be the vector space generated by  $P$ . Note that there exists a vector  $\mathbf{b} \in \mathbb{Z}^n$  such that  $X \subseteq \mathbf{b} + P^*$ . From Lemma 7.1 we have  $\dim(\mathbf{b} + P^*) = \text{rank}(V)$ . In particular  $\dim(X) \leq \text{rank}(V)$ . Conversely, let us consider an interior vector  $\mathbf{a} \in \mathcal{I}(P^*)$  and observe that  $R = \{\mathbf{a}\} \cup (\mathbf{a} + P) \subseteq \mathcal{I}(P^*)$ . As  $X$  is pseudo-linear, there exists  $\mathbf{x} \in X$  such that  $\mathbf{x} + R^* \subseteq X$ . Note that the vector space generated by  $R$  is equal to  $V$ . Thus, from Lemma 7.1 we deduce that  $\dim(\mathbf{x} + R^*) = \text{rank}(V)$ . In particular  $\dim(X) \geq \text{rank}(V)$ . We have proved the equality  $\dim(X) = \text{rank}(V)$ .  $\square$

## 7.2 Pseudo-linear sets intersection

In this section we prove that linearizations  $L_1, L_2$  of two pseudo-linear sets  $X_1, X_2$  with an empty intersection  $X_1 \cap X_2 = \emptyset$  satisfy  $\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$ .

We first characterize the intersection of two linear sets.

**Lemma 7.3** *For any set of periods  $P_1, P_2$  there exists a set of periods  $P$  such that  $P_1^* \cap P_2^* = P^*$ . Moreover, for any  $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{Z}^n$ , there exists a finite set  $B \subseteq \mathbb{Z}^n$  such that  $(\mathbf{b}_1 + P_1^*) \cap (\mathbf{b}_2 + P_2^*) = B + (P_1^* \cap P_2^*)$ .*

*Proof* : Let us consider an enumeration  $\mathbf{p}_{i,1}, \dots, \mathbf{p}_{i,k_i}$  of the  $k_i \geq 0$  vectors in  $P_i$  where  $i \in \{1, 2\}$ . If  $k_1 = 0$  or if  $k_2 = 0$  then  $P_1^* = \{\mathbf{0}\}$  or  $P_2^* = \{\mathbf{0}\}$  and the lemma is immediate. Thus, we can assume that  $k_1, k_2 \geq 1$ .

Let us consider the set  $X$  of vectors  $(\lambda_1, \lambda_2) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$  such that  $\mathbf{b}_1 + \sum_{j=1}^{k_1} \lambda_1[j] \mathbf{p}_{1,j} = \mathbf{b}_2 + \sum_{j=1}^{k_2} \lambda_2[j] \mathbf{p}_{2,j}$ . Let us also consider the set  $X_0$  of vectors  $(\lambda_1, \lambda_2) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$  such that  $\sum_{j=1}^{k_1} \lambda_1[j] \mathbf{p}_{1,j} = \sum_{j=1}^{k_2} \lambda_2[j] \mathbf{p}_{2,j}$ . Observe that  $X = Z + X_0$  where  $Z$  is the finite set  $Z = \min(X)$  and  $X_0 = Z_0^*$  where  $Z_0$  is the finite set  $Z_0 = \min(X_0 \setminus \{\mathbf{0}\})$ .

Let us denote by  $B$  the finite set of vectors  $\mathbf{b} \in \mathbb{Z}^n$  such that there exists  $(\lambda_1, \lambda_2) \in Z$  satisfying  $\mathbf{b}_1 + \sum_{j=1}^{k_1} \lambda_1[j] \mathbf{p}_{1,j} = \mathbf{b} = \mathbf{b}_2 + \sum_{j=1}^{k_2} \lambda_2[j] \mathbf{p}_{2,j}$ . Let us also denote by  $P$  the finite set of vectors  $\mathbf{p} \in \mathbb{Z}^n$  such that there exists  $(\lambda_1, \lambda_2) \in Z_0$  satisfying  $\sum_{j=1}^{k_1} \lambda_1[j] \mathbf{p}_{1,j} = \mathbf{p} = \sum_{j=1}^{k_2} \lambda_2[j] \mathbf{p}_{2,j}$ . Remark that  $(\mathbf{b}_1 + P_1^*) \cap (\mathbf{b}_2 + P_2^*) = B + P^*$  and  $P_1^* \cap P_2^* = P^*$ .  $\square$

In order to prove the following proposition, we introduce the definition of *groups*. A *group*  $G$  of  $\mathbb{Z}^n$  is a monoïd of  $\mathbb{Z}^n$  such that any element admits an inverse  $-G \subseteq G$ . Observe that for any set  $X \subseteq \mathbb{Z}^n$ , the set  $G = X^* - X^*$  is the unique minimal for the inclusion sub-group of  $(\mathbb{Z}^n, +)$  that contains  $X$ . This group is called the sub-group of  $\mathbb{Z}^n$  generated by  $X$ . Now, let us consider the group  $G = M - M$  generated by a monoïd  $M$  and observe that a vector  $\mathbf{a}$  is in the interior of  $M$  if and only if for any  $\mathbf{g} \in G$  there exists  $N \in \mathbb{N}$  such that  $\mathbf{g} + N\mathbf{a} \in M$ .

**Lemma 7.4** *For any vector  $\mathbf{v} \in V$  where  $V$  is the vector space generated by a group  $G$ , there exists an integer  $d \geq 1$  such that  $d\mathbf{v} \in G$ .*

*Proof* : A vector  $\mathbf{v} \in V$  can be decomposed into a sum  $\mathbf{v} = \sum_{i=1}^k \lambda_i \mathbf{g}_i$  with  $k \in \mathbb{N}$ ,  $\lambda_i \in \mathbb{Q}$  and  $\mathbf{g}_i \in G$ . Let us consider an integer  $d \geq 1$  such that  $d\lambda_i \in \mathbb{Z}$  and observe that  $d\mathbf{v} \in G$ .  $\square$

Now, we prove the main result of this section.

**Proposition 7.5** *Let  $L_1, L_2$  be linearizations of pseudo-linear sets  $X_1, X_2 \subseteq \mathbb{Z}^n$  with an empty intersection  $X_1 \cap X_2 = \emptyset$ . We have:*

$$\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$$

*Proof* : Let  $L_1, L_2$  be linearizations of two pseudo-linear sets  $X_1, X_2 \subseteq \mathbb{Z}^n$ . For the moment, we do not assume that  $X_1 \cap X_2$  is empty. There exists some linearizators  $P_1, P_2$  of the pseudo-linear sets  $X_1, X_2$  and vectors  $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{Z}^n$



such that  $L_1 = \mathbf{b}_1 + P_1^*$  and  $L_2 = \mathbf{b}_2 + P_2^*$  are linearizations of  $X_1, X_2$ . Let us denote by  $V_1, V_2$  the vector spaces generated by  $P_1, P_2$ . Lemma 7.2 shows that  $\dim(X_1) = \text{rank}(V_1)$  and  $\dim(X_2) = \text{rank}(V_2)$ . From Lemma 7.3 there exists a set of periods  $P$  and a finite set  $B \subseteq \mathbb{Z}^n$  such that  $P_1^* \cap P_2^* = P^*$  and  $L_1 \cap L_2 = B + P^*$ . Observe that if  $B = \emptyset$  the proposition is immediate. Thus, we can assume that there exists  $\mathbf{b} \in B$ . Let  $V$  be the vector space generated by  $P$ . Lemma 7.1 shows that  $\dim(B + P^*) = \text{rank}(V)$ . Observe that  $V \subseteq V_1 \cap V_2$ . Thus, if there exists  $j \in \{1, 2\}$  such that  $V$  is strictly included in  $V_j$  then  $\text{rank}(V) < \text{rank}(V_j)$  and in this case  $\dim(L_1 \cap L_2) < \max\{\dim(X_1), \dim(X_2)\} = \dim(X_1 \cup X_2)$ .

So we can assume that  $V_1 = V = V_2$ . We prove in the sequel that  $X_1 \cap X_2 \neq \emptyset$ . We denote by  $G_1, G, G_2$  the groups generated respectively by  $P_1, P, P_2$ . Note that the vector spaces generated by  $G_1, G, G_2$  are equal to  $V_1, V, V_2$ .

Let  $\mathbf{a}$  be an interior vector of  $P^*$  and let us prove that  $\mathbf{a} \in \mathcal{I}(P_1^*) \cap \mathcal{I}(P_2^*)$ . Let  $j \in \{1, 2\}$ . Note that  $\mathbf{a} \in P^* \subseteq P_j^*$ . Let  $\mathbf{p} \in \mathcal{I}(P_j^*)$ . Since  $-\mathbf{p} \in V$  and  $V$  is the vector space generated by  $G$ , Lemma 7.4 shows that there exists an integer  $d \geq 1$  such that  $-d\mathbf{p} \in G$ . From  $\mathbf{a} - d\mathbf{p} \in G$  and  $\mathbf{a} \in \mathcal{I}(P^*)$  we deduce that there exists  $N \in \mathbb{N}$  such that  $\mathbf{a} - d\mathbf{p} + N\mathbf{a} \in P^*$ . From  $P^* \subseteq P_j^*$  we deduce that  $\mathbf{a} \in \frac{1}{1+N}(d\mathbf{p} + P_j^*)$ . From  $\mathbf{p} \in \mathcal{I}(P_j^*)$  and Lemma 3.1 we get  $\mathbf{a} \in \mathcal{I}(P_j^*)$ .

Let  $R_j = \{\mathbf{a}\} \cup (\mathbf{a} + P_j)$ . From  $\mathbf{a} \in \mathcal{I}(P_j^*)$ , Lemma 3.1 shows that  $R_j \subseteq \mathcal{I}(P_j^*)$ . As  $X_j$  is pseudo-linear, there exists  $\mathbf{x}_j \in X_j$  such that  $\mathbf{x}_j + R_j^* \subseteq X_j$ . From  $\mathbf{b}, \mathbf{x}_j \in \mathbf{b}_j + P_j^*$  we deduce that  $\mathbf{x}_j - \mathbf{b} \in G_j$ . As the group generated by  $R_j$  is equal to  $G_j$ , there exists  $\mathbf{r}_j, \mathbf{r}'_j \in R_j^*$  such that  $\mathbf{x}_j + \mathbf{r}_j = \mathbf{b} + \mathbf{r}'_j$ .

As  $V$  is the vector space generated by  $G_1$  and  $\mathbf{r}'_2 \in R_2^* \subseteq V_2 = V$ , Lemma 7.4 shows that there exists an integer  $d_1 \geq 1$  such that  $d_1\mathbf{r}'_2 \in G_1$ . As  $\mathbf{a} \in \mathcal{I}(P_1^*)$ , there exists an integer  $N_1 \geq 0$  such that  $d_1\mathbf{r}'_2 + N_1\mathbf{a} \in P_1^*$ . As  $P_1^* \subseteq R_1^* - \mathbb{N}\mathbf{a}$ , we deduce that there exists an integer  $N'_1 \geq 0$  such that  $d_1\mathbf{r}'_2 + (N_1 + N'_1)\mathbf{a} \in R_1^*$ . We denote by  $\mathbf{r}''_1$  this vector. Symmetrically, there exist some integers  $d_2 \geq 1$  and  $N_2, N'_2 \geq 0$  such that the vector  $d_2\mathbf{r}'_1 + (N_2 + N'_2)\mathbf{a}$  denoted by  $\mathbf{r}''_2$  is in  $R_2^*$ . We get:

$$\begin{aligned} \mathbf{x}_1 + \mathbf{r}_1 + (d_2 - 1)\mathbf{r}'_1 + \mathbf{r}''_1 + (N_2 + N'_2)\mathbf{a} \\ = \mathbf{b} + d_2\mathbf{r}'_1 + d_1\mathbf{r}'_2 + (N_1 + N'_1 + N_2 + N'_2)\mathbf{a} \\ \mathbf{x}_2 + \mathbf{r}_2 + (d_1 - 1)\mathbf{r}'_2 + \mathbf{r}''_2 + (N_1 + N'_1)\mathbf{a} \\ = \mathbf{b} + d_1\mathbf{r}'_2 + d_2\mathbf{r}'_1 + (N_2 + N'_2 + N_1 + N'_1)\mathbf{a} \end{aligned}$$

We have proved that these two last vectors are equal. We deduce that  $(\mathbf{x}_1 + R_1^*) \cap (\mathbf{x}_2 + R_2^*)$  is non empty. In particular  $X_1 \cap X_2 \neq \emptyset$ .  $\square$

## 8 Separators

The reachability problem for a tuple  $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$  consists to decide if  $\mathbf{s} \rightarrow_{\mathcal{V}} \mathbf{s}'$ . This problem can be reformulated by introducing the definition of separators. A pair  $(S, S')$  of configuration sets is called a *separator* for  $\mathcal{V}$  if  $\text{post}_{\mathcal{V}}^*(S) \cap \text{pre}_{\mathcal{V}}^*(S') = \emptyset$  where  $\text{post}_{\mathcal{V}}^*(S)$  and  $\text{pre}_{\mathcal{V}}^*(S')$  are respectively the set of *reachable states from S* and the set of *co-reachable states from S'* formally defined by:

$$\begin{aligned} \text{post}_{\mathcal{V}}^*(S) &= \{\mathbf{s}' \in \mathbb{N}^n \mid \exists \mathbf{s} \in S \quad \mathbf{s} \rightarrow_{\mathcal{V}} \mathbf{s}'\} \\ \text{pre}_{\mathcal{V}}^*(S') &= \{\mathbf{s} \in \mathbb{N}^n \mid \exists \mathbf{s}' \in S' \quad \mathbf{s} \rightarrow_{\mathcal{V}} \mathbf{s}'\} \end{aligned}$$

Naturally, a pair  $(\mathbf{s}, \mathbf{s}')$  is in the complement of the reachability relation  $\rightarrow_{\mathcal{V}}$  if and only if the pair  $(\{\mathbf{s}\}, \{\mathbf{s}'\})$  is a separator. A separator  $(I, I')$  is said *inductive* if  $I$  is a *forward invariant*  $\text{post}_{\mathcal{V}}^a(I) \subseteq I$  and  $I'$  is a *backward invariant*  $\text{pre}_{\mathcal{V}}^a(I') \subseteq I'$  for any  $a \in \Sigma$  where  $\text{post}_{\mathcal{V}}^a(S)$  and  $\text{pre}_{\mathcal{V}}^a(S')$  are defined for any  $S, S' \subseteq \mathbb{N}^n$  by:

$$\begin{aligned} \text{post}_{\mathcal{V}}^a(S) &= \{\mathbf{s}' \in \mathbb{N}^n \mid \exists \mathbf{s} \in S \quad \mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'\} \\ \text{pre}_{\mathcal{V}}^a(S') &= \{\mathbf{s} \in \mathbb{N}^n \mid \exists \mathbf{s}' \in S' \quad \mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'\} \end{aligned}$$

As  $(\text{post}_{\mathcal{V}}^*(S), \text{pre}_{\mathcal{V}}^*(S'))$  is an inductive separator for any separator  $(S, S')$ , we deduce that separators are included into inductive separators.

We are interested in inductive separators definable in the decidable Presburger logic  $\text{FO}(\mathbb{N}, +, \leq)$ . Note that a pair  $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$  of Presburger formulas denotes an inductive separator  $(I, I')$  if and only if  $\psi(\mathbf{x}) \wedge \psi'(\mathbf{x})$  and the following formulas are unsatisfiable for any  $a \in \Sigma$ . In particular we can effectively decide if  $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$  denotes an inductive separator.

$$\begin{aligned} \psi(\mathbf{x}) \quad \wedge \quad \mathbf{x}' = \mathbf{x} + \delta(a) \quad \wedge \quad \neg\psi(\mathbf{x}') \\ \psi'(\mathbf{x}') \quad \wedge \quad \mathbf{x}' = \mathbf{x} + \delta(a) \quad \wedge \quad \neg\psi'(\mathbf{x}) \end{aligned}$$

That means a pair  $(\phi(\mathbf{x}), \phi(\mathbf{x}'))$  of Presburger formulas denoting an inductive separators provides a *checkable certificate of non-reachability* for any pair  $(c, c')$  of configurations satisfying  $(\phi(\mathbf{x}), \phi(\mathbf{x}'))$ .

---

```

1 Reachability( $\mathbf{s} \in \mathbb{N}^n, \mathcal{V} = (\Sigma, n, T)$  a VAS,  $\mathbf{s}' \in \mathbb{N}^n$ )
2   repeat forever
3     fairly select  $\sigma \in \Sigma^*$ 
4       if  $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'$ 
5         return "reachable"
6     fairly select  $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ 
7       formulas in  $\text{FO}(\mathbb{N}, +, \leq)$ 
8     if  $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ 
9       inductive separator for  $(\{\mathbf{s}\}, \{\mathbf{s}'\})$ 
10      return "unreachable"

```

---

In this section we prove that Presburger separators are included in Presburger inductive separators. We deduce that algorithm  $\text{Reachability}(s, \mathcal{V}, s')$  decides the reachability problem. The termination is guaranteed by the previous result. Note [5] that in general  $(\text{post}_{\mathcal{V}}^*(S), \text{pre}_{\mathcal{V}}^*(S'))$  is not Presburger even if  $S$  and  $S'$  are reduced to single vectors  $S = \{s\}$  and  $S' = \{s'\}$ . That means, this inductive separator must be over-approximated by another inductive separator. Intuitively, the approximation is obtained by observing that for any Presburger sets  $S, S'$  of configurations, the sets  $\text{post}_{\mathcal{V}}^*(S) \cap S'$  and  $S \cap \text{pre}_{\mathcal{V}}^*(S')$  are *semi-pseudo-linear*. This property is proved in sub-section 8.1. In the next sub-section 8.2, we provide an induction to compute Presburger inductive separators that over-approximate Presburger separators.

## 8.1 Reachability sets

We prove that  $\text{post}_{\mathcal{V}}^*(S) \cap S'$  and  $S \cap \text{pre}_{\mathcal{V}}^*(S')$  are semi-pseudo-linear for any semi-linear sets  $S, S' \subseteq \mathbb{N}^n$ .

Since semi-linear sets are finite unions of linear sets we only prove this result for the special case of two linear sets  $S = s + P^*$  and  $S' = s' + (P')^*$  where  $s, s' \in \mathbb{N}^n$  and where  $P, P'$  are two set of periods of  $\mathbb{N}^n$ . We consider two alphabets  $\Sigma_P, \Sigma_{P'}$  disjoint of  $\Sigma$  and a displacement function  $\bar{\delta}$  defined over  $\bar{\Sigma} = \Sigma_P \cup \Sigma \cup \Sigma_{P'}$  that extends  $\delta$  such that:

$$P = \{\bar{\delta}(a) \mid a \in \Sigma_P\} \quad P' = \{-\bar{\delta}(a) \mid a \in \Sigma_{P'}\}$$

We consider the VAS  $\bar{\mathcal{V}} = (\bar{\Sigma}, n, \bar{\delta})$ . Let us consider the displacement functions  $\bar{\delta}_P$  and  $\bar{\delta}_{P'}$  defined over  $\bar{\Sigma}$  by:

$$\bar{\delta}_P(a) = \begin{cases} \bar{\delta}(a) & \text{if } a \in \Sigma_P \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$$\bar{\delta}_{P'}(a) = \begin{cases} -\bar{\delta}(a) & \text{if } a \in \Sigma_{P'} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Just observe that  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S' = s' + \bar{\delta}_{P'}(\mathcal{L}(s, \bar{\mathcal{V}}, s'))$  and  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = s + \bar{\delta}_P(\mathcal{L}(s, \bar{\mathcal{V}}, s'))$ . In particular these two sets are images by linear functions of Parikh images of  $\mathcal{L}(s, \bar{\mathcal{V}}, s')$ . Theorem 5.13 shows that Parikh images of  $\mathcal{L}(s, \bar{\mathcal{V}}, s')$  are semi-pseudo-linear. From Proposition 4.1 we deduce the following theorem:

**Theorem 8.1** *For any semi-linear sets  $S, S' \subseteq \mathbb{N}^n$ , the sets  $\text{post}_{\bar{\mathcal{V}}}^*(S) \cap S'$  and  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S')$  are semi-pseudo-linear.*

## 8.2 Induction with domains

Given a pair  $(S, S') \subseteq (\mathbb{N}^n, \mathbb{N}^n)$  of disjoint sets, the set  $D = \mathbb{N}^n \setminus (S \cup S')$  is called the *domain* of  $(S, S')$ . Note that

there exists inductive separators with non-empty domains. However, a separator with an empty domain is necessary inductive.

The Presburger inductive separator that over-approximate a Presburger separator  $(S_0, S'_0)$  is obtain inductively. We build a non-decreasing sequence  $(S_j, S'_j)_{j \geq 0}$  of Presburger separators starting from the initial Presburger separator  $(S_0, S'_0)$  such that the dimension of the domain  $D_j = \mathbb{N}^n \setminus (S_j \cup S'_j)$  is strictly decreasing. In order to obtain this sequence, observe that it is sufficient to show that for any Presburger separator  $(S_0, S'_0)$  with a non-empty domain  $D_0$ , there exists a Presburger separator  $(S, S') \supseteq (S_0, S'_0)$  with a domain  $D$  such that  $\dim(D) < \dim(D_0)$ .

We first define a set  $S'$  that over-approximates  $S'_0$  and such that  $(S_0, S')$  remains a separator. As  $S_0$  and  $D_0$  are Presburger, Theorem 8.1 shows that  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap D_0$  is equal to a finite union of pseudo-linear sets  $X_1, \dots, X_k$ . Let us consider some linearizations  $L_1, \dots, L_k$  of these pseudo-linear sets and let us define the following Presburger set  $S'$ .

$$S' = S'_0 \cup (D_0 \setminus (\bigcup_{j=1}^k L_j))$$

We observe that  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap S' = \emptyset$  since  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap S'_0 = \emptyset$  and  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap D_0 \subseteq \bigcup_{j=1}^k L_j$ . Thus  $\text{post}_{\bar{\mathcal{V}}}^*(S_0) \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = \emptyset$  and we have proved that  $S'$  contains  $S'_0$  and  $(S_0, S')$  is a separator.

Now we define symmetrically a set  $S$  that over-approximates  $S_0$  and such that  $(S, S')$  remains a separator. As  $D_0$  and  $S'$  are Presburger, Theorem 8.1 shows that  $D_0 \cap \text{pre}_{\bar{\mathcal{V}}}^*(S')$  is equal to a finite union of pseudo-linear sets  $X'_1, \dots, X'_{k'}$ . Let us consider some linearizations  $L'_1, \dots, L'_{k'}$  of these pseudo-linear sets and let us define the following Presburger set  $S$ .

$$S = S_0 \cup (D_0 \setminus (\bigcup_{j'=1}^{k'} L'_{j'}))$$

Once again, note that  $S \cap \text{pre}_{\bar{\mathcal{V}}}^*(S') = \emptyset$ . Thus  $S$  contains  $S_0$  and  $(S, S')$  is a separator.

Let  $D$  be the domain of the separator  $(S, S')$ . From  $D_0 = \mathbb{N}^n \setminus (S_0 \cup S'_0)$ , we get the following equality.

$$D = D_0 \cap \left( \bigcup_{\substack{1 \leq j \leq k \\ 1 \leq j' \leq k'}} (L_j \cap L'_{j'}) \right)$$

From  $X_j, X'_{j'} \subseteq D_0$  we get  $\dim(X_j \cup X'_{j'}) \leq \dim(D_0)$ . As  $X_j \subseteq \text{post}_{\bar{\mathcal{V}}}^*(S_0) \subseteq \text{post}_{\bar{\mathcal{V}}}^*(S)$  and  $X'_{j'} \subseteq \text{pre}_{\bar{\mathcal{V}}}^*(S')$

and  $(S, S')$  is a separator, we deduce that  $X_j$  and  $X'_j$  are two pseudo-linear sets with an empty intersection. From the main result proved in section 3, we get  $\dim(L_j \cap L'_j) < \dim(X_j \cup X'_j)$ . We deduce  $\dim(D) < \dim(D_0)$ . We have proved the following theorem.

**Theorem 8.2** *Presburger separators are included in Presburger inductive separators.*

## 9 Conclusion

Thanks to the classical KLMST decomposition we have proved that Parikh Images of languages accepted by VASs are semi-pseudo-linear.

As application, we have proved the termination of a simple algorithm for deciding the reachability problem for VAS. Even though the proof of termination is based on the classical KLMST decomposition, the complexity of the algorithm does not depend on this decomposition. In fact, the complexity depends on the size of the minimal pair of Presburger formulas denoting an inductive separator when  $(\{c\}, \{c'\})$  is separable and the size of a minimal  $\sigma \in \Sigma^*$  such that  $c \xrightarrow{\sigma} c'$  otherwise. This algorithm is the *very first one* that does not require the KLMST decomposition for its implementation.

We left as an open question the problem of computing a lower bound and an upper bound of the size of a pair of Presburger formulas denoting an inductive separator. Note that the VAS exhibiting a large (Ackermann size) but finite reachability set given in [4] does not directly provide a lower-bound for this size since inductive separators can over-approximate reachability sets.

We also left as an open question the problem of adapting such an algorithm to obtain a complete *Counter Example Guided Abstract Refinement* approach [1] for the VAS reachability problem based for instance on *interpolants* [6] for FO  $(\mathbb{N}, +, \leq)$ . In practice, such an algorithm should be more efficient than the enumeration-based algorithm provided in this paper.

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## A Proofs of Section 5.1

**Lemma 5.4** *There exists a non-negative integral solution  $\xi = (s_j, (\mu_{j,t})_t, s'_j)_j$  of the characteristic system of a perfect MGVS such that  $\mu_{j,t} > 0$  for any  $0 \leq j \leq k$  and  $t \in T_j$  and such that  $s_j \xrightarrow{w_j} \nu$  and  $s'_j \xrightarrow{w'_j} \nu$  for any  $0 \leq j \leq k$ .*

*Proof* : The definition of perfect MGVS requires that there exists an integral solution  $\xi = (s_j, (\mu_{j,t})_t, s'_j)_j$  of its characteristic system. This solution is non-necessary non-negative. However, there exists a non-negative rational solution  $\xi_0 = (s_{0,j}, (\mu_{0,j,t})_t, s'_{0,j})_j$  of the homogeneous characteristic system satisfying the perfect MGVS condition. Naturally, by replacing  $\xi_0$  by a sequence in  $(\mathbb{N} \setminus \{0\})\xi_0$  we can assume that  $\xi_0$  is a non-negative integral solution also satisfying the perfect MGVS condition. Now, just observe that there exists an integer  $c_0 \geq 0$  large enough such that  $\xi + c_0\xi_0$  is a non-negative integral solution of the characteristic system satisfying  $\mu_{j,t} + c_0\mu_{0,j,t} > 0$  for any  $t \in T_j$  and for any  $0 \leq j \leq k$ . Moreover, as  $\lim_{c \rightarrow +\infty} (s_j + cs_{0,j}) = m_j$  and  $\lim_{c \rightarrow +\infty} (s'_j + cs'_{0,j}) = m'_j$ , the relations  $m_j \xrightarrow{\sigma_j} \nu$  and  $m'_j \xrightarrow{\sigma'_j} \nu$  and Lemma 5.2 shows that there exists an integer  $c \geq c_0$  large enough such that  $(s_j + cs_{0,j}) \xrightarrow{w_j} \nu$  and  $(s'_j + cs'_{0,j}) \xrightarrow{w'_j} \nu$ . Therefore  $\xi + c\xi_0$  is a non-negative integral solution of the characteristic system satisfying the lemma.  $\square$

**Lemma 5.5** *There exists a non-negative integral solution  $\xi_0 = (s_{0,j}, (\mu_{0,j,t})_t, s'_{0,j})_j$  of the homogeneous characteristic system of a perfect MGVS  $\mathcal{U}$  such that  $\mu_{0,j,t} > |\theta_j|_t + |\theta'_j|_t$  for any  $0 \leq j \leq k$  and  $t \in T_j$ , and such that for any  $0 \leq j \leq k$  and  $1 \leq i \leq n$ :*

- $s_{0,j} \geq 0$  and  $s_{0,j}[i] > 0$  if and only if  $m_j[i] = \top$ .
- $s'_{0,j} \geq 0$  and  $s'_{0,j}[i] > 0$  if and only if  $m'_j[i] = \top$ .
- $s_{0,j} + \delta(w_j) \geq 0$  and  $(s_{0,j} + \delta(w_j))[i] > 0$  if and only if  $x_j[i] = \top$ .
- $s'_{0,j} - \delta(w'_j) \geq 0$  and  $(s'_{0,j} - \delta(w'_j))[i] > 0$  if and only if  $x'_j[i] = \top$ .

*Proof* : Let  $\xi_0 = (s_{0,j}, (\mu_{0,j,t})_t, s'_{0,j})_j$  be a non-negative rational solution of the homogeneous characteristic system satisfying the perfect MGVS condition. By replacing  $\xi_0$  by  $(\mathbb{N} \setminus \{0\})\xi_0$  we can assume that  $\xi_0$  is a non-negative integral solution satisfying the perfect condition. We are going to prove that there exists an integer  $c \in \mathbb{N}$  large enough such that  $c\xi_0$  satisfies the lemma.

First of all, observe that for any  $c \geq 1$  and for any  $1 \leq i \leq n$ , we have:

- $cs_{0,j} \geq 0$  and  $cs_{0,j}[i] > 0$  if and only if  $m_j[i] = \top$ .
- $cs'_{0,j} \geq 0$  and  $cs'_{0,j}[i] > 0$  if and only if  $m'_j[i] = \top$ .

Let us consider  $1 \leq i \leq n$  and let us prove that there exists an integer  $c_i \geq 0$  such that for any  $c \geq c_i$  we have  $(cs_{0,j} + \delta(w_j))[i] \geq 0$  and  $(cs_{0,j} + \delta(w_j))[i] > 0$  if and only if  $x_j[i] = \top$ . Note that  $m_j[i] \leq x_j[i]$  thus either  $m_j[i] = x_j[i] \in \mathbb{N}$ , or  $(m_j[i], x_j[i]) \in \mathbb{N} \times \{\top\}$ , or  $m_j[i] = x_j[i] = \top$ . We separate the proof following these three cases. Let us first consider the case  $m_j[i] = x_j[i] \in \mathbb{N}$ . As  $m_j[i] \in \mathbb{N}$  and  $\xi_0$  is solution of the homogeneous characteristic system, we get  $s_{0,j}[i] = 0$ . Moreover the cycle  $\theta_j = (x_j \xrightarrow{w_j}_{G_j} x_j)$  shows that  $x_j + \delta(w_j) = x_j$ . From  $x_j[i] \in \mathbb{N}$  we deduce that  $\delta(w_j)[i] = 0$ . In particular  $(cs_{0,j} + \delta(w_j))[i] = 0$  and we have proved the case  $m_j[i] = x_j[i] \in \mathbb{N}$  by considering  $c_i = 0$ . Let us consider the second case  $(m_j[i], x_j[i]) \in \mathbb{N} \times \{\top\}$ . As in the previous case, since  $m_j[i] \in \mathbb{N}$  we deduce that  $s_{0,j}[i] = 0$ . Note that the perfect condition shows that  $\delta(w_j)[i] > 0$  in this case. In particular for any  $c \geq 0$  we have  $(cs_{0,j} + \delta(w_j))[i] > 0$  and we have proved the case  $(m_j[i], x_j[i]) \in \mathbb{N} \times \{\top\}$  by considering  $c_i = 0$ . Finally, let us consider the case  $m_j[i] = x_j[i] = \top$ . As  $m_j[i] = \top$  we deduce that  $s_{0,j}[i] > 0$  in particular there exists an integer  $c_i \geq 0$  large enough such that  $(cs_{0,j} + \delta(w_j))[i] > 0$  for any  $c \geq c_i$ . We have proved the three cases.

Symmetrically, for any  $1 \leq i \leq n$ , there exists an integer  $c'_i \geq 0$  such that for any  $c \geq c'_i$  we have  $(cs'_{0,j} - \delta(w'_j))[i] \geq 0$  and  $(cs'_{0,j} - \delta(w'_j))[i] > 0$  if and only if  $x'_j[i] = \top$ .

Finally, as  $\mu_{0,j,t} > 0$  for any  $t \in T_j$  and for any  $0 \leq j \leq k$ , we deduce that there exists an integer  $c \geq 0$  large enough such that  $c\mu_{0,j,t} > |\theta_j|_t + |\theta'_j|_t$  for any  $t \in T_j$  for any  $0 \leq j \leq k$ . Naturally, we can also assume that  $c \geq 1$ ,  $c \geq c_i$  and  $c \geq c'_i$  for any  $1 \leq i \leq n$ . We deduce that  $c\xi_0$  satisfies the lemma.  $\square$

**Lemma 5.6** *For any  $c \geq 0$  we have:*

$$\begin{aligned} s_j + cs_{0,j} &\xrightarrow{w_j^c} \nu & s_j + c(s_{0,j} + \delta(w_j)) \\ s'_j + c(s'_{0,j} - \delta(w'_j)) &\xrightarrow{(w'_j)^c} \nu & s'_j + cs'_{0,j} \end{aligned}$$

*Proof* : Since the two relations are symmetrical, we just prove the first one. The choice of  $\xi$  satisfying Lemma 5.4 shows that  $s_j \xrightarrow{w_j} \nu$ . The conditions  $s_{0,j} \geq 0$ ,  $s_{0,j} + \delta(w_j) \geq 0$  and  $s_j \xrightarrow{w_j} \nu$  with an immediate induction on the integer  $c \geq 0$  provides the required relation.  $\square$

**Lemma 5.7** *There exists  $c_0 \geq 0$  such that for any  $c \geq c_0$ :*

$$s_j + c(s_{0,j} + \delta(w_j)) \xrightarrow{\sigma_{0,j}^c} \nu \quad s_j + c(s'_{0,j} - \delta(w'_j))$$

*Proof* : Let us recall that  $\mathbf{x}_i = \mathbf{x}'_i$ . We denote by  $\mathbf{u}$  the vector in  $\{0, 1\}^n$  satisfying  $\mathbf{u}[i] = 1$  if  $\mathbf{x}_i[i] = \top = \mathbf{x}'_i[i]$  and satisfying  $\mathbf{u}[i] = 0$  otherwise. From the choice of  $\xi_0$  satisfying Lemma 5.5, we observe that  $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{u}$  and  $\mathbf{s}'_{0,j} - \delta(w'_j) \geq \mathbf{u}$ . Note that  $\lim_{c \rightarrow +\infty} (\mathbf{s}_j + c\mathbf{u}) = \mathbf{x}_j$ . As  $\mathbf{x}_j \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x}'_j$ , Lemma 5.2 proves that there exists an integer  $c_0 \geq 0$  such that  $\mathbf{s}_j + c_0\mathbf{u} \xrightarrow{\sigma_{0,j}}_{\mathcal{V}}$ . Now, let us consider integers  $c \geq 1$  and  $c' \geq 0$  such that  $c + c' \geq c_0$  and let us prove the relation:

$$\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + c'(\mathbf{s}'_{0,j} - \delta(w'_j)) \xrightarrow{\sigma_{0,j}}_{\mathcal{V}}$$

$$\mathbf{s}_j + (c-1)(\mathbf{s}_{0,j} + \delta(w_j)) + (c'+1)(\mathbf{s}'_{0,j} - \delta(w'_j))$$

From  $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{u}$  and  $\mathbf{s}'_{0,j} - \delta(w'_j) \geq \mathbf{u}$  we deduce that  $c(\mathbf{s}_{0,j} + \delta(w_j)) + c'(\mathbf{s}'_{0,j} - \delta(w'_j)) \geq (c+c')\mathbf{u} \geq c_0\mathbf{u}$ . Thus, the previous relation directly comes from  $\mathbf{s}_j + c_0\mathbf{u} \xrightarrow{\sigma_{0,j}}_{\mathcal{V}}$  and  $\mathbf{s}_{0,j} + \delta(w_j) + \delta(\sigma_{0,j}) + \delta(w'_j) = \mathbf{s}'_{0,j}$ . Now, an immediate induction provides  $\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) \xrightarrow{\sigma_{0,j}^c}_{\mathcal{V}}$   $\mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$  for any  $c \geq c_0$ .  $\square$

**Lemma 5.8** *There exists  $c' \geq 0$  such that for any  $c \geq c'$ :*

$$\mathbf{s}_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)) \xrightarrow{\sigma_j}_{\mathcal{V}} \mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))$$

*Proof* : As  $\lim_{c \rightarrow +\infty} (\mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j))) = \mathbf{x}'_j$  and  $\mathbf{x}_j \xrightarrow{\sigma_j}_{G_j} \mathbf{x}'_j$ , Lemma 5.2 proves that there exists  $c' \geq 0$  such that  $\xrightarrow{\sigma_j}_{\mathcal{V}} (\mathbf{s}'_j + c(\mathbf{s}'_{0,j} - \delta(w'_j)))$  for any  $c \geq c'$ . Since  $\mathbf{s}_j + \delta(\sigma_j) = \mathbf{s}'_j$  we are done.  $\square$

## B Proofs of Section 5.2

**Lemma 5.9** *There exists  $\xi \in H$  and  $\xi_0 \in H_0$  such that  $\xi_0 + H \subseteq \xi + H_0$ .*

*Proof* : As the MGVS  $\mathcal{U}$  is perfect the set  $H$  is non empty. Let us consider the set  $I$  of components  $i$  such that  $\xi[i]$  is independent of  $\xi \in H$ . As the MGVS is perfect we deduce that for any integer  $c \geq 0$  there exists  $\xi \in H$  such that  $\xi[i] \geq c$  for any  $i \notin I$ . As  $\min(H)$  is finite, we deduce that there exists  $\xi \in H$  such that  $\xi \geq \xi'$  for any  $\xi' \in \min(H)$ . In particular  $\xi_0 = \sum_{\xi' \in \min(H)} (\xi - \xi')$  is in  $H_0$ . Let us prove that  $\xi_0 + H \subseteq \xi + H_0$ . Consider  $\xi'' \in H$ . By definition of  $\min(H)$ , there exists  $\xi''' \in \min(H)$  such that  $\xi''' \leq \xi''$ . The definition of  $\xi_0$  shows that  $\xi_0 - (\xi - \xi''')$  is equal to a sum of terms  $(\xi - \xi')$  indexed by  $\xi' \in \min(H) \setminus \{\xi'''\}$ . Therefore  $\xi_0 - (\xi - \xi''') \in H_0$ . As  $\xi'' - \xi''' \in H_0$  we have proved that the sum of  $\xi_0 - (\xi - \xi''')$  and  $\xi'' - \xi'''$  is also in  $H_0$ . Note that this sum is equal to  $\xi_0 - \xi + \xi''$ . We have proved that  $\xi_0 + \xi'' \in \xi + H_0$ . Therefore  $\xi_0 + H \subseteq \xi + H_0$ .  $\square$

**Lemma 5.10** *For any  $\xi_i = ((\mathbf{s}_{i,j})_j, (\mu_{i,j,t})_{j,t}, (\mathbf{s}'_{i,j})_j)$  interior vector of  $H_0$  there exists a cycle  $\pi_{i,j} = (\mathbf{x}_j \xrightarrow{\sigma_{i,j}}_{G_j} \mathbf{x}'_j)$  such that  $\mu_{i,j,t} = |\pi_{i,j}|_t$  for any  $t \in T_j$  and any  $0 \leq j \leq k$ .*

*Proof* : Since  $\mathcal{U}$  is perfect, for any  $t \in T_j$ , there exists a solution  $\xi_0 = (\mathbf{s}_{0,j}, (\mu_{0,j,t})_t, \mathbf{s}'_{0,j})_j$  in  $H_0$  such that  $\mu_{0,j,t} > 0$ . As  $H_0 = P_0^*$ , for any  $t \in T_j$  there exists  $\xi_0 \in P_0$  satisfying the same property. Lemma 3.1 shows that  $\xi_i = (\mathbf{s}_{i,j}, (\mu_{i,j,t})_t, \mathbf{s}'_{i,j})_j$  is a sum over all solutions  $\xi_0 \in P_0$  of terms of the form  $\lambda \xi_0$  where  $\lambda > 0$  is a rational value that naturally depends on  $\xi_0$ . In particular we deduce that  $\mu_{i,j,t} > 0$  for any  $t \in T_j$  and for any  $0 \leq j \leq k$ . Lemma 5.1 shows that there exists a cycle  $\pi_{i,j} = (\mathbf{x}_j \xrightarrow{\sigma_{i,j}}_{G_j} \mathbf{x}'_j)$  such that  $\mu_{i,j,t} = |\pi_{i,j}|_t$  for any  $t \in T_j$  and any  $1 \leq j \leq k$ .  $\square$

**Lemma 5.11** *There exists  $c_i \geq 0$  such that for any  $1 \leq j \leq k$  and  $c \geq c_i$ :*

$$\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + \mathbf{s}_{i,j} \xrightarrow{\sigma_{i,j}}_{\mathcal{V}} \mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j)) + \mathbf{s}'_{i,j}$$

*Proof* : As  $\lim_{c \rightarrow +\infty} (\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j))) = \mathbf{x}_j$  and  $\mathbf{x}_j \xrightarrow{\sigma_{i,j}}_{G_j} \mathbf{x}'_j$ , Lemma 5.2 proves that there exists an integer  $c_i \geq 0$  such that  $(\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j))) \xrightarrow{\sigma_{i,j}}_{\mathcal{V}}$  for any  $c \geq c_i$  and for any  $0 \leq j \leq k$ . As  $\mathbf{s}_{i,j} \geq \mathbf{0}$  and  $\mathbf{s}_{i,j} + \delta(\sigma_{i,j}) = \mathbf{s}'_{i,j} \geq \mathbf{0}$  we deduce the lemma.  $\square$