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An Easy Algorithm For The General Vector Addition System Reachability Problem

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ABSTRACT

The reachability problem for Vector Addition Systems (VAS) or equivalently for Petri Nets is a central problem of net theory. The general problem is known decidable by algorithms exclusively based on the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition. This decomposition is difficult and it just has a non-primitive recursive upper-bound complexity. In this paper, we prove that if a configuration is not reachable from an initial configuration, there exists a semi-linear inductive invariant that proves this property. We deduce an easy algorithm for deciding the reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by fairly enumerating the possible paths and a second one that tries to prove the non-reachability by fairly enumerating semi-linear inductive invariants. This algorithm is the very first one that does not require the KLMST decomposition. In particular, this algorithm is the first candidate to obtain a precise (eventually elementary) upper-bound complexity for the VAS reachability problem.

1. INTRODUCTION

Vector Addition Systems (VAS) or equivalently Petri Nets are one of the most popular formal methods for the representation and the analysis of parallel processes [2]. The reachability problem is central since many computational problems (even outside the parallel processes) reduce to the reachability problem. Sacerdote and Tenney provided in [10] a partial proof of the decidability of this problem. The proof was completed in 1981 by Mayr [9] and simplified by Kosaraju [7] from [10, 9]. Ten years later [8], Lambert provided a more simplified version based on [7]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known non-primitive recursive. Nowadays, the exact complexity of the reachability problem for VAS is still an open-problem. Even an elementary upper-bound complexity is open. In fact, the known general reachability algorithms are exclusively based on the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition.

In this paper, we prove that if a configuration is not reachable from an initial configuration, there exists a semi-linear set or equivalently a formula in the Presburger arithmetic [3] that denotes an inductive invariant proving this property. We deduce an easy algorithm for deciding the reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by fairly enumerating the possible paths and a second one that tries to prove the non-reachability by fairly enumerating semi-linear inductive invariants. This algorithm is the very first one that does not require the KLMST decomposition. In particular, this algorithm should be a good candidate to obtain a precise (eventually elementary) upper-bound complexity for the VAS reachability problem. Note [5] that in general, reachability sets are not semi-linear. Semi-linear inductive invariants are obtained by over-approximating reachability sets by semi-linear sets. In fact, we observe that reachability sets can be precisely over-approximated by semi-linear sets.

Outline of the paper: In section 2, the non-reachability problem for VAS is reduced to the existence of separators (a pair of inductive invariants). In section 3 we introduce the class of semi-pseudo-linear sets, a class of sets that can be precisely over-approximated by semi-linear sets. In section 4, reachability sets are proved semi-pseudo-linear. Finally in section 5 we show the existence of semi-linear separators proving the non-reachability of a pair of configurations. *In order to simplify the presentation of this paper, the independent parts of sections 3 and 4 with the remaining of the*

paper are presented in some sub-sections. That means the reader may safely skip these sub-sections in order to read other sections of the paper.

2. VECTOR ADDITION SYSTEMS

In this section, the non-reachability problem for Vector Addition Systems is reduced to the existence of *separators*.

```

1 Reachability ( $\mathbf{s} \in \mathbb{N}^n$ ,  $\mathcal{V} = (\Sigma, n, T)$  a VAS,  $\mathbf{s}' \in \mathbb{N}^n$ )
2   repeat forever
3     fairly select  $\sigma \in \Sigma^*$ 
4     if  $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'$ 
5       return "reachable"
6     fairly select  $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$  formulas in  $\text{FO}(\mathbb{N}, +, \leq)$ 
7     if  $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$  denotes separator for  $(\{\mathbf{s}\}, \{\mathbf{s}'\})$ 
8       return "unreachable"

```

A *Vector Addition System (VAS)* is a tuple $\mathcal{V} = (\Sigma, n, \delta)$ where Σ is a non-empty finite alphabet, $n \in \mathbb{N}$ is the *dimension* and $\delta \in \Sigma \rightarrow \mathbb{Z}^n$ is the *displacement function*. A *configuration* is a vector in \mathbb{N}^n . The binary relation $\xrightarrow{a}_{\mathcal{V}}$ where $a \in \Sigma$ over the set of configurations is defined by $\mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'$ if and only if $\mathbf{s}' = \mathbf{s} + \delta(a)$. Given a word $\sigma = a_1 \dots a_k$ of $k \in \mathbb{N}$ elements $a_i \in \Sigma$, we denote by $\xrightarrow{\sigma}_{\mathcal{V}}$ the binary relation over the set of configurations equals to the concatenation $\xrightarrow{a_1}_{\mathcal{V}} \dots \xrightarrow{a_k}_{\mathcal{V}}$ if $k \geq 1$ and equals to the identity binary relation if $k = 0$. We also denote by $\rightarrow_{\mathcal{V}}$ the *reachability binary relation* over the set of configurations defined by $\mathbf{s} \rightarrow_{\mathcal{V}} \mathbf{s}'$ if and only if there exists $\sigma \in \Sigma^*$ such that $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'$. Given two sets S, S' of configurations, we denote by $\text{post}_{\mathcal{V}}^*(S)$ and $\text{pre}_{\mathcal{V}}^*(S')$ respectively the set of *reachable states from S* and the set of *co-reachable states from S'* formally defined by:

$$\begin{aligned} \text{post}_{\mathcal{V}}^*(S) &= \{\mathbf{s}' \in \mathbb{N}^n \mid \exists \mathbf{s} \in S \quad \mathbf{s} \rightarrow_{\mathcal{V}} \mathbf{s}'\} \\ \text{pre}_{\mathcal{V}}^*(S') &= \{\mathbf{s} \in \mathbb{N}^n \mid \exists \mathbf{s}' \in S' \quad \mathbf{s} \rightarrow_{\mathcal{V}} \mathbf{s}'\} \end{aligned}$$

The reachability problem for a tuple $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ where \mathbf{s}, \mathbf{s}' are two configurations of a VAS \mathcal{V} consists to decide if $\mathbf{s} \rightarrow_{\mathcal{V}} \mathbf{s}'$. This problem can be reformulated by introducing the definition of separators. A pair (S, S') of configuration sets is said *separable* if $\text{post}_{\mathcal{V}}^*(S) \cap \text{pre}_{\mathcal{V}}^*(S') = \emptyset$. Naturally, a pair $(\mathbf{s}, \mathbf{s}')$ is in the complement of the reachability relation $\rightarrow_{\mathcal{V}}$ if and only if the pair $(\{\mathbf{s}\}, \{\mathbf{s}'\})$ is separable. A *separator* for a pair (S, S') of configuration sets is a pair (I, I') such that $I \cap I' = \emptyset$ and such that I is a *forward invariant* $\text{post}_{\mathcal{V}}^*(I) = I$ containing $S \subseteq I$ and I' is a *backward invariant* $\text{pre}_{\mathcal{V}}^*(I') = I'$ containing $S' \subseteq I'$. Observe that in this case (S, S') is separable. As $(\text{post}_{\mathcal{V}}^*(S), \text{pre}_{\mathcal{V}}^*(S'))$ is a separator for any separable pair (S, S') , we deduce that a pair (S, S') is separable if and only if there exists a separator (I, I') for (S, S') .

We are interested in separators definable in the decidable logic $\text{FO}(\mathbb{N}, +, \leq)$. Note that a pair $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ of formulas in this logic denotes a separator (I, I') if and only if $\psi(\mathbf{x}) \wedge \psi'(\mathbf{x})$ and the following formulas are unsatisfiable for any $a \in \Sigma$. In particular we can effectively decide if

$(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ denotes a separator.

$$\begin{aligned} \psi(\mathbf{x}) \quad \wedge \quad \mathbf{x}' = \mathbf{x} + \delta(a) \quad \wedge \quad \neg\psi(\mathbf{x}') \\ \psi'(\mathbf{x}') \quad \wedge \quad \mathbf{x}' = \mathbf{x} + \delta(a) \quad \wedge \quad \neg\psi'(\mathbf{x}) \end{aligned}$$

In this paper we prove that there exists a separator (I, I') definable in $\text{FO}(\mathbb{N}, +, \leq)$ for any pair (S, S') of separable sets definable in $\text{FO}(\mathbb{N}, +, \leq)$. We deduce that algorithm *Reachability* $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ decides the reachability problem. The termination is guaranty by the previous result. Note [5] that in general, the separator $(\text{post}_{\mathcal{V}}^*(S), \text{pre}_{\mathcal{V}}^*(S'))$ is not definable in $\text{FO}(\mathbb{N}, +, \leq)$ even if S and S' are reduced to single vectors $S = \{\mathbf{s}\}$ and $S' = \{\mathbf{s}'\}$. That means, this separator must be over-approximated by another separator (I, I') definable in $\text{FO}(\mathbb{N}, +, \leq)$. Intuitively, the approximation is obtained by observing that $\text{post}_{\mathcal{V}}^*(S) \cap S'$ and $S \cap \text{pre}_{\mathcal{V}}^*(S')$ are *semi-pseudo-linear* for any pair (S, S') of sets definable in $\text{FO}(\mathbb{N}, +, \leq)$, a class of sets that can be precisely over-approximated by semi-linear sets.

3. SEMI-PSEUDO-LINEAR SETS

In this section we introduce the class of *pseudo-linear sets* and *semi-pseudo-linear sets*. We show that a pseudo-linear set X can be precisely over-approximated by a *semi-linear set* S called a *linearization* of X . We also introduce a monotonic function $\text{dim} : (P(\mathbb{Z}^n), \subseteq) \rightarrow (\{-\infty, 0, \dots, n\}, \leq)$ that associates to any set $X \subseteq \mathbb{Z}^n$ a *dimension* $\text{dim}(X)$. We prove that a linearization S of a pseudo-linear set X satisfies $\text{dim}(S) = \text{dim}(X)$ and linearizations S_1, S_2 of pseudo-linear sets X_1, X_2 with an empty intersection $X_1 \cap X_2 = \emptyset$ satisfy $\text{dim}(S_1 \cap S_2) < \max\{\text{dim}(X_1), \text{dim}(X_2)\}$.

We first associate a dimension to sets $X \subseteq \mathbb{Z}^n$. The *dimension* $\text{dim}(X)$ of a non-empty set $X \subseteq \mathbb{Z}^n$ is the minimal integer $d \in \{0, \dots, n\}$ such that:

$$\sup_{k \geq 0} \frac{|X \cap \{-k, \dots, k\}^n|}{(1 + 2k)^d} < +\infty$$

The dimension of the empty-set set is denoted by $\text{dim}(\emptyset) = -\infty$. Let us observe some immediate properties satisfied by the dimension function. First of all, we have $\text{dim}(X) \leq 0$ if and only if X is finite. The dimension function is monotonic $\text{dim}(X_1) \leq \text{dim}(X_2)$ for any $X_1 \subseteq X_2$. Moreover it satisfies $\text{dim}(X_1 \cup X_2) = \max\{\text{dim}(X_1), \text{dim}(X_2)\}$ and $\text{dim}(X_1 + X_2) \leq \text{dim}(X_1) + \text{dim}(X_2)$. In particular $\text{dim}(\mathbf{v} + X) = \text{dim}(X)$ for any $\mathbf{v} \in \mathbb{Z}^n$ and for any $X \subseteq \mathbb{Z}^n$.

Now, we introduce the pseudo-linear sets. A set of *periods* is a finite set $P \subseteq \mathbb{N}^n$. We denote by $P^* = \sum_{\mathbf{p} \in P} \mathbb{N}\mathbf{p}$ and by $P^+ = \sum_{\mathbf{p} \in P} (\mathbb{N} \setminus \{0\})\mathbf{p}$. Note that $\mathbf{0} \in P^*$ for any set of periods P even if $P = \emptyset$ whereas $P^+ = \emptyset$ in this case. A set $S \subseteq \mathbb{Z}^n$ is said *linear* if there exists a vector $\mathbf{b} \in \mathbb{Z}^n$ and a set of periods $P \subseteq \mathbb{Z}^n$ such that $S = \mathbf{b} + P^*$. A *semi-linear set* is a finite union of linear sets. Recall that sets definable in $\text{FO}(\mathbb{Z}, \mathbb{N}, +, \leq)$ are exactly the semi-linear sets and sets definable in $\text{FO}(\mathbb{N}, +, \leq)$ are exactly the non-negative semi-linear sets [3]. A set $X \subseteq \mathbb{Z}^n$ is said *pseudo-linear* if there exists $\mathbf{b} \in \mathbb{Z}^n$ and a set of periods $P \subseteq \mathbb{Z}^n$ such that $X \subseteq \mathbf{b} + P^*$ and such that for any finite set $P_0 \subseteq P^+$ there exists $\mathbf{x} \in X$ such that $\mathbf{x} + P_0^* \subseteq X$.

In this case, P is called a *linearizator* of X and $X + P^*$ is called a *linearization* of X . Observe that if $X \subseteq \mathbb{N}^n$ is a non-negative pseudo-linear set then a linearizator P is also non-negative. In particular the linearization $X + P^*$ is also non-negative. The following lemma 1 shows that a linearization is semi-linear. A *semi-pseudo-linear set* is a finite union of *pseudo-linear sets*.

LEMMA 1. *For any $\mathbf{b} \in \mathbb{Z}^n$, for any set of periods $P \subseteq \mathbb{Z}^n$, and for any $X \subseteq \mathbf{b} + P^*$, there exists a finite set $B \subseteq X$ such that $X + P^* = B + P^*$.*

PROOF. Let us consider a sequence $\mathbf{p}_1, \dots, \mathbf{p}_d \in \mathbb{Z}^n$ denoting elements in $P = \{\mathbf{p}_1, \dots, \mathbf{p}_d\}$. The function $f : \mathbb{N}^d \rightarrow \mathbf{b} + P^*$ defined by $f(\mathbf{x}) = \mathbf{b} + \sum_{i=1}^d x[i]\mathbf{p}_i$ is surjective. In particular the set $H = f^{-1}(X)$ satisfies $f(H) = X$. Since (\mathbb{N}^d, \leq) is a well partially-ordered set, we deduce that $H_0 = \min(H)$ is finite and $H \subseteq H_0 + \mathbb{N}^d$. We deduce that $f(H) \subseteq B + P^*$ with $B = f(H_0)$. From $H_0 \subseteq H$ we get $f(H_0) \subseteq f(H)$. We have proved that $X \subseteq B + P^*$ where $B \subseteq X$. Therefore $X + P^* = B + P^*$. \square

All other results and notations introduced in this section are not used in the sequel. The reader may safely skip the remaining of this section to read the other ones. In sub-section 3.1 we introduce a geometrical characterization of the pseudo-linear sets based on vector spaces, groups and cones. This characterization is used in the next sub-section 3.2 to prove results presented in the introduction of this section.

3.1 Geometrical Characterization

In this sub-section, we provide a geometrical characterization of the pseudo-linear sets based on vector spaces, groups and cones.

Some notations : Given two vectors $\alpha, \mathbf{x} \in \mathbb{Q}^n$, we denote by $\langle \alpha, \mathbf{x} \rangle$ the usual *dot product* $\langle \alpha, \mathbf{x} \rangle = \sum_{i=1}^n \alpha[i]\mathbf{x}[i]$. Given a vector $\mathbf{x} \in \mathbb{Q}^n$, we denote by $\|\mathbf{x}\|_\infty = \max_i |\mathbf{x}[i]|$.

A *vector space* $V \subseteq \mathbb{Q}^n$ is a set of the form $V = \mathbb{Q}\mathbf{v}_1 + \dots + \mathbb{Q}\mathbf{v}_d$ where $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{Q}^n$. In this case V is called the vector space generated by $\mathbf{v}_1, \dots, \mathbf{v}_d$. A set $V \subseteq \mathbb{Q}^n$ is a vector space if and only if V is the set of rational solutions $\mathbf{x} \in \mathbb{Q}^n$ of a conjunction of *equality constraints* $\bigwedge_{j=1}^k \langle \alpha_j, \mathbf{x} \rangle = 0$ where $\alpha_j \in \mathbb{Q}^n$. In the sequel V denotes a vector space.

A *V-group* $G \subseteq \mathbb{Z}^n$ is a set of the form $G = \mathbb{Z}\mathbf{g}_1 + \dots + \mathbb{Z}\mathbf{g}_d$ where $\mathbf{g}_1, \dots, \mathbf{g}_d \in \mathbb{Z}^n$ generates the vector space V . In this case G is called the group *generated* by $\mathbf{g}_1, \dots, \mathbf{g}_d$. A set $G \subseteq \mathbb{Z}^n$ is a *V-group* if and only if G is the set of integral solutions $\mathbf{x} \in \mathbb{Z}^n \cap V$ of a conjunction of *modular constraints* $\bigwedge_{j=1}^k \langle \alpha_j, \mathbf{x} \rangle \in m_j \mathbb{Z}$ where $\alpha_j \in \mathbb{Z}^n$ and $m_j \in \mathbb{N} \setminus \{0\}$. This characterization is used in the following Lemma 2. This lemma shows that the intersection $G = G_1 \cap G_2$ of two *V-groups* G_1, G_2 is a *V-group*.

LEMMA 2. *For any vector $\mathbf{v} \in V$ and for any V-group G the set $(\mathbb{N} \setminus \{0\})\mathbf{v}$ contains a vector in G .*

PROOF. Let us consider a sequence $\alpha_1, \dots, \alpha_k \in \mathbb{Z}^n$ and a sequence $m_1, \dots, m_k \in \mathbb{N} \setminus \{0\}$ such that G is the set of solutions $\mathbf{x} \in \mathbb{Z}^n \cap V$ such that $\bigwedge_{j=1}^k \langle \alpha_j, \mathbf{x} \rangle \in m_j \mathbb{Z}$. There exists $\lambda \in \mathbb{N} \setminus \{0\}$ such that $\lambda \mathbf{v} \in \mathbb{Z}^n$. The product $m = m_1 \dots m_k$ shows that $m\lambda \mathbf{v} \in G$. \square

A *V-cone* $C \subseteq \mathbb{Q}^n$ is a set of the form $C = \mathbb{Q}_+\mathbf{r}_1 + \dots + \mathbb{Q}_+\mathbf{r}_d$ where $\mathbf{r}_1, \dots, \mathbf{r}_d \in \mathbb{Q}^n$ is a sequence that generates the vector space V . A sequence $\alpha_1, \dots, \alpha_k \in V \setminus \{\mathbf{0}\}$ is said *V-satisfiable* if there exists a vector $\mathbf{v} \in V$ satisfying the strict constraints $\bigwedge_{j=1}^k \langle \alpha_j, \mathbf{v} \rangle > 0$. A set $C \subseteq \mathbb{Q}^n$ is a *V-cone* if and only if C is the set of solutions $\mathbf{x} \in V$ of a conjunction of *half-space constraints* $\bigwedge_{j=1}^k \langle \alpha_j, \mathbf{x} \rangle \geq 0$ where $\alpha_1, \dots, \alpha_k \in V \setminus \{\mathbf{0}\}$ is *V-satisfiable*. A vector $\mathbf{v}_0 \in V$ is said *attractor* for a *V-cone* C if $(\mathbf{x} + \mathbb{Q}_+\mathbf{v}_0) \cap C \neq \emptyset$ for any $\mathbf{x} \in V$. Observe that any *V-cone* has at least one attractor and $\mathbf{0}$ is an attractor if and only if $C = V$. Attractor vectors can be easily characterized. Let us consider a *V-cone* C that is generated by a sequence $\mathbf{r}_1, \dots, \mathbf{r}_d$ and defined by a *V-satisfiable* sequence $\alpha_1, \dots, \alpha_k \in V \setminus \{\mathbf{0}\}$. We observe that the following two sets are equal to the set of attractors:

$$(\mathbb{Q}_+ \setminus \{0\})\mathbf{r}_1 + \dots + (\mathbb{Q}_+ \setminus \{0\})\mathbf{r}_d = \left\{ \mathbf{v} \in V \mid \bigwedge_{j=1}^k \langle \alpha_j, \mathbf{v} \rangle > 0 \right\}$$

LEMMA 3. *Let \mathbf{v}_0 be an attractor of a V-cone C . There exists an attractor in $\mathbf{v} + \mathbb{N}\mathbf{v}_0$ for any $\mathbf{v} \in V$.*

PROOF. Let us consider a *V-satisfiable* sequence $\alpha_1, \dots, \alpha_k$ that defines the *V-cone* C . As \mathbf{v}_0 is an attractor we have $\langle \alpha_j, \mathbf{v}_0 \rangle > 0$ for any j . Thus there exists $\lambda \in \mathbb{N}$ larger enough such that $\langle \alpha_j, \mathbf{v} + \lambda \mathbf{v}_0 \rangle > 0$ for any j . Therefore $\mathbf{v} + \lambda \mathbf{v}_0$ is an attractor for C . \square

The following two Lemmas 4 and 5 provides an equivalence between sets of the form P^* where P is a set of periods that generates a vector space V and sets of the form $G \cap C$ where G is a *V-group* and C is a *V-cone*.

LEMMA 4. *For any V-group G and any V-cone C there exists a set of periods P such that $G \cap C = P^*$. Moreover for any set of periods P such that $G \cap C = P^*$, the sets V , G and C are respectively the vector space, the group and the cone generated by P .*

PROOF. Let us first prove that there exists a set of periods P such that $G \cap C = P^*$. Let us consider a sequence $\mathbf{r}_1, \dots, \mathbf{r}_k$ that generates C . From Lemma 2, we can assume that $\mathbf{r}_1, \dots, \mathbf{r}_k \in G$. Let us consider the finite set $B = \{\mathbf{b} \in G \cap C \mid \|\mathbf{b}\|_\infty \leq \sum_{j=1}^k \|\mathbf{r}_j\|_\infty\}$, the finite set $R = \{\mathbf{r}_1, \dots, \mathbf{r}_k\}$ and the set of periods $P = B \cup R$. Since $P \subseteq G \cap C$ and $G \cap C$ is stable by addition we have $P^* \subseteq G \cap C$. For the other inclusion, note that for any $\mathbf{x} \in G \cap C$, from $\mathbf{x} \in C$ we deduce that there exists a sequence $\lambda_1, \dots, \lambda_k \in \mathbb{Q}_+$ such that $\mathbf{x} = \sum_{j=1}^k \lambda_j \mathbf{r}_j$. Note that $\lambda_j = z_j + \mu_j$ with $z_j \in \mathbb{N}$ and $0 \leq \mu_j < 1$. We deduce that $\mathbf{x} = \mathbf{b} + \mathbf{r}$ with $\mathbf{b} = \sum_{j=1}^k z_j \mathbf{r}_j$ and $\mathbf{r} = \sum_{j=1}^k \mu_j \mathbf{r}_j$. Observe

that $\mathbf{b} \in B$ and $\mathbf{r} \in R^*$. Thus $\mathbf{x} \in P^*$ and we have proved that $G \cap C = P^*$.

Now let us consider a set of periods P such that $G \cap C = P^*$ and let us consider the sets V' , G' and C' respectively the vector space, the group and the cone generated by P . Let us prove that $V = V'$, $G = G'$ and $C = C'$. From $P \subseteq V$, $P \subseteq G'$ and $P \subseteq C'$ we get $V' \subseteq V$, $G' \subseteq G$ and $C' \subseteq C$. Let us show that $G \subseteq G'$. Let us consider a sequence $\mathbf{g}_1, \dots, \mathbf{g}_d \in G$ that generates the group G . Let us consider a vector $\mathbf{v}_0 \in V$ that is attractor for C . Lemma 2 shows that $\mathbf{v}_0 \in G$ can be assumed. From Lemma 3 we deduce that there exists an integer $\lambda \in \mathbb{N}$ such that $\mathbf{g}_i + \lambda \mathbf{v}_0$ is an attractor for any $1 \leq i \leq d$. In particular $\mathbf{g}_i + \lambda \mathbf{v}_0 \in G \cap C \subseteq P^*$. As the group generated by $\{\mathbf{g}_i + \lambda \mathbf{v}_0 \mid 1 \leq i \leq d\}$ is equal to G we deduce that $G \subseteq P^*$. As the vector space generated by G and G' are equal to V and V' , we also deduce that $V \subseteq V'$. Finally, let us show that $C \subseteq C'$. Let us consider a sequence $\mathbf{r}_1, \dots, \mathbf{r}_d$ of vectors that generates C . From Lemma 2, we can assume that $\mathbf{r}_i \in G \cap C$. Therefore $\mathbf{r}_1, \dots, \mathbf{r}_d \in G \cap C \subseteq P^*$. Thus the cone generated by P contains C and we have proved that $C \subseteq C'$. \square

LEMMA 5. *For any set of periods $P \subseteq \mathbb{Z}^n$ there exists a vector $\mathbf{p} \in P^*$ such that the group G and the cone C generated by P satisfy:*

$$\mathbf{p} + (G \cap C) \subseteq P^* \subseteq G \cap C$$

PROOF. Let us consider a sequence $\mathbf{p}_1, \dots, \mathbf{p}_d \in \mathbb{Z}^n$ and let $P = \{\mathbf{p}_1, \dots, \mathbf{p}_d\}$. We denote by G and C respectively the group and the cone generated by P . From $P^* \subseteq G$ and $P^* \subseteq C$ we deduce $P^* \subseteq G \cap C$. Let us consider $\mathbf{r} \in G \cap C$. As $\mathbf{r} \in C$ there exists $\lambda_1, \dots, \lambda_d \in \mathbb{Q}_+$ such that $\mathbf{r} = \sum_{j=1}^d \lambda_j \mathbf{p}_j$. Note that $\lambda_j = z_j + \mu_j$ with $z_j \in \mathbb{N}$ and $0 \leq \mu_j < 1$. In particular $\mathbf{r} \in \mathbf{b} + P^*$ where $\mathbf{b} = \sum_{j=1}^d \mu_j \mathbf{p}_j$. Note that $\mathbf{b} \in B$ where $B = \{\mathbf{b} \in G \mid \|\mathbf{b}\|_\infty \leq \sum_{j=1}^d \|\mathbf{p}_j\|_\infty\}$. Observe that for any $\mathbf{b} \in G$, there exists $\mathbf{b}_+, \mathbf{b}_- \in P^*$ such that $\mathbf{b} = \mathbf{b}_+ - \mathbf{b}_-$. Let us consider $\mathbf{p} = \sum_{\mathbf{b} \in B} \mathbf{b}_-$ and observe that $\mathbf{p} + B \subseteq P^*$. From $G \cap C \subseteq B + P^*$ we get $\mathbf{p} + (G \cap C) \subseteq P^*$. \square

A cone C_0 is said *properly included* in a V -cone C if it is generated by attractor vectors of C . Observe that in this case any vector $\mathbf{v} \in C_0 \setminus \{\mathbf{0}\}$ is an attractor vector for C .

PROPOSITION 1 (EQUIVALENT CHARACTERIZATION). *A set $X \subseteq \mathbb{Z}^n$ is pseudo-linear if and only if there exists $\mathbf{b} \in \mathbb{Z}^n$, a vector space $V \subseteq \mathbb{Q}^n$, a V -group G , a V -cone C such that $X \subseteq \mathbf{b} + (G \cap C)$ and such that for any cone C_0 properly included in C there exists $\mathbf{x} \in X$ satisfying $\mathbf{x} + (G \cap C_0) \subseteq X$. Moreover in this case for any linearizator P for X , the sets V , G and C are respectively the vector space, the group, and the cone generated by P .*

PROOF. Let us first consider a set $X \subseteq \mathbb{Z}^n$ such that there exists $\mathbf{b} \in \mathbb{Z}^n$, a vector space $V \subseteq \mathbb{Q}^n$, a V -group G , and a V -cone C such that $X \subseteq \mathbf{b} + (G \cap C)$ and such that for any cone C_0 properly included in C there exists $\mathbf{x} \in X$

satisfying $\mathbf{x} + (G \cap C_0) \subseteq X$. We consider a V -satisfiable sequence $\alpha_1, \dots, \alpha_k \in V \setminus \{\mathbf{0}\}$ defining C . Let us prove that there exists a linearizator P for X . From Lemma 4, there exists a finite set $P \subseteq G \cap C$ such that $P^* = G \cap C$. From $X \subseteq \mathbf{b} + (G \cap C)$ we deduce that $X \subseteq \mathbf{b} + P^*$. Next, consider a finite set $P_0 \subseteq P^+$. Note that any vector in P_0 is attractor for C . In particular the cone C_0 generated by P_0 is properly included in C . We deduce that there exists $\mathbf{x} \in X$ such that $\mathbf{x} + (G \cap C_0) \subseteq X$. Observe that $P_0 \subseteq G \cap C_0$ and thus $P_0^* \subseteq G \cap C_0$ and we deduce that $\mathbf{x} + P_0^* \subseteq X$. We have proved that X is pseudo-linear. Conversely, let us consider a pseudo-linear set X and a linearizator P for X . There exists a vector $\mathbf{b} \in \mathbb{Z}^n$ such that $X \subseteq \mathbf{b} + P^*$. We denote by V the vector space generated by P , we also denote by G the V -group generated by P and we denote by C the V -cone generated by P . As $P^* \subseteq G \cap C$ we deduce that $X \subseteq \mathbf{b} + (G \cap C)$. Let us consider a cone C_0 properly included in C . Note that if P is empty then $C_0 = \{\mathbf{0}\}$ and the property is immediate. Thus, we can assume that $P \neq \emptyset$ and in this case there exists $\mathbf{p}_+ \in P^+$. Lemma 4 shows that there exists a finite set R such that $G \cap C_0 = R^*$. By replacing R by $R \setminus \{\mathbf{0}\}$ we can assume that $R \subseteq (G \cap C_0) \setminus \{\mathbf{0}\}$. Note that any vector $\mathbf{r} \in R$ is an attractor for C . Thus there exists $\lambda \in \mathbb{N} \setminus \{0\}$ such that $\lambda \mathbf{r} \in P^+$. As R is finite we deduce that there exists a uniform $\lambda \in \mathbb{N} \setminus \{0\}$ such that $\lambda R \subseteq P^+$. Let $P_0 = \lambda R \cup \{\mathbf{p}_+\} \cup (\mathbf{p}_+ + P)$. As $P_0 \subseteq P^+$ and P is a linearizator for X , there exists $\mathbf{x}_0 \in X$ such that $\mathbf{x}_0 + P_0^* \subseteq X$. Note that the group generated by P_0 is equal to G and the cone generated by P_0 contains C_0 . From Lemma 5 we deduce that there exists $\mathbf{p}_0 \in P_0^*$ such that $\mathbf{p}_0 + (G \cap C_0) \subseteq P_0^*$. From $\mathbf{x}_0 + P_0^* \subseteq X$ we get $\mathbf{x} + (G \cap C_0) \subseteq X$ where $\mathbf{x} = \mathbf{x}_0 + \mathbf{p}_0$. We have proved that for any cone C_0 properly included in C there exists $\mathbf{x} \in X$ satisfying $\mathbf{x} + (G \cap C_0) \subseteq X$.

Finally, let us consider a linearizator P for X , a vector space $V \subseteq \mathbb{Q}^n$, a V -group G , a V -cone C such that $X \subseteq \mathbf{b} + (G \cap C)$ and such that for any cone C_0 properly included in C there exists $\mathbf{x} \in X$ satisfying $\mathbf{x} + (G \cap C_0) \subseteq X$, and let us consider the sets V' , G' , C' respectively equals to the vector space, the group and the cone generated by P . We are going to prove that $V = V'$, $G = G'$ and $C = C'$. As P is a linearizator for X , there exists \mathbf{b}' such that $X \subseteq \mathbf{b}' + P^*$. Let us consider an attractor \mathbf{v}_0 for C . From Lemma 2, we can assume that $\mathbf{v}_0 \in G$. For any $\mathbf{v} \in G \cap C$ observe that $\mathbf{v}_0 + \mathbf{v} \in G$ is an attractor for C . Therefore $C_0 = \mathbb{Q}_+(\mathbf{v}_0 + \mathbf{v})$ is properly included in C . Thus there exists $\mathbf{x} \in X$ such that $\mathbf{x} + (G \cap \mathbb{Q}_+(\mathbf{v}_0 + \mathbf{v})) \subseteq X$. From $X \subseteq \mathbf{b}' + P^*$ we deduce that $\mathbf{x} - \mathbf{b}' + \mathbb{N}(\mathbf{v}_0 + \mathbf{v}) \subseteq P^*$. Thus $\mathbf{v}_0 + \mathbf{v} \in V'$, $\mathbf{v}_0 + \mathbf{v} \in G'$ and $\mathbf{v}_0 + \mathbf{v} \in C'$. We deduce that $\mathbf{v}_0 + (G \cap C) \subseteq V'$, $\mathbf{v}_0 + (G \cap C) \subseteq G'$ and $\mathbf{v}_0 + (G \cap C) \subseteq C'$. From Lemma 4 we have $V \subseteq V'$, $G \subseteq G'$ and $C \subseteq C'$. Since the other inclusion are immediate, we are done. \square

3.2 Dimension

Thanks to the geometrical characterization provided in the previous sub-section, we prove in this one that a linearization S of a pseudo-linear set X satisfies $\dim(S) = \dim(X)$ and linearizations S_1, S_2 of pseudo-linear sets X_1, X_2 with an empty intersection $X_1 \cap X_2 = \emptyset$ satisfy $\dim(S_1 \cap S_2) < \max\{\dim(X_1), \dim(X_2)\}$.

A *basis* of a vector space V is a sequence $\mathbf{v}_1, \dots, \mathbf{v}_d \in V$ such that for any $\mathbf{v} \in V$ there exists a unique sequence $\lambda_1, \dots, \lambda_d \in \mathbb{Q}$ such that $\mathbf{v} = \sum_{j=1}^d \lambda_j \mathbf{v}_j$. Any vector space V generated by a set X has at least one basis $\mathbf{v}_1, \dots, \mathbf{v}_d \in X$. The integer d is unique and it is called the *rank* of V . It is denoted by $\text{rank}(V)$. Since $\mathbb{Z}^n \cap V$ is a V -group and $C = V$ is a V -cone, the following Lemma 6 shows in particular $\dim(\mathbb{Z}^n \cap V) = \text{rank}(V)$.

LEMMA 6. *We have $\dim(G \cap C) = \text{rank}(V)$ for any V -group G and for any V -cone C .*

PROOF. Let us first prove that $\dim(G \cap C) \geq \text{rank}(V)$. From Lemma 4 there exists a set of periods P that generated the vector space V and such that $G \cap C = P^*$. As P generates V , there exists a basis $\mathbf{p}_1, \dots, \mathbf{p}_d \in P$ of the vector space V with $d = \text{rank}(V)$. In particular, the function $f : \mathbb{N}^d \rightarrow P^*$ defined by $f(\mathbf{x}) = \sum_{i=1}^d \mathbf{x}[i] \mathbf{p}_i$ is an injective function. We deduce that the cardinal of $f(\{0, \dots, k\}^d)$ is equal to $(1+k)^d$. Moreover, note that a vector \mathbf{y} in this set satisfies $\|\mathbf{y}\|_\infty \leq k \sum_{i=1}^d \|\mathbf{p}_i\|_\infty$. We deduce that $\dim(P^*) \geq d$ and we have proved that $\dim(G \cap C) \geq \text{rank}(V)$.

Now, let us prove that $\dim(G \cap C) \leq \text{rank}(V)$. Since $G \cap C \subseteq \mathbb{Z}^n \cap V$ it is sufficient to prove that $\dim(\mathbb{Z}^n \cap V) \leq \text{rank}(V)$. Let us consider a basis $\mathbf{v}_1, \dots, \mathbf{v}_d$ of V with $d = \text{rank}(V)$. Let us consider the function $f : \mathbb{Q}^n \rightarrow V$. Since for any matrix, the rank of the column vectors is equal to the rank of the line vectors, there exists a sequence $1 \leq j_1 < \dots < j_d \leq n$ such that the linear function $g : \mathbb{Q}^n \rightarrow \mathbb{Q}^d$ defined by $g(\mathbf{x}) = (\mathbf{x}[j_1], \dots, \mathbf{x}[j_d])$ satisfies $h = g \circ f$ is a bijective linear function. In particular we deduce that for any $\mathbf{v} \in \mathbb{Z}^n \cap V \cap \{-k, \dots, k\}^n$ there exists a vector $\mathbf{x} = g(\mathbf{v}) \in \{-k, \dots, k\}^d$ such that $\mathbf{v} = f \circ h^{-1}(\mathbf{x})$. Therefore $|\mathbb{Z}^n \cap V \cap \{-k, \dots, k\}^n| \leq (1+2k)^d$ for any $k \in \mathbb{N}$. We have proved that $\dim(\mathbb{Z}^n \cap V) \leq d$ and we deduce that $\dim(G \cap C) \leq \text{rank}(V)$. \square

We now provide the following technical Lemma 7.

LEMMA 7. *Let G be a V -group and let $c_1, \dots, c_k \in \mathbb{Q}$. A sequence $\alpha_1, \dots, \alpha_k \in V \setminus \{0\}$ is V -satisfiable if and only if:*

$$\dim(\{\mathbf{x} \in G \mid \bigwedge_{j=1}^k \langle \alpha_j, \mathbf{x} \rangle \geq c_j\}) = \text{rank}(V)$$

PROOF. Let G be a V -group, let $c_1, \dots, c_k \in \mathbb{Q}$ and let $\alpha_1, \dots, \alpha_k \in V \setminus \{0\}$. We denote by X the set $X = \{\mathbf{x} \in G \mid \bigwedge_{j=1}^k \langle \alpha_j, \mathbf{x} \rangle \geq c_j\}$ and we denote by $C = \{\mathbf{x} \in V \mid \bigwedge_{j=1}^k \langle \alpha_j, \mathbf{x} \rangle \geq 0\}$. As $X \subseteq G$, Lemma 6 proves that $\dim(X) \leq \text{rank}(V)$.

Assume first that $\alpha_1, \dots, \alpha_k$ is V -satisfiable and let us prove that $\dim(X) \geq \text{rank}(V)$. There exists a vector $\mathbf{v}_0 \in V$ such that $\langle \alpha_j, \mathbf{v}_0 \rangle > 0$ for any j . From Lemma 2 we can assume that $\mathbf{v}_0 \in G$. Note that there exists $\lambda \in \mathbb{N}$ larger enough such that $\lambda \mathbf{v}_0 \in X$. In particular X is not empty and there exists $\mathbf{x}_0 \in X$. Now, let us observe that $\mathbf{x}_0 + (G \cap C) \subseteq X$. From lemma 6 we get $\dim(X) \geq \text{rank}(V)$.

Next, assume that $\alpha_1, \dots, \alpha_k \in V \setminus \{0\}$ is not V -satisfiable and let us prove that $\dim(X) < \text{rank}(V)$. Naturally without loss of generality, we can assume that $\alpha_j \in \mathbb{Z}^n$. Note that if X is empty then the relation is immediate. Thus, we can assume that X is non empty and in this case there exists $\mathbf{x}_0 \in X$. Observe that if $\mathbf{x} \in X \setminus (\mathbf{x}_0 + (G \cap C))$ there exists $1 \leq j \leq k$ such that $\langle \alpha_j, \mathbf{x} \rangle \in Z_j$ where $Z_j = \{c_j, \dots, \langle \alpha_j, \mathbf{x}_0 \rangle - 1\}$. In particular we have proved the following inclusion:

$$X \subseteq (\mathbf{x}_0 + (G \cap C)) \bigcup_{\substack{1 \leq j \leq k \\ z \in Z_j}} X_{j,z}$$

where $X_{j,z} = \{\mathbf{x} \in \mathbb{Z}^n \cap V \mid \langle \alpha_j, \mathbf{x} \rangle = z\}$. Since C is not a V -cone, it is included in a vector space V_0 strictly included in V . Lemma 6 shows that $\dim(\mathbf{x}_0 + (\mathbb{Z}^n \cap V_0)) < \text{rank}(V)$ and in particular $\dim(\mathbf{x}_0 + (G \cap C)) < \text{rank}(V)$. Let us prove that $\dim(X_{j,z}) < \text{rank}(V)$. Naturally if $X_{j,z} = \emptyset$ then $\dim(X_{j,z}) < \text{rank}(V)$. Otherwise there exists $\mathbf{x}_{j,z} \in X_{j,z}$. Note that in this case $X_{j,z} = \mathbf{x}_{j,z} + (\mathbb{Z}^n \cap V_j)$ where $V_j = \{\mathbf{x} \in V \mid \langle \alpha_j, \mathbf{x} \rangle = 0\}$. As $\alpha_j \in V \setminus V_j$ we deduce that V_j is strictly included in V . Thus $\text{rank}(V_j) < \text{rank}(V)$. Lemma 6 provides $\dim(X_{j,z}) < \text{rank}(V)$. We have proved that $\dim(X) < \text{rank}(V)$. \square

We deduce the following Corollary 1.

COROLLARY 1. *Let G be a V -group, $\mathbf{v}_1, \mathbf{v}_2 \in V$ and let $C = C_1 \cap C_2$ where C_1 and C_2 are two V -cones. We have $\dim(G \cap (\mathbf{v}_1 + C_1) \cap (\mathbf{v}_2 + C_2)) = \text{rank}(V)$ if and only if C is a V -cone.*

PROOF. Let $\alpha_{i,1}, \dots, \alpha_{i,k_i} \in V \setminus \{0\}$ be a V -satisfiable sequence defining the V -cone C_i for any $i \in \{1, 2\}$. Note that $\mathbf{x} \in \mathbf{v}_i + C_i \Leftrightarrow \mathbf{x} - \mathbf{v}_i \in C_i$. Moreover, as $\mathbf{v}_i \in V$ we have $\mathbf{v}_i + V = V$. Therefore, we have the following equality where $c_{i,j} = \langle \alpha_{i,j}, \mathbf{v}_i \rangle$:

$$\mathbf{v}_i + C_i = \{\mathbf{x} \in V \mid \bigwedge_{j=1}^{k_i} \langle \alpha_{i,j}, \mathbf{x} \rangle \geq c_{i,j}\}$$

From lemma 7 we are done. \square

LEMMA 8. *Let P be a linearizator for a pseudo-linear set X , and let V be the vector space generated by P , let G be the V -group generated by P and let C be the V -cone generated by C . We have:*

$$\dim(X) = \dim(X + P^*) = \dim(P^*) = \dim(G \cap C) = \text{rank}(V)$$

PROOF. Since $X \subseteq X + P^*$ we get $\dim(X) \leq \dim(X + P^*)$. As there exists $\mathbf{b} \in \mathbb{Z}^n$ such that $X \subseteq \mathbf{b} + P^*$ we deduce that $X + P^* \subseteq \mathbf{b} + P^*$. Thus $\dim(X + P^*) \leq \dim(P^*)$. From $P^* \subseteq G \cap C$ we deduce that $\dim(P^*) \leq \dim(G \cap C)$. From Lemma 6 we get $\dim(G \cap C) = \text{rank}(V)$. Observe that there exists a V -cone C_0 properly included in C . As X is pseudo-linear, Proposition 1 shows that there exists $\mathbf{x} \in X$ such that $\mathbf{x} + (G \cap C_0) \subseteq X$. From Lemma 7 we deduce that $\dim(G \cap C_0) = \text{rank}(V)$. Therefore $\text{rank}(V) \leq \dim(X)$. We have proved that $\dim(X) \leq \dim(X + P^*) \leq \dim(P^*) \leq \dim(G \cap C) \leq \text{rank}(V) \leq \dim(X)$. \square

We deduce the following Proposition 2.

PROPOSITION 2. *A linearization S of a pseudo-linear set X satisfies $\dim(X) = \dim(S)$.*

Finally, we get the proof of the following Proposition 3.

PROPOSITION 3. *Let S_1, S_2 be linearizations of pseudo-linear sets $X_1, X_2 \subseteq \mathbb{Z}^n$ with an empty intersection $X_1 \cap X_2 = \emptyset$. We have:*

$$\dim(S_1 \cap S_2) < \max\{\dim(X_1), \dim(X_2)\}$$

PROOF. Let $j \in \{1, 2\}$ and let us denote by P_j a linearizer for X_j such that $S_j = X_j + P_j^*$. Let us consider the vector space V_j generated by P_j , the V_j -group G_j generated by P_j and the V_j -cone C_j generated by P_j . From proposition 1 there exists a vector $\mathbf{b}_j \in \mathbb{Z}^n$ such that $S_j \subseteq \mathbf{b}_j + (G_j \cap C_j)$. From lemma 8 we get $\dim(S_j) = \dim(X_j) = \text{rank}(V_j)$. Let us consider the set $Z = S_1 \cap S_2$. Note that if $(\mathbf{b}_1 + G_1) \cap (\mathbf{b}_2 + G_2) = \emptyset$ then $Z = \emptyset$ and the proposition is proved. Thus, we can assume that there exists $\mathbf{b}_{12} \in (\mathbf{b}_1 + G_1) \cap (\mathbf{b}_2 + G_2)$. Note that this set is then equal to $\mathbf{b}_{12} + G$ where $G = G_1 \cap G_2$. Note that G is a V -group with $V = V_1 \cap V_2$ and $\mathbf{b}_j - \mathbf{b}_{12} \in V$ for any $j \in \{1, 2\}$. If $V_1 \neq V_2$ then $\text{rank}(V) < \max\{\text{rank}(V_1), \text{rank}(V_2)\}$ and we get the proposition. Thus we can assume that $V_1 = V_2$. In this case $V_1 = V = V_2$ and we have proved the following inclusion:

$$Z \subseteq \mathbf{b}_{12} + (G \cap (\mathbf{b}_1 - \mathbf{b}_{12} + C_1) \cap (\mathbf{b}_2 - \mathbf{b}_{12} + C_2))$$

We deduce that $\dim(Z) \leq \text{rank}(V)$. Assume by contradiction that $\dim(Z) = \text{rank}(V)$. From Corollary 1 we deduce that $C = C_1 \cap C_2$ is a V -cone. Thus, there exists a V -cone C_0 properly included in C . Let $j \in \{1, 2\}$. As C_0 is properly included in C and $C \subseteq C_j$ we deduce that C_0 is properly included in C_j . Thus, there exists $\mathbf{x}_j \in X_j$ such that $\mathbf{x}_j + (G_j \cap C_0) \subseteq X_j$. From $\mathbf{x}_j \in X_j$ we deduce that $\mathbf{x}_j \in \mathbf{b}_j + G_j$. In particular $\mathbf{x}_j + G_j = \mathbf{b}_j + G_j$ and $\mathbf{x}_j - \mathbf{b}_j \in V$. Thus $(\mathbf{x}_1 + G_1) \cap (\mathbf{x}_2 + G_2) = (\mathbf{b}_1 + G_1) \cap (\mathbf{b}_2 + G_2)$. Since this last set is equal to $\mathbf{b}_{12} + G$, we have proved that the following set Z_0 is included in $X_1 \cap X_2$:

$$Z_0 = \mathbf{b}_{12} + (G \cap (\mathbf{x}_1 - \mathbf{b}_{12} + C_0) \cap (\mathbf{x}_2 - \mathbf{b}_{12} + C_0))$$

From $\mathbf{x}_j - \mathbf{b}_j \in V$ and $\mathbf{b}_j - \mathbf{b}_{12} \in V$ we deduce that $\mathbf{x}_j - \mathbf{b}_{12} \in V$. From Corollary 1, we deduce that $\dim(Z_0) = \text{rank}(V)$. In particular $Z_0 \neq \emptyset$ and we get a contradiction with $X_1 \cap X_2 = \emptyset$. \square

4. REACHABILITY SETS

In this section we prove that $\text{post}_V^*(S) \cap S'$ and $S \cap \text{pre}_V^*(S')$ are semi-pseudo-linear for any semi-linear sets $S, S' \subseteq \mathbb{N}^n$. All other results and notations introduced in this section are not used in the sequel. The reader may safely skip the remaining of this section in order to read the other ones. In sub-section 4.1 we recall the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition. This decomposition is used in the next sub-section 4.2 to establish the semi-pseudo-linearity of the reachability sets.

4.1 Languages Accepted

The *language accepted* $\mathcal{L}(\mathbf{b}, \mathcal{V}, \mathbf{b}')$ by a tuple $(\mathbf{b}, \mathcal{V}, \mathbf{b}')$ where $(\mathbf{b}, \mathbf{b}')$ are two configurations of a VAS \mathcal{V} is the set of words $\sigma \in \Sigma^*$ such that $\mathbf{b} \xrightarrow{\sigma} \mathbf{b}'$. In this section we recall the KLMST decomposition by following notations introduced by Lambert [8].

We first extends the set of non-negative integers \mathbb{N} with an additional element \top . In the sequel, this element is either interpreted as a ‘‘very large integer’’ or a ‘‘dontcare integer’’. More formally, we denote by \mathbb{N}_\top the set $\mathbb{N} \cup \{\top\}$. The total order \leq over \mathbb{N} is extended over \mathbb{N}_\top by $x_1 \leq x_2$ if and only if $x_2 = \top \vee (x_1, x_2 \in \mathbb{N} \wedge x_1 \leq x_2)$. We also introduce the partial order \trianglelefteq over \mathbb{N}_\top by $x_1 \trianglelefteq x_2$ if and only if $x_1 = x_2 \vee x_2 = \top$. Intuitively the elements \top denotes a ‘‘very large integer’’ for the total order \leq whereas it denotes a ‘‘dontcare integer’’ for the partial order \trianglelefteq . Given a sequence $(x_i)_{i \geq 0}$ in \mathbb{N}_\top , we denote by $x = \lim_{i \rightarrow +\infty} x_i$ the element $x = \top$ if for any $r \in \mathbb{N}$ there exists $i_0 \geq 0$ such that $x_i \geq r$ for any $i \geq i_0$ and the element $x \in \mathbb{N}$ if there exists $i_0 \geq 0$ such that $x_i = x$ for any $i \geq i_0$.

We also extends the semantics of VAS. A vector in \mathbb{N}_\top^n is called an *extended configuration* of \mathcal{V} . The addition function $+: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ is extended to the totally-defined function in $(\mathbb{Z} \cup \{\top\}) \times (\mathbb{Z} \cup \{\top\}) \rightarrow (\mathbb{Z} \cup \{\top\})$ satisfying $x_1 + x_2 = \top$ if $x_1 = \top$ or $x_2 = \top$. With slightly abusing notations, the binary relation \xrightarrow{a}_V where $a \in \Sigma$ over the set of extended configurations is defined by $\mathbf{x} \xrightarrow{a}_V \mathbf{x}'$ if and only if $\mathbf{x}' = \mathbf{x} + \delta(a)$. Given a word $\sigma = a_1 \dots a_k$ of $k \in \mathbb{N}$ elements $a_i \in \Sigma$, we denote by $\xrightarrow{\sigma}_V$ the binary relation over the set of extended configurations equals to the concatenation $\xrightarrow{a_1}_V \dots \xrightarrow{a_k}_V$ if $k \geq 1$ and equals to the identity binary relation if $k = 0$. Given an extended configuration \mathbf{x} we denote by $\mathbf{x} \xrightarrow{\sigma}_V$ if there exists an extended configuration \mathbf{x}' such that $\mathbf{x} \xrightarrow{\sigma}_V \mathbf{x}'$ and symmetrically for any extended configuration \mathbf{x}' we denote by $\xrightarrow{\sigma}_V \mathbf{x}'$ if there exists an extended configuration \mathbf{x} such that $\mathbf{x} \xrightarrow{\sigma}_V \mathbf{x}'$.

Next we recall some elements of graph theory. An *alphabet* Σ is a non-empty finite set. We denote by Σ^* the set of finite words over Σ . A *graph* G is a tuple $G = (Q, \Sigma, T)$ where Q is a finite set of *states*, Σ is a finite set of *labels*, and $T \subseteq Q \times \Sigma \times Q$ is a finite set of *transitions*. A *path* π is a word $\pi = t_1 \dots t_k$ of $k \in \mathbb{N}$ transitions $t_i \in T$ such that there exists $q_0, \dots, q_k \in Q$ and $a_1, \dots, a_k \in \Sigma$ such that $t_i = (q_{j-1}, a_j, q_j)$ for any $1 \leq j \leq k$. In this case we say that π is a path from q_0 to q_k labelled by $\sigma = a_1 \dots a_k$ and we denote π by $q_0 \xrightarrow{\sigma}_G q_k$ or simply $q_0 \rightarrow_G q_k$. When $q_0 = q_k$, the path π is called a *cycle*. Given a transition $t \in T$, we denote by $|\pi|_t$ the number of occurrences of t in π . Let us recall the following lemma.

LEMMA 9 (EULER CYCLE). *Let $G = (Q, \Sigma, T)$ be a strongly connected graph and let $(\mu_t)_{t \in T}$ be a sequence in $\mathbb{N} \setminus \{0\}$. There exists a cycle π in G such that $|\pi|_t = \mu_t$ for any transition $t \in T$ if and only if for any state $q_0 \in Q$ we have the following equality:*

$$\sum_{t=(q,a,q_0) \in T} \mu_t = \sum_{t'=(q_0,a,q') \in T} \mu_{t'}$$

A *graph vector* $G = (Q, \Sigma, T)$ for \mathcal{V} is a graph such that $Q \subseteq \mathbb{N}_+^n$ is a finite set of extended configurations, and $T \subseteq Q \times \Sigma \times Q$ is a finite set of transitions $(\mathbf{x}, a, \mathbf{x}')$ such that $\mathbf{x} \xrightarrow{a}_{\mathcal{V}} \mathbf{x}'$. Even if the proof of the following lemma is immediate by induction over the length of σ , it is central in the KLMST decomposition.

LEMMA 10 (GRAPH VECTOR PATHS). *Let us consider a path $\mathbf{x} \xrightarrow{\sigma}_G \mathbf{x}'$ in a graph vector G for \mathcal{V} . For any sequences $(\mathbf{x}_c)_{c \in \mathbb{N}}$ and $(\mathbf{x}'_c)_{c \in \mathbb{N}}$ of extended configurations that converge toward $\mathbf{x} = \lim_{c \rightarrow +\infty} \mathbf{x}_c$ and $\mathbf{x}' = \lim_{c \rightarrow +\infty} \mathbf{x}'_c$, there exists $c_0 \in \mathbb{N}$ such that $\mathbf{x}_c \xrightarrow{\sigma}_{\mathcal{V}}$ and $\xrightarrow{\sigma}_{\mathcal{V}} \mathbf{x}'_c$ for any $c \geq c_0$.*

A *marked graph vector* is a tuple $(\mathbf{m}, G, \mathbf{x}, \mathbf{m}')$ where G is a strongly connected graph vector, \mathbf{x} is a state of this graph vector and $\mathbf{m}, \mathbf{m}' \preceq \mathbf{x}$ are two extended configurations. A *marked graph vector sequences (MGVS)* is an alternating sequence of marked graph vectors and actions:

$$\mathcal{U} = (\mathbf{m}_0, G_0, \mathbf{x}_0, \mathbf{m}'_0), a_1, \dots, a_k, (\mathbf{m}_k, G_k, \mathbf{x}_k, \mathbf{m}'_k)$$

satisfying $\mathbf{b} = \mathbf{m}_0$, $(\mathbf{m}'_{j-1} \xrightarrow{a_j}_{\mathcal{V}} \mathbf{m}_j)_{1 \leq j \leq k}$ and $\mathbf{m}'_k = \mathbf{b}'$. The *language accepted* by a MGVS \mathcal{U} is the set $\mathcal{L}(\mathcal{U})$ of words of the form $\sigma_0 a_1 \dots a_k \sigma_k$ such that for any $0 \leq j \leq k$ there exists a cycle $\mathbf{x}_j \xrightarrow{\sigma_j}_{G_j} \mathbf{x}_j$ and there exists two configurations $\mathbf{s}_j \preceq \mathbf{m}_j$ and $\mathbf{s}'_j \preceq \mathbf{m}'_j$ such that:

$$\mathbf{s}_0 \xrightarrow{\sigma_0}_{\mathcal{V}} \mathbf{s}'_0 \xrightarrow{a_1}_{\mathcal{V}} \mathbf{s}_1 \dots \mathbf{s}'_{k-1} \xrightarrow{a_k}_{\mathcal{V}} \mathbf{s}_k \xrightarrow{\sigma_k}_{\mathcal{V}} \mathbf{s}'_k$$

We observe that $\mathcal{L}(\mathcal{U}) \subseteq \mathcal{L}(\mathbf{b}, \mathcal{V}, \mathbf{b}')$ since $(\mathbf{s}_0, \mathbf{s}'_k) = (\mathbf{b}, \mathbf{b}')$.

We now associate a characteristic linear system to a MGVS \mathcal{U} . Denoting by $\mu_{j,t}$ the number of occurrences of a transition $t \in T_j$ in the cycle $\mathbf{x}_j \xrightarrow{\sigma_j}_{G_j} \mathbf{x}_j$ we get a non-negative sequence $(\mu_{j,t})_{j,t}$ indexed by $0 \leq j \leq k$ and $t \in T_j$. We also obtain a tuple $\xi = ((\mathbf{s}_j)_j, (\mathbf{s}'_j)_j, (\mu_{j,t})_{j,t})$ said *associated* to σ . We observe that ξ is a non-negative solution of the following linear system called the *characteristic system* of the MGVS \mathcal{U} :

$$\left\{ \begin{array}{l} \text{for all } 1 \leq j \leq k \\ \mathbf{s}'_{j-1} + \delta(a_j) = \mathbf{s}_j \\ \\ \text{for all } 0 \leq j \leq k \\ \mathbf{s}_j + \sum_{t=(q,a,q') \in T_j} \mu_{j,t} \delta(a) = \mathbf{s}'_j \\ \\ \text{for all } 0 \leq j \leq k \text{ and for all } 1 \leq i \leq n \\ \mathbf{s}_j[i] = \mathbf{m}_j[i] \text{ if } \mathbf{m}_j[i] \in \mathbb{N} \\ \mathbf{s}'_j[i] = \mathbf{m}'_j[i] \text{ if } \mathbf{m}'_j[i] \in \mathbb{N} \\ \\ \text{for all } 0 \leq j \leq k \text{ and for all } q_j \in Q_j \\ \sum_{t=(q,a,q_j) \in T} \mu_{j,t} = \sum_{t'=(q_j,a,q') \in T} \mu_{j,t'} \end{array} \right.$$

Naturally there exists non-negative solutions ξ of the characteristic system that are not associated to an accepted word. In particular even if there exists non-negative solutions of the characteristic linear system we cannot conclude that $\mathcal{L}(\mathcal{U}) \neq \emptyset$. However, under the following *perfect* condition, we can prove that $\mathcal{L}(\mathcal{U}) \neq \emptyset$.

A MGVS \mathcal{U} is said *perfect* if for any integer $c \geq 0$, there exists a non-negative solution $((\mathbf{s}_j)_j, (\mathbf{s}'_j)_j, (\mu_{j,t})_{j,t})$ of the characteristic system of \mathcal{U} such that for any $0 \leq j \leq k$ and for any $1 \leq i \leq n$:

- $\mathbf{s}_j[i] \geq c$ if $\mathbf{m}_j[i] = \top$, and
- $\mathbf{s}'_j[i] \geq c$ if $\mathbf{m}'_j[i] = \top$, and
- $\mu_{j,t} \geq c$ for any $t \in T_j$.

and such that for any $0 \leq j \leq k$:

- there exists a cycle $\theta_j = (\mathbf{x}_j \xrightarrow{w_j}_{G_j} \mathbf{x}_j)$ such that $\mathbf{m}_j \xrightarrow{w_j}_{\mathcal{V}}$ and such that $\mathbf{m}_j + \delta(w_j) \geq \mathbf{m}_j$ and $\delta(w_j)[i] > 0$ if $\mathbf{m}_j[i] \in \mathbb{N}$ and $\mathbf{x}_j[i] = \top$.
- there exists a cycle $\theta'_j = (\mathbf{x}_j \xrightarrow{w'_j}_{G_j} \mathbf{x}_j)$ such that $\mathbf{m}'_j \xrightarrow{w'_j}_{\mathcal{V}}$ and such that $\mathbf{m}'_j - \delta(w'_j) \geq \mathbf{m}'_j$ and $-\delta(w'_j)[i] > 0$ if $\mathbf{m}'_j[i] \in \mathbb{N}$ and $\mathbf{x}_j[i] = \top$.

Let us recall without proof the fundamental decomposition theorem.

THEOREM 1 (FUNDAMENTAL DECOMPOSITION[8]). *For any tuple $(\mathbf{b}, \mathcal{V}, \mathbf{b}')$, we can effectively compute a finite sequence of perfect MGVS $\mathcal{U}_1, \dots, \mathcal{U}_l$ for this tuple such that:*

$$\mathcal{L}(\mathbf{b}, \mathcal{V}, \mathbf{b}') = \mathcal{L}(\mathcal{U}_1) \cup \dots \cup \mathcal{L}(\mathcal{U}_l)$$

Let us now recall why $\mathcal{L}(\mathcal{U}) \neq \emptyset$ for any perfect MGVS \mathcal{U} . The homogeneous form of the characteristic system, obtained by replacing constants by zero, is called the *homogeneous characteristic system* of \mathcal{U} . A solution of the homogeneous form is denoted by $\xi_0 = ((\mathbf{s}_{0,j})_j, (\mathbf{s}'_{0,j})_j, (\mu_{0,j,t})_{j,t})$ in the sequel.

LEMMA 11. *There exists $\xi_0 = ((\mathbf{s}_{0,j})_j, (\mathbf{s}'_{0,j})_j, (\mu_{0,j,t})_{j,t})$ a solution of the homogeneous characteristic system and for any $0 \leq j \leq k$ there exists a cycle $\pi_{0,j} = (\mathbf{x}_j \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x}_j)$ and such that:*

- $\mu_{0,j,t} = |\theta_j|_t + |\pi_{0,j}|_t + |\theta'_j|_t$ for any $t \in T_j$,
- $\mathbf{s}_{0,j} \geq \mathbf{0}$ and $\mathbf{s}_{0,j}[i] > 0$ if and only if $\mathbf{m}_j[i] = \top$,
- $\mathbf{s}'_{0,j} \geq \mathbf{0}$ and $\mathbf{s}'_{0,j}[i] > 0$ if and only if $\mathbf{m}'_j[i] = \top$,
- $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{0}$ and $(\mathbf{s}_{0,j} + \delta(w_j))[i] > 0$ if and only if $\mathbf{x}_j[i] = \top$,
- $\mathbf{s}'_{0,j} - \delta(w'_j) \geq \mathbf{0}$ and $(\mathbf{s}'_{0,j} - \delta(w'_j))[i] > 0$ if and only if $\mathbf{x}_j[i] = \top$,

PROOF. As \mathcal{U} is perfect, for any integer $c \geq 0$, there exists a solution $\xi(c) = ((\mathbf{s}_j(c))_j, (\mathbf{s}'_j(c))_j, (\mu_{j,t}(c))_{j,t})$ of the characteristic system of \mathcal{U} such that for any $0 \leq j \leq k$:

- $\mathbf{s}_j(c)[i] \geq c$ if $\mathbf{m}_j[i] = \top$, and
- $\mathbf{s}'_j(c)[i] \geq c$ if $\mathbf{m}'_j[i] = \top$, and
- $\mu_{j,t}(c) \geq c$ for any $t \in T_j$.

Observe that $\xi_0(c) = ((\mathbf{s}_{0,j}(c))_j, (\mathbf{s}'_{0,j}(c))_j, (\mu_{0,j,t}(c))_{j,t})$ defined by $\xi_0(c) = \xi(c) - \xi(0)$ is solution of the homogeneous characteristic system. Observe that there exists an integer $c_0 \geq 0$ larger enough such that $\mathbf{s}_{0,j}(c)[i] > 0$ if $\mathbf{m}_j[i] = \top$, such that $\mathbf{s}'_{0,j}(c)[i] > 0$ if $\mathbf{m}'_j[i] = \top$, and such that $|\mu_{0,j,t}(c)| > |\theta_j|_t + |\theta_{j'}|_t$ for any $t \in T_j$ and for any integer $c \geq c_0$.

Let us prove that we can also assume the following properties for any $c \geq c_0$:

- $\mathbf{s}_{0,j}(c) + \delta(w_j) \geq 0$ and $(\mathbf{s}_{0,j}(c) + \delta(w_j))[i] > 0$ if $\mathbf{x}_j[i] = \top$.
- $\mathbf{s}_{0,j'}(c) - \delta(w'_j) \geq 0$ and $(\mathbf{s}_{0,j'}(c) - \delta(w'_j))[i] > 0$ if $\mathbf{x}_j[i] = \top$.

The two conditions are symmetrical. Let us just show the first one. Assume that $\mathbf{x}_j[i] = \top$. Consider the case $\mathbf{m}_j[i] \in \mathbb{N}$. The homogeneous characteristic system shows $\mathbf{s}_{0,j}(c)[i] = 0$. Moreover, by definition of θ_j , as $\mathbf{m}_j[i] < \mathbf{x}_j[i]$ we get $\delta(w_j)[i] > 0$. Thus $(\mathbf{s}_{0,j}(c) + \delta(w_j))[i] > 0$ in this case. Now, assume that $\mathbf{m}_j[i] = \top$. In this case $\mathbf{s}_{0,j}(c)[i] \geq c - \mathbf{s}_{0,j}(0)[i]$. Thus by replacing c_0 by a larger integer, we can assume that $(\mathbf{s}_{0,j}(c) + \delta(w_j))[i] > 0$ for any $c \geq c_0$.

Let us consider $\xi_0 = ((\mathbf{s}_{0,j})_j, (\mathbf{s}'_{0,j})_j, (\mu_{0,j,t})_{j,t})$ defined by fixing $c = c_0$ and formally given by $\xi_0 = \xi_0(c_0)$. Observe that if $\mathbf{m}_j[i] \in \mathbb{N}$ then $\mathbf{s}_{0,j}[i] = 0$. Thus $\mathbf{s}_{0,j} \geq \mathbf{0}$ and $\mathbf{s}_{0,j}[i] > 0$ if and only if $\mathbf{m}_j[i] = \top$. Symmetrically we deduce that $\mathbf{s}_{0,j'} \geq \mathbf{0}$ and $\mathbf{s}_{0,j'}[i] > 0$ if and only if $\mathbf{m}'_j[i] = \top$. Observe also that if $\mathbf{x}_j[i] \in \mathbb{N}$ then $\mathbf{s}_{0,j}[i] = 0$ and $\delta(w_j)[i] = 0$. In particular $(\mathbf{s}_{0,j} + \delta(w_j))[i] = 0$. We deduce that $\mathbf{s}_{0,j} + \delta(w_j) \geq \mathbf{0}$ and we have $(\mathbf{s}_{0,j} + \delta(w_j))[i] > 0$ if and only if $\mathbf{x}_j[i] = \top$. Symmetrically $\mathbf{s}_{0,j'} - \delta(w'_j) \geq \mathbf{0}$ and we have $(\mathbf{s}_{0,j'} - \delta(w'_j))[i] > 0$ if and only if $\mathbf{x}_j[i] = \top$.

From the Euler Lemma 9, there exists a cycle $\pi_{0,j} = (\mathbf{x}_j \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x}_j)$ such that $|\pi_{0,j}|_t = |\mu_{0,j,t}| - (|\theta_j|_t + |\theta_{j'}|_t)$ for any $t \in T_j$. \square

As the MGVS \mathcal{U} is perfect, Lemma 10 shows that there exists a non-negative solution $\xi = ((\mathbf{s}_j)_j, (\mathbf{s}'_j)_j, (\mu_{j,t})_{j,t})$ of the characteristic system such that $\mu_{j,t} \geq 1$ for any $0 \leq j \leq k$ and $t \in T_j$ and such that $\mathbf{s}_j \xrightarrow{w_j}_{\mathcal{V}}$ and $\frac{\sigma_{0,j} w'_j}{\mathcal{V}} \mathbf{s}'_j$ for any $0 \leq j \leq k$. Let $0 \leq j \leq k$. From Lemma 9, there exists a cycle $\pi_j = (\mathbf{x}_j \xrightarrow{\sigma_j}_{G_j} \mathbf{x}_j)$ such that $\mu_{j,t} = |\pi_j|_t$ for any $t \in T_j$. Let us observe that for any $c \in \mathbb{N}$ and for any $0 \leq j \leq k$ we have:

$$(\mathbf{s}_j + c\mathbf{s}_{0,j}) \xrightarrow{w_j^c}_{\mathcal{V}} \frac{\sigma_{0,j}^c (w'_j)^c}{\mathcal{V}} (\mathbf{s}'_j + c\mathbf{s}'_{0,j})$$

As $\lim_{c \rightarrow +\infty} (\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j))) = \mathbf{x}_j$, lemma 10 proves that there exists an integer $c_0 \geq 0$ such that for any integer $c \geq c_0$

we have $(\mathbf{s}_j + c(\mathbf{s}_{0,j} + \delta(w_j))) \xrightarrow{\sigma_j}_{\mathcal{V}}$. As ξ is a solution of the characteristic system we get $\mathbf{s}_j + \delta(\sigma_j) = \mathbf{s}'_j$. Moreover, as ξ_0 is a solution of the homogeneous characteristic system we get $\mathbf{s}_{0,j} + \delta(w_j \sigma_{0,j} w'_j) = \mathbf{s}_{0,j}$. In particular we deduce that $(\mathbf{s}_j + c\mathbf{s}_{0,j}) + c\delta(w_j) + \delta(\sigma_j) = (\mathbf{s}'_j + c\mathbf{s}'_{0,j}) - c\delta(w'_j) - c\delta(\sigma_{0,j})$. We deduce that for any $c \geq c_0$ we have:

$$(\mathbf{s}_j + c\mathbf{s}_{0,j}) \xrightarrow{w_j^c \sigma_{0,j}^c (w'_j)^c}_{\mathcal{V}} (\mathbf{s}'_j + c\mathbf{s}'_{0,j})$$

Therefore there exists a word in $\mathcal{L}(\mathcal{U})$ associated to $\xi + c\xi_0$ for any $c \geq c_0$. In particular we have proved that $\mathcal{L}(\mathcal{U}) \neq \emptyset$.

4.2 Semi-pseudo-linear Reachability Sets

In this section we prove that $\text{post}_{\mathcal{V}}^*(S) \cap S'$ and $S \cap \text{pre}_{\mathcal{V}}^*(S')$ are semi-pseudo-linear for any semi-linear sets $S, S' \subseteq \mathbb{N}^n$.

We first reduce this problem to the forward case with $S = \{\mathbf{b}\}$ and $S' = \mathbf{b} + Z^*$ where Z is a set of periods. Since $\text{post}_{\mathcal{V}}^*(X) = \text{pre}_{-\mathcal{V}}^*(X)$ for any set $X \subseteq \mathbb{N}^n$ where $-\mathcal{V} = (\Sigma, n, -\delta)$, the problem reduces to prove that $\text{post}_{\mathcal{V}}^*(S) \cap S'$ is semi-pseudo-linear. Note also that a semi-linear set is a finite union of linear sets. As $\text{post}_{\mathcal{V}}^*(X_1 \cup \dots \cup X_k) = \text{post}_{\mathcal{V}}^*(X_1) \cup \dots \cup \text{post}_{\mathcal{V}}^*(X_k)$ for any $X_1, \dots, X_k \subseteq \mathbb{N}^n$, we have reduced the problem to prove that $\text{post}_{\mathcal{V}}^*(S) \cap S'$ is semi-pseudo-linear for any linear sets S, S' . Let us consider a linear set $S = \mathbf{b} + P^*$ where $\mathbf{b} \in \mathbb{N}^n$ and $P \subseteq \mathbb{N}^n$ a finite set of periods. Without loss of generality we can assume that $P \cap \Sigma = \emptyset$. Let us consider the VAS $\mathcal{V}' = (\Sigma', n, \delta')$ where $\Sigma' = \Sigma \cup P$ and δ' extends δ by $\delta(\mathbf{p}) = \mathbf{p}$ for any $\mathbf{p} \in P$. We observe that $\text{post}_{\mathcal{V}}^*(S) = \text{post}_{\mathcal{V}'}^*(\{\mathbf{b}\})$. In particular we can assume that $S = \{\mathbf{b}\}$. We have proved the proposed reduction.

So, let us consider two configurations $\mathbf{b}, \mathbf{b}' \in \mathbb{N}^n$, a set of periods $Z \subseteq \mathbb{N}^n$ and let us prove that $\text{post}_{\mathcal{V}}^*(\{\mathbf{b}\}) \cap (\mathbf{b}' + Z^*)$ is semi-pseudo-linear. We can assume that $Z \cap \Sigma = \emptyset$. As expected, we consider the VAS \mathcal{V}_Z obtained from \mathcal{V} by considering the alphabet $\Sigma \cup Z$ and the displacement function that extends δ by $\delta(\mathbf{z}) = -\mathbf{z}$ for any $\mathbf{z} \in Z$. We also consider the displacement function δ_Z also defined over $\Sigma \cup Z$ by $\delta_Z(a) = \mathbf{0}$ if $a \in \Sigma$ and $\delta_Z(\mathbf{z}) = \mathbf{z}$ if $\mathbf{z} \in Z$. Observe that we have the following equality:

$$\text{post}_{\mathcal{V}}^*(\{\mathbf{b}\}) \cap (\mathbf{b}' + Z^*) = \mathbf{b}' + \delta_Z(\mathcal{L}(\mathbf{b}, \mathcal{V}_Z, \mathbf{b}'))$$

From theorem 1, it is sufficient to prove that $\delta_Z(\mathcal{L}(\mathcal{U}))$ is pseudo-linear for any perfect MGVS \mathcal{U} for $(\mathbf{b}, \mathcal{V}_Z, \mathbf{b}')$. So let us consider such a perfect MGVS \mathcal{U} of the following form:

$$\mathcal{U} = (\mathbf{m}_0, G_0, \mathbf{x}_0, \mathbf{m}'_0), a_1, \dots, a_k, (\mathbf{m}_k, G_k, \mathbf{x}_k, \mathbf{m}'_k)$$

Given a solution $\xi = ((\mathbf{s}_j)_j, (\mathbf{s}'_j)_j, (\mu_{j,t})_{j,t})$ of the characteristic system and a solution $\xi_0 = ((\mathbf{s}_{0,j})_j, (\mathbf{s}'_{0,j})_j, (\mu_{0,j,t})_{j,t})$ of the homogeneous characteristic system, we consider the vectors $\beta_0(\xi_0)$ and $\beta(\xi)$ defined by the following equalities:

$$\beta(\xi) = \delta_Z(a_1 \dots a_k) + \sum_{\substack{\mathbf{z} \in Z \\ 0 \leq j \leq k \\ t = (q, \mathbf{z}, q') \in T_j}} \mu_{j,t} \mathbf{z}$$

$$\beta_0(\xi_0) = \sum_{\substack{\mathbf{z} \in Z \\ 0 \leq j \leq k \\ t = (q, \mathbf{z}, q') \in T_j}} \mu_{0,j,t} \mathbf{z}$$

We observe that $\xi + \xi_0$ is a solution of the characteristic system that satisfies $\beta(\xi + \xi_0) = \beta(\xi) + \beta_0(\xi_0)$. Moreover, note that if ξ is associated to a word $\sigma \in \mathcal{L}(\mathcal{U})$ then $\delta_Z(\sigma) = \beta(\xi)$.

Intuitively, the linearizator P for $\delta_Z(\mathcal{L}(\mathcal{U}))$ is obtained from the homogeneous characteristic system of \mathcal{U} . Given a non-negative integer h , recall that (\mathbb{N}^h, \leq) is a well-quasi partially-ordered set. Thus any $X \subseteq \mathbb{N}^h$ has a finite set $\min(X)$ of minimal for \leq elements satisfying $X \subseteq \min(X) + \mathbb{N}^h$. We denote by H_0 the non-negative solutions of the homogeneous characteristic system. We consider the following set of periods:

$$P = \beta_0(\min(H_0 \setminus \{\mathbf{0}\}))$$

Let us denote by H the set of non-negative solutions of the characteristic system and let ξ be any element in this set (since the MGVS is perfect the set H is non empty). Note that $\xi' - \xi$ is a solution of the homogeneous characteristic system for any $\xi' \in H$. Naturally this solution is not necessary non-negative. However, there exists ξ'_+ and ξ'_- in H_0 such that $\xi' - \xi = \xi'_+ - \xi'_-$. As $\min(H)$ is finite we deduce that $\xi_0 = \sum_{\xi' \in \min(H)} \xi'_-$ is a tuple in H_0 . Observe that $\min(H) - \xi + \xi_0 \subseteq H_0$. From $H = \min(H) + H_0$, we deduce that $H \subseteq \xi - \xi_0 + H_0$. From $H_0 = \min(H_0 \setminus \{\mathbf{0}\})^*$ we deduce the following inclusion:

$$\delta_Z(\mathcal{L}(\mathcal{U})) \subseteq \beta(\xi) - \beta_0(\xi_0) + P^*$$

Let us consider a finite set $R \subseteq P^+$ and let $\mathbf{r}_1, \dots, \mathbf{r}_d$ be a sequence in R such that $R = \{\mathbf{r}_1, \dots, \mathbf{r}_d\}$. We first prove the following lemma.

LEMMA 12. *For any vector $\mathbf{r} \in P^+$ there exists a non-negative solution $\xi_{\mathbf{r}} = ((\mathbf{s}_{\mathbf{r},j})_j, (\mathbf{s}'_{\mathbf{r},j})_j, (\mu_{\mathbf{r},j,t})_{j,t})$ of the homogeneous system and for any $0 \leq j \leq k$ there exists a cycle $\pi_{\mathbf{r},j} = (\mathbf{x}_j \xrightarrow{\sigma_{\mathbf{r},j}}_{G_j} \mathbf{x}_j)$ such that:*

- $\mu_{\mathbf{r},j,t} = |\pi_{\mathbf{r},j}|_t$ for any $t \in T_j$ and for any $0 \leq j \leq k$,
- $\mathbf{r} = \delta_Z(\sigma_{\mathbf{r},0} \dots \sigma_{\mathbf{r},k})$.

PROOF. Let us consider $\xi_{\mathbf{r}} = ((\mathbf{s}_{\mathbf{r},j})_j, (\mathbf{s}'_{\mathbf{r},j})_j, (\mu_{\mathbf{r},j,t})_{j,t})$ in $\min(H_0 \setminus \{\mathbf{0}\})^+$ such that $\mathbf{r} = \beta_0(\xi_{\mathbf{r}})$. As \mathcal{U} is perfect, we deduce that $\mu_{\mathbf{r},j,t} > 1$ for any $t \in T_j$. Lemma 9 shows that there exists a cycle $\pi_{\mathbf{r},j} = (\mathbf{x}_j \xrightarrow{\sigma_{\mathbf{r},j}}_{G_j} \mathbf{x}_j)$ such that $\mu_{\mathbf{r},j,t} = |\pi_{\mathbf{r},j}|_t$ for any $t \in T_j$. Note that $\beta_0(\xi_{\mathbf{r}}) = \delta_Z(\sigma_{\mathbf{r},0} \dots \sigma_{\mathbf{r},k})$. \square

Now, let us consider a solution ξ of the characteristic system and a solution ξ_0 of the homogeneous characteristic system satisfying the same properties that the ones given at the end of the previous section. As $\lim_{c \rightarrow +\infty} (\mathbf{s}_j + c(\mathbf{s}_{\mathbf{0},j} + \delta(w_j))) = \mathbf{x}_j$, lemma 10 proves that by replacing c_0 by a larger integer, we can assume that $(\mathbf{s}_j + c(\mathbf{s}_{\mathbf{0},j} + \delta(w_j))) \xrightarrow{\sigma_{\mathbf{r},j}}_{\mathcal{V}}$ for any $c \geq c_0$, for any $0 \leq j \leq k$ and for any $\mathbf{r} \in R$. Let us consider a sequence $n_1, \dots, n_d \in \mathbb{N}$ and an integer $c \geq c_0$. Observe

that for any $0 \leq j \leq k$ we have:

$$\frac{\mathbf{s}_j + c\mathbf{s}_{\mathbf{0},j} + \sum_{i=1}^d n_i \mathbf{s}_{\mathbf{r}_i,j}}{w_j^c \sigma_j \sigma_{\mathbf{r}_1,j}^{n_1} \dots \sigma_{\mathbf{r}_d,j}^{n_d} \sigma_{\mathbf{0},j}^c (w'_j)^c} \xrightarrow{\mathcal{V}} \mathbf{s}'_j + c\mathbf{s}'_{\mathbf{0},j} + \sum_{i=1}^d n_i \mathbf{s}'_{\mathbf{r}_i,j}$$

We have proved that there exists a word $\sigma \in \mathcal{L}(\mathcal{U})$ associated with $\xi + c\xi_0 + \sum_{i=1}^d n_i \xi_{\mathbf{r}_i}$. By applying β to the previous tuple, we deduce that $\delta_Z(\sigma) = \beta(\xi) + c\beta_0(\xi_0) + \sum_{i=1}^d n_i \mathbf{r}_i$. We have proved the following inclusion:

$$\beta(\xi) + c\beta_0(\xi_0) + R^* \subseteq \delta_Z(\mathcal{L}(\mathcal{U}))$$

We have proved that P is a linearizator for $\delta_Z(\mathcal{L}(\mathcal{U}))$. Therefore, we have proved the following theorem:

THEOREM 2. *For any pair $S, S' \subseteq \mathbb{N}^n$ of semi-linear sets, the sets $\text{post}_{\mathcal{V}}^*(S) \cap S'$ and $S \cap \text{pre}_{\mathcal{V}}^*(S')$ are semi-pseudo-linear.*

5. SEMI-LINEAR SEPARATORS

In this section we prove that there exists a semi-linear separator for any pair of semi-linear separable pair.

Given a pair (S, S') of separable sets of configurations, the set $D = \mathbb{N}^n \setminus (S \cup S')$ is called the *domain* of (S, S') . Note that a separable pair (S, S') with an empty domain D is a separator.

The semi-linear separator for a semi-linear separable pair (S_0, S'_0) is obtain inductively. We build a non-decreasing sequence $(S_j, S'_j)_{j \geq 0}$ of semi-linear separable sets starting from the initial semi-linear separable pair (S_0, S'_0) such that the dimension of the domain $D_j = \mathbb{N}^n \setminus (S_j \cup S'_j)$ is strictly decreasing. In order to obtain this sequence, observe that it is sufficient to show that for any semi-linear separable pair (S_0, S'_0) of configuration sets with a non-empty domain D_0 , there exists a semi-linear separable pair $(S, S') \supseteq (S_0, S'_0)$ with a domain D such that $\dim(D) < \dim(D_0)$.

As D_0 and S_0 are semi-linear, we deduce that $\text{post}_{\mathcal{V}}^*(S_0) \cap D_0$ is equal to a finite union of pseudo-linear sets X_1, \dots, X_k . Let us consider some linearizators P_1, \dots, P_k for these pseudo-linear sets and let us consider the following semi-linear set:

$$S' = S'_0 \cup (D_0 \setminus (\bigcup_{j=1}^k X_j + P_j^*))$$

Observe that $\text{post}_{\mathcal{V}}^*(S_0) \cap S' = \emptyset$. Thus $\text{post}_{\mathcal{V}}^*(S_0) \cap \text{pre}_{\mathcal{V}}^*(S') = \emptyset$ and we deduce that $X_j \cap \text{pre}_{\mathcal{V}}^*(S') = \emptyset$ for any $1 \leq j \leq k$. From the following Lemma 13, we deduce that there exists a semi-linear set $Z_j \subseteq X_j + P_j^*$ such that $\dim(Z_j) < \dim(X_j)$ and such that $(X_j + P_j^*) \cap \text{pre}_{\mathcal{V}}^*(S') \subseteq Z_j$. We consider the following set:

$$S = S_0 \cup \bigcup_{j=1}^k ((X_j + P_j^*) \setminus Z_j)$$

Rennes, France) and for his work on semi-linear VAS.

LEMMA 13. *Let X be a pseudo-linear set, let P be a linearizer for X , and let S' be a semi-linear set such that $X \cap \text{pre}_V^*(S') = \emptyset$. There exists a semi-linear set $Z \subseteq X + P^*$ such that $\dim(Z) < \dim(X)$ and such that $(X + P^*) \cap \text{pre}_V^*(S') \subseteq Z$.*

PROOF. As $X + P^*$ is semi-linear, we deduce that $(X + P^*) \cap \text{pre}_V^*(S')$ is semi-pseudo-linear. Thus this set is equal to a finite union of pseudo-linear sets X_1, \dots, X_k . We denote by P_j a precise linearizer for X_j . As $X \cap \text{pre}_V^*(S') = \emptyset$ we deduce that $X \cap X_j = \emptyset$ for any $1 \leq j \leq k$. Thus the dimension of the semi-linear set $Z_j = (X + P^*) \cap (X_j + P_j^*)$ is strictly less than $\max\{\dim(X), \dim(X_j)\}$. From $X_j \subseteq X + P^*$ we also deduce that $\dim(X_j) \leq \dim(X + P^*) = \dim(X)$. Thus $\dim(X_j) \leq \dim(X)$ and we get $\dim(Z_j) < \dim(X)$. Denoting by $Z = \bigcup_j Z_j$ we are done. \square

Observe that $S \cap \text{pre}_V^*(S') = \emptyset$. Thus $\text{post}_V^*(S) \cap \text{pre}_V^*(S') = \emptyset$ and we deduce that (S, S') is separable. The domain D of this separable pair satisfies the following inclusion:

$$D \subseteq \bigcup_{j=1}^k Z_j$$

As $X_j \subseteq D_0$ we have $\dim(X_j) \leq \dim(D_0)$. From $\dim(Z_j) < \dim(X_j)$ we get $\dim(Z_j) < \dim(D_0)$. Therefore $\dim(D) < \dim(D_0)$. We have proved the following theorem.

THEROEM 3. *There exists a semi-linear separator for any pair of semi-linear separable sets.*

6. CONCLUSION

We have proved the termination of the algorithm **Reachability**. Even though the proof is based on the classical KLMST decomposition, its complexity does not depend on this decomposition. In fact, the complexity of this algorithm depends on the size of the minimal pair of Presburger formulas denoting a separator for a separable pair $(\{s\}, \{s'\})$. This algorithm is the *very first one* that does not require the KLMST decomposition. In particular, this algorithm is the first candidate to obtain a precise (eventually elementary) upper-bound complexity for the VAS reachability problem. We left as an open question the problem of computing a lower bound and an upper bound of the size of a pair of Presburger formulas denoting a separator for a separable pair $(\{s\}, \{s'\})$. Note that the VAS exhibiting a large (Ackermann size) but finite reachability set given in [4] does not directly provide a lower-bound for this size since separators can over-approximate reachability sets.

We also left as an open question the problem of adapting such an algorithm to obtain a complete *Counter Example Guided Abstract Refinement* approach [1] for the VAS reachability problem based on *interpolants* [6] for FO($\mathbb{N}, +, \leq$). In practice, such an algorithm should be more efficient than the enumeration-based algorithm provided in this paper.

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