The General Vector Addition System Reachability Problem by Presburger Inductive Invariants

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Abstract

The reachability problem for Vector Addition Systems (VASs) is a central problem of net theory. The general problem is known decidable by algorithms exclusively based on the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney decomposition. This decomposition is used in this paper to prove that the Parikh images of languages accepted by VASs are semi-pseudo-linear*; a class that extends the semi-linear sets, a.k.a. the sets definable in the Presburger arithmetic. We provide an application of this result; we prove that a final configuration is not reachable from an initial one if and only if there exists a Presburger formula denoting a forward inductive invariant that contains the initial configuration but not the final one. Since we can decide if a Preburger formula denotes an inductive invariant, we deduce that there exist checkable certificates of non-reachability. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas.*

I. Introduction

Vector Addition Systems (VASs) or equivalently Petri Nets are one of the most popular formal methods for the representation and the analysis of parallel processes [2]. The reachability problem is central since many computational problems (even outside the parallel processes) reduce to the reachability problem. Sacerdote and Tenney provided in [10] a partial proof of decidability of this problem. The proof was completed in 1981 by Mayr [7] and simplified by Kosaraju [5] from [7], [10]. Ten years later [6], Lambert provided a more simplified version based on [5]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known non-primitive recursive. Nowadays, the exact complexity of the reachability problem for VASs is still an open-problem. Even an elementary upper-bound complexity is open. In fact, the known general reachability algorithms are exclusively based on the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition.

In this paper, by using the KLMST decomposition we prove that the Parikh images of languages accepted by VASs are semi-pseudo-linear, a class that extends the semi-linear sets, a.k.a. the sets definable in the Presburger arithmetic [3]. We provide an application of this result; we prove that a final configuration is not reachable from an initial one if and only if there exists a Presburger formula denoting a forward inductive invariant that contains the initial configuration but not the final one. Since we can decide if a Presburger formula denotes an inductive invariant, we deduce that there exist checkable certificates of non-reachability. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas.

Outline of the paper: Section II introduces the class of *Vector Addition Systems (VASs)*. Section III recalls the class of *MRGSs* and the KLMST decomposition of languages accepted by VASs into finite unions of languages accepted by *perfect MRGSs*. Semi-pseudo-linear sets are introduced in Section IV. In Section V, Parikh images of languages accepted by perfect MRGSs are proved pseudo-linear. In Section VI we introduce the class of *locally semi-pseudo-linear sets* a subclass of the semi-pseudo-linear sets stable by intersection with any semi-linear set. Reachability sets of VASs from semi-linear sets are proved locally semi-pseudo-linear in this section. In Section VII

we study approximations of two pseudo-linear sets having an empty intersection. Finally in Section VIII we deduce that if a final configuration is not reachable from an initial one, there exists a Presburger formula denoting a forward inductive invariant that contains the initial configuration but not the final one.

II. Vector Addition Systems

We denote by $\mathbb{Q}, \mathbb{Q}_+, \mathbb{Z}, \mathbb{N}$ respectively the set of rational values, non-negative rational values, the set of integers and the set of non-negative integers. The *components of a vector* $\mathbf{x} \in \mathbb{Q}^n$ are denoted by $(\mathbf{x}[1], \ldots, \mathbf{x}[n])$. Let $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x} \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$. The sum $\mathbf{x_1} + \mathbf{x_2}$ and the product $r\mathbf{x}$ are naturally defined component wise. Given a function $f : E \to F$ where E, F are sets, we denote by $f(X) = \{f(x) \mid x \in X\}$ for any subset $X \subseteq E$. This definition naturally defines sets $X_1 + X_2$ and RX where $X_1, X_2, X \subseteq \mathbb{Q}^n$ and $R \subseteq \mathbb{Q}$. With slight abuse of notation, $\{\mathbf{x_1}\} + X_2$, $X_1 + \{\mathbf{x_2}\}$, $\{r\}X$ and $R\{\mathbf{x}\}$ are simply denoted by $\mathbf{x_1} + X_2$, $X_1 + \mathbf{x_2}$, rX and $R\mathbf{x}$.

The lattice $(\mathbb{N}, \leq)$ is completed with an additional element $\top$ such that $k \leq \top$ for any $k \in \mathbb{N} \cup \{\top\}$. The set $\mathbb{N} \cup \{\top\}$ is denoted by $\mathbb{N}_\top$. Given a non-decreasing sequence $(x_i)_{i \geq 0}$ in $(\mathbb{N}_\top, \leq)$ we denote by $\lim_{i \to +\infty}(x_i)$ the *least upper bound* in $\mathbb{N}_\top$. The $\top$ element is interpreted as a "don't care value" by introducing the partial order $\trianglelefteq$ over $\mathbb{N}_\top$ defined by $x_1 \trianglelefteq x_2$ if and only if $x_1 = x_2$ or $x_2 = \top$. Orders $\leq$ and $\trianglelefteq$ are extended component-wise over $\mathbb{N}_\top^n$. The set of minimal elements for $\leq$ of a set $X \subseteq \mathbb{N}^n$ is denoted by $\min(X)$. As $(\mathbb{N}^n, \leq)$ is a *well partially ordered set*, note that $\min(X)$ is finite and $X \subseteq \min(X) + \mathbb{N}^n$ for any $X \subseteq \mathbb{N}^n$.

An *alphabet* is a non-empty finite set Σ. The set of words over Σ is denoted by Σ^*. The empty word is denoted by ϵ. The concatenation of two words σ_1 and σ_2 is simply denoted by $\sigma_1\sigma_2$. The concatenation of $r \geq 1$ times a word σ is denoted by σ^r. By definition $\sigma^0 = \epsilon$. The number of occurrences of an element $a \in \Sigma$ in a word $\sigma \in \Sigma^*$ is denoted by $|\sigma|_a$. The *Parikh image* of a word σ over Σ is the function $||\sigma||_\Sigma : \Sigma \to \mathbb{N}$ defined by $||\sigma||_\Sigma(a) = |\sigma|_a$ for any $a \in \Sigma$. This function is simply denoted by $||\sigma||$ when Σ is known without any ambiguity. The *Parikh image* $||\mathcal{L}||$ of a language $\mathcal{L} \subseteq \Sigma^*$ is defined as the set of functions $||\sigma||$ over the words $\sigma \in \mathcal{L}$.

A *Vector Addition System (VAS)* is a tuple $\mathcal{V} = (\Sigma, n, \delta)$ where Σ is an alphabet, $n \in \mathbb{N}$ is the *dimension*, and $\delta : \Sigma \to \mathbb{Z}^n$. Functions $\delta : \Sigma \to \mathbb{Z}^n$ are called *displacement functions*. These functions are naturally extended to functions $\delta : \Sigma^* \to \mathbb{Z}^n$ satisfying $\delta(\epsilon) = \mathbf{0}$ and $\delta(\sigma) = \sum_{i=1}^k \delta(a_i)$ for any word $\sigma = a_1 \ldots a_k$ of $k \geq 1$ elements $a_i \in \Sigma$. A *configuration* is a vector in $\mathbb{N}^n$ and an *extended configuration* is a vector in $\mathbb{N}_\top^n$. For $a \in \Sigma$, the binary relation $\xrightarrow{a}_\mathcal{V}$ is defined over the set of extended configurations by $\mathbf{x} \xrightarrow{a}_\mathcal{V} \mathbf{x}'$ if and only if $\mathbf{x}' = \mathbf{x} + \delta(a)$ with $\top + z = \top$ by definition for any $z \in \mathbb{Z}$. Let $k \geq 1$. Given a word $\sigma = a_1 \ldots a_k$ of elements $a_i \in \Sigma$, we denote by $\xrightarrow{\sigma}_\mathcal{V}$ the concatenation $\xrightarrow{a_1}_\mathcal{V} \cdots \xrightarrow{a_k}_\mathcal{V}$. By definition $\xrightarrow{\epsilon}_\mathcal{V}$ is the identity binary relation over the set of extended configurations. We denote by $\xrightarrow{*}_\mathcal{V}$ the *reachability binary relation* over the set of extended configurations defined by $\mathbf{x} \xrightarrow{*}_\mathcal{V} \mathbf{x}'$ if and only if there exists $\sigma \in \Sigma^*$ such that $\mathbf{x} \xrightarrow{\sigma}_\mathcal{V} \mathbf{x}'$. Observe that in this case $\mathbf{x}[i] = \top$ if and only if $\mathbf{x}'[i] = \top$. Intuitively the $\top$ element provides a simple way to get rid of some components of a VAS since these components remain equal to $\top$. The *reachability problem for* $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ where $(\mathbf{s}, \mathbf{s}')$ are two configurations of a VAS $\mathcal{V}$ consists to decide if $\mathbf{s} \xrightarrow{*}_\mathcal{V} \mathbf{s}'$. Let $\mathbf{m}, \mathbf{m}'$ be two extended configurations. The *language accepted* by $(\mathbf{m}, \mathcal{V}, \mathbf{m}')$ is the set $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}') = \{\sigma \in \Sigma^* \mid \exists \mathbf{s}, \mathbf{s}' \in \mathbb{N}^n \quad \mathbf{s} \trianglelefteq \mathbf{m} \quad \mathbf{s} \xrightarrow{\sigma}_\mathcal{V} \mathbf{s}' \quad \mathbf{s}' \trianglelefteq \mathbf{m}'\}$. Given two sets S, S' of configurations, the set $\mathrm{post}^*_\mathcal{V}(S)$ of *reachable configurations from* S and the set $\mathrm{pre}^*_\mathcal{V}(S')$ of *co-reachable configurations from* S' are formally defined by:

$$\mathrm{post}^*_\mathcal{V}(S) = \{\mathbf{s}' \in \mathbb{N}^n \mid \exists \mathbf{s} \in S \quad \mathbf{s} \xrightarrow{*}_\mathcal{V} \mathbf{s}'\}$$
$$\mathrm{pre}^*_\mathcal{V}(S') = \{\mathbf{s} \in \mathbb{N}^n \mid \exists \mathbf{s}' \in S' \quad \mathbf{s} \xrightarrow{*}_\mathcal{V} \mathbf{s}'\}$$

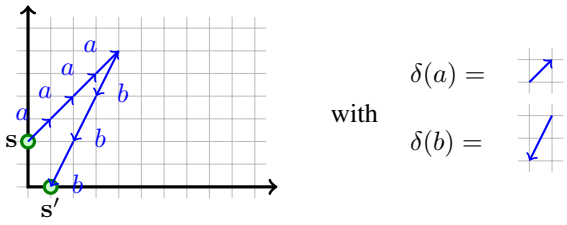

Figure 1. A Vector Addition System.

Example II.1 *A VAS* $\mathcal{V} = (\Sigma, n, \delta)$ *with* $\Sigma = \{a, b\}$, $n = 2$, $\delta(a) = (1, 1)$ *and* $\delta(b) = (-1, -2)$ *is depicted in Figure 1. Observe that* $\mathbf{s} \xrightarrow{a^4b^3}_\mathcal{V} \mathbf{s}'$ *with* $\mathbf{s} = (0, 2)$ *and* $\mathbf{s}' = (1, 0)$. *Note that* $\mathrm{post}^*_\mathcal{V}(\{\mathbf{s}\}) = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[2] \leq \mathbf{x}[1]+2\}$ *and* $\mathrm{pre}^*_\mathcal{V}(\{\mathbf{s}'\}) = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[2] \geq 2(\mathbf{x}[1]-1)\}$.

A *graph* is a tuple $G = (Q, \Sigma, T)$ where Q is a finite set of *states*, Σ is an alphabet, $T \subseteq Q \times \Sigma \times Q$ is a finite set of *transitions*. A *path* π is a word $\pi = t_1 \ldots t_k$ of $k \in \mathbb{N}$ transitions $t_i \in T$ such that there exists $q_0, \ldots, q_k \in Q$ and there exists $a_1, \ldots, a_k \in \Sigma$ such that $t_i = (q_{j-1}, a_j, q_j)$ for any $1 \leq j \leq k$. In this case we say that π *is a path labeled by* $\sigma = a_1 \ldots a_k$ *from* q_0 *to* q_k. We also denote π by $q_0 \xrightarrow{\sigma}_G q_k$. When the states q_0 and q_k are equal, the path π is called a *cycle on this state*. As usual a graph is said *strongly connected* if for any pair of states $(q, q') \in Q \times Q$, there exists a path for q to q'.

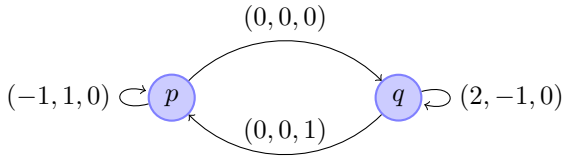

Figure 2. A VASS taken from [4].

Remark II.2 *A* Vector Addition System with States (VASS) *is a tuple* $(Q, \Sigma, T, n, \delta)$ *where* $G = (Q, \Sigma, T)$ *is a graph and* $\mathcal{V} = (\Sigma, n, \delta)$ *is a VAS. A couple in* $Q \times \mathbb{N}^n$ *is called a* VASS configuration. *Let* $\sigma \in \Sigma^*$. *The VASS semantics is defined over the VASS configurations by* $(q, \mathbf{s}) \xrightarrow{\sigma} (q', \mathbf{s}')$ *if and only if* $q \xrightarrow{\sigma}_G q'$ *and* $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'$. *Note [4] that* n*-dim VASSs can be simulated by* $(n+3)$*-dim VASs.*

Example II.3 *Recall [4] that sets* $\mathrm{post}^*_{\mathcal{V}}(S)$ *and* $\mathrm{pre}^*_{\mathcal{V}}(S')$ *are definable in the Presburger arithmetic* $\mathrm{FO}(\mathbb{N}, +, \leq)$ *when* S *and* S' *are definable in this logic and* $n \leq 5$. *Moreover from [4] we deduce an example of* 6*-dim VAS* $\mathcal{V}$ *and a pair of configurations* $(\mathbf{s}, \mathbf{s}') \notin \xrightarrow{*}_{\mathcal{V}}$ *such that neither* $\mathrm{post}^*_{\mathcal{V}}(\{\mathbf{s}\})$ *nor* $\mathrm{pre}^*_{\mathcal{V}}(\{\mathbf{s}'\})$ *are definable in the Presburger arithmetic. This example is obtained by considering the VASS depicted at Figure 2. This VASS has a loop on state* p *and another loop on state* q. *Intuitively iterating loop on state* p *transfers the content of the first counter to the second counter whereas iterating the loop on state* q *transfers and multiply by two the content of the second counter to the first counter. The third counter is incremented each time we come back to state* p *from* q. *In [4] the set of reachable configurations from* $(p, (1, 0, 0))$ *is proved equal to* $(\{p\} \times \{\mathbf{x} \in \mathbb{N}^3 \mid \mathbf{x}[1] + \mathbf{x}[2] \leq 2^{\mathbf{x}[3]}\}) \cup (\{q\} \times \{\mathbf{x} \in \mathbb{N}^3 \mid \mathbf{x}[1] + 2\mathbf{x}[2] \leq 2^{\mathbf{x}[3]+1}\})$.

III. The KLMST decomposition

The emptiness of $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}')$ can be decided with the *Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST)* decomposition. This decomposition shows that $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}')$ is effectively decomposable as a finite union $\bigcup_{\mathcal{U} \in F} \mathcal{L}(\mathcal{U})$ where $\mathcal{L}(\mathcal{U})$ is the language accepted by a *perfect MRGS* $\mathcal{U}$. We provide in Section III-A a new definition of *perfect MRGS* that does not require complex constructions. This definition is proved equivalent to the original one [6] in Section III-B. Finally in Section III-C we recall the KLMST decomposition.

A. The Perfect MRGSs

In this section we introduce the class of *Marked Reachability Graph Sequences (MRGSs)* by following notations introduced by Lambert [6]. We also provide a new definition for the class of MRGS said to be *perfect* [6].

A *reachability graph* for a VAS $\mathcal{V} = (\Sigma, n, \delta)$ is a graph $G = (Q, \Sigma, T)$ with $Q \subseteq \mathbb{N}^n_\top$ and $T \subseteq \{(q, a, q') \in Q \times \Sigma \times Q \mid q \xrightarrow{a}_{\mathcal{V}} q'\}$. A *marked reachability graph* $\mathcal{M} = (\mathbf{m}, \mathbf{x}, G, \mathbf{x}', \mathbf{m}')$ for $\mathcal{V}$ is a strongly connected reachability graph G for $\mathcal{V}$ equipped with two extended configurations $\mathbf{x}, \mathbf{x}' \in Q$ respectively called the *input state* and the *output state*, and equipped with two extended configurations $\mathbf{m}, \mathbf{m}'$ satisfying $\mathbf{m} \trianglelefteq \mathbf{x}$ and $\mathbf{m}' \trianglelefteq \mathbf{x}'$ respectively called the *input constraint* and the *output constraint*. An *accepted tuple* for $\mathcal{M}$ is a tuple $(\mathbf{s}, \pi, \mathbf{s}')$ where $\pi = (\mathbf{x} \xrightarrow{\sigma}_G \mathbf{x}')$ is a path in G labeled by σ from the input state $\mathbf{x}$ to the output state $\mathbf{x}'$ and where $\mathbf{s}, \mathbf{s}' \in \mathbb{N}^n$ are two configurations such that $\mathbf{s} \trianglelefteq \mathbf{m}$, $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'$ and $\mathbf{s}' \trianglelefteq \mathbf{m}'$. Intuitively the graph G and the input/output states enforce σ to label a path in G from $\mathbf{x}$ to $\mathbf{x}'$. The *input/output constraints* enforce $\mathbf{s}[i]$ and $\mathbf{s}'[i]$ to be equal to $\mathbf{m}[i]$ and $\mathbf{m}'[i]$ respectively when $\mathbf{m}[i]$ and $\mathbf{m}'[i]$ are not equal to the "don't care value" $\top$.

Remark III.1 *As* $\top + z = \top$ *for any* $z \in \mathbb{Z}$, *there exists a set* $I \subseteq \{1, \ldots, n\}$ *such that for any state* $\mathbf{y} \in Q$ *we have* $\mathbf{y}[i] \in \mathbb{N}$ *if and only if* $i \in I$. *This set* I *is called the set of* rigid components *[5]. Observe that for any decomposition of* π *into* $\pi = (\mathbf{x} \xrightarrow{w}_G \mathbf{y} \xrightarrow{w'}_G \mathbf{x}')$ *the unique configuration* $\mathbf{r}$ *satisfying* $\mathbf{s} \xrightarrow{w}_{\mathcal{V}} \mathbf{r} \xrightarrow{w'}_{\mathcal{V}} \mathbf{s}'$ *also satisfies* $\mathbf{r}[i] = \mathbf{y}[i]$ *for any* $i \in I$.

A *marked reachability graph sequence (MRGS)* $\mathcal{U}$ for $(\mathbf{m}, \mathcal{V}, \mathbf{m}')$ is a sequence $\mathcal{U} = \mathcal{M}_0 a_1 \mathcal{M}_1 \ldots a_k \mathcal{M}_k$ that alternates elements $a_j \in \Sigma$ and marked reachability graphs $\mathcal{M}_j = (\mathbf{m_j}, \mathbf{x_j}, G_j, \mathbf{x'_j}, \mathbf{m'_j})$ such that $\mathbf{m_0} \trianglelefteq \mathbf{m}$ and $\mathbf{m'_k} \trianglelefteq \mathbf{m}'$. An *accepted sequence* for $\mathcal{U}$ is a sequence $(\mathbf{s_j}, \pi_j, \mathbf{s'_j})_{0 \leq j \leq k}$ such that $(\mathbf{s_j}, \pi_j, \mathbf{s'_j})$ is an accepted tuple for $\mathcal{M}_j$ for any $0 \leq j \leq k$ and such that $\mathbf{s'_{j-1}} \xrightarrow{a_j}_{\mathcal{V}} \mathbf{s_j}$ for any $1 \leq j \leq k$. The *language accepted* by $\mathcal{U}$ is the set of words of the form $\sigma = \sigma_0 a_1 \sigma_1 \ldots a_k \sigma_k$ such that there exists an accepted sequence $(\mathbf{s_j}, \pi_j, \mathbf{s'_j})_{0 \leq j \leq k}$ where π_j is labeled by σ_j. This set is denoted by $\mathcal{L}(\mathcal{U})$. Since $\mathbf{m_0} \trianglelefteq \mathbf{m}$ and $\mathbf{m'_k} \trianglelefteq \mathbf{m}'$, relations $\mathbf{s_0} \trianglelefteq \mathbf{m_0}$ and $\mathbf{s'_k} \trianglelefteq \mathbf{m'_k}$ imply $\mathbf{s_0} \trianglelefteq \mathbf{m}$ and $\mathbf{s'_k} \trianglelefteq \mathbf{m}'$. In particular the inclusion $\mathcal{L}(\mathcal{U}) \subseteq \mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}')$ holds.

Definition III.2 *A MRGS* $\mathcal{U}$ *is said to be* perfect *if for any* $c \in \mathbb{N}$, *there exists an accepted sequence* $(\mathbf{s_j}, \pi_j, \mathbf{s'_j})_{0 \leq j \leq k}$ *for* $\mathcal{U}$ *such that for any* $0 \leq j \leq k$:

- $\mathbf{s_j}[i] \geq c$ *for any* i *such that* $\mathbf{m_j}[i] = \top$,
- *there exists a prefix* $\mathbf{x_j} \xrightarrow{w_j}_{G_j} \mathbf{x_j}$ *of* π_j *and a configuration* $\mathbf{r_j}$ *such that* $\mathbf{s_j} \xrightarrow{w_j}_{\mathcal{V}} \mathbf{r_j}$ *and such that* $\mathbf{r_j}[i] \geq c$ *for any* i *such that* $\mathbf{x_j}[i] = \top$, *and*
- $|\pi_j|_t \geq c$ *for any* $t \in T_j$,

- *there exists a suffix* $\mathbf{x'_j} \xrightarrow{w'_j}_{G_j} \mathbf{x'_j}$ *of* π_j *and a configuration* $\mathbf{r'_j}$ *such that* $\mathbf{r'_j} \xrightarrow{w'_j}_{\mathcal{V}} \mathbf{s'_j}$ *and such that* $\mathbf{r'_j}[i] \geq c$ *for any* i *such that* $\mathbf{x'_j}[i] = \top$.
- $\mathbf{s'_j}[i] \geq c$ *for any* i *such that* $\mathbf{m'_j}[i] = \top$,

B. Original perfect condition

The perfect condition given in Definition III.2 is proved equivalent to the original one [6]. The original definition requires additional results recalled in this section. These results are also used in Section V to establish the pseudo-linearity of Parikh images of language accepted by perfect MRGSs.

Let $\mathcal{M} = (\mathbf{m}, \mathbf{x}, G, \mathbf{x'}, \mathbf{m'})$ by a marked reachability graph. We say that $\mathcal{M}$ satisfies the *input loop condition* if there exists a sequence $(\mathbf{x} \xrightarrow{w_c}_G \mathbf{x})_c$ of cycles and a non-decreasing sequence $(\mathbf{m_c})_c$ of extended configurations such that $\mathbf{m} \xrightarrow{w_c}_{\mathcal{V}} \mathbf{m_c}$ for any c and $\lim_{c\to+\infty} \mathbf{m_c} = \mathbf{x}$. Symmetrically, we say that $\mathcal{M}$ satisfies the *output loop condition* if there exists a sequence $(\mathbf{x'} \xrightarrow{w'_c}_G \mathbf{x'})_c$ of cycles and a non-decreasing sequence $(\mathbf{m'_c})_c$ of extended configurations such that $\mathbf{m'_c} \xrightarrow{w'_c}_{\mathcal{V}} \mathbf{m'}$ for any c and $\lim_{c\to+\infty} \mathbf{m'_c} = \mathbf{x'}$. Following Lemma III.3 and Lemma III.4 show that these conditions are decidable in EXPSPACE since they reduce to *covering problems* [9].

Lemma III.3 *The input loop condition is satisfied by* $\mathcal{M}$ *iff there exist a cycle* $\mathbf{x} \xrightarrow{w}_G \mathbf{x}$ *and an extended configuration* $\mathbf{y}$ *satisfying* $\mathbf{m} \xrightarrow{w}_{\mathcal{V}} \mathbf{y}$ *and satisfying* $\mathbf{y}[i] > \mathbf{m}[i]$ *for any* i *such that* $\mathbf{m}[i] < \mathbf{x}[i]$.

Lemma III.4 *The output loop condition is satisfied by* $\mathcal{M}$ *iff there exist a cycle* $\mathbf{x'} \xrightarrow{w'}_G \mathbf{x'}$ *and an extended configuration* $\mathbf{y'}$ *satisfying* $\mathbf{y'} \xrightarrow{w'}_{\mathcal{V}} \mathbf{m'}$ *and satisfying* $\mathbf{y'}[i] > \mathbf{m'}[i]$ *for any* i *such that* $\mathbf{m'}[i] < \mathbf{x'}[i]$.

The Parikh image $||\pi||$ of a path π from a state q to a state q' in a graph $G = (Q, \Sigma, T)$ provides a function $\mu = ||\pi||$ that satisfies the following linear system where $e : Q \times Q \to \{0, 1\}$ denotes the function that takes the one value iff its two arguments are equal:

$$\chi_{q,G,q'}(\mu) := \bigwedge_{p\in Q} \left(\sum_{t=(p_0,a,p)\in T} \mu(t) + e(q,p) = \sum_{t=(p,a,p_1)\in T} \mu(t) + e(p,q') \right)$$

The *Euler lemma* provides a converse result. In fact, if G is strongly connected then any solution $\mu : T \to \mathbb{N}\backslash\{0\}$ of $\chi_{q,G,q'}$ is the Parikh image of a path from q to q'. Since $\chi_{q,G,q}$ does not depend on $q \in Q$, this linear system is simply denoted by χ_G in the sequel. Naturally, the Parikh image of a cycle satisfies this linear system.

$$\begin{cases} \frac{\text{for all } 1 \leq j \leq k}{\mathbf{s'_{j-1}} + \delta(a_j) = \mathbf{s_j}} \\ \frac{\text{for all } 0 \leq j \leq k}{\mathbf{s_j} + \sum_{t\in T_j} \mu_j(t)\delta(t) = \mathbf{s'_j}} \\ \frac{\text{for all } 0 \leq j \leq k,\ 1 \leq i \leq n}{\mathbf{s_j}[i] = \mathbf{m_j}[i] \text{ if } \mathbf{m_j}[i] \in \mathbb{N} \quad \mathbf{s'_j}[i] = \mathbf{m'_j}[i] \text{ if } \mathbf{m'_j}[i] \in \mathbb{N}} \\ \frac{\text{for all } 0 \leq j \leq k}{\chi_{\mathbf{x_j},G_j,\mathbf{x'_j}}(\mu_j)} \end{cases} \quad \begin{cases} \frac{\text{for all } 1 \leq j \leq k}{\mathbf{s'_{0,j-1}} = \mathbf{s_{0,j}}} \\ \frac{\text{for all } 0 \leq j \leq k}{\mathbf{s_{0,j}} + \sum_{t\in T_j} \mu_{0,j}(t)\delta(t) = \mathbf{s'_{0,j}}} \\ \frac{\text{for all } 0 \leq j \leq k,\ 1 \leq i \leq n}{\mathbf{s_{0,j}}[i] = 0 \text{ if } \mathbf{m_j}[i] \in \mathbb{N} \quad \mathbf{s'_{0,j}}[i] = 0 \text{ if } \mathbf{m'_j}[i] \in \mathbb{N}} \\ \frac{\text{for all } 0 \leq j \leq k}{\chi_{G_j}(\mu_{0,j})} \end{cases}$$

Figure 3. On the left the characteristic system. On the right the homogeneous characteristic system.

Let $(\mathbf{s_j}, \pi_j, \mathbf{s'_j})_{0\leq j\leq k}$ be an accepted sequence of a MRGS $\mathcal{U}$. Observe that $\xi = (\mathbf{s_j}, \mu_j, \mathbf{s'_j})_{0\leq j\leq k}$ with $\mu_j = ||\pi_j||$ is a solution of the linear system given in Figure 3 where $\delta(t)$ denotes $\delta(a)$ for any transition $t = (q, a, q')$. This linear system is called the *characteristic system* of $\mathcal{U}$. A solution ξ of the characteristic system is said *concretizable* if there exists an accepted sequence $(\mathbf{s_j}, \pi_j, \mathbf{s'_j})_{0\leq j\leq k}$ such that $\xi = (\mathbf{s_j}, ||\pi_j||, \mathbf{s'_j})_j$. The homogeneous form of the characteristic system, obtained by replacing constant terms by zero is called the *homogeneous characteristic system of* $\mathcal{U}$. This system is given in Figure 3. In the sequel, a solution of the homogeneous characteristic system is denoted by $\xi_{\mathbf{0}} = (\mathbf{s_{0,j}}, \mu_{0,j}, \mathbf{s_{0,j}}')_j$.

We say that $\mathcal{U}$ satisfies the *large solution condition* if there exists a non-decreasing sequence $(\xi_{\mathbf{c}})_{c\in\mathbb{N}}$ of solutions $\xi_{\mathbf{c}} = (\mathbf{s_{j,c}}, \mu_{j,c}, \mathbf{s'_{j,c}})_j$ with components in $\mathbb{N}$ of the characteristic system such that:

- $\lim_{c\to+\infty} \mathbf{s_{j,c}} = \mathbf{m_j}$ for any j,
- $\lim_{c\to+\infty} \mu_{j,c}(t) = \top$ for any j and $t \in T_j$, and
- $\lim_{c\to+\infty} \mathbf{s_{j,c}} = \mathbf{m'_j}$ for any j.

The following lemma shows that the large solution condition is decidable in polynomial time since condition (i) is decidable in polynomial time with the *Hermite decomposition* and condition (ii) is decidable in polynomial time with the *interior points method*.

Lemma III.5 *The large solution condition is satisfied by* $\mathcal{U}$ *iff the following conditions (i) and (ii) hold:*

(i) *Its characteristic system has a solution* ξ *with components in* $\mathbb{Z}$,

(ii) *Its homogeneous characteristic system has a solution* $\xi_{\mathbf{0}} = (\mathbf{s_{0,j}}, \mu_{0,j}, \mathbf{s_{0,j}}')_j$ *with components in* $\mathbb{Q}$ *satisfying for any* j*:*

- $\mathbf{s_{0,j}}[i] > 0$ *for any* i *such that* $\mathbf{m_j}[i] = \top$,
- $\mu_{0,j}(t) > 0$ *for any* $t \in T_j$*, and*
- $\mathbf{s'_{0,j}}[i] > 0$ *for any* i *such that* $\mathbf{m'_j}[i] = \top$.

By adapting [6], we deduce that the perfect condition given in Definition III.2 is equivalent to the original one [6] (also equivalent to the θ-condition [5]). More formally, we prove the following Proposition III.6.

Proposition III.6 *A MRGS* $\mathcal{U}$ *is* perfect *if and only if it satisfies the large solution condition and if its marked reachability graphs satisfy the input and output loop conditions.*

C. The KLMST decomposition

We provide an informal presentation of the algorithm deciding the emptiness of $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}')$. This algorithm is based on a partial order $\sqsubseteq$ over the MRGSs that does not admit infinite decreasing sequence. During its execution, a finite set F of MRGSs is computed. This set satisfies the invariant $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}') = \bigcup_{\mathcal{U} \in F} \mathcal{L}(\mathcal{U})$. Initially, the algorithm consider the set F reduced to a single MRGS syntactically obtained from $(\mathbf{m}, \mathcal{V}, \mathbf{m}')$. Recursively, while there exist MRGSs in F that do not satisfy the *perfect condition*, such a MRGS $\mathcal{U}$ is picked up from F. Since $\mathcal{U}$ is not perfect, Proposition III.6 shows that either it does not satisfy the large solution condition or one of its marked reachability graphs does not satisfies the input or the output loop condition. Considering separately these cases, the algorithm computes a finite set F' of MRGSs satisfying $\mathcal{U}' \sqsubset \mathcal{U}$ for any $\mathcal{U}' \in F'$ and $\mathcal{L}(\mathcal{U}) = \bigcup_{\mathcal{U}' \in F'} \mathcal{L}(\mathcal{U}')$. Then, the algorithm replaces F by $F \backslash \{\mathcal{U}\} \cup F'$ and it restarts the while loop. Since there does not exist infinite decreasing sequence of MRGSs for $\sqsubseteq$, the loop termination is guaranteed. When the loop terminates, the set F only contains perfect MRGSs. If F is non empty the algorithm decides that $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}')$ is non empty, otherwise it decides that $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}')$ is empty. The correctness of the algorithm is obtained by observing that the language accepted by a perfect MRGS is always non empty. This algorithm provides the following Theorem III.7.

Theorem III.7 (Fundamental Decomposition [5], [6]) *For any tuple* $(\mathbf{m}, \mathcal{V}, \mathbf{m}')$*, we can effectively compute a finite set* F *of perfect MRGSs for* $(\mathbf{m}, \mathcal{V}, \mathbf{m}')$ *such that:*

$$\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}') = \bigcup_{\mathcal{U} \in F} \mathcal{L}(\mathcal{U})$$

IV. Semi-Pseudo-Linear Sets

We introduce the class of semi-pseudo-linear sets.

A *monoïd* of $\mathbb{Z}^n$ is a set $M \subseteq \mathbb{Z}^n$ such that $\mathbf{0} \in M$ and $M + M \subseteq M$. Observe that for any $X \subseteq \mathbb{Z}^n$, the set $M = \{\mathbf{0}\} \cup \{\sum_{i=1}^{k} \mathbf{x_i} \mid k \geq 1\ \mathbf{x_i} \in X\}$ is the minimal for the inclusion monoïd that contains X. This monoïd is called the *monoïd generated* by X and denoted X^*. A monoïd is said to be *finitely generated* if it can be generated by a finite set.

Let M be a monoïd. A vector $\mathbf{a} \in M$ is said to be *interior* to M if for any $\mathbf{x} \in M$ there exists an integer $N \geq 1$ satisfying $N\mathbf{a} \in \mathbf{x} + M$. The *interior of a monoïd* M is the set of interior vectors to M. It is denoted by $\mathcal{I}(M)$.

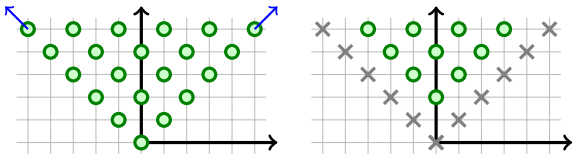

Figure 4. On the left a monoïd M. On the right its interior $\mathcal{I}(M)$.

Example IV.1 *Let* $P = \{(1,1), (-1,1)\}$*. The monoïd* $M = P^*$ *and its interior are depicted in Figure 4.*

The following Lemma IV.2 characterizes the set $\mathcal{I}(P^*)$ where P is a finite set.

Lemma IV.2 *Let* $P = \{\mathbf{p_1}, \ldots, \mathbf{p_k}\} \subseteq \mathbb{Z}^n$ *with* $k \in \mathbb{N}$*. We have* $\mathcal{I}(P^*) = \{\mathbf{0}\}$ *if* $k = 0$ *and* $\mathcal{I}(P^*) = P^* \cap ((\mathbb{Q}_+ \backslash \{0\})\mathbf{p_1} + \cdots + (\mathbb{Q}_+ \backslash \{0\})\mathbf{p_k})$ *if* $k \geq 1$*.*

A set $L \subseteq \mathbb{Z}^n$ is said to be *linear* [3] if there exists a vector $\mathbf{b} \in \mathbb{Z}^n$ and a finitely generated monoïd $M \subseteq \mathbb{Z}^n$ such that $L = \mathbf{b} + M$. A *semi-linear set* $S \subseteq \mathbb{Z}^n$ is a finite union of linear sets $L_i \subseteq \mathbb{Z}^n$. Recall [3] that sets definable in FO $(\mathbb{N}, +, \leq)$ also called *Presburger sets* are exactly the non-negative semi-linear sets. By observing that integers are differences of two non-negative integers, we deduce that sets definable in FO $(\mathbb{Z}, +, \leq)$ are exactly the semi-linear sets.

Let us now introduce the class of *pseudo-linear sets* and *semi-pseudo-linear sets*. Intuitively, the pseudo-linear sets extend the *linear sets*, and the semi-pseudo-linear sets extend the semi-linear sets. More formally, a set $X \subseteq \mathbb{Z}^n$ is said to be *pseudo-linear* if there exists $\mathbf{b} \in \mathbb{Z}^n$ and a finitely generated monoïd $M \subseteq \mathbb{Z}^n$ such that $X \subseteq \mathbf{b} + M$ and such that for any finite set R of interior vectors to M,

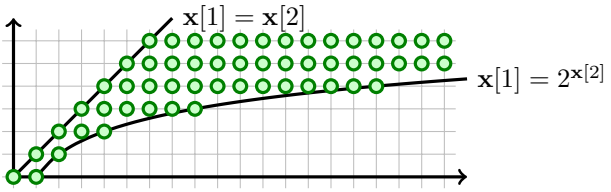

Figure 5. A pseudo-linear set.

there exists $\mathbf{x} \in X$ such that $\mathbf{x} + R^* \subseteq X$. In this case, M is called a *linearizator* for X and the linear set $L = \mathbf{b} + M$ is called a *linearization* of X. A *semi-pseudo-linear set* is a finite union of *pseudo-linear sets*.

Example IV.3 *The set $X = \{\mathbf{x} \in \mathbb{Z}^2 \mid 0 \leq \mathbf{x}[2] \leq \mathbf{x}[1] \leq 2^{\mathbf{x}[2]}\}$ is depicted in Figure 5. Observe that X is pseudo-linear and $L = \{\mathbf{x} \in \mathbb{Z}^2 \mid 0 \leq \mathbf{x}[2] \leq \mathbf{x}[1]\}$ is a linearization of X. The set $Y = \{(2^k, 0) \mid k \in \mathbb{N}\}$ is not semi-pseudo-linear. However $Z = X \cup Y$ is pseudo-linear since L is still a linearization of Z.*

Remark IV.4 *Any linear set $L = \mathbf{b} + M$ is pseudo-linear. M is a linearizator for L and L is a linearization of L. Any semi-linear set is semi-pseudo-linear.*

Remark IV.5 *Semi-pseudo-linear sets can be empty whereas pseudo-linear sets cannot be empty.*

As expected, the class of pseudo-linear sets is stable by linear function images. A function $f : \mathbb{Z}^n \to \mathbb{Z}^{n'}$ is said *linear* if there exists a matrix $A \in \mathbb{Z}^{n \times n'}$ and a vector $\mathbf{v} \in \mathbb{Z}^{n'}$ such that $f(\mathbf{x}) = A\mathbf{x} + \mathbf{v}$ for any $\mathbf{x} \in \mathbb{Z}^n$.

Proposition IV.6 *Images $X' = f(X)$ of pseudo-linear sets X by a linear function f are pseudo-linear. Moreover $L' = f(L)$ is a linearization of X' for any linearization L of X.*

V. The Parikh Images of Perfect MRGSs

The Parikh images of languages accepted by perfect MRGSs are proved pseudo-linear in this section. From the KLMST decomposition, we deduce the semi-pseudo-linearity of the Parikh image of $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m}')$.

Let us consider a perfect MRGS $\mathcal{U}$ for $(\mathbf{m}, \mathcal{V}, \mathbf{m}')$. We denote by H the solutions with components in $\mathbb{N}$ of the characteristic system of $\mathcal{U}$. We consider the set of concretizable solutions H'. Since the Parikh image of $\mathcal{L}(\mathcal{U})$ is the image by a linear function of H', from Proposition IV.6 it is sufficient to prove that H' is pseudo-linear. Let us introduce the set H_0 of solutions with components in $\mathbb{N}$ of the homogeneous characteristic system. We prove in the sequel that H_0 is a linearizator for H'. First of all observe that H_0 is a monoïd finitely generated since $H_0 = P_0^*$ where $P_0 = \min(H_0 \backslash \{\mathbf{0}\})$.

Since $H' \subseteq H$, the following Lemma V.1 shows that H' is included in the linear set $(\xi - \xi_\mathbf{0}) + H_0$.

Lemma V.1 *There exists $\xi \in H$ and $\xi_\mathbf{0} \in H_0$ such that $H \subseteq (\xi - \xi_\mathbf{0}) + H_0$.*

Proof: As $\mathcal{U}$ satisfies the large solution condition there exists $\xi \in H$. Moreover, Lemma III.5 shows that there exists a solution $\xi_\mathbf{0}$ with components in $\mathbb{Q}$ of the homogeneous characteristic system satisfying the additional conditions $\mathbf{s}_{\mathbf{0},\mathbf{j}}[i] > 0$ if $\mathbf{m}_\mathbf{j}[i] = \top$, $\mathbf{s}'_{\mathbf{0},\mathbf{j}}[i] > 0$ if $\mathbf{m}'_\mathbf{j}[i] = \top$, and $\mu_{0,j}(t) > 0$ for any $t \in T_j$. By multiplying $\xi_\mathbf{0}$ by a positive integer, we can assume that the components of $\xi_\mathbf{0}$ are in $\mathbb{Z}$. Note that for any $\xi' \in H$, there exists $c \in \mathbb{N}$ such that $\xi' + c\xi_\mathbf{0} \geq \xi$. As $\min(H)$ is finite, by multiplying $\xi_\mathbf{0}$ by a positive integer we can assume that $\xi' + \xi_\mathbf{0} \geq \xi$ for any $\xi' \in H$. That means $H \subseteq (\xi - \xi_\mathbf{0}) + H_0$. ■

Now, let us consider a finite set $R_0 = \{\xi_\mathbf{1}, \ldots, \xi_\mathbf{d}\}$ included in the interior of H_0. We are going to prove that there exists $\xi \in H$ such that $\xi + R_0^* \subseteq H'$. We first prove the following lemma.

Lemma V.2 *For any $\xi_\mathbf{l} = (\mathbf{s}_{\mathbf{l},\mathbf{j}}, \mu_{l,j}, \mathbf{s}'_{\mathbf{l},\mathbf{j}})_j$ interior vector of H_0, the function $\mu_{l,j}$ is the Parikh image of a cycle $\pi_{l,j} = (\mathbf{x_j} \xrightarrow{\sigma_{l,j}}_{G_j} \mathbf{x_j})$.*

Proof: Since $\mathcal{U}$ satisfies the large solution condition, Lemma III.5 shows for any $t \in T_j$, there exists a solution $\xi_\mathbf{0} = (\mathbf{s}_{\mathbf{0},\mathbf{j}}, \mu_{0,j}, \mathbf{s}'_{\mathbf{0},\mathbf{j}})_j$ in H_0 such that $\mu_{0,j}(t) > 0$. As $H_0 = P_0^*$, for any $t \in T_j$ there exists $\xi_\mathbf{0} \in P_0$ satisfying the same property. Lemma IV.2 shows that $\xi_\mathbf{l}$ is a sum over all solutions $\xi_\mathbf{0} \in P_0$ of terms of the form $\lambda\xi_\mathbf{0}$ where $\lambda > 0$ is a value in $\mathbb{Q}$ that naturally depends on $\xi_\mathbf{0}$ and $\xi_\mathbf{l}$. In particular we deduce that $\mu_{l,j}(t) > 0$ for any $t \in T_j$ and for any $0 \leq j \leq k$. Euler lemma shows that $\mu_{l,j}$ is the Parikh image of a cycle $\pi_{l,j} = (\mathbf{x_j} \xrightarrow{\sigma_{l,j}}_{G_j} \mathbf{x_j})$. ■

Since $\mathbf{x_j} \xrightarrow{\sigma_{l,j}}_{\mathcal{V}}$, there exists an integer $c \geq 0$ such that for any $0 \leq j \leq k$ and for any configuration $\mathbf{r_j}$ satisfying $\mathbf{r_j}[i] \geq c$ if $\mathbf{x_j}[i] = \top$ and $\mathbf{r_j}[i] = \mathbf{x_j}[i]$ otherwise, we have $\mathbf{r_j} \xrightarrow{\sigma_{l,j}}_{\mathcal{V}}$.

As $\mathcal{U}$ is perfect, there exists an accepted tuple $(\mathbf{s_j}, \pi_j, \mathbf{s'_j})_{0 \leq j \leq k}$ such that for any j, π_j can be decomposed into:

$$\pi_j = (\mathbf{x_j} \xrightarrow{w_j}_{G_j} \mathbf{x_j} \xrightarrow{\sigma_j}_{G_j} \mathbf{x'_j} \xrightarrow{w'_j}_{G_j} \mathbf{x'_j})$$

and such that the pair of configurations $(\mathbf{r_j}, \mathbf{r'_j})$ satisfying the following relations:

$$\mathbf{s_j} \xrightarrow{w_j}_{\mathcal{V}} \mathbf{r_j} \xrightarrow{\sigma_j}_{\mathcal{V}} \mathbf{r'_j} \xrightarrow{w'_j}_{\mathcal{V}} \mathbf{s'_j}$$

also satisfy:

- $\mathbf{r_j}[i] \geq c$ if $\mathbf{x_j}[i] = \top$ and $\mathbf{r_j}[i] = \mathbf{x_j}[i]$ otherwise,
- $\mathbf{r'_j}[i] \geq c$ if $\mathbf{x'_j}[i] = \top$ and $\mathbf{r'_j}[i] = \mathbf{x'_j}[i]$ otherwise.

In particular we have $\mathbf{r_j} \xrightarrow{\sigma_{l,j}}_{\mathcal{V}}$ for any $0 \leq j \leq k$ and for any $1 \leq l \leq d$.

As $\mathbf{s_{l,j}} \geq \mathbf{0}$ and $\mathbf{r_j} \xrightarrow{\sigma_{l,j}}_{\mathcal{V}}$ we deduce that $\mathbf{r_j} + \mathbf{s_{l,j}} \xrightarrow{\sigma_{l,j}}_{\mathcal{V}}$. Moreover, from $\mathbf{s_{l,j}} + \delta(\sigma_{l,j}) = \mathbf{s'_{l,j}}$ we get:

$$\mathbf{r_j} + \mathbf{s_{l,j}} \xrightarrow{\sigma_{l,j}}_{\mathcal{V}} \mathbf{r_j} + \mathbf{s'_{l,j}}$$

As $\mathbf{s_{l,j}}, \mathbf{s'_{l,j}} \geq \mathbf{0}$, an immediate induction shows that for any sequence $n_1, \ldots, n_d \in \mathbb{N}$ we have the following relation:

$$\mathbf{r_j} + \sum_{l=1}^{d} n_l \mathbf{s_{l,j}} \xrightarrow{\sigma_{1,j}^{n_1} \ldots \sigma_{d,j}^{n_d}}_{\mathcal{V}} \mathbf{r_j} + \sum_{l=1}^{d} n_l \mathbf{s'_{l,j}}$$

Let $\xi = (\mathbf{s_j}, ||\pi_j||, \mathbf{s'_j})_{0 \leq j \leq k}$. We have proved that $\xi + \sum_{l=1}^{d} n_l \xi_{\mathbf{l}}$ is concretizable. Thus $\xi + R_0^* \subseteq H'$. Therefore H' is pseudo-linear and H_0 is a linearizator for H'. We have proved the following Theorem V.3.

Theorem V.3 *The Parikh image of $\mathcal{L}(\mathcal{U})$ is pseudo-linear for any perfect MRGS $\mathcal{U}$.*

From Theorem III.7 and Theorem V.3 we deduce the following Corollary V.4.

Corollary V.4 *The Parikh image of $\mathcal{L}(\mathbf{m}, \mathcal{V}, \mathbf{m'})$ is semi-pseudo-linear.*

VI. Locally Semi-Pseudo-Linear Sets

A set $X \subseteq \mathbb{Z}^n$ is said to be *locally semi-pseudo-linear* if $X \cap S$ is semi-pseudo-linear for any semi-linear set $S \subseteq \mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a linear set, locally semi-pseudo-linear sets are semi-pseudo-linear. However the converse is not true in general (see Example VI.1). In this section, $\mathrm{post}^*_{\mathcal{V}}(S)$ and $\mathrm{pre}^*_{\mathcal{V}}(S')$ are proved locally semi-pseudo-linear for any semi-linear sets $S, S' \subseteq \mathbb{N}^n$. This result is used in Section VIII to get a *local analysis* of $\mathrm{post}^*_{\mathcal{V}}(S)$ and $\mathrm{pre}^*_{\mathcal{V}}(S')$ with respect to some semi-linear sets.

Example VI.1 *Let us consider the pseudo-linear set $Z = X \cup Y$ introduced in Example IV.3 and observe that Z is not locally semi-pseudo-linear since $Y = Z \cap S$ is not semi-pseudo-linear with $S = (1,0) + \{(1,0)\}^*$.*

Let us prove that $\mathrm{post}^*_{\mathcal{V}}(S) \cap S'$ and $S \cap \mathrm{pre}^*_{\mathcal{V}}(S')$ are semi-pseudo-linear for any semi-linear sets $S, S' \subseteq \mathbb{N}^n$. Since semi-linear sets are finite unions of linear sets we only prove this result for the special case of two linear sets $S = \mathbf{s} + P^*$ and $S' = \mathbf{s'} + (P')^*$ where $\mathbf{s}, \mathbf{s'} \in \mathbb{N}^n$ and $P, P' \subseteq \mathbb{N}^n$ are two finite sets. We consider two alphabets $\Sigma_P, \Sigma_{P'}$ disjoint of Σ and a displacement function $\bar{\delta}$ defined over $\bar{\Sigma} = \Sigma_P \cup \Sigma \cup \Sigma_{P'}$ that extends δ such that:

$$P = \{\bar{\delta}(a) \mid a \in \Sigma_P\} \qquad P' = \{-\bar{\delta}(a) \mid a \in \Sigma_{P'}\}$$

We consider the VAS $\bar{\mathcal{V}} = (\bar{\Sigma}, n, \bar{\delta})$. Intuitively, since $\bar{\delta}(\Sigma_P) \subseteq \mathbb{N}^n$ and $\bar{\delta}(\Sigma_{P'}) \subseteq -\mathbb{N}^n$, words in $\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s'})$ can be reordered into words in $(\Sigma_P^* \Sigma^* \Sigma_{P'}^*) \cap \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s'})$.

Let us consider the displacement functions f and f' defined over $\bar{\Sigma}$ by:

$$f(a) = \begin{cases} \bar{\delta}(a) & \text{if } a \in \Sigma_P \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$$f'(a) = \begin{cases} -\bar{\delta}(a) & \text{if } a \in \Sigma_{P'} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Lemma VI.2 *We have $\mathrm{post}^*_{\mathcal{V}}(S) \cap S' = \mathbf{s'} + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s'}))$ and $S \cap \mathrm{pre}^*_{\mathcal{V}}(S') = \mathbf{s} + f(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s'}))$.*

Observe that sets $\mathbf{s'} + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s'}))$ and $\mathbf{s} + f(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s'}))$ are images by linear functions of the Parikh image of $\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s'})$. Corollary V.4 shows that the Parikh image of $\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s'})$ is semi-pseudo-linear. From Proposition IV.6 we deduce the following Theorem VI.3.

Theorem VI.3 $\mathrm{post}^*_{\mathcal{V}}(S)$ *and* $\mathrm{pre}^*_{\mathcal{V}}(S')$ *are locally semi-pseudo-linear for any semi-linear sets $S, S' \subseteq \mathbb{N}^n$.*

VII. Pseudo-Linear Sets Intersections

Let X_1, X_2 be two pseudo-linear sets with an empty intersection $X_1 \cap X_2$ and let L_1, L_2 be linearizations of X_1, X_2. Since L_1, L_2 over-approximate X_1, X_2, the intersection $L_1 \cap L_2$ is not empty in general. In this section we introduce a dimension function that satisfies $\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$. This dimension function is defined in Section VII-A and the strict inequality is proved in Section VII-B.

A. Dimension

The classical *(mass) dimension function* is introduced in this section. We associate to any set $X \subseteq \mathbb{Z}^n$ the sequence $(r_k)_{k \in \mathbb{N}}$ defined by the following equality (by definition $\ln(0) = -\infty$ and $|.|$ denotes the *cardinal function*):

$$r_k = \frac{\ln(|X \cap \{-k, \ldots, k\}^n|)}{\ln(2k+1)}$$

Observe that r_k is either $-\infty$ or a real value such that $0 \leq r_k \leq n$. We denote by $\dim_L(X)$ and $\dim_U(X)$ respectively the *limit-inf* and the *limit-sup* of $(r_k)_{r\in\mathbb{N}}$. In this paper we consider the dimension function $\dim = \dim_L$. The other choice is also possible since the sets considered in this paper satisfy $\dim_L(X) = \dim_U(X)$.

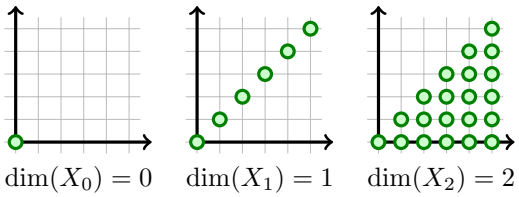

Figure 6. Dimension of some sets.

Example VII.1 *Let* $X_0 = \{(0,0)\}$, $X_1 = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[1] = \mathbf{x}[2]\}$ *and* $X_2 = \{\mathbf{x} \in \mathbb{N}^2 \mid \mathbf{x}[2] \leq \mathbf{x}[1]\}$ *be the sets depicted in Figure 6. As* $|X_0 \cap \{-k,\ldots,k\}^2| = 1$, $|X_1 \cap \{-k,\ldots,k\}^2| = k+1$, *and* $|X_2 \cap \{-k,\ldots,k\}^2| = \frac{1}{2}(k+1)(k+2)$ *we get* $\dim(X_0) = 0$, $\dim(X_1) = 1$ *and* $\dim(X_2) = 2$.

Let us show some immediate properties satisfied by the dimension function. Observe that $\dim(X) = -\infty$ if and only if X is empty. The dimension function is monotonic $\dim(X_1) \leq \dim(X_2)$ for any $X_1 \subseteq X_2$. Moreover it satisfies $\dim(X_1 \cup X_2) = \max\{\dim(X_1), \dim(X_2)\}$ and $\dim(X_1 + X_2) \leq \dim(X_1) + \dim(X_2)$. In particular $\dim(\mathbf{v} + X) = \dim(X)$ for any $\mathbf{v} \in \mathbb{Z}^n$.

Remark VII.2 *The dimension of any non-empty semi-linear set is integral.*

As expected, the dimension of a pseudo-linear set is equal to the dimension of any linearization.

Lemma VII.3 *We have* $\dim(X) = \dim(L)$ *for any linearization* L *of a pseudo-linear set* $X \subseteq \mathbb{Z}^n$.

B. Pseudo-linear sets with empty intersections

In this section we prove that linearizations L_1, L_2 of two pseudo-linear sets X_1, X_2 with an empty intersection $X_1 \cap X_2 = \emptyset$ satisfy the strict inequality $\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$. Note that even if $X_1 \cap X_2 = \emptyset$, the intersection $L_1 \cap L_2$ may be non empty since L_1, L_2 are over-approximations of X_1, X_2.

Example VII.4 *Let us consider the pseudo-linear set* X *described in Example IV.3 and a linearization* $L = \{\mathbf{x} \in \mathbb{Z}^2 \mid 0 \leq \mathbf{x}[2] \leq \mathbf{x}[1]\}$ *of* X. *We also consider the linear set* $X' = (8,2) + \{(1,0),(3,-1)\}^*$. *Sets* X *and* X' *are depicted together in Figure 7. Note that* $L' = X'$ *is a linearization of the linear set* X'. *Notice that* $X \cap X' = \emptyset$. *The set* $L \cap L'$ *is depicted in gray in Figure 7. Observe that* $L \cap L' = \{(8,2),(11,1),(14,0)\} + \{(1,0)\}^*$. *Therefore* $1 = \dim(L \cap L') < \dim(X \cup X') = 2$.

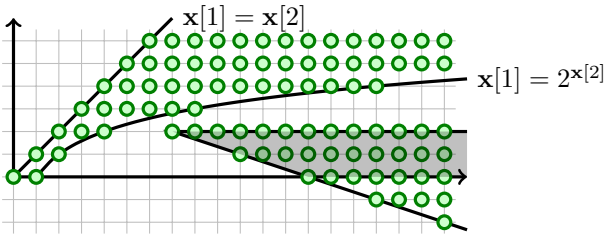

Figure 7. Two pseudo-linear sets with an empty intersection.

We say that two linear sets L_1, L_2 have a *non-degenerate intersection* if $\dim(L_1) = \dim(L_1 \cap L_2) = \dim(L_2)$.

Lemma VII.5 *Let* $L_1 = \mathbf{b_1} + M_1$ *and* $L_2 = \mathbf{b_2} + M_2$ *be two linear sets with a non-degenerate intersection. There exist finite sets* $R_1 \subseteq \mathcal{I}(M_1)$ *and* $R_2 \subseteq \mathcal{I}(M_2)$ *such that* $(\mathbf{x_1} + R_1^*) \cap (\mathbf{x_2} + R_2^*) \neq \emptyset$ *for any* $(\mathbf{x_1}, \mathbf{x_2}) \in (L_1, L_2)$.

Proposition VII.6 *Let* L_1, L_2 *be linearizations of pseudo-linear sets* $X_1, X_2 \subseteq \mathbb{Z}^n$ *with an empty intersection* $X_1 \cap X_2 = \emptyset$. *We have:*

$$\dim(L_1 \cap L_2) < \dim(X_1 \cup X_2)$$

Proof: Let us consider linearizations L_1, L_2 of two pseudo-linear sets X_1, X_2 such that $\dim(L_1 \cap L_2) \geq \dim(X_1 \cup X_2)$ and let us prove that $X_1 \cap X_2 \neq \emptyset$. Lemma VII.3 shows that $\dim(X_1) = \dim(L_1)$ and $\dim(X_2) = \dim(L_2)$. By monotonicity of the dimension function, we deduce that $\dim(L_1) = \dim(L_1 \cap L_2) = \dim(L_2)$. Thus L_1 and L_2 have a non-degenerate intersection. As L_1, L_2 are two linear sets, there exists $\mathbf{b_1}, \mathbf{b_2} \in \mathbb{Z}^n$ and two finitely generated monoïds M_1, M_2 such that $L_1 = \mathbf{b_1} + M_1$ and $L_2 = \mathbf{b_2} + M_2$. Lemma VII.5 shows that there exist finite sets $R_1 \subseteq \mathcal{I}(M_1)$ and $R_2 \subseteq \mathcal{I}(M_2)$ such that $(\mathbf{x_1} + R_1^*) \cap (\mathbf{x_2} + R_2^*) \neq \emptyset$ for any $(\mathbf{x_1}, \mathbf{x_2}) \in (L_1, L_2)$. As L_1, L_2 are linearizations of the pseudo-linear sets X_1, X_2 there exists $(\mathbf{x_1}, \mathbf{x_2}) \in (X_1, X_2)$ such that $\mathbf{x_1} + R_1^* \subseteq X_1$ and $\mathbf{x_2} + R_2^* \subseteq X_2$. As $(\mathbf{x_1}, \mathbf{x_2}) \in (L_1, L_2)$ we deduce that $(\mathbf{x_1} + R_1^*) \cap (\mathbf{x_2} + R_2^*) \neq \emptyset$. We have proved that $X_1 \cap X_2 \neq \emptyset$. ■

VIII. Presburger Closed Separators

The VAS reachability problem can be reformulated by introducing the definition of separators. A pair (S, S') of configuration sets is called a *separator* for a VAS $\mathcal{V}$ if $S \times S'$ has an empty intersection with the reachability binary relation $\xrightarrow{*}_{\mathcal{V}}$. The set $D = \mathbb{N}^n \backslash (S \cup S')$ is called the *(free) domain* of (S, S').

Let us consider the following sets for any pair (S, S') of configurations sets and for any $a \in \Sigma$:

$$\mathrm{post}^a_{\mathcal{V}}(S) = \{\mathbf{s}' \in \mathbb{N}^n \mid \exists \mathbf{s} \in S \quad \mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'\}$$
$$\mathrm{pre}^a_{\mathcal{V}}(S') = \{\mathbf{s} \in \mathbb{N}^n \mid \exists \mathbf{s}' \in S' \quad \mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s}'\}$$

A set $S \subseteq \mathbb{N}^n$ is called a *forward invariant* if $\mathrm{post}^a_{\mathcal{V}}(S) \subseteq S$ for any $a \in \Sigma$. A set $S' \subseteq \mathbb{N}^n$ is called a *backward invariant* if $\mathrm{pre}^a_{\mathcal{V}}(S') \subseteq S'$ for any $a \in \Sigma$. Let (S, S') be a pair of configuration sets such that S is a forward invariant and S' is a backward invariant and observe that (S, S') is a separator if and only if $S \cap S' = \emptyset$. In fact in this case the condition $(S \times S') \cap \xrightarrow{*}_{\mathcal{V}} = \emptyset$ reduces to $S \cap S' = \emptyset$.

A separator (S, S') is said to be *closed (with respect to the reachability relation)* if S is a forward invariant and if S' is a backward invariant. As $(\mathrm{post}^*_{\mathcal{V}}(S), \mathrm{pre}^*_{\mathcal{V}}(S'))$ is a closed separator for any separator (S, S'), we deduce that separators are included into closed separators. We are interested in closed separators definable in the Presburger arithmetic $\mathrm{FO}(\mathbb{N}, +, \leq)$. Let us consider a pair $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ of Presburger formulas denoting a pair (S, S') of configurations sets. Note that (S, S') is a closed separator if and only if $\psi(\mathbf{x}) \wedge \psi'(\mathbf{x})$ and the following formulas are unsatisfiable for any $a \in \Sigma$.

$$\begin{array}{ccccc} \psi(\mathbf{x}) & \wedge & \mathbf{x}' = \mathbf{x} + \delta(a) & \wedge & \neg\psi(\mathbf{x}') \\ \psi'(\mathbf{x}') & \wedge & \mathbf{x}' = \mathbf{x} + \delta(a) & \wedge & \neg\psi'(\mathbf{x}) \end{array}$$

In particular we can effectively decide if $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ denotes a closed separator. That means pairs $(\psi(\mathbf{x}), \psi'(\mathbf{x}))$ of Presburger formulas denoting closed separators provide *checkable certificates of non-reachability*.

In this section we prove that Presburger separators are included in Presburger closed separators. In general $(\mathrm{post}^*_{\mathcal{V}}(S), \mathrm{pre}^*_{\mathcal{V}}(S'))$ is not Presburger (see Example II.3). That means, this closed separator must be over-approximated by another closed separator.

A Presburger closed separator that over-approximates a Presburger separator (S_0, S'_0) is obtained inductively. We build a non-decreasing finite sequence $(S_j, S'_j)_j$ of Presburger separators starting from the initial Presburger separator (S_0, S'_0) such that the dimension of the domain $D_j = \mathbb{N}^n \backslash (S_j \cup S'_j)$ is strictly decreasing toward $-\infty$. In order to obtain this sequence, observe that it is sufficient to show that for any Presburger separator (S_0, S'_0) with a non-empty domain D_0, there exists a Presburger separator $(S, S') \supseteq (S_0, S'_0)$ with a domain D such that $\dim(D) < \dim(D_0)$.

Remark VIII.1 *In the sequel, we often use the fact that $(S, S') \subseteq (\mathbb{N}^n, \mathbb{N}^n)$ is a separator if and only if $\mathrm{post}^*_{\mathcal{V}}(S) \cap \mathrm{pre}^*_{\mathcal{V}}(S') = \emptyset$ if and only if $\mathrm{post}^*_{\mathcal{V}}(S) \cap S' = \emptyset$ if and only if $S \cap \mathrm{pre}^*_{\mathcal{V}}(S') = \emptyset$.*

We first define a set S' that over-approximates S'_0 and such that (S_0, S') is a separator. As S_0 is semi-linear, Theorem VI.3 shows that $\mathrm{post}^*_{\mathcal{V}}(S_0)$ is locally semi-pseudo-linear. As D_0 is semi-linear, we deduce that $\mathrm{post}^*_{\mathcal{V}}(S_0) \cap D_0$ is equal to a finite union of pseudo-linear sets $X_1, \ldots, X_k$. Let us consider some linearizations $L_1, \ldots, L_k$ of these pseudo-linear sets and let us define the following Presburger set S'.

$$S' = S'_0 \cup (D_0 \backslash (\bigcup_{j=1}^{k} L_j))$$

We observe that $\mathrm{post}^*_{\mathcal{V}}(S_0) \cap S' = \emptyset$ since $\mathrm{post}^*_{\mathcal{V}}(S_0) \cap S'_0 = \emptyset$ and $\mathrm{post}^*_{\mathcal{V}}(S_0) \cap D_0 \subseteq \bigcup_{j=1}^{k} L_j$. We have proved that S' contains S'_0 and (S_0, S') is a separator.

Now we define symmetrically a set S that over-approximates S_0 and such that (S, S') is a separator. As S' is semi-linear, Theorem VI.3 shows that $\mathrm{pre}^*_{\mathcal{V}}(S')$ is locally semi-pseudo-linear. As D_0 is semi-linear we deduce that $D_0 \cap \mathrm{pre}^*_{\mathcal{V}}(S')$ is equal to a finite union of pseudo-linear sets $X'_1, \ldots, X'_{k'}$. Let us consider some linearizations $L'_1, \ldots, L'_{k'}$ of these pseudo-linear sets and let us define the following Presburger set S.

$$S = S_0 \cup (D_0 \backslash (\bigcup_{j'=1}^{k'} L'_{j'}))$$

Once again, note that $S \cap \mathrm{pre}^*_{\mathcal{V}}(S') = \emptyset$. Thus S contains S_0 and (S, S') is a separator.

Let D be the domain of the separator (S, S'). From $D_0 = \mathbb{N}^n \backslash (S_0 \cup S'_0)$, we get the following equality:

$$D = D_0 \cap \left(\bigcup_{\substack{1 \leq j \leq k \\ 1 \leq j' \leq k'}} (L_j \cap L'_{j'}) \right)$$

From $X_j, X'_{j'} \subseteq D_0$ we get $\dim(X_j \cup X'_{j'}) \leq \dim(D_0)$. As $X_j \subseteq \mathrm{post}^*_{\mathcal{V}}(S_0) \subseteq \mathrm{post}^*_{\mathcal{V}}(S)$ and $X'_{j'} \subseteq \mathrm{pre}^*_{\mathcal{V}}(S')$ and (S, S') is a separator, we deduce that X_j and $X'_{j'}$ are two pseudo-linear sets with an empty intersection. Proposition VII.6 provides $\dim(L_j \cap L'_{j'}) < \dim(X_j \cup X'_{j'})$. We deduce $\dim(D) < \dim(D_0)$.

Since the dimensions of non-empty Presburger sets are non-negative integers (see Remark VII.2), we have proved the following Theorem VIII.2.

Theorem VIII.2 *Presburger separators are included in Presburger closed separators.*

As $(\{\mathbf{s}\}, \{\mathbf{s}'\})$ is a Presburger separator if $(\mathbf{s}, \mathbf{s}') \notin \xrightarrow{*}_{\mathcal{V}}$, the previous theorem shows that there exists a Presburger closed separator (S, S') such that $(\mathbf{s}, \mathbf{s}') \in (S, S')$. By considering $I = S$, following Corollary VIII.3 is proved.

Corollary VIII.3 *Let (s, s') be a pair of configurations of a VAS $\mathcal{V}$. We have $(\mathbf{s}, \mathbf{s}') \notin \xrightarrow{*}_{\mathcal{V}}$ if and only if there exists a Presburger formula denoting a forward invariant I such that $\mathbf{s} \in I$ and $\mathbf{s}' \notin I$.*

IX. Conclusion

Thanks to the classical KLMST decomposition we have proved that the Parikh Images of languages accepted by VASs are semi-pseudo-linear. As an application, we have proved that for any pair $(\mathbf{s}, \mathbf{s}')$ of configurations in the complement of the reachability relation there exists a Presburger formula $\psi(\mathbf{x})$ denoting a forward invariant I such that $\mathbf{s} \in I$ and $\mathbf{s}' \notin I$. We deduce that the following algorithm decides the reachability problem.

```
1  Reachability( s , V , s' )
2     k ← 0
3     repeat forever
4        for each word σ ∈ Σ^k
5           if s --σ-->_V s'
6              return ''reachable''
7        for each Presburger formula ψ(x) of length k
8           if ψ(s) and ¬ψ(s') are true and
9              ψ(x) ∧ y = x + δ(a) ∧ ¬ψ(y) unsat ∀a ∈ Σ
10             return ''unreachable''
11       k ← k + 1
```

The correctness is immediate and the termination is guaranteed by Corollary VIII.3. This algorithm is the *very first one* that does not require the classical KLMST decomposition for its implementation. Even though the termination proof is based on the KLMST decomposition, the complexity of the algorithm does not depend on this decomposition. In fact, the complexity depends on the minimal size of a word $\sigma \in \Sigma^*$ such that $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s}'$ if $\mathbf{s} \xrightarrow{*}_{\mathcal{V}} \mathbf{s}'$, and the minimal size of a Presburger formula $\psi(\mathbf{x})$ denoting a forward invariant I such that $\mathbf{s} \in I$ and $\mathbf{s}' \in I$ otherwise. We left as an open question the problem of computing lower and upper bounds for these sizes. Note that the VAS exhibiting a large (Ackermann size) but finite reachability set given in [8] does not directly provide an Ackerman lower-bound for these sizes since inductive separators can over-approximate reachability sets.

We also left as an open question the problem of adapting the *Counter Example Guided Abstract Refinement* approach [1] to obtain an algorithm for the VAS reachability problem with termination guarantee. In practice, such an algorithm should be more efficient than the previously given enumeration-based algorithm.

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Appendix A.
Proofs of Lemma III.3

Lemma III.3 *The input loop condition is satisfied by $\mathcal{M}$ iff there exist a cycle $\mathbf{x} \xrightarrow{w}_G \mathbf{x}$ and an extended configuration $\mathbf{y}$ satisfying $\mathbf{m} \xrightarrow{w}_{\mathcal{V}} \mathbf{y}$ and satisfying $\mathbf{y}[i] > \mathbf{m}[i]$ for any i such that $\mathbf{m}[i] < \mathbf{x}[i]$.*

Proof: Assume first that $\mathcal{U}$ satisfies the input loop condition. There exist a sequence $(\mathbf{x} \xrightarrow{w_c}_G \mathbf{x})_c$ of cycles and a non-decreasing sequence $(\mathbf{m_c})_c$ of extended configurations such that $\mathbf{m} \xrightarrow{w_c}_{\mathcal{V}} \mathbf{m_c}$ for any c and $\lim_{c\to+\infty} \mathbf{m_c} = \mathbf{x}$. Let us consider the set I of integers i such that $\mathbf{m}[i] < \mathbf{x}[i]$. Let us prove that for any $i \in I$ there exists an integer c_i such that $\mathbf{m_c}[i] > \mathbf{m}[i]$ for any $c \geq c_i$. Let $i \in I$. Since $\mathbf{m}[i] \trianglelefteq \mathbf{x}[i]$ we deduce that $\mathbf{m}[i] \in \mathbb{N}$ and $\mathbf{x}[i] = \top$. From $\lim_{c\to+\infty} \mathbf{m_c}[i] = \mathbf{x}[i]$ we deduce that there exists an integer $c_i \geq 0$ such that $\mathbf{m_c}[i] > \mathbf{m}[i]$ for any $c \geq c_i$. Now let us consider an integer c such that $c \geq c_i$ for any $i \in I$. Observe that $\mathbf{m_c}[i] > \mathbf{m}[i]$ for any $i \in I$. We have proved that there exist a cycle $\mathbf{x} \xrightarrow{w}_G \mathbf{x}$ with $w = w_c$ and an extended configuration $\mathbf{y} = \mathbf{m_c}$ satisfying $\mathbf{m} \xrightarrow{w}_{\mathcal{V}} \mathbf{y}$ and satisfying $\mathbf{y}[i] > \mathbf{m}[i]$ for any i such that $\mathbf{m}[i] < \mathbf{x}[i]$.

Next, assume that there exist a cycle $\mathbf{x} \xrightarrow{w}_G \mathbf{x}$ and an extended configuration $\mathbf{y}$ satisfying $\mathbf{m} \xrightarrow{w}_{\mathcal{V}} \mathbf{y}$ and satisfying $\mathbf{y}[i] > \mathbf{m}[i]$ for any i such that $\mathbf{m}[i] < \mathbf{x}[i]$.

Let us prove that for any i such that $\mathbf{m}[i] \geq \mathbf{x}[i]$ we have $\mathbf{y}[i] = \mathbf{m}[i]$. The relation $\mathbf{m}[i] \trianglelefteq \mathbf{x}[i]$ implies $\mathbf{m}[i] \leq \mathbf{x}[i]$. Thus $\mathbf{m}[i] = \mathbf{x}[i]$. The paths $\mathbf{m} \xrightarrow{w}_{\mathcal{V}} \mathbf{y}$ and $\mathbf{x} \xrightarrow{w}_{\mathcal{V}} \mathbf{x}$ with $\mathbf{m}[i] = \mathbf{x}[i]$ provides $\mathbf{y}[i] = \mathbf{x}[i]$. We have proved that $\mathbf{y}[i] = \mathbf{m}[i]$.

Therefore $\mathbf{y} \geq \mathbf{m}$ and an immediate induction shows that there exists a non-decreasing sequence $(\mathbf{m_c})_c$ of extended configurations such that $\mathbf{m} \xrightarrow{w^c}_{\mathcal{V}} \mathbf{m_c}$. Finaly, just observe that $(\mathbf{x} \xrightarrow{w^c}_G \mathbf{x})$ is a cycle and $\lim_{c\to+\infty} \mathbf{m_c} = \mathbf{x}$. ∎

Appendix B.
Proofs of Lemma III.4

The proof is symmetrical to the one of Lemma III.3.

Appendix C.
Proofs of Lemma III.5

Lemma III.5 *The large solution condition is satisfied by $\mathcal{U}$ iff the following conditions (i) and (ii) hold:*

(i) *Its characteristic system has a solution ξ with components in $\mathbb{Z}$,*

(ii) *Its homogeneous characteristic system has a solution $\xi_{\mathbf{0}} = (\mathbf{s_{0,j}}, \mu_{0,j}, \mathbf{s_{0,j}}')_j$ with components in $\mathbb{Q}$ satisfying for any j:*

- $\star$ $\mathbf{s_{0,j}}[i] > 0$ *for any i such that* $\mathbf{m_j}[i] = \top$,
- $\star$ $\mu_{0,j}(t) > 0$ *for any $t \in T_j$, and*
- $\star$ $\mathbf{s'_{0,j}}[i] > 0$ *for any i such that* $\mathbf{m'_j}[i] = \top$.

Proof: Let us consider ξ and $\xi_{\mathbf{0}}$ satisfying condition (i) and (ii). By multiplying $\xi_{\mathbf{0}}$ by a positive integer, its components can be assumed in $\mathbb{Z}$. Note that in this case the components are in fact in $\mathbb{N}$. Since there exists an integer $c \geq 0$ such that $\xi + c\xi_{\mathbf{0}}$ has its components in $\mathbb{N}$, by replacing ξ by $\xi + c\xi_{\mathbf{0}}$ we can assume that the components of ξ are in $\mathbb{N}$. Now, just observe that $\xi_{\mathbf{c}} = \xi + c\xi_{\mathbf{0}}$ provides a sequence $(\xi_{\mathbf{c}})_c$ that proves that $\mathcal{U}$ satisfies the large solution condition.

Next assume that $\mathcal{U}$ satisfies the large solution condition. There exists a sequence $(\xi_{\mathbf{c}})_c$ proving the large solution condition of $\mathcal{U}$. Let us denote by $\xi = (\mathbf{s_j}, \mu_j, \mathbf{s'_j})_j$ the first solution of this sequence. This solution naturally satisfies (i). Observe that there exists an integer $c \geq 0$ such that for any j:

- $\mathbf{s_{j,c}}[i] > \mathbf{s_j}[i]$ for any i such that $\mathbf{m_j}[i] = \top$,
- $\mu_{j,c}(t) > \mu_j(t)$ for any $t \in T_j$, and
- $\mathbf{s'_{j,c}}[i] > \mathbf{s'_j}[i]$ for any i such that $\mathbf{m'_j}[i] = \top$.

Notice that $\xi_{\mathbf{0}} = \xi_{\mathbf{c}} - \xi$ provides a solution of the homogeneous characteristic system satisfying condition (ii). ∎

Appendix D.
Proofs of Proposition III.6

A MRGS is said to be *original-perfect* if it satisfies the large solution condition and its marked reachability graphs satisfy the input and output loop conditions.

Even if the proof of the following lemma is immediate by induction over the length of w, w', it is central in the KLMST decomposition.

Lemma D.1 (Continuity)

- *For any $\mathbf{x} \xrightarrow{w}_{\mathcal{V}}$ there exists an integer $c \geq 0$ such that $\mathbf{y} \xrightarrow{w}_{\mathcal{V}}$ for any extended configuration $\mathbf{y}$ satisfying $\mathbf{y}[i] \geq c$ if $\mathbf{x}[i] = \top$ and $\mathbf{y}[i] = \mathbf{x}[i]$ otherwise for any i.*
- *For any $\xrightarrow{w'}_{\mathcal{V}}$ $\mathbf{x}'$ there exists an integer $c' \geq 0$ such that $\xrightarrow{w'}_{\mathcal{V}} \mathbf{y}'$ for any extended configuration $\mathbf{y}'$ satisfying $\mathbf{y}'[i] \geq c'$ if $\mathbf{x}'[i] = \top$ and $\mathbf{y}'[i] = \mathbf{x}'[i]$ otherwise for any i.*

Lemma D.2 *Perfect MRGSs are original-perfect.*

Proof: Let us consider a perfect MRGS $\mathcal{U}$. Notice that $\mathcal{U}$ satisfies the large solution condition since from any accepted sequence $(\mathbf{s_j}, \pi_j, \mathbf{s_j})_j$ we deduce a solution

$(\mathbf{s_j}, ||\pi_j||, \mathbf{s'_j})_c$. Since the input loop condition and the output loop condition are symmetrical, we just prove that the marked reachability graph $\mathcal{M}_j$ satisfies the input loop condition. We consider an integer $c \in \mathbb{N}$ satisfying $c > \mathbf{m_j}[i]$ for any i such that $\mathbf{m_j}[i] < \mathbf{x_j}[i]$. Since $\mathcal{U}$ is perfect, there exists an accepted sequence $(\mathbf{s_j}, \pi_j, \mathbf{s'_j})_{0 \leq j \leq k}$, a prefix $\mathbf{x_j} \xrightarrow{w_j}_{\mathcal{V}} \mathbf{x_j}$ of π_j, an extended configuration $\mathbf{r_j}$ such that $\mathbf{s_j} \xrightarrow{w_j}_{\mathcal{V}} \mathbf{r_j}$ and such that $\mathbf{r_j}[i] \geq c$ for any i such that $\mathbf{x_j}[i] = \top$. Since $\mathbf{s_j} \trianglelefteq \mathbf{m_j}$ we deduce that $\mathbf{s_j} \leq \mathbf{m_j}$. As $\mathbf{s_j} \xrightarrow{w_j}_{\mathcal{V}} \mathbf{r_j}$ and $\mathbf{s_j} \leq \mathbf{m_j}$ there exists an extended configuration $\mathbf{y_j}$ such that $\mathbf{m_j} \xrightarrow{w_j}_{\mathcal{V}} \mathbf{y_j}$. Let us prove that $\mathbf{y_j}[i] > \mathbf{m_j}[i]$ for any i such that $\mathbf{m_j}[i] < \mathbf{x_j}[i]$. Let i be such an integer. Since $\mathbf{m_j} \trianglelefteq \mathbf{x_j}$ and $\mathbf{m_j}[i] < \mathbf{x_j}[i]$ we deduce that $\mathbf{m_j}[i] \in \mathbb{N}$ and $\mathbf{x_j}[i] = \top$. From $\mathbf{x_j}[i] = \top$ we deduce that $\mathbf{r_j}[i] \geq c$. From $\mathbf{m_j}[i] \in \mathbb{N}$ we deduce that $\mathbf{s_j}[i] = \mathbf{m_j}[i]$. Thus $\mathbf{y_j}[i] = \mathbf{r_j}[i] \geq c > \mathbf{s_j}[i] = \mathbf{m_j}[i]$. Lemma III.3 shows that $\mathcal{M}_j$ satisfies the input loop condition. ■

Now, let us consider an original-perfect MRGS $\mathcal{U}$ and let us prove that $\mathcal{U}$ is perfect. Since $\mathcal{M}_j$ satisfies the input and output loop conditions, Lemma III.3 and Lemma III.4 show that:

- there exist a cycle $\theta_j = (\mathbf{x_j} \xrightarrow{w_j}_{G_j} \mathbf{x_j})$ and an extended configuration $\mathbf{y_j}$ satisfying $\mathbf{m_j} \xrightarrow{w_j}_{\mathcal{V}} \mathbf{y_j}$ and satisfying $\mathbf{y_j}[i] > \mathbf{m_j}[i]$ for any i such that $\mathbf{m_j}[i] < \mathbf{x_j}[i]$,
- there exist a cycle $\theta'_j = (\mathbf{x'_j} \xrightarrow{w'_j}_{G_j} \mathbf{x'_j})$ and an extended configuration $\mathbf{y'_j}$ satisfying $\mathbf{y'_j} \xrightarrow{w'_j}_{\mathcal{V}} \mathbf{m'_j}$ and satisfying $\mathbf{y'_j}[i] > \mathbf{m'_j}[i]$ for any i such that $\mathbf{m'_j}[i] < \mathbf{x'_j}[i]$.

The proof that $\mathcal{U}$ is perfect is obtained by first exhibiting a solution ξ with components in $\mathbb{N}$ of the characteristic system and a solution $\xi_{\mathbf{0}}$ with components in $\mathbb{N}$ of the homogeneous characteristic system satisfying some particular properties. These two solutions ξ and $\xi_{\mathbf{0}}$ are respectively defined in Lemma D.3 and Lemma D.4.

Lemma D.3 *There exists a solution $\xi = (\mathbf{s_j}, \mu_j, \mathbf{s'_j})_j$ of the characteristic system such that for any j:*

- *$\mathbf{s_j}$ is a configuration satisfying $\mathbf{s_j} \xrightarrow{w_j}_{\mathcal{V}}$,*
- *μ_j is the Parikh image of a path $\pi_j = (\mathbf{x_j} \xrightarrow{\sigma_j}_{G_j} \mathbf{x'_j})$,*
- *$\mathbf{s'_j}$ is a configuration satisfying $\xrightarrow{w'_j}_{\mathcal{V}} \mathbf{s'_j}$.*

Proof: As $\mathbf{m_j} \xrightarrow{w_j}_{\mathcal{V}}$, Lemma D.1 shows that there exists an integer $c \geq 0$ such that $\mathbf{s_j} \xrightarrow{w_j}_{\mathcal{V}}$ for any configuration $\mathbf{s_j}$ satisfying $\mathbf{s_j}[i] \geq c$ if $\mathbf{m_j}[i] = \top$ and $\mathbf{s_j}[i] = \mathbf{m_j}[i]$ otherwise for any i. Symmetrically, as $\xrightarrow{w'_j}_{\mathcal{V}} \mathbf{m'_j}$, Lemma D.1 shows that there exists an integer $c' \geq 0$ such that $\xrightarrow{w'_j}_{\mathcal{V}} \mathbf{s'_j}$ for any configuration $\mathbf{s'_j}$ satisfying $\mathbf{s'_j}[i] \geq c'$ if $\mathbf{m'_j}[i] = \top$ and $\mathbf{s'_j}[i] = \mathbf{m'_j}[i]$ otherwise for any i. Since $\mathcal{U}$ satisfies the large solution condition there exists a solution $\xi = (\mathbf{s_j}, \mu_j, \mathbf{s'_j})_j$ with components in $\mathbb{N}$ of the characteristic system such that $\mathbf{s_j}$ and $\mathbf{s'_j}$ satisfies the previous conditions and such that $\mu_j(t) \geq 1$ for any $t \in T_j$. As G_j is strongly connected, Euler lemma shows that μ_j is the Parikh image of a path $\pi_j = (\mathbf{x_j} \xrightarrow{\sigma_j}_{G_j} \mathbf{x'_j})$. ■

Lemma D.4 *There exists a solution $\xi_{\mathbf{0}} = (\mathbf{s_{0,j}}, \mu_{0,j}, \mathbf{s'_{0,j}})$ of the homogeneous characteristic system such that for any j:*

- *the value $\mathbf{s_{0,j}}[i]$ is strictly positive if $\mathbf{m_j}[i] = \top$ and it is equal to 0 otherwise for any i,*
- *the value $(\mathbf{s_{0,j}} + \delta(w_j))[i]$ is strictly positive if $\mathbf{x_j}[i] = \top$ and it is equal to 0 otherwise for any i,*
- *$\mu_{0,j} - (||\theta_j|| + ||\theta'_j||)$ is the Parikh image of a cycle $\pi_{0,j} = (\mathbf{x_j} \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x_j})$ and $|\pi_{0,j}|_t > 0$ for any $t \in T_j$,*
- *the value $(\mathbf{s_{0,j}}' - \delta(w'_j))[i]$ is strictly positive if $\mathbf{x'_j}[i] = \top$ and it is equal to 0 otherwise for any i, and*
- *the value $\mathbf{s'_{0,j}}[i]$ is strictly positive if $\mathbf{m'_j}[i] = \top$ and it is equal to 0 otherwise for any i.*

Proof: As $\mathcal{U}$ satisfies the large solution condition, Lemma III.5 shows that there exists a solution $\xi_{\mathbf{0}} = (\mathbf{s_{0,j}}, \mu_{0,j}, \mathbf{s'_{0,j}})_j$ with components in $\mathbb{Q}$ of the homogeneous characteristic system satisfying the additional constraints $\mathbf{s_{0,j}}[i] > 0$ if $\mathbf{m_j}[i] = \top$, $\mathbf{s'_{0,j}}[i] > 0$ if $\mathbf{m'_j}[i] = \top$, and $\mu_{0,j}(t) > 0$ for any $t \in T_j$. By multiplying $\xi_{\mathbf{0}}$ by a positive integer, we can assume that $\xi_{\mathbf{0}}$ is a solution with components in $\mathbb{Z}$ satisfying the additional constraints. We are going to prove that there exists a positive integer $c \geq 1$ such that $c\xi_{\mathbf{0}}$ satisfies the lemma.

First of all, observe that for any $c \geq 1$ and for any j:

- the value $c\mathbf{s_{0,j}}[i]$ is strictly positive if $\mathbf{m_j}[i] = \top$ and it is equal to 0 otherwise for any i,
- the value $c\mathbf{s'_{0,j}}[i]$ is strictly positive if $\mathbf{m'_j}[i] = \top$ and it is equal to 0 otherwise for any i.

Let us consider $1 \leq i \leq n$.

Let us prove that there exists a positive integer $c_i \geq 1$ such that for any $c \geq c_i$ the value $(c\mathbf{s_{0,j}} + \delta(w_j))[i]$ is strictly positive if $\mathbf{x_j}[i] = \top$ and it is equal to 0 otherwise. Note that $\mathbf{m_j}[i] \trianglelefteq \mathbf{x_j}[i]$ thus either $\mathbf{m_j}[i] = \mathbf{x_j}[i] \in \mathbb{N}$, or $(\mathbf{m_j}[i], \mathbf{x_j}[i]) \in \mathbb{N} \times \{\top\}$, or $\mathbf{m_j}[i] = \mathbf{x_j}[i] = \top$. We separate the proof following these three cases. Let us first consider the case $\mathbf{m_j}[i] = \mathbf{x_j}[i] \in \mathbb{N}$. As $\mathbf{m_j}[i] \in \mathbb{N}$ and $\xi_{\mathbf{0}}$ is a solution of the homogeneous characteristic system, we get $\mathbf{s_{0,j}}[i] = 0$. The cycle θ_j shows that $\mathbf{x_j} + \delta(w_j) = \mathbf{x_j}$. From $\mathbf{x_j}[i] \in \mathbb{N}$ we deduce that $\delta(w_j)[i] = 0$. In particular $(c\mathbf{s_{0,j}} + \delta(w_j))[i] = 0$ and we have proved the case $\mathbf{m_j}[i] = \mathbf{x_j}[i] \in \mathbb{N}$ by considering $c_i = 1$. Let us consider

the second case $(\mathbf{m_j}[i], \mathbf{x_j}[i]) \in \mathbb{N} \times \{\top\}$. As $\mathbf{m_j}[i] \in \mathbb{N}$ we deduce that $\mathbf{s_{0,j}}[i] = 0$. Since $\mathbf{m_j}[i] < \mathbf{x_j}[i]$ the condition satisfied by the loop θ_j shows that $\mathbf{y_j}[i] > \mathbf{m_j}[i]$. As $\mathbf{y_j}[i] = \mathbf{m_j}[i] + \delta(w_j)[i]$, we deduce that $\delta(w_j)[i] > 0$. In particular for any $c \geq 1$ we have $(c\mathbf{s_{0,j}} + \delta(w_j))[i] > 0$ and we have proved the case $(\mathbf{m_j}[i], \mathbf{x_j}[i]) \in \mathbb{N} \times \{\top\}$ by considering $c_i = 1$. Finally, let us consider the case $\mathbf{m_j}[i] = \mathbf{x_j}[i] = \top$. As $\mathbf{m_j}[i] = \top$ we deduce that $\mathbf{s_{0,j}}[i] > 0$ in particular there exists an integer $c_i \geq 1$ large enough such that $(c\mathbf{s_{0,j}} + \delta(w_j))[i] > 0$ for any $c \geq c_i$. We have proved the three cases.

Symmetrically, for any $1 \leq i \leq n$, there exists an integer $c'_i \geq 0$ such that for any $c \geq c'_i$ the value $(c\mathbf{s'_{0,j}} - \delta(w'_j))[i]$ is strictly positive if $\mathbf{x'_j}[i] = \top$ and it is equal to 0 otherwise.

Finally, as $\mu_{0,j}(t) > 0$ for any $t \in T_j$ and for any $0 \leq j \leq k$, we deduce that there exists an integer $c \geq 1$ large enough such that $c\mu_{0,j}(t) > |\theta_j|_t + |\theta_{j'}|_t$ for any $t \in T_j$ and for any $0 \leq j \leq k$. Naturally, we can also assume that $c \geq c_i$ and $c \geq c'_i$ for any $1 \leq i \leq n$. Let us replace $\xi_\mathbf{0}$ by $c\xi_\mathbf{0}$. As $\mu_{0,j}(t) - |\theta_j|_t + |\theta_{j'}|_t > 0$ for any $t \in T_j$, Euler lemma shows that $\mu_{0,j} - (||\theta_j|| + ||\theta'_j||)$ is the Parikh image of a cycle $\pi_{0,j} = (\mathbf{x_j} \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x_j})$. ■

Let us fix notations satisfying both Lemma D.3 and Lemma D.4. We now provide technical lemmas that prove together that $\mathcal{U}$ is perfect.

Lemma D.5 *For any $c \geq 0$ we have:*

$$\mathbf{s_j} + c\mathbf{s_{0,j}} \xrightarrow{w_j^c} \mathcal{V} \quad \mathbf{s_j} + c(\mathbf{s_{0,j}} + \delta(w_j))$$
$$\mathbf{s'_j} + c(\mathbf{s'_{0,j}} - \delta(w'_j)) \xrightarrow{(w'_j)^c} \mathcal{V} \quad \mathbf{s'_j} + c\mathbf{s'_{0,j}}$$

Proof: Since the two relations are symmetrical, we just prove the first one. The choice of ξ satisfying Lemma D.3 shows that $\mathbf{s_j} \xrightarrow{w_j} \mathcal{V}$. Let us consider $c \in \mathbb{N}$ and let us prove by induction over c' that for any $0 \leq c' \leq c$ we have:

$$\mathbf{s_j} + c\mathbf{s_{0,j}} \xrightarrow{w_j^{c'}} \mathcal{V} \mathbf{s_j} + (c - c')\mathbf{s_{0,j}} + c'(\mathbf{s_{0,j}} + \delta(w_j))$$

Naturally, the case $c' = 0$ is immediate. The induction is obtained just by observing that $\mathbf{s_{0,j}} \geq \mathbf{0}$, $\mathbf{s_{0,j}} + \delta(w_j) \geq \mathbf{0}$ and $\mathbf{s_j} \xrightarrow{w_j} \mathcal{V}$. ■

Lemma D.6 *There exists $c_0 \geq 0$ such that for any $c \geq c_0$:*

$$\mathbf{s_j} + c(\mathbf{s_{0,j}} + \delta(w_j)) \xrightarrow{\sigma_{0,j}^c} \mathcal{V} \quad \mathbf{s_j} + c(\mathbf{s'_{0,j}} - \delta(w'_j))$$

Proof: Since there exists a path in G_j from $\mathbf{x_j}$ to $\mathbf{x'_j}$ we deduce that $\mathbf{x_j}[i] = \top$ if and only if $\mathbf{x'_j}[i] = \top$. We denote by $\mathbf{u_j}$ the vector in $\{0,1\}^n$ satisfying $\mathbf{u_j}[i] = 1$ if $\mathbf{x_i}[i] = \top = \mathbf{x'_i}[i]$ and satisfying $\mathbf{u_j}[i] = 0$ otherwise. From the choice of $\xi_\mathbf{0}$ satisfying Lemma D.4, we observe that $\mathbf{s_{0,j}} + \delta(w_j) \geq \mathbf{u_j}$ and $\mathbf{s'_{0,j}} - \delta(w'_j) \geq \mathbf{u_j}$. Note that $\lim_{c \to +\infty}(\mathbf{s_j} + c\mathbf{u_j}) = \mathbf{x_j}$. As $\mathbf{x_j} \xrightarrow{\sigma_{0,j}}_{G_j} \mathbf{x_j}$, Lemma D.1 proves that there exists an integer $c_0 \geq 0$ such that $\mathbf{s_j} + c_0\mathbf{u_j} \xrightarrow{\sigma_{0,j}} \mathcal{V}$. Now, let us consider an integer $c \geq c_0$. Let us prove by induction over c' that for any $0 \leq c' \leq c$, we have:

$$\mathbf{s_j} + c(\mathbf{s_{0,j}} + \delta(w_j)) \xrightarrow{\sigma_{0,j}^{c'}} \mathcal{V} \mathbf{s_j} + (c - c')(\mathbf{s_{0,j}} + \delta(w_j)) + c'(\mathbf{s'_{0,j}} - \delta(w'_j))$$

Naturally, the case $c' = 0$ is immediate. Assume the previous relation holds for an integer c' such that $0 \leq c' < c$ and let us consider $c'' = c' + 1$. From $\mathbf{s_{0,j}} + \delta(w_j) \geq \mathbf{u_j}$ and $\mathbf{s'_{0,j}} - \delta(w'_j) \geq \mathbf{u_j}$ we deduce that $(c - c')(\mathbf{s_{0,j}} + \delta(w_j)) + c'(\mathbf{s'_{0,j}} - \delta(w'_j)) \geq c\mathbf{u_j} \geq c_0\mathbf{u_j}$. Thus, the induction directly comes from $\mathbf{s_j} + c_0\mathbf{u_j} \xrightarrow{\sigma_{0,j}} \mathcal{V}$ and $\mathbf{s_{0,j}} + \delta(w_j) + \delta(\sigma_{0,j}) + \delta(w'_j) = \mathbf{s'_{0,j}}$. ■

Lemma D.7 *There exists $c' \geq 0$ such that for any $c \geq c'$:*

$$\mathbf{s_j} + c(\mathbf{s'_{0,j}} - \delta(w'_j)) \xrightarrow{\sigma_j} \mathcal{V} \quad \mathbf{s'_j} + c(\mathbf{s'_{0,j}} - \delta(w'_j))$$

Proof: As $\lim_{c \to +\infty}(\mathbf{s'_j} + c(\mathbf{s'_{0,j}} - \delta(w'_j))) = \mathbf{x'_j}$ and $\mathbf{x_j} \xrightarrow{\sigma_j}_{G_j} \mathbf{x'_j}$, Lemma D.1 proves that there exists $c' \geq 0$ such that $\xrightarrow{\sigma_j} \mathcal{V}$ $(\mathbf{s'_j} + c(\mathbf{s'_{0,j}} - \delta(w'_j)))$ for any $c \geq c'$. Since $\mathbf{s_j} + \delta(\sigma_j) = \mathbf{s'_j}$ we are done. ■

Now, let us consider an integer $c \geq 0$ satisfying $c \geq c_0$ and $c \geq c'$ where c_0 and c' are respectively defined by Lemma D.6 and Lemma D.7. For each $0 \leq j \leq k$, we consider the following path:

$$\pi_{j,c} = (\mathbf{x_j} \xrightarrow{w_j^c}_{G_j} \mathbf{x_j} \xrightarrow{\sigma_{0,j}^c \sigma_j}_{G_j} \mathbf{x'_j} \xrightarrow{(w'_j)^c}_{G_j} \mathbf{x'_j})$$

We have proved that $(\mathbf{s_j} + c\mathbf{s_{0,j}}, \pi_{j,c}, \mathbf{s'_j} + c\mathbf{s'_{0,j}})_j$ is an accepted sequence for $\mathcal{U}$. Thus $\mathcal{U}$ is perfect.

Appendix E. Proof of Lemma IV.2

Lemma IV.2 *Let $P = \{\mathbf{p_1}, \ldots, \mathbf{p_k}\} \subseteq \mathbb{Z}^n$ with $k \in \mathbb{N}$. We have $\mathcal{I}(P^*) = \{\mathbf{0}\}$ if $k = 0$ and $\mathcal{I}(P^*) = P^* \cap ((\mathbb{Q}_+ \backslash \{0\})\mathbf{p_1} + \cdots + (\mathbb{Q}_+ \backslash \{0\})\mathbf{p_k})$ if $k \geq 1$.*

Proof: Since the case $k = 0$ is immediate, we assume that $k \geq 1$. Let us first consider an interior vector $\mathbf{a} \in \mathcal{I}(P^*)$. As $\sum_{j=1}^k \mathbf{p_j} \in P^*$ and $\mathbf{a} \in \mathcal{I}(P^*)$, there exists $N \geq 1$ such that $N\mathbf{a} \in (\sum_{j=1}^k \mathbf{p_j}) + P^*$. Let $\mathbf{p} \in P^*$ such that $N\mathbf{a} = \sum_{j=1}^k \mathbf{p_j} + \mathbf{p}$. As $\mathbf{p} \in P^*$, there exists a sequence $(N_j)_{1 \leq j \leq k}$ of elements in $\mathbb{N}$ such that $\mathbf{p} = \sum_{j=1}^k N_j \mathbf{p_j}$. Combining this equality with the

previous one provides $\mathbf{a} = \sum_{j=1}^{k} \frac{1+N_j}{N}\mathbf{p_j}$. Thus $\mathbf{a} \in (\mathbb{Q}_+\backslash\{0\})\mathbf{p_1} + \cdots + (\mathbb{Q}_+\backslash\{0\})\mathbf{p_k}$. Conversely, let us consider $\mathbf{a} \in P^* \cap ((\mathbb{Q}_+\backslash\{0\})\mathbf{p_1} + \cdots + (\mathbb{Q}_+\backslash\{0\})\mathbf{p_k})$. Observe that there exists an integer $d \geq 1$ large enough such that $d\mathbf{a} \in (\mathbb{N}\backslash\{0\})\mathbf{p_1} + \cdots + (\mathbb{N}\backslash\{0\})\mathbf{p_k}$. In particular for any $\mathbf{x} \in P^*$ there exists $N \geq 1$ such that $Nd\mathbf{a} \in \mathbf{x} + P^*$. ■

Appendix F.
Proof of Proposition IV.6

Proposition IV.6 *Images $X' = f(X)$ of pseudo-linear sets X by a linear function f are pseudo-linear. Moreover $L' = f(L)$ is a linearization of X' for any linearization L of X.*

Proof: Let us consider a linear function $f : \mathbb{Z}^n \to \mathbb{Z}^{n'}$ defined by a matrix $A \in \mathbb{Z}^{n\times n'}$ and a vector $\mathbf{v} \in \mathbb{Z}^{n'}$. Let us consider a pseudo-linear set $X \subseteq \mathbb{Z}^n$. As X is pseudo-linear, there exists a linearizator M of X and a vector $\mathbf{b} \in \mathbb{Z}^n$ such that $X \subseteq \mathbf{b} + M$. As M is finitely generated there exists a finite set P such that $M = P^*$. We are going to prove that $L' = f(L)$ is a linearization of $X' = f(X)$. Let us consider $\mathbf{b}' = f(\mathbf{b})$ and $P' = \{A\mathbf{p} \mid \mathbf{p} \in P\}$ and observe that $L' = \mathbf{b}' + (P')^*$. In particular L' is a linear set. Since $X \subseteq L$ we deduce that $X' \subseteq L'$. Let us consider a set $R' = \{\mathbf{r'_1}, \ldots, \mathbf{r_d}'\}$ included in the interior of $(P')^*$. As $\mathbf{r'_i} \in (P')^*$ there exists $\mathbf{p_i} \in P^*$ such that $\mathbf{r'_i} = A\mathbf{p_i}$. Lemma IV.2 shows that $\mathbf{r'_i}$ is a sum of vectors of the form $\lambda_{i,\mathbf{p}} A\mathbf{p}$ over all $\mathbf{p} \in P$ where $\lambda_{i,\mathbf{p}} > 0$ is a value in $\mathbb{Q}$. There exists an integer $n_i \geq 1$ large enough such that $n_i\lambda_{i,\mathbf{p}} \in \mathbb{N}\backslash\{0\}$ for any $\mathbf{p} \in P$. We deduce that $\mathbf{r_i} = \sum_{\mathbf{p}\in P} n_i\lambda_{i,\mathbf{p}}\mathbf{p}$ is a vector in P^*. Moreover, form Lemma IV.2 we deduce that $\mathbf{r_i}$ is in the interior of P^*. Let us consider the set R of vectors $\mathbf{r_i} + k_i\mathbf{p_i}$ where k_i is an integer such that $0 \leq k_i < n_i$. As $\mathbf{r_i} \in \mathcal{I}(P^*)$ and $\mathbf{p_i} \in P^*$ we deduce that $\mathbf{r_i} + k_i\mathbf{p_i} \in \mathcal{I}(P^*)$. We have proved that $R \subseteq \mathcal{I}(P^*)$. As L is a linearization of X, there exists $\mathbf{x} \in X$ such that $\mathbf{x} + R^* \subseteq X$. We deduce that $f(\mathbf{x}) + AR^* \subseteq X'$. Let us consider $\mathbf{x}' = f(\mathbf{x}) + A(\sum_{i=1}^{d} \mathbf{r_i})$ and let us prove that $\mathbf{x}' + (R')^* \subseteq X'$. Consider $\mathbf{r}' \in (R')^*$. There exists a sequence $(\mu'_i)_{1\leq i\leq d}$ of integers in $\mathbb{N}$ such that $\mathbf{r}' = \sum_{i=1}^{d} \mu'_i\mathbf{r'_i}$. The Euclid division of μ'_i by n_i shows that $\mu'_i = k_i + n_i\mu_i$ where $\mu_i \in \mathbb{N}$ and $0 \leq k_i < n_i$. From $n_i\mathbf{r'_i} = A\mathbf{r_i}$ we deduce that $\mathbf{x}' + \mathbf{r}' = f(\mathbf{x}) + A(\sum_{i=1}^{d}(\mathbf{r_i} + k_i\mathbf{p_i}) + \sum_{i=1}^{d} \mu_i\mathbf{r_i})$. Observe that $\mathbf{r_i} + k_i\mathbf{p_i}$ and $\mathbf{r_i}$ are both in R. We have proved that $\mathbf{x}' + \mathbf{r}' \in f(\mathbf{x}) + AR^*$. Thus $\mathbf{x}' + (R')^* \subseteq X'$. We have proved that L' is a linearization of X'. ■

Appendix G.
Proofs of Lemma VI.2

The following lemma formally explains why words in $\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$ can be reordered into words in $(\Sigma_P^*\Sigma^*\Sigma_{P'}^*) \cap \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$.

Lemma G.1 *Assume that $\mathbf{s} \xrightarrow{\sigma a \sigma'}_{\mathcal{V}} \mathbf{s}'$ holds with $\sigma, \sigma' \in \Sigma^*$, and $a \in \Sigma$. We have:*

- $\mathbf{s} \xrightarrow{a\sigma\sigma'}_{\mathcal{V}} \mathbf{s}'$ *if* $\delta(a) \geq \mathbf{0}$.
- $\mathbf{s} \xrightarrow{\sigma\sigma' a}_{\mathcal{V}} \mathbf{s}'$ *if* $\delta(a) \leq \mathbf{0}$.

Proof: We only consider the case $\delta(a) \geq \mathbf{0}$ since the other case is symmetrical by replacing $(\mathbf{s}, \mathcal{V}, \mathbf{s}')$ by $(\mathbf{s}', -\mathcal{V}, \mathbf{s})$ where $-\mathcal{V} = (\Sigma, n, -\delta)$. Let us consider the pair of configurations $(\mathbf{r}, \mathbf{r}')$ such that $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{r} \xrightarrow{a}_{\mathcal{V}} \mathbf{r}' \xrightarrow{\sigma'}_{\mathcal{V}} \mathbf{s}'$. Since $\delta(a) \geq \mathbf{0}$ we have $\mathbf{s} \xrightarrow{a}_{\mathcal{V}} \mathbf{s} + \delta(a)$. As $\mathbf{s} + \delta(a) \geq \mathbf{s}$ and $\mathbf{s} \xrightarrow{\sigma}_{\mathcal{V}}$ we deduce that $\mathbf{s} + \delta(a) \xrightarrow{\sigma}_{\mathcal{V}} \mathbf{s} + \delta(a) + \delta(\sigma)$. From $\mathbf{r}' = \mathbf{s} + \delta(\sigma) + \delta(a)$ we deduce the lemma. ■

Lemma VI.2 *We have $\mathrm{post}^*_{\mathcal{V}}(S) \cap S' = \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$ and $S \cap \mathrm{pre}^*_{\mathcal{V}}(S') = \mathbf{s} + f(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$.*

Proof: Let us consider $\mathbf{c}' \in \mathrm{post}^*_{\mathcal{V}}(S) \cap S'$ and let us prove that $\mathbf{c}' \in \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$. There exists $\mathbf{c} \in S$ and a word $v \in \Sigma^*$ such that $\mathbf{c} \xrightarrow{v}_{\mathcal{V}} \mathbf{c}'$. In particular $\mathbf{c} \xrightarrow{v}_{\bar{\mathcal{V}}} \mathbf{c}'$. Since $S = \mathbf{s} + P^*$ we observe that there exists a word $u \in \Sigma_P^*$ such that $\mathbf{s} \xrightarrow{u}_{\bar{\mathcal{V}}} \mathbf{c}$. Symmetrically since $S' = \mathbf{s}' + (P')^*$ there exists $u' \in \Sigma_{P'}^*$ such that $\mathbf{c}' \xrightarrow{u'}_{\bar{\mathcal{V}}} \mathbf{s}'$. We have proved that $uvu' \in \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$. Note that $f'(uvu') = -\bar{\delta}(v)$. From $\mathbf{s}' = \mathbf{c}' + \bar{\delta}(v)$ we have proved that $\mathbf{c}' \in \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$.

Conversely, let us consider a vector $\mathbf{c}' \in \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$ and let us prove that $\mathbf{c}' \in \mathrm{post}^*_{\mathcal{V}}(S) \cap S'$. There exists a word $\sigma \in \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}')$ such that $\mathbf{c}' = \mathbf{s}' + f'(\sigma)$. Since $\delta(\Sigma_P) \subseteq \mathbb{N}^n$ and $\delta(\Sigma_{P'}) \subseteq -\mathbb{N}^n$, Lemma G.1 shows that σ can be reordered into a word $\sigma_0 \in \mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}') \cap (\Sigma_P^*\Sigma^*\Sigma_{P'}^*)$. As σ_0 and σ have the same Parikh image we deduce that $f'(\sigma) = f'(\sigma_0)$. In particular, we can assume without loss of generality that $\sigma = uvu'$ with $u \in \Sigma_P^*$, $v \in \Sigma^*$ and $u' \in \Sigma_{P'}^*$. Let us consider the two configurations $\mathbf{c}, \mathbf{c}'$ such that $\mathbf{s} \xrightarrow{u}_{\bar{\mathcal{V}}} \mathbf{c} \xrightarrow{v}_{\bar{\mathcal{V}}} \mathbf{c}' \xrightarrow{u'}_{\bar{\mathcal{V}}} \mathbf{s}'$. Since $u \in \Sigma_P^*$ we deduce that $\mathbf{c} \in S$ and since $u' \in \Sigma_{P'}^*$ we get $\mathbf{c}' \in S'$. Moreover from $v \in \Sigma^*$ we deduce $\mathbf{c} \xrightarrow{v}_{\mathcal{V}} \mathbf{c}'$. We have proved that $\mathbf{c}' \in \mathrm{post}^*_{\mathcal{V}}(S) \cap S'$.

Thus $\mathrm{post}^*_{\mathcal{V}}(S) \cap S' = \mathbf{s}' + f'(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$. Symmetrically we get $S \cap \mathrm{pre}^*_{\mathcal{V}}(S') = \mathbf{s} + f(\mathcal{L}(\mathbf{s}, \bar{\mathcal{V}}, \mathbf{s}'))$. ■

Appendix H.
Proof of Remark VII.2

A *vector space* V of $\mathbb{Q}^n$ is a set $V \subseteq \mathbb{Q}^n$ such that $\mathbf{0} \in V$, $V + V \subseteq V$ and $\mathbb{Q}V \subseteq V$. Observe that for

any set $X \subseteq \mathbb{Q}^n$ the set $V = \{\mathbf{0}\} \cup \{\sum_{i=1}^k \lambda_i \mathbf{x_i} \mid k \geq 1\ \lambda_i \in \mathbb{Q}\ \mathbf{x_i} \in X\}$ is the unique minimal for the inclusion vector space that contains X. This vector space is called the *vector space generated* by X. Recall that for any vector space V there exists a finite set $B \subseteq V$ that generates V. The minimal integer $d \in \mathbb{N}$ such that there exists a finite set $B \subseteq V$ with d elements that generates V is called the *rank* of V and denoted $\mathrm{rank}(V)$. Note that for any set $X \subseteq \mathbb{Q}^n$ there exists a finite set $B \subseteq X$ such that the vector space V generated by B is equal to the vector space generated by X and such that $|B| = \mathrm{rank}(V)$.

Proposition H.1 *We have* $\dim(M) = rank(V)$ *where* V *is the vector space generated by a monoïd* M.

Proof: Since $M \subseteq \mathbb{Z}^n \cap V$ it is sufficient to prove that $\dim(M) \geq \mathrm{rank}(V)$ and $\dim(\mathbb{Z}^n \cap V) \leq \mathrm{rank}(V)$. Let us denote by $||\mathbf{x}||_\infty = \max\{|\mathbf{x}[1]|, \ldots, |\mathbf{x}[k]|\}$ the usual ∞-*norm* of a vector $\mathbf{x} \in \mathbb{Q}^n$. As M generates the vector space V, there exists a sequence $\mathbf{m_1}, \ldots, \mathbf{m_d} \in M$ with $d = \mathrm{rank}(V)$ that generates V. Since the case $d = 0$ is immediate we assume that $d \geq 1$. We denote by $f : \mathbb{Q}^d \to V$ the function $f(\mathbf{x}) = \sum_{i=1}^d \mathbf{x}[i]\mathbf{m_i}$.

Let us first prove that $\dim(M) \geq d$. By minimality of $d = \mathrm{rank}(V)$ note that f is injective. In particular the cardinal of $f(\{0, \ldots, k\}^d)$ is equal to $(1+k)^d$. Observe that a vector $\mathbf{m}$ in this set satisfies $||\mathbf{m}||_\infty \leq k \sum_{i=1}^d ||\mathbf{m_i}||_\infty$ and $\mathbf{m} \in M$. We deduce that $\dim(M) \geq d$.

Now, let us prove that $\dim(\mathbb{Z}^n \cap V) \leq d$. Since for any matrix, the rank of the vector space generated by the column vectors is equal to the rank of the vector space generated by the line vectors, there exists a sequence $1 \leq j_1 < \cdots < j_d \leq n$ such that the function $g : \mathbb{Q}^n \to \mathbb{Q}^d$ defined by $g(\mathbf{x}) = (\mathbf{x}[j_1], \ldots, \mathbf{x}[j_d])$ satisfies $h = g \circ f$ is a bijective function. In particular we deduce that for any $\mathbf{v} \in \mathbb{Z}^n \cap V \cap \{-k, \ldots, k\}^n$ there exists a vector $\mathbf{x} = g(\mathbf{v}) \in \{-k, \ldots, k\}^d$ such that $\mathbf{v} = f \circ h^{-1}(\mathbf{x})$. Therefore $|\mathbb{Z}^n \cap V \cap \{-k, \ldots, k\}^n| \leq (1+2k)^d$ for any $k \in \mathbb{N}$. We deduce that $\dim(\mathbb{Z}^n \cap V) \leq d$. ■

Since semi-linear sets X are finite unions of *linear sets*, i.e. sets of the form $\mathbf{b} + M$ where M is a finitely generated monoïd, we deduce that $\dim(X)$ is integral.

Appendix I.
Proof of Lemma VII.3

This proof is based on results given in section H.
Lemma VII.3 *We have* $\dim(X) = \dim(L)$ *for any linearization* L *of a pseudo-linear set* $X \subseteq \mathbb{Z}^n$.

Proof: There exists $\mathbf{b} \in \mathbb{Z}^n$ and a linearizator M for X such that $L = \mathbf{b} + M$. From $X \subseteq L$ we deduce that $\dim(X) \leq \dim(L)$. Let us prove the converse. Let us consider an interior vector $\mathbf{a} \in \mathcal{I}(M)$. Since M is finitely generated, there exists a finite set P such that $M = P^*$. Observe that $R = \{\mathbf{a}\} \cup (\mathbf{a} + P)$ is a finite subset of $\mathcal{I}(M)$. As X is pseudo-linear, there exists $\mathbf{x} \in X$ such that $\mathbf{x} + R^* \subseteq X$. Note that the vector space generated by R is equal to the vector space generated by P. Thus, from Proposition H.1 we deduce that $\dim(R^*) = \dim(P^*)$. As $\dim(\mathbf{x}+R^*) = \dim(R^*)$ and $\dim(\mathbf{b}+P^*) = \dim(P^*)$ we deduce that $\dim(\mathbf{x} + R^*) = \dim(L)$. Since $\mathbf{x} + R^* \subseteq X$ we deduce that $\dim(L) \leq \dim(X)$. ■

Appendix J.
Proof of Lemma VII.5

This proof is based on results given in section H.

A *group* of $\mathbb{Z}^n$ is a set $Z \subseteq \mathbb{Z}^n$ such that $\mathbf{0} \in Z$, $Z + Z \subseteq Z$ and $-Z \subseteq Z$. Observe that for any $X \subseteq \mathbb{Z}^n$, the set $G = X^* - X^*$ is the minimal for the inclusion group that contains X. This group is said to be generated by X. Let us consider the group $G = M - M$ generated by a monoïd M and $\mathbf{a} \in \mathbb{Z}^n$. Observe that $\mathbf{a} \in \mathcal{I}(M)$ if and only if for any $\mathbf{g} \in G$ there exists an integer $N \geq 1$ such that $\mathbf{g} + N\mathbf{a} \in M$.

Lemma J.1 *For any vector* $\mathbf{v} \in V$ *where* V *is the vector space generated by a group* G*, there exists an integer* $d \geq 1$ *such that* $d\mathbf{v} \in G$.

Proof: As $\mathbf{v} \in V$, either $\mathbf{v} = \mathbf{0}$ or $\mathbf{v}$ can be decomposed into a finite sum $\mathbf{v} = \sum_{i=1}^k \lambda_i \mathbf{g_i}$ with $k \geq 1$, $\lambda_i \in \mathbb{Q}$ and $\mathbf{g_i} \in G$. The case $\mathbf{v} = \mathbf{0}$ is immediate with $d = 1$ and the second case is obtained by consider an integer $d \geq 1$ such that $d\lambda_i \in \mathbb{Z}$ for any i. ■

Lemma J.2 ([3]) *For any finite sets* $P_1, P_2 \subseteq \mathbb{Z}^n$ *there exists a finite set* $P \subseteq \mathbb{Z}^n$ *such that* $P_1^* \cap P_2^* = P^*$. *Moreover, for any* $\mathbf{b_1}, \mathbf{b_2} \in \mathbb{Z}^n$, *there exists a finite set* $B \subseteq \mathbb{Z}^n$ *such that* $(\mathbf{b_1} + P_1^*) \cap (\mathbf{b_2} + P_2^*) = B + (P_1^* \cap P_2^*)$.

Proof: Let us consider an enumeration $\mathbf{p_{i,1}}, \ldots, \mathbf{p_{i,k_i}}$ of the $k_i \geq 0$ vectors in P_i where $i \in \{1, 2\}$. If $k_1 = 0$ or if $k_2 = 0$ then $P_1^* = \{\mathbf{0}\}$ or $P_2^* = \{\mathbf{0}\}$ and the lemma is immediate. Thus, we can assume that $k_1, k_2 \geq 1$.

Let us consider the set X of vectors $(\lambda_\mathbf{1}, \lambda_\mathbf{2}) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ such that $\mathbf{b_1} + \sum_{j=1}^{k_1} \lambda_\mathbf{1}[j]\mathbf{p_{1,j}} = \mathbf{b_2} + \sum_{j=1}^{k_2} \lambda_\mathbf{2}[j]\mathbf{p_{2,j}}$. Let us also consider the set X_0 of vectors $(\lambda_\mathbf{1}, \lambda_\mathbf{2}) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ such that $\sum_{j=1}^{k_1} \lambda_\mathbf{1}[j]\mathbf{p_{1,j}} = \sum_{j=1}^{k_2} \lambda_\mathbf{2}[j]\mathbf{p_{2,j}}$. Observe that $X = Z + X_0$ where Z is the finite set $Z = \min(X)$ and $X_0 = Z_0^*$ where Z_0 is the finite set $Z_0 = \min(X_0 \backslash \{\mathbf{0}\})$.

Let us denote by B the finite set of vectors $\mathbf{b} \in \mathbb{Z}^n$ such that there exists $(\lambda_\mathbf{1}, \lambda_\mathbf{2}) \in Z$ satisfying $\mathbf{b_1} +$

$\sum_{j=1}^{k_1} \lambda_{\mathbf{1}}[j]\mathbf{p_{1,j}} = \mathbf{b} = \mathbf{b_2} + \sum_{j=1}^{k_2} \lambda_{\mathbf{2}}[j]\mathbf{p_{2,j}}$. Let us also denote by P the finite set of vectors $\mathbf{p} \in \mathbb{Z}^n$ such that there exists $(\lambda_{\mathbf{1}}, \lambda_{\mathbf{2}}) \in Z_0$ satisfying $\sum_{j=1}^{k_1} \lambda_{\mathbf{1}}[j]\mathbf{p_{1,j}} = \mathbf{p} = \sum_{j=1}^{k_2} \lambda_{\mathbf{2}}[j]\mathbf{p_{2,j}}$. Remark that $(\mathbf{b_1} + P_1^*) \cap (\mathbf{b_2} + P_2^*) = B + P^*$ and $P_1^* \cap P_2^* = P^*$. ■

Lemma VII.5 *Let $L_1 = \mathbf{b_1} + M_1$ and $L_2 = \mathbf{b_2} + M_2$ be two linear sets with a non-degenerate intersection. There exist finite sets $R_1 \subseteq \mathcal{I}(M_1)$ and $R_2 \subseteq \mathcal{I}(M_2)$ such that $(\mathbf{x_1} + R_1^*) \cap (\mathbf{x_2} + R_2^*) \neq \emptyset$ for any $(\mathbf{x_1}, \mathbf{x_2}) \in (L_1, L_2)$.*

Proof: As M_1, M_2 are finitely generated, there exists some finite sets $P_1, P_2 \subseteq \mathbb{Z}^n$ such that $M_1 = P_1^*$ and $M_2 = P_2^*$. From Lemma J.2 there exists a finite set $P \subseteq \mathbb{Z}^n$ and a finite set $B \subseteq \mathbb{Z}^n$ such that $P_1^* \cap P_2^* = P^*$ and $L_1 \cap L_2 = B + P^*$. Note that $B = \emptyset$ is not possible since in this case $\dim(L_1 \cap L_2) = -\infty$. Thus there exists a vector $\mathbf{b} \in B$.

Let us denote by V_1, V, V_2 the vector spaces generated respectively by P_1, P, P_2 and let us prove that $V_1 = V = V_2$. Proposition H.1 shows that $\dim(L_1) = V_1$, $\dim(L_1 \cap L_2) = \mathrm{rank}(V)$ and $\dim(L_2) = \mathrm{rank}(V_2)$. From $\dim(L_1 \cap L_2) = \dim(L_1)$ we deduce that $\mathrm{rank}(V) = \mathrm{rank}(V_1)$. Moreover as $P^* \subseteq P_1^*$ we deduce that $V \subseteq V_1$. The inclusion $V \subseteq V_1$ and the relation $\mathrm{rank}(V) = \mathrm{rank}(V_1)$ prove together that $V = V_1$. Symmetrically we deduce that $V = V_2$.

We denote by G_1, G, G_2 the groups generated respectively by P_1, P, P_2. Note that the vector spaces generated by G_1, G, G_2 are equal to V_1, V, V_2.

Let $\mathbf{a}$ be an interior vector of P^* and let us prove that $\mathbf{a} \in \mathcal{I}(P_1^*) \cap \mathcal{I}(P_2^*)$. Let $j \in \{1, 2\}$. Note that $\mathbf{a} \in P^* \subseteq P_j^*$. Let $\mathbf{p} \in \mathcal{I}(P_j^*)$. Since $-\mathbf{p} \in V$ and V is the vector space generated by G, Lemma J.1 shows that there exists an integer $d \geq 1$ such that $-d\mathbf{p} \in G$. From $\mathbf{a} \in \mathcal{I}(P^*)$ we deduce that there exists $N \geq 1$ such that $-d\mathbf{p} + N\mathbf{a} \in P^*$. From $P^* \subseteq P_j^*$ we deduce that $\mathbf{a} \in \frac{1}{N}(d\mathbf{p} + P_j^*)$. From $\mathbf{p} \in \mathcal{I}(P_j^*)$ and Lemma IV.2 we get $\mathbf{a} \in \mathcal{I}(P_j^*)$.

We define R_1 and R_2 by $R_j = \{\mathbf{a}\} \cup (\mathbf{a} + P_j)$ for $j \in \{1, 2\}$. From $\mathbf{a} \in \mathcal{I}(P_j^*)$, Lemma IV.2 shows that $R_j \subseteq \mathcal{I}(P_j^*)$. Let us consider $\mathbf{x_1} \in L_1$ and $\mathbf{x_2} \in L_2$ and let us prove that $(\mathbf{x_1} + R_1^*) \cap (\mathbf{x_2} + R_2^*) \neq \emptyset$.

From $\mathbf{b}, \mathbf{x_j} \in \mathbf{b_j} + P_j^*$ we deduce that $\mathbf{x_j} - \mathbf{b} \in G_j$. As the group generated by R_j is equal to G_j, there exists $\mathbf{r_j}, \mathbf{r'_j} \in R_j^*$ such that $\mathbf{x_j} + \mathbf{r_j} = \mathbf{b} + \mathbf{r'_j}$.

As V is the vector space generated by G_1 and $\mathbf{r'_2} \in R_2^* \subseteq V_2 = V$, Lemma J.1 shows that there exists an integer $d_1 \geq 1$ such that $d_1\mathbf{r'_2} \in G_1$. As $\mathbf{a} \in \mathcal{I}(P_1^*)$, there exists an integer $N_1 \geq 1$ such that $d_1\mathbf{r'_2} + N_1\mathbf{a} \in P_1^*$. As $P_1^* \subseteq R_1^* - \mathbb{N}\mathbf{a}$, we deduce that there exists an integer $N'_1 \geq 0$ such that $d_1\mathbf{r'_2} + (N_1 + N'_1)\mathbf{a} \in R_1^*$. We denote by $\mathbf{r''_1}$ this vector. Symmetrically, there exist some integers $d_2 \geq 1$, $N_2 \geq 1$ and $N'_2 \geq 0$ such that the vector $d_2\mathbf{r'_1} + (N_2 + N'_2)\mathbf{a}$ denoted by $\mathbf{r''_2}$ is in R_2^*. We get:

$$\begin{aligned}
&\mathbf{x_1} + \mathbf{r_1} + (d_2 - 1)\mathbf{r'_1} + \mathbf{r''_1} + (N_2 + N'_2)\mathbf{a} \\
&= \mathbf{b} + d_2\mathbf{r'_1} + d_1\mathbf{r'_2} + (N_1 + N'_1 + N_2 + N'_2)\mathbf{a} \\
&\mathbf{x_2} + \mathbf{r_2} + (d_1 - 1)\mathbf{r'_2} + \mathbf{r''_2} + (N_1 + N'_1)\mathbf{a} \\
&= \mathbf{b} + d_1\mathbf{r'_2} + d_2\mathbf{r'_1} + (N_2 + N'_2 + N_1 + N'_1)\mathbf{a}
\end{aligned}$$

We have proved that these vectors are equal. Therefore $(\mathbf{x_1} + R_1^*) \cap (\mathbf{x_2} + R_2^*) \neq \emptyset$. ■