The Symmetric Sugeno Integral

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Abstract

We propose an extension of the Sugeno integral for negative numbers, in the spirit of the symmetric extension of Choquet integral, also called Šipoš integral. Our framework is purely ordinal, since the Sugeno integral has its interest when the underlying structure is ordinal. We begin by defining negative numbers on a linearly ordered set, and we endow this new structure with a suitable algebra, very close to the ring of real numbers. In a second step, we introduce the Möbius transform on this new structure. Lastly, we define the symmetric Sugeno integral, and show its similarity with the symmetric Choquet integral.

Keywords: fuzzy measure, ordinal scale, Sugeno integral, Möbius transform, symmetric integral

1 Introduction

In the field of fuzzy measure theory, the Sugeno integral [30] has been the first proposed to compute an average value of some function with respect to a fuzzy measure, and early applications of fuzzy measures in multicriteria evaluation have used this integral. Later, Murofushi and Sugeno proposed the use of a functional defined by Choquet [5] as a better definition of an integral with respect to a fuzzy measure. In decision making, the pioneering

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works of Schmeidler [26, 27] brought also into light the so-called Choquet integral, as a generalization of expected utility.

This attention focused on the Choquet integral gave rise to a rapid progress, both on a pure mathematical point of view and in decision making. As a consequence, the Sugeno integral has remained a little bit in the background until recently, where it has been (re)discovered that the Sugeno integral could have some interest in qualitative decision making, and more generally, whenever qualitative or ordinal information is used. This is due to the ordinal nature of its definition, which uses only *min* and *max*. See e.g. [9] and [21] for recent results on Sugeno integral.

This paper brings new results for the Sugeno integral, essentially motivated by qualitative decision making, although having their own mathematical interest. The starting point is the following: the Choquet integral can be defined in two different ways for functions taking negative values, named after Denneberg as *symmetric* and *asymmetric* Choquet integrals [6]. The symmetric integral, which integrates separately positive and negative values, is of particular interest in decision making, since it reflects well the symmetry in behaviour of human decision making (gains and losses are treated separately and differently). Cumulative Prospect Theory [31] precisely models this symmetric behaviour, and is based on the symmetric Choquet integral (see [18] for a comparative study of symmetric and asymmetric models in decision making). However, until now, there was no proposal for defining the Sugeno integral over functions taking negative values.

The aim of this paper is precisely to fill this gap, i.e. to define a symmetric Sugeno integral, in a general way. A fundamental requirement here is to keep the ordinal nature of the Sugeno integral, and our construction should work for any function and fuzzy measure valued on a (possibly finite) totally ordered set (ordinal scale), where no other operation than supremum and infimum are defined. Our construction is in three steps:

- definition of the concept of negative value for an ordinal scale, leading to a symmetric ordinal scale. Then we endow it with suitable operations so as to build a structure close to a ring, and to mimic real numbers with usual ring operations.

- definition of an ordinal Möbius transform. It is known that the Möbius transform is a powerful tool for the analysis and representation of fuzzy measures, or even more general set functions: it is a basic ingredient for Dempster-Shafer theory [28], upper and lower probabilities [4], co-operative game theory (where it is called *dividend*), and also pseudo-Boolean functions [20]. Also, the Choquet integral has a particularly simple form using the Möbius transform. Since fuzzy measures become valued on an ordinal scale, one has to redefine the Möbius transform accordingly.

- definition of a symmetric Sugeno integral, based on the preceding two steps.

We will make an emphasis on the third step, since the first and second steps have been solved in a previous paper by the author [15], and we will recall here the main results, limited to what is necessary for defining the symmetric Sugeno integral.

Preliminary works on symmetric Sugeno integral by the author can be found in [14].

2 Basic concepts

We provide in this section the necessary material for the sequel. Since our results are established on ordered structures, and will be compared to the corresponding ones in the numerical case, we present in a first part definition in the (classical) numerical setting. Then, is a second part, we present the definitions on ordered structures.

Let us consider a finite set $N = \{1, \dots, n\}$. A *fuzzy measure* or *capacity* on N is a set function $v : \mathcal{P}(N) \longrightarrow [0, 1]$ such that $v(\emptyset) = 0$, $v(N) = 1$, and $v(A) \leq v(B)$ whenever $A \subset B$.

The *conjugate* fuzzy measure of v is defined by $\bar{v}(A) = 1 - v(A^c)$, where A^c denotes the complement set of A.

We present some examples of fuzzy measures which will be useful in the sequel. A *unanimity game* u_B is a fuzzy measure defined by:

$$u_B(A) := \begin{cases} 1, & \text{if } A \supset B, A \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

for any $B \subset N$. A *possibility measure* Π [8] is a fuzzy measure satisfying:

$$\Pi(A \cup B) = \Pi(A) \vee \Pi(B), \quad \forall A, B \subset N.$$

Due to this property, Π can be defined unambiguously by giving its value on singletons only. We write $\pi(i) := \Pi(\{i\})$, $i \in N$, and call π the *possibility distribution* associated to or generating Π.

A *necessity measure* N is the conjugate of a possibility measure.

We introduce now integrals with respect to fuzzy measures. Let us consider a function $f : N \longrightarrow \mathbb{R}^+$. We write for simplicity $f_i := f(i)$, for $i \in N$.

The *Choquet integral* [5] of f w.r.t v is defined by:

$$\mathcal{C}_v(f) := \sum_{i=1}^{n}[f_{(i)} - f_{(i-1)}]v(A_{(i)}), \tag{1}$$

where $\cdot_{(i)}$ indicates a permutation on N so that $f_{(1)} \leq f_{(2)} \leq \cdots \leq f_{(n)}$, and $A_{(i)} := \{(i), \ldots, (n)\}$. Also $f_{(0)} := 0$.

We introduce now the extension of Choquet integral for real-valued functions. Let us consider $f : N \longrightarrow \mathbb{R}$, the two usual definitions are given below:

$$\check{\mathcal{C}}_v(f) = \mathcal{C}_v(f^+) - \mathcal{C}_v(f^-) \tag{2}$$
$$\mathcal{C}_v(f) = \mathcal{C}_v(f^+) - \mathcal{C}_{\bar{v}}(f^-) \tag{3}$$

where f^+, f^- are respectively the positive and the negative parts of f, that is $f^+ = (f_1^+, \ldots, f_n^+)$, $f^- = (f_1^-, \ldots, f_n^-)$, $f_i^+ = f_i \vee 0$, $f_i^- = -f_i \vee 0$.

These extensions are respectively named symmetric and asymmetric integrals by Denneberg [6]. The first one was in fact proposed by Šipoš [32], and the second one is the usual definition of the Choquet integral for real-valued functions (hence we keep the same symbol). The terms symmetric and asymmetric come from the following property:

$$\mathcal{C}_v(-f) = -\mathcal{C}_{\bar{v}}(f) \tag{4}$$
$$\check{\mathcal{C}}_v(-f) = -\check{\mathcal{C}}_v(f), \tag{5}$$

for any f in $\mathbb{R}^n$. The explicit expression of $\check{\mathcal{C}}_v$ is:

$$\begin{aligned}\check{\mathcal{C}}_\mu(f) = &\sum_{i=1}^{p-1}(f_{(i)} - f_{(i+1)})\mu(\{(1), \ldots, (i)\}) \\ &+ f_{(p)}\mu(\{(1), \ldots, (p)\}) \\ &+ f_{(p+1)}\mu(\{(p+1), \ldots, (n)\}) \\ &+ \sum_{i=p+2}^{n}(f_{(i)} - f_{(i-1)})\mu(\{(i), \ldots, (n)\})\end{aligned} \tag{6}$$

where $f_{(1)} \leq \cdots \leq f_{(p)} < 0 \leq f_{(p+1)} \leq \cdots \leq f_{(n)}$.

We turn now to the ordinal case. We consider a linearly ordered set L^+, with bottom and top elements denoted by $\mathbb{O}$ and $\mathbb{1}$. A *negation* is a mapping $n : L^+ \longrightarrow L^+$ such that $n(n(x)) = x$ and $x \leq y$ implies $n(x) \geq n(y)$. Note that if L^+ is finite, then n is unique and is simply the reverse order of L^+.

A L^+-valued fuzzy measure is a set function $v : \mathcal{P}(N) \longrightarrow L^+$, which assigns $\mathbb{O}$ to the empty set and $\mathbb{1}$ to N, and satisfies monotonicity as above.

The conjugation is defined thanks to the negation by $\overline{v}(A) = n(v(A^c))$. The definition of unanimity game, possibility and necessity measures are left unchanged (just replace 0,1 by $\mathbb{O}, \mathbb{1}$).

The corresponding integral in the ordinal case is the Sugeno integral [30]. We consider functions $f : N \longrightarrow L^+$. The *Sugeno integral* of f with respect to v is defined by:

$$\mathcal{S}_v(f) := \bigvee_{i=1}^{n} [f_{(i)} \wedge v(\{(i), \ldots, (n)\})] \tag{7}$$

with same notations as above.

3 Symmetric ordered structures

We consider a linearly ordered set $(L^+, \leq)$, with bottom and top elements denoted by $\mathbb{O}$ and $\mathbb{1}$. We introduce $L^- := \{-a | a \in L^+\}$, with the reversed order, i.e. $-a \leq -b$ iff $a \geq b$ in L^+. The bottom and top of L^- are respectively $-\mathbb{1}$ and $-\mathbb{O}$.

We denote by L the union of L^+ and L^-, with identification of $-\mathbb{O}$ with $\mathbb{O}$. Top and bottom are respectively $\mathbb{1}$ and $-\mathbb{1}$. We call L a *symmetric linearly ordered set*.

We introduce some mappings on L. The *reflection* maps $a \in L$ to $-a$, and $-(-a) = a$ for any $a \in L$. We have:

$$(-a) \vee (-b) = -(a \wedge b), \quad (-a) \wedge (-b) = -(a \vee b).$$

The *absolute value* of $a \in L$ is denoted $|a|$, and $|a| := a$ if $a \in L^+$, and $|a| = -a$ otherwise. The *sign function* is defined by:

$$\text{sign} : L \to L\,, \quad \text{sign}\, x = \begin{cases} -\mathbb{1} & \text{for } x < \mathbb{O} \\ \mathbb{O} & \text{for } x = \mathbb{O} \\ \mathbb{1} & \text{for } x > \mathbb{O} \end{cases}.$$

Our aim is to endow L with operations similar to usual operations $+, \cdot$ on $\mathbb{R}$, so that the algebraic structure is close to a ring. However, since our aim is to extend the Sugeno integral which is based on minimum and maximum, we require that the restriction to L^+ of these new operations are precisely minimum and maximum. We call *symmetric maximum* and *symmetric minimum* these new operations, which we denote $\varovee$ and $\varowedge$ respectively.

In [15], it is shown that, based on the above requirements, the "best" possible definition (in the sense of being close to a ring) is given as follows:

$$a \varovee b := \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and either } |a| \vee |b| = -a \text{ or } = -b \\ \mathbb{O} & \text{if } b = -a \\ |a| \vee |b| & \text{else.} \end{cases} \tag{8}$$

Observe that, except for the case $b = -a$, $a \varovee b$ equals the absolutely larger one of the two elements a and b. Figure 1 gives the constant level curves of this operation.

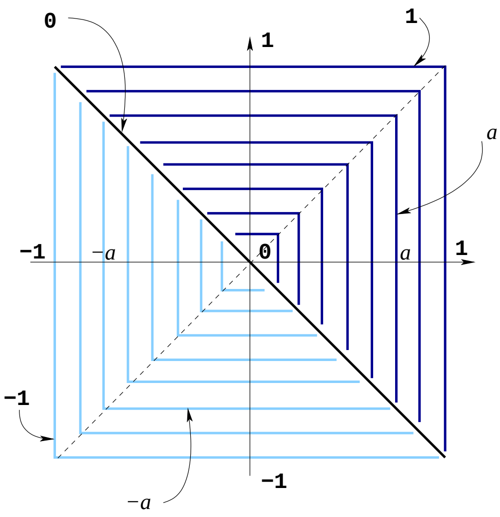

Figure 1: Constant level curves of the symmetric maximum

The symmetric minimum is defined as follows.

$$a \varowedge b := \begin{cases} -(|a| \wedge |b|) & \text{if sign } a \neq \text{sign } b \\ |a| \wedge |b| & \text{else.} \end{cases} \tag{9}$$

The absolute value of $a \varowedge b$ equals $|a| \wedge |b|$ and $a \varowedge b < \mathbb{O}$ iff the two elements a and b have opposite signs. Figure 2 shows the constant level curves of the symmetric minimum. Another equivalent formulation of these two operations is due to Marichal [22], when L is a symmetric real interval. It clearly shows the relationship with the ring of real numbers.

$$a \varovee b = \text{sign}\,(a + b)(|a| \vee |b|) \tag{10}$$

$$a \varowedge b = \text{sign}\,(a \cdot b)(|a| \wedge |b|). \tag{11}$$

The properties of $(L, \varovee, \varowedge)$, which is *not* a ring, are summarized below.

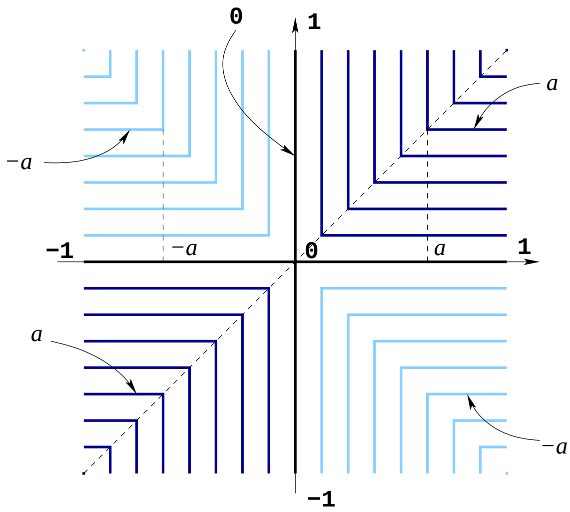

Figure 2: Constant level curves of the symmetric minimum

Proposition 1 *The structure $(L, \varovee, \varowedge)$ has the following properties.*

(i) $\varovee$ *is commutative.*

(ii) $\mathbb{O}$ *is the unique neutral element of* $\varovee$*, and the unique absorbant element of* $\varowedge$.

(iii) $a \varovee -a = \mathbb{O}$*, for all* $a \in L$.

(iv) $-(a \varovee b) = (-a) \varovee (-b)$.

(v) $\varovee$ *is associative for any expression involving* $a_1, \dots, a_n$*,* $a_i \in L$*, such that* $\bigvee_{i=1}^n a_i \neq -\bigwedge_{i=1}^n a_i$.

(vi) $\varowedge$ *is commutative.*

(vii) $\mathbb{1}$ *is the unique neutral element of* $\varowedge$*, and the unique absorbant element of* $\varovee$.

(viii) $\varowedge$ *is associative on* L.

(ix) $\varowedge$ *is distributive w.r.t* $\varovee$ *in* L^+ *and* L^-.

(v) shows that $\varovee_{i=1}^n a_i$ is unambiguously defined iff $\bigvee_{i=1}^n a_i \neq -\bigwedge_{i=1}^n a_i$. If equality occurs, we can propose several *rules of computation* which ensure uniqueness, among which the following ones [15]:

1. Put $\varovee_{i=1}^{n} a_i = 0$. This corresponds to combine separately positive and negative values. We denote this rule by $\lfloor \varovee_{i=1}^{n} a_i \rfloor$, and it is defined by:

$$\Big\lfloor \varovee_{i=1}^{n} a_i \Big\rfloor := \Big(\varovee_{a_i \geq \mathbb{0}} a_i \Big) \varovee \Big(\varovee_{a_i < \mathbb{0}} a_i \Big).$$

2. Discard the pair(s) of opposite extremal values, successively until Condition (v) is satisfied. We denote this rule by $\lceil \varovee_{i=1}^{n} a_i \rceil$, defined formally by:

$$\Big\lceil \varovee_{a_i \in A} a_i \Big\rceil := \varovee_{a_i \in A \setminus \bar{A}} a_i,$$

where $A := a_1, \ldots, a_n$, and $\bar{A} := \bar{a}_1, \ldots, \bar{a}_{2k}$ is the sequence of pairs of maximal opposite terms.

3. Discard as before pair(s) of maximal opposite terms, but with duplicates, i.e. the set $\bar{A}$ contains in addition all duplicates of maximal opposite terms. This rule is denoted by $\langle \varovee_{i=1}^{n} a_i \rangle$.

Taking for example $L = \mathbb{Z}$ and the sequence of numbers $3, 3, 3, 2, 1, 0, -2, -3, -3$, for which associativity does not hold, the result for rule $\lfloor \cdot \rfloor$ is 0, while we have:

$$\lceil 3 \varovee 3 \varovee 3 \varovee 2 \varovee 1 \varovee 0 \varovee -2 \varovee -3 \varovee -3 \rceil = 3 \varovee 2 \varovee 1 \varovee 0 \varovee -2 = 3$$
$$\langle 3 \varovee 3 \varovee 3 \varovee 2 \varovee 1 \varovee 0 \varovee -2 \varovee -3 \varovee -3 \rangle = 1 \varovee 0 = 1.$$

We will use and comment on these rules in Section 6 (see also [15] for a detailed study of their properties).

The symmetric maximum with the $\lceil \cdot \rceil$ rule coincides with the limit of some family of uni-norms proposed by Mesiar and Komorniková [25]. We refer the reader to [16] for details.

4 The Möbius transform on symmetric ordered structures

Let us recall briefly some facts on the classical Möbius transform (see e.g. [1, 2]). Let $(X, \leq)$ be a locally finite (i.e. any segment $[u, v] := \{x \in X | u \leq x \leq v\}$ is finite) partially ordered set (poset for short) possessing a unique minimal element, denoted 0, and consider f, g two real-valued functions on X such that

$$g(x) = \sum_{y \leq x} f(y). \tag{12}$$

A fundamental question in combinatorics is to solve this equation, i.e. to recover f from g. The solution is given through the *Möbius function* $\mu(x,y)$ by

$$f(x)=\sum_{y\leq x}\mu(y,x)g(y) \tag{13}$$

where μ is defined inductively by

$$\mu(x,y)=\begin{cases}1, & \text{if } x=y\\ -\sum_{x\leq t<y}\mu(x,t), & \text{if } x<y\\ 0, & \text{otherwise.}\end{cases}$$

Taking for X the Boolean lattice of subsets of a finite set N, f and g are now set functions. In this case, for any $A\subset B\subset N$ we have $\mu(A,B)=(-1)^{|B\setminus A|}$, and denoting set functions by v,m, formulas (12) and (13) become

$$v(A)=\sum_{B\subset A}m(B) \tag{14}$$

$$m(A)=\sum_{B\subset A}(-1)^{|A\setminus B|}v(B). \tag{15}$$

The set function m is called the *Möbius transform* of v. When necessary, we write m^v to stress the fact it is the Möbius transform of v.

It is well known that the Möbius transform is the coordinate vector of the set function in the basis of unanimity games:

$$v(A)=\sum_{B\subset N}m^v(B)u_B(A),\quad \forall A\subset N. \tag{16}$$

Considering conjugate fuzzy measures $\overline{v}$, we can obtain a decomposition with respect to $\overline{u_A}$ [19]:

$$v(A)=\sum_{B\subset N}m^{\overline{v}}(B)\overline{u_B}(A),\quad \forall A\subset N. \tag{17}$$

Noting that $\overline{u_B}(A)=1$ if $A\cap B\neq\emptyset$ and 0 otherwise, we obtain

$$v(A)=\sum_{B\mid A\cap B\neq\emptyset}m^{\overline{v}}(B)=1-\sum_{B\mid A\cap B=\emptyset}m^{\overline{v}}(B). \tag{18}$$

The second expression comes from the fact that $v(N)=1=\sum_{B\subset N}m(B)$. The first expression is well known in Dempster-Shafer theory [28], which deals with fuzzy measures having non negative Möbius transforms, called *belief functions*, whose conjugate are called *plausibility*.

Lastly, we recall the expressions of symmetric and asymmetric Choquet integrals in terms of the Möbius transform [4, 19].

$$\mathcal{C}_v(f) = \sum_{A\subset N} m^v(A) \bigwedge_{i\in A} f_i \tag{19}$$

$$\check{\mathcal{C}}_v(f) = \sum_{A\subset N} m^v(A) \left[\bigwedge_{i\in A} f_i^+ - \bigwedge_{i\in A} f_i^-\right], \tag{20}$$

for any $f \in \mathbb{R}^n$.

We turn now to our ordinal framework. Let $(L, \geq)$ be a symmetric linearly ordered set. The *successor* of any $x \in L$ is an element $y \in L$ such that $y > x$ and there is no z such that $x < z < y$. We write $y \succ x$. We consider two L-valued functions f, g on X satisfying the equation:

$$g(x) = \mathop{\otimes}_{y\leq x} f(y). \tag{21}$$

Note that the above expression is well defined only if we use some rule of computation, as the three rules proposed above. The study of the existence of solutions to this equation is a difficult topic, partly solved in [15]. We just mention here that if $|g|$ is isotone (i.e. $x \leq y$ implies $|g(x)| \leq |g(y)|$) and if either the $\lfloor\cdot\rfloor$ or the $\langle\cdot\rangle$ rule is used, then there exists many solutions, among which the canonical one, which is defined as follows: the *canonical ordinal* Möbius function is defined by:

$$\mu(x, y) := \begin{cases} \mathbb{1}, & \text{if } x = y \\ -\mathbb{1}, & \text{if } x \prec y \\ \mathbb{O}, & \text{otherwise.} \end{cases} \tag{22}$$

which leads to the *canonical ordinal Möbius transform* of g, defined by:

$$m^g(x) := g(x) \otimes \Big[- \mathop{\otimes}_{y\prec x} g(y)\Big],$$

where in this expression the same computation rule used in (21) has to be applied.

This result is no more true for the $\lceil\cdot\rceil$ rule, which has no solution in many cases.

It is possible to get the whole set of non negative solutions when f, g are valued on L^+ (so that computation rules become useless), and g is isotone (fuzzy measures correspond to this case, hence its interest).

Proposition 2 *[15] For any non negative isotone function g, any non negative solution of the equation (21) lays in the interval $[m_*, m^*]$, defined by:*

$$
\begin{aligned}
m^*(x) = & g(x), \quad \forall x \in X \\
m_*(x) = m^g(x) = & \begin{cases} g(x), & \textit{if } g(x) > g(y), \quad \forall y \prec x \\ \mathbb{O}, & \textit{otherwise} \end{cases}, \quad \forall x \in X.
\end{aligned}
$$

Note that negative solutions exist. It is easy to see that

$$
m_*(x) = \begin{cases} g(x), & \text{if } g(x) > g(y), \quad \forall y \prec x \\ \text{any } e \in L, e \succ -g(x), & \text{otherwise} \end{cases},
$$

$\forall x \in X$ is the least solution. However, negative solutions do not possess good properties. It is to be noted that m_* has been first proposed as the (ordinal) Möbius transform of a fuzzy measure by Marichal [23] and Mesiar [24] independently. See also preliminary works of the author in [13], and related works by De Baets [3].

If there is no fear of ambiguity, we denote simply m^g by m. Moreover, since our framework is ordinal in the rest of the paper, we will omit to call it "ordinal", and will use the term "classical" Möbius transform when referring to the usual definition. We denote by $[m]$ the interval $[m_*, m^*]$, and with some abuse of notation, any function in this interval.

From now on, we restrict to the case of fuzzy measures on a finite set $N = \{1, \dots, n\}$.

The Möbius transform possesses many interesting properties, some of which are listed below.

(i) the Möbius transform can be written in a way which is very similar to formula (15):

$$
m(A) := \bigvee_{B \subset A, |A \setminus B| \text{even}} v(B) \oslash \left(- \bigvee_{B \subset A, |A \setminus B| \text{odd}} v(B) \right) \tag{23}
$$

for any $A \subset N$ (see [13]).

(ii) the Möbius transform is still the coordinate vector of any fuzzy measure in the basis of unanimity games, but with a different decomposition:

$$
v(A) = \bigvee_{B \subset N} \Big([m](B) \wedge u_B(A) \Big), \quad \forall A \subset N, \tag{24}
$$

where $[m]$ stands for any function in the interval $[m_*, m^*]$. But since the decomposition is not unique, we have no more a basis.

(iii) Let π a possibility distribution defined on N, such that $\mathbb{O} < \pi(1) < \cdots < \pi(n) = \mathbb{1}$, and consider the associated possibility and necessity measures Π, N. Then their Möbius transforms are given by:

$$m^{\Pi}(A) = \begin{cases} \Pi(\{i\}), & \text{if } A = \{i\}, i \in N \\ \mathbb{O}, & \text{otherwise.} \end{cases}$$

$$m^{\mathrm{N}}(A) = \begin{cases} n(\Pi(\{i\}), & \text{if } A = \{i+1, \ldots, n\}, i \in N \\ \mathbb{O}, & \text{otherwise.} \end{cases}$$

If for some i, we have $\pi(i) = \pi(i+1)$, then the result is unchanged for m^{Π}, and for m^{N}, we have $m^{\mathrm{N}}(\{i+1, \ldots, n\}) = \mathbb{O}$.

See [15] for a more general result, giving the Möbius transform of a conjugate fuzzy measure.

Note that the "focal elements" (i.e., in the terminology of Shafer, the subsets where the Möbius transform is non zero) are singletons in the case of the possibility measure, and are nested subsets for the necessity measure. This result needs some comments. In the classical case, the Möbius transform is non zero only on singletons if and only if the fuzzy measure is a probability measure, i.e. an additive measure. In our algebra based on min and max, the corresponding notion is a "maxitive" measure, in other words, a possibility measure. This shows the consistency of the construction. Now, it is also known that the Möbius transform of a necessity measure, in the classical case, is non zero only for a chain (i.e. a set of nested subsets). It is very surprising to get the same result here, and moreover, the chains are identical. Here there is a discrepancy with the classical case, since probability measures are self-conjugate, and possibility measures are not.

(iv) it is possible to derive a decomposition of a fuzzy measure in terms of the conjugate of unanimity games, as in (17). Specifically, using (24):

$$\begin{aligned}\overline{v}(A) =& n\Big(\bigovee_{B\subset N} ([m^v](B) \wedge u_B(A^c))\Big) \\ =& n\Big(\bigovee_{B\subset N} ([m^v](B) \wedge n(\overline{u_B}(A)))\Big) \\ =& \bigowedge_{B\subset N} \Big[n([m^v](B)) \vee \overline{u_B}(A)\Big]\end{aligned}$$

since $n(a \vee b) = n(a) \wedge n(b)$ and $n(a \wedge b) = n(a) \vee n(b)$, for any $a, b \in L^+$. Hence we get:

$$v(A) = \bigowedge_{B\subset N} \Big[n([m^{\overline{v}}](B)) \vee \overline{u_B}(A)\Big]. \tag{25}$$

It can be shown [15] that $n([m^{\overline{v}}](B)) = m^v(B^c)$ when $[m^v] \equiv v$. This shows that

$$v(A) = \bigotimes_{B \subset N} \Big[v(B^c) \vee \overline{u_B}(A) \Big].$$

Now observe that $\overline{u_B}(A) = \mathbb{1}$ if $A \cap B \neq \emptyset$, and $\mathbb{O}$ otherwise. Thus,

$$v(A) = n\Big[\bigovee_{B \cap A = \emptyset} [m^{\overline{v}}](B) \Big] \tag{26}$$

which is the exact counterpart of (18).

(v) the author has proposed some time ago the notion of k-additive measure [12], i.e. a fuzzy measure whose (classical) Möbius transform vanishes for subsets of more than k elements. As remarked by Mesiar [24], the concept can be extended to the ordinal Möbius transform. The author called this *k-possibility measures* (k-maxitive measure in the terminology of Mesiar), since this defines possibility distributions on subsets of at most k elements (see [13] for some properties of k-possibility measures).

Finally, we indicate that, contrary to the classical case, the Möbius transform is not a "linear" operator on the set of fuzzy measures, where of course "linear" is to be taken in the sense of "maxitive". This is shown by the following example:

Example 1: Let us take X to be the Boolean lattice 2^2 whose elements are denoted $\emptyset, \{1\}, \{2\}, \{1,2\}$, and consider two functions g_1, g_2 defined as follows:

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
g_1	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{1}$
g_2	$\mathbb{O}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$

The computation of the Möbius transform m_* gives

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$m_*[g_1]$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{1}$
$m_*[g_2]$	$\mathbb{O}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{O}$

Clearly, $g_1 \bigovee g_2 = g_2$, but $m_*^{g_1} \bigovee m_*^{g_2} \neq m_*^{g_2}$.

5 The symmetric Sugeno integral

It is possible to express the Sugeno integral with respect to the Möbius transform. Indeed, the following can be shown.

Proposition 3 *For any function $f : N \longrightarrow L^+$ and any L^+-valued fuzzy measure v on N, the Sugeno integral of f w.r.t. v can be written as:*

$$\mathcal{S}_v(f) := \bigvee_{A\subset N}\Big(\bigwedge_{i\in A} f_i \wedge [m](A)\Big)$$

where $[m]$ is any function in $[m_*, m^*]$.

Proof: It suffices to prove that the relation holds for m^* (i) and m_* (ii).

(i) We have, using distributivity of $\wedge, \vee$ and monotonicity of v:

$$\begin{aligned}
\bigvee_{A\subset N}\Big(\bigwedge_{i\in A} f_i \wedge v(A)\Big) &= \bigvee_{\substack{A\subset N\\ A\ni(1)}}\big(f_{(1)} \wedge v(A)\big) \vee \bigvee_{\substack{A\subset N\setminus(1)\\ A\ni(2)}}\big(f_{(2)} \wedge v(A)\big) \vee \\
&\quad \cdots \vee (f_{(n)} \wedge v(\{(n)\}) \\
&= \Big(f_{(1)} \wedge \bigvee_{\substack{A\subset N\\ A\ni(1)}} v(A)\Big) \vee \Big(f_{(2)} \wedge \bigvee_{\substack{A\subset N\setminus(1)\\ A\ni(2)}} v(A)\Big) \vee \\
&\quad \cdots \vee (f_{(n)} \wedge v(\{(n)\}) \\
&= (f_{(1)} \wedge v(N)) \vee (f_{(2)} \wedge v(N \setminus (1)) \vee \cdots \\
&\quad \vee(f_{(n)} \wedge v(\{(n)\}) \\
&= \bigvee_{i=1}^{n}\big(f_{(i)} \wedge v(\{(i),\dots,(n)\})\big).
\end{aligned}$$

(ii) For a given non empty $A \subset N$, if it exists some $j \in A$ such that $v(A) = v(A \setminus j)$, then $v(A) \wedge \bigwedge_{i\in A} f_i \leq v(A \setminus j) \wedge \bigwedge_{i\in A\setminus j} f_i$, hence the corresponding term in the supremum over N (in the expression with m^*) can be deleted, or equivalently, $v(A)$ can be replaced by $\mathbb{O}$. But $m_*(A) = \mathbb{O}$ if $v(A) = v(A \setminus j)$ for some j, hence the result. ■

The result with m^* is already in the original work of Sugeno [30]. Also, Marichal has shown the above proposition using min-max Boolean functions [21]. Note the analogy with the expression of the Choquet integral using the Möbius transform (see (19)).

We address now the problem of extending the definition of Sugeno integral for functions which are L-valued, i.e. they may take "negative" values. We focus on the symmetric definition. Following what is done in the numerical case for the Choquet integral, we propose the following definition for the symmetric Sugeno integral:

$$\check{\mathcal{S}}_v(f) = \mathcal{S}_v(f^+) \varovee (-\mathcal{S}_v(f^-)) \tag{27}$$

where $f^+ := f \vee \mathbb{O}$, $f^- := (-f) \vee \mathbb{O}$. From the definition, it is immediate that:

$$\check{\mathcal{S}}_v(-f) = -\check{\mathcal{S}}_v(f) \tag{28}$$

which justifies the name "symmetric".

Let us express the symmetric integral in an explicit form, using the fuzzy measure and its Möbius transform.

Proposition 4 *For any f valued in L and any fuzzy measure v on N,*

$$\begin{aligned}\check{\mathcal{S}}_v(f) := &\left[\mathop{\varovee}_{i=1}^{p} \big(f_{(i)} \varowedge v(\{(1),\dots,(i)\})\big)\right] \varovee \\ &\left[\mathop{\varovee}_{i=p+1}^{n} \big(f_{(i)} \varowedge v(\{(i),\dots,(n)\})\big)\right],\end{aligned} \tag{29}$$

where $-\mathbb{1} \leq f_{(1)} \leq \cdots \leq f_{(p)} < \mathbb{O}$, and $\mathbb{O} \leq f_{(p+1)} \leq \cdots \leq f_{(n)} \leq \mathbb{1}$.

$$\begin{aligned}\check{\mathcal{S}}_v(f) = &\left[\mathop{\varovee}_{A\subset N^+} \left(m(A) \varowedge \left[\bigwedge_{i\in A} f_i^+ \varovee \Big(-\bigwedge_{i\in A} f_i^-\Big)\right]\right)\right] \varovee \\ &\left[\mathop{\varovee}_{A\subset N^-} \left(m(A) \varowedge \left[\bigwedge_{i\in A} f_i^+ \varovee \Big(-\bigwedge_{i\in A} f_i^-\Big)\right]\right)\right] \varovee \\ &\left[\mathop{\varovee}_{A^+,A^-\neq\emptyset} \left(m(A) \varowedge \left[\bigwedge_{i\in A} f_i^+ \varovee \Big(-\bigwedge_{i\in A} f_i^-\Big)\right]\right)\right]\end{aligned} \tag{30}$$

where $N^+ := \{i \in N | f_i \geq \mathbb{O}\}$, $N^- := N \setminus N^+$, $A^+ := A \cap N^+$, and $A^- := A \cap N^-$.

Proof: Let us show the first formula. By Prop. 3, we have:

$$\Big[\bigvee_{A\subset N} (m(A) \wedge \bigwedge_{i\in A} f_i^+)\Big] = \bigvee_{i=p+1}^{n} [f_{(i)} \wedge v(\{(i),\dots,(n)\})]$$

since $f_{(i)} = \mathbb{O}$ for $i < p$. Similarly, since $f^-_{(1)} \geq \cdots \geq f^-_{(p)}$, we get:

$$
\begin{aligned}
-\bigvee_{A\subset N}\Big(m(A)\wedge\bigwedge_{i\in A} f_i^-\Big) &= -\bigvee_{i=1}^{p}\Big(f^-_{(i)}\wedge v(\{(1),\ldots,(i)\})\Big)\\
&= \mathop{\varovee}_{i=1}^{p}\Big(-\,(f^-_{(i)}\wedge v(\{(1),\ldots,(i)\}))\Big)\\
&= \mathop{\varovee}_{i=1}^{p}\Big((-f^-_{(i)})\varowedge v(\{(1),\ldots,(i)\})\Big)\\
&= \mathop{\varovee}_{i=1}^{p}\Big(f_{(i)}\varowedge v(\{(1),\ldots,(i)\})\Big).
\end{aligned}
$$

Since the symmetric Sugeno integral is the "sum" of these two terms, the result is proven.

Let us show the second formula. By definition of $\check{\mathcal{S}}_v$ and Prop. 3, we have:

$$
\begin{aligned}
\check{\mathcal{S}}_v(f) =& \Big[\bigvee_{A\subset N}\Big(m(A)\wedge\bigwedge_{i\in A} f_i^+\Big)\Big]\varovee\Big[-\bigvee_{A\subset N}\Big(m(A)\wedge\bigwedge_{i\in A} f_i^-\Big)\Big]\\
=& \Big[\bigvee_{A\subset N^+}\Big(m(A)\wedge\bigwedge_{i\in A} f_i^+\Big)\Big]\varovee\Big[-\bigvee_{A\subset N^-}\Big(m(A)\wedge\bigwedge_{i\in A} f_i^-\Big)\Big]\\
=& \Big[\mathop{\varovee}_{A\subset N^+}\Big(m(A)\varowedge\Big[\bigwedge_{i\in A} f_i^+\varovee\Big(-\bigwedge_{i\in A} f_i^-\Big)\Big]\Big)\Big]\varovee\\
& \Big[\mathop{\varovee}_{A\subset N^-}\Big(m(A)\varowedge\Big[\bigwedge_{i\in A} f_i^+\varovee\Big(-\bigwedge_{i\in A} f_i^-\Big)\Big]\Big)\Big].
\end{aligned}
$$

Observe that

$$
\mathop{\varovee}_{A^+,A^-\neq\emptyset}\Big[m(A)\varowedge\Big(\bigwedge_{i\in A} f_i^+\varovee\Big(-\bigwedge_{i\in A} f_i^-\Big)\Big)\Big] = \mathbb{O}
$$

hence this last term can be added without changing the result. ■

Let us make some comments on these results.

(i) Both formulas are unambiguous with respect to possible associativity problem, since positive terms and negative terms are separately combined.

(ii) Formula (29) is very similar to (6), which is the expression of the symmetric Choquet integral.

(iii) Formula (30) has in fact no computation interest, since it is more complicated than necessary (see proof). Its interest lies in the fact that it is a formula which is very close to the corresponding one (20) for the symmetric Choquet integral. Indeed, a summation over $A \subset N$ can be partitioned into $A \subset N^+$, $A \subset N^-$, and $A^+, A^- \neq \emptyset$. However, due to the lack of associativity, we cannot write, as it was wrongly claimed in [14], that

$$\check{\mathcal{S}}_v(f) = \underset{A\subset N}{\circledvee} \left(m(A) \owedge \left[\bigwedge_{i\in A} f_i^+ \circledvee \left(-\bigwedge_{i\in A} f_i^- \right) \right] \right).$$

6 Alternative definitions for the symmetric Sugeno integral

While our definition of symmetric Sugeno integral (Eq. (27)) seems to be natural with respect to what is done for the Choquet integral, we may think of other definitions, provided the symmetry property (28) is preserved. We propose the following ones, which satisfy the symmetry requirement.

$$\check{\mathcal{S}}^1_v(f) = \langle \underset{A\subset N}{\circledvee} \left(m(A) \owedge \left[\bigwedge_{i\in A} f_i^+ \circledvee \left(-\bigwedge_{i\in A} f_i^- \right) \right] \right) \rangle \tag{31}$$

$$\begin{aligned}\check{\mathcal{S}}^2_v(f) = \langle & \left[\underset{i=1}{\overset{p}{\circledvee}} \left(f_{(i)} \owedge v(\{(1), \dots, (i)\}) \right) \right] \circledvee \\ & \left[\underset{i=p+1}{\overset{n}{\circledvee}} \left(f_{(i)} \owedge v(\{(i), \dots, (n)\}) \right) \right] \rangle.\end{aligned} \tag{32}$$

The first one is suggested by Remark (iii) in Section 5, in order to avoid a complicated expression with the Möbius transform. The second one puts all terms of (29) together and apply the rule of computation. The following example shows that these formulas and the original one are indeed different.

Example 2: Let us take $N = \{1, 2, 3\}$, $L = [-1, +1]$, v and f defined in the following tables.

$v(\{1\}) = 0.3$	$v(\{1,2\}) = 0.4$
$v(\{2\}) = 0.25$	$v(\{1,3\}) = 0.3$
$v(\{3\}) = 0.2$	$v(\{2,3\}) = 0.6$

$f(1) = -1$
$f(2) = 0.3$
$f(3) = 1$

Observe that the Möbius transform $[m]$ is reduced to v, except for subset $\{1, 3\}$, where $[m](\{1, 3\}) = [0, 0.3]$. Let us compute

$\mathcal{S}_v(f^+)$ and $\mathcal{S}_v(f^-)$. We have:

$$\begin{aligned}\mathcal{S}_v(f^+) =&(0.3 \wedge 0.6) \vee (1 \wedge 0.2) = 0.3 \vee 0.2 = 0.3\\ \mathcal{S}_v(f^-) =&(1 \wedge 0.3) = 0.3.\end{aligned}$$

Hence, according to original definition (eq. (27)), we get $\check{\mathcal{S}}_v(f) = 0$. If we compute from (32), we obtain:

$$\begin{aligned}\check{\mathcal{S}}^2_v(f) &= \langle(-1 \oslash 0.3) \varovee (0.3 \wedge 0.6) \varovee (1 \wedge 0.2)\rangle\\ &= \langle -0.3 \varovee 0.3 \varovee 0.2\rangle = 0.2.\end{aligned}$$

Now, if we compute from (31), we obtain:

$$\begin{aligned}\check{\mathcal{S}}^1_v(f) &= \langle(0.25 \wedge 0.3) \varovee (0.2 \wedge 1) \varovee (0.6 \wedge 0.3) \varovee (-1 \oslash 0.3)\rangle\\ &= \langle 0.25 \varovee 0.2 \varovee 0.3 \varovee -0.3\rangle = 0.25.\end{aligned}$$

Hence, all expressions lead to different results.

Let us comment about these formulas.

- Clearly, $\check{\mathcal{S}}^1_v$ is not the expression of $\check{\mathcal{S}}^2_v$ with the Möbius transform, as one could have expected, and it remains difficult to interpret.

- Comparing the original formula and (32) is easier since they can be viewed as the same expression using different rules of computation, which are $\langle\quad\rangle$ and $\lfloor\quad\rfloor$. This suggests a third variant, using the rule denoted by $\lceil\quad\rceil$:

$$\begin{aligned}\check{\mathcal{S}}^3_v(f) =&\lceil\left[\mathop{\varovee}_{i=1}^{p}\big(f_{(i)} \oslash v(\{(1),\dots,(i)\})\big)\right] \varovee\\ &\left[\mathop{\varovee}_{i=p+1}^{n}\big(f_{(i)} \oslash v(\{(i),\dots,(n)\})\big)\right]\rceil. \qquad (33)\end{aligned}$$

  $\check{\mathcal{S}}_v$ takes separately the maximum of positive and negative values, and then compare the results. We obtain 0 as soon as the best positive value equals in absolute value the worst negative one. $\check{\mathcal{S}}^2_v$ is more discriminating than $\check{\mathcal{S}}_v$ in the sense that many cases where $\check{\mathcal{S}}_v$ gives 0 are distinguished by $\check{\mathcal{S}}^2_v$, since maximal opposite values are discarded until they become different, in a way which is similar to the "discrimin" proposed by Dubois *et al.* [7]. This is also the case for $\check{\mathcal{S}}^3_v$, except that multiple occurrences are not removed.

- $\check{\mathcal{S}}_v^2$ is not monotonic in the sense that if $f \geq f'$, it may happen that $\check{\mathcal{S}}_v^2(f) < \check{\mathcal{S}}_v^2(f')$, since the rule $\langle\rangle$ is not monotonic, as shown by the following example:

  EXAMPLE 3: Let us consider the following values $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, 5$.

i	1	2	3	4	5
a_i	-5	-5	-1	2	5
b_i	-5	-4	-1	2	5

  Clearly $a_i \leq b_i, \forall i$, but $\langle \varovee_{i=1}^5 a_i \rangle = 2$ while $\langle \varovee_{i=1}^5 b_i \rangle = -4$.

  On the contrary $\check{\mathcal{S}}_v$ and $\check{\mathcal{S}}_v^3$ are monotonic. Since in decision making monotonicity is a mandatory property, $\check{\mathcal{S}}_v^2$ cannot be used. Hence, we recommend the use of either $\check{\mathcal{S}}_v$ or $\check{\mathcal{S}}_v^3$.

7 Related works

We are not aware of similar attempts to build algebraic structures on symmetric ordered sets. We did an extension of this work in [16], where L was restricted to $[-1, 1]$, but the operator to be extended was any t-conorm. The result is that an Abelian group can be built when the t-conorm is strict, but a ring is not possible. There are connections of this result with uninorms [33], and in a more general way, with partially ordered groups and rings (see Fuchs [11]).

Taking the viewpoint of decision making, as explained in the introduction, the symmetric Sugeno integral may be used as the main ingredient of an ordinal Cumulative Prospect Theory. Specifically, we aim at finding a representation of preference over a set of alternatives or acts, when acts have as consequence gains as well as losses, or put differently, when one can imagine the *symmetric* of a given act. There exist some works along this line, under the name of *signed orders*. We present briefly the concept of signed order [10].

Let X be a set of alternatives (possibly multidimensional), and $\succeq$ a transitive complete relation on X. We consider a copy X^* of X, whose elements are denoted x^* (*reflection* of x). The relation $\succeq$ extended on $X \cup X^*$ is assumed to be self-reflecting:

$$r \succeq s \Leftrightarrow s^* \succeq r^*.$$

$(X \cup X^*, \succeq)$ is called a *self-reflecting signed order*. Fishburn studies under which conditions a numerical representation u of $\succeq$ can be found, i.e. such that $r \succeq s \Leftrightarrow u(r) \geq u(s)$. It amounts that u has to be skew symmetric, i.e. $u(r) + u(r^*) = 0$.

A similar attempt has been done by Suck [29], who proposed *compensatory structures*. A being a set of alternatives, a compensatory structure is a triplet (A, L, K), where L, K are binary relations, L is complete and transitive, K is symmetric such that there exists $a' \in A$ for each $a \in A$ so that aKa', and if aLb, aKa' and bKb', then $b'La'$. Clearly, K is a reflection which reverses the order, like for Fishburn's signed order. The representation theorem is indeed similar, and includes the case of skew symmetry.

Our approach and aim are however rather different, since we are dealing with *scores* or *utilities*, on which we want to build some structure, and not on the alternatives. It remains to embed the use of the symmetric Sugeno integral in a decision making framework. We have already proposed such an attempt for multicriteria decision making [17].

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