Measure and integral with purely ordinal scales

Dieter Denneberg, Michel Grabisch

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Dieter Denneberg, Universität Bremen
denneberg@math.uni-bremen.de

Michel Grabisch, Université de Paris I Panthéon-Sorbonne
michel.grabisch@lip6.fr

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Abstract

We develop a purely ordinal model for aggregation functionals for lattice valued functions, comprising as special cases quantiles, the Ky Fan metric and the Sugeno integral. For modeling findings of psychological experiments like the reflection effect in decision behaviour under risk or uncertainty, we introduce reflection lattices. These are complete linear lattices endowed with an order reversing bijection like the reflection at 0 on the real interval $[-1, 1]$. Mathematically we investigate the lattice of non-void intervals in a complete linear lattice, then the class of monotone interval-valued functions and their inner product.

1 Motivation and survey

Measuring and aggregation or integration techniques have a very long tradition. Here numbers play an important role. But how do humans perceive numbers? The numbers, say the set $\mathbb{R}$ of reals, support two basic structures, the algebraic structure defined by $+$ and $\times$, and the ordinal structure given by $\leq$. There are many situations where only order is relevant, cardinals being used merely by tradition and convenience. During the last years the interest in ordinal aggregation has increased, see e.g. [4, 5, 6, 14, 19, 20].

We are interested in the question if aggregation or integration can be done in purely ordinal terms and what results can be obtained. Of course many partial results are already available. Since often they are formulated in terms of numbers, we ask what can be sustained if one ignores the algebraic structure or what weaker additional structure has to be imposed on the linear ordinal scale in order to formulate some well known important issues. It turns out that enough structure is given by an order reversing bijection of the scale leaving one point fixed. Thus the scale decomposes into two symmetric parts. This can be interpreted as the first step to numbers. Since, repeating the procedure with each resulting part of the scale infinitely often, one ends up with the binary representation of the numbers in the unit interval $[0, 1] \subset \mathbb{R}$ or some superstructure.

There are several ordinal concepts for aggregating values with respect to a measure. The oldest and best known selects a certain quantile, say the median, of a sample as the aggregated value. Next Ky Fan's [8] metric on the space $L_0(\mu)$ of μ-measurable functions is essentially ordinal. More recently and independently Sugeno [23] developed his integral which employs the same idea as Ky Fan. One aim of the present paper is to develop a common purely ordinal model for these three examples. This is done with a complete linear lattice M as scale, comprising the classical case $M = [0, 1] \subset \mathbb{R}$. For the Sugeno integral the scales used for functions and the measure are identical. But in general we allow separate scales for the functions and the measure and the two scales are related by a commensurability application, as we call it.

The structure of a linear lattice seems not to be sufficient to model elementary human behaviour in the presence of risk or uncertainty. There is some empirical psychological evidence (cf. reflection effect, inverse S-shaped decision weights, etc.) that in certain decision situations humans have a point $\mathbb{O}$ of reference (often the status quo) on their scale which allows to distinguish good and bad or gains and losses, i.e. values above, respectively below, the reference point [17, 18]. Then the attainable gains and the attainable losses are aggregated separately and finally these two aggregated values are compared to reach the final decision. In the cardinal models this behaviour can be modeled with the symmetric Choquet integral. We define the symmetric Sugeno integral in order to model the essentials of this behaviour in purely ordinal terms.

We also define the analogue of the asymmetric Choquet integral in our context. This can be done in introducing two commensurability functions, one for the positive part of the scale, the other one for the negative part.

Finally we comment on the new technical tools and the organisation of the paper. In Section 2 we model the scale with neutral reference point as a complete linear reflection lattice R corresponding to $[-1, 1] \subset \mathbb{R}$. On R we use the binary relations from [14] to get operations corresponding to addition and multiplication in $\mathbb{R}$.

In Section 3 we develop a theory of increasing interval valued functions and their inverse. Introducing these tools is motivated as follows. Mathematically, the idea of Fan and Sugeno for the aggregated value of a random variable is very simple, just take the argument at which the decreasing distribution function intersects a preselected increasing function, the identity function in their case. As already the quantiles show, the aggregated values are intervals rather than points on the linear scale L. So we look for a suitable ordering on the family $\mathcal{I}_L$ of nonempty intervals in L. The ordering which had been introduced by Topkis (see [24]) on the family of nonempty sublattices of an arbitrary lattice turns out to be the right one to handle monotonic functions (Proposition 3.1). This ordering, restricted to $\mathcal{I}_L$, is only a partial one, but we show that $\mathcal{I}_L$ is a completely distributive lattice in our situation. This structure is needed for an ordinal analogue of the inner product of vectors, which is introduced in Section 4. This product will, in Section 7, formalise the idea of Fan and Sugeno in our general context. Still in Section 4 the product provides a convenient tool to fill the gaps in the domain of the inverse of a monotone function, we call this saturation.

In close analogy to probability theory we introduce lattice valued measures in Section 5 and, in Section 6, the distribution function and its saturated inverse, the quantile function. In Section 7 we define an aggregated value of a function f as the product of a preselected commensurability function with the quantile function of f. To meet the empirical findings w.r.t. a neutral reference point mentioned above, we generalise our model in Section 8 to functions having values in a reflection lattice as introduced in Section 1.

In Section 9 we define the ordinal analogue of a metric induced by a Fan-Sugeno functional and use this to define the ordinal Ky-Fan 'norm' $\|\cdot\|_0$ and the ordinal supremum-norm $\|\cdot\|_\infty$.

2 The reflection lattice

The order structure of the real line allows for the order reversing reflection at the null. This structure will be generalised in this section to be the range of the functions to be integrated. That is, we apply isomorphic scales for gains and for losses.

Throughout this section L will denote a complete distributive lattice with bottom $\mathbb{O}$ and top $\mathbb{I}$. We endow the set $L_- := \{-a \mid a \in L\}$ with the reversed order from L, i.e. $-a \leq -b$ in L_- iff $a \geq b$ in L. The bottom of L_- is now $-\mathbb{I}$ and the top $-\mathbb{O}$. The disjoint union R of L and L_-, with $-\mathbb{O}$ identified with $\mathbb{O}$,

$$\mathbb{O} = -\mathbb{O}\,,$$

and setting $a \leq b$ for $a \in L_-$, $b \in L_+$, is called a **reflection lattice** and $\mathbb{O}$ the **neutral** or **reference point**. If, in addition, R is totally ordered, we call it a **linear reflection lattice**. R is again a (totally ordered) complete distributive lattice and it has bottom $-\mathbb{I}$ and top $\mathbb{I}$. For emphasising the symmetry we often write L_+ for $L \subset R$, hence $L_+ = \{a \in R \mid a \geq \mathbb{O}\}$, $L_- = \{a \in R \mid a \leq \mathbb{O}\}$. On the reflection lattice R we have the **reflection** at $\mathbb{O}$

$$\mathrm{refl} : R \to R\,, \quad x \mapsto -x\,,$$

where $-(-a) := a$ for $a \in R$. The reflection reverses the ordering of R,

$$a < b \ \text{ iff } \ -a > -b\,,$$

$$(-a) \vee (-b) = -(a \wedge b)\,, \quad (-a) \wedge (-b) = -(a \vee b)\,.$$

We also define the **absolute value**

$$\mathrm{abs} : R \to L_+\,, \quad x \mapsto |x| := \begin{cases} x & \text{if } x \geq \mathbb{O} \\ -x & \text{if } x < \mathbb{O} \end{cases}$$

and the **sign function**

$$\mathrm{sign} : R \to R\,, \quad \mathrm{sign}\, x = \begin{cases} \mathbb{I} & \text{for } x > \mathbb{O} \\ \mathbb{O} & \text{for } x = \mathbb{O} \\ -\mathbb{I} & \text{for } x < \mathbb{O} \end{cases} .$$

Example 2.1 The standard example for a linear reflection lattice is $\mathbb{R} \cup \{-\infty, \infty\}$ with the usual ordering $\leq$ and inf, sup as lattice operations $\wedge$, $\vee$. Here $-\mathbb{I} = -\infty$, $\mathbb{I} = \infty$ and $\mathbb{O} = 0$. Any closed, reflection invariant subset R of this reflection lattice containing 0 is again a reflection lattice with $\mathbb{I} = \sup_{x \in L} x$. □

Two new operations $\bigtriangledown$ and $\bigtriangleup$ are defined on a reflection lattice R which will play the roles of addition and multiplication for numbers (introduced in [11] for linear reflection lattices). On L_+ they coincide with the lattice operations $\vee$, $\wedge$.

$$x \bigtriangledown y := \begin{cases} x \vee y & \text{if } x, y \in L_+ \\ x \wedge y & \text{if } x, y \in L_- \\ x & \text{if } \mathrm{sign}\, x \neq \mathrm{sign}\, y,\ |x| > |y| \\ y & \text{if } \mathrm{sign}\, x \neq \mathrm{sign}\, y,\ |x| < |y| \\ \mathbb{O} & \text{if } \mathrm{sign}\, x \neq \mathrm{sign}\, y,\ |x| = |y| \text{ or } |x|, |y| \text{ are incomparable.} \end{cases}$$

[1] Except for the last case, $x \bigtriangledown y$ equals the absolutely larger one of the two elements x and y.

$$x \bigtriangleup y := \begin{cases} |x| \wedge |y| & \text{if } \mathrm{sign}\, x = \mathrm{sign}\, y \\ -(|x| \wedge |y|) & \text{else.} \end{cases}$$

The absolute value of $x \bigtriangleup y$ equals $|x| \wedge |y|$ and $x \bigtriangleup y < \mathbb{O}$ iff the two elements x and y have opposite signs.

Proposition 2.1 *For a,b,c in a reflection lattice $R = L_+ \cup L_-$ we have*

(i) $\bigtriangledown$ *and* $\bigtriangleup$ *are commutative operations;*

(ii) $\mathbb{O}$ *is the unique neutral element of* $\bigtriangledown$ *in* R *and* $\mathbb{I}$ *the unique neutral element of* $\bigtriangleup$*.*

(iii) $-(a \bigtriangledown b) = (-a) \bigtriangledown (-b)$ *and* $-(a \bigtriangleup b) = (-a) \bigtriangleup b$*;*

(iv) $\bigtriangledown$ *is associative on* L_+ *and on* L_-*,* $\bigtriangleup$ *is associative on* R*;*

(v) if $a, b, c \in L_+$ *or* $a, b, c \in L_-$ *then* $a \bigtriangleup (b \bigtriangledown c) = (a \bigtriangleup b) \bigtriangledown (a \bigtriangleup c)$*.*

(vi) $a \leq b$ *implies* $a \bigtriangledown c \leq b \bigtriangledown c$ *and* $a \bigtriangleup c \leq b \bigtriangleup c$*.*

[1] Quoting Rubos [22]: *. . . the preorder on preferences is partial, and individuals stay with the status quo when unable to make a comparison*, we see that the definition of $\bigtriangledown$ for incomparable $|x|, |y|$ matches with experimental evidence as summarised by Rubos.

The straightforward proofs are omitted. Only the proof of (vi) is cumbersome because of the many cases to be checked. $\bigtriangledown$ is not associative since $\mathbb{I} \bigtriangledown (\mathbb{I} \bigtriangledown (-\mathbb{I})) = \mathbb{I} \neq \mathbb{O} = (\mathbb{I} \bigtriangledown \mathbb{I}) \bigtriangledown (-\mathbb{I})$. This happens already in the smallest non-trivial reflection lattice which is a special case of Example 2.1.

Example 2.2 In $R = \{-1, 0, 1\} \subset \mathbb{R}$ the operation $\bigtriangleup$ coincides with multiplication in $\mathbb{R}$ (which is associative) and the operation $\bigtriangledown$ is similar to addition with the exception that sums greater than 1 are pulled down to 1 and sums below -1 are replaced by -1. □

Non-associativity of $\bigtriangledown$ will not bother us in the present paper, since we have only to combine two elements with $\bigtriangledown$. For a detailed study of associativity in a linear reflection lattice see [11].

$\bigtriangledown$ playing the role of addition, we call $a \bigtriangledown (-b)$ the **(pseudo) difference** of a and b in R. Then we define the **distance** of a and b in R by

$$\operatorname{dist}(x, y) := |x \bigtriangledown (-y)| = \begin{cases} \mathbb{O} & \text{if } x = y \\ |x| \vee |y| & \text{else} \end{cases} .$$

This distance has the usual properties if $+$ is replaced by $\vee$

$$\begin{aligned} \operatorname{dist}(x, y) &\geq \mathbb{O} \quad \text{and } \operatorname{dist}(x, y) = \mathbb{O} \text{ iff } x = y\,, \\ \operatorname{dist}(x, y) &= \operatorname{dist}(y, x)\,, \\ \operatorname{dist}(x, z) &\leq \operatorname{dist}(x, y) \vee \operatorname{dist}(y, z)\,. \end{aligned}$$

Also $\operatorname{dist}(x, \mathbb{O}) = |x|$.

3 Monotone interval-valued functions

A weakly monotone function is not invertible, but pseudo-inverses are often used in probability theory. Likewise they are important in our purely ordinal approach. As the pseudo inverse of a monotone function can be perceived as an interval-valued function, we use the terminology of set-valued mappings, also called correspondences. We give two equivalent definitions of isotonicity. For one of them an ordering for intervals is introduced.

Let M and L be arbitrary sets. A **correspondence** φ from M to L is a mapping $M \to 2^L$ which assigns to $x \in M$ a subset $\varphi(x) \subset L$. An ordinary function is a correspondence with $\varphi(x)$ being a singleton for all $x \in M$. Since $\varphi(x)$ may be empty for a correspondence φ, the **domain** of φ is defined as

$$\operatorname{dom}(\varphi) := \{x \in M \mid \varphi(x) \neq \emptyset\}\,.$$

There is a bijection between correspondences φ from M to L and subsets of $M \times L$ assigning to φ its **graph**,

$$\operatorname{graph}(\varphi) := \{(x, y) \in M \times L \mid y \in \varphi(x)\}\,.$$

The **inverse correspondence** $\varphi^{-1} : L \to 2^M$ of a correspondence $\varphi : M \to 2^L$ is defined by

$$\varphi^{-1}(y) := \{x \in M \mid (x, y) \in \operatorname{graph}(\varphi)\}, \quad y \in L\,,$$

i.e. φ and φ^{-1} have the same graph (modulo the natural bijection $M \times L \to L \times M$). φ is called **surjective** if its **image** $\operatorname{im}(\varphi) := \operatorname{dom}(\varphi^{-1})$ equals total L.

For defining order preserving correspondences we suppose that M and L are (partially) ordered sets. A subset I of L will be called a **(preference) interval**[2] iff for any two points $a, b \in I$ the **closed interval**

$$[a, b] := \{x \in L \mid a \leq x \leq b\}$$

also belongs to I. Notice that $[a, b] = \emptyset$ if $a > b$. The intersection of intervals is again an interval and, if L is a linearly ordered lattice, also the union of two intersecting intervals is an interval. We denote with $\mathcal{I}_L$ the set of all nonempty preference intervals in L.

A correspondence φ from M to L is called **isotonic** or **increasing** if for all (x_1, y_2), $(x_2, y_1) \in \operatorname{graph}(\varphi)$ with $x_1 \leq x_2$, $y_1 \leq y_2$ the whole rectangle $[x_1, x_2] \times [y_1, y_2]$ is contained in $\operatorname{graph}(\varphi)$. If all such rectangles in $\operatorname{graph}(\varphi)$ are degenerate, i.e. $x_1 = x_2$ or $y_1 = y_2$, then φ is called **sharply increasing**. Clearly an increasing correspondence φ from M to L is interval valued, so it can be perceived as an application

$$\varphi : D \to \mathcal{I}_L \quad \text{with } D := \operatorname{dom}(\varphi) \subset M\,.$$

Our definition is symmetric in both coordinates of $M \times L$, so the inverse correspondence φ^{-1} of an increasing correspondence φ is increasing, too. Especially φ^{-1} is interval valued,

$$\varphi^{-1} : E \to \mathcal{I}_M \quad \text{with } E := \operatorname{dom}(\varphi^{-1}) = \operatorname{im}(\varphi) \subset L\,. \tag{1}$$

If the ordered set L is a lattice our definition can be formulated more naturally in introducing an ordering on $\mathcal{I}_L$. Topkis (see [24]) deduces from the ordering of L the **relation** $\sqsubseteq$ on $2^L \setminus \{\emptyset\}$ in defining

$$Y_1 \sqsubseteq Y_2 \quad \text{iff} \quad y_1 \wedge y_2 \in Y_1 \text{ and } y_1 \vee y_2 \in Y_2 \text{ for all } y_1 \in Y_1, y_2 \in Y_2\,.$$

He shows ([24] Lemma 2.4.1) that the relation $\sqsubseteq$ is transitive and antisymmetric. Furthermore it is reflexive, hence an ordering, if it is restricted to the set of nonempty sublattices of L ([24] Theorem 2.4.1). We excluded the empty set from $\mathcal{I}_L$ in order to get an order relation on this set, provided L is linearly ordered (recall that an open interval in a non-linear lattice is no sublattice in general). $\mathcal{I}_L$ does not inherit from L the linearity of the ordering. If L has at least three elements $y_1 < y_2 < y_3$ then the intervals $[y_2, y_2]$ and $[y_1, y_3]$ are incomparable w.r.t. $\sqsubseteq$ in $\mathcal{I}_L$.

[2] This notion is due to [9], the usual definition of (closed, open or semiclosed) intervals, which is common in order theory, is not sufficient in our context.

Proposition 3.1 *Let M be an ordered set and L a linearly ordered lattice. $\varphi : M \to \mathcal{I}_L$ is increasing iff $x_1 \le x_2$ implies $\varphi(x_1) \sqsubseteq \varphi(x_2)$.*

Proof Let us call the last condition $\sqsubseteq$-increasing. First suppose φ is increasing. Let $x_1 \le x_2$ and $y_1 \in \varphi(x_1)$, $y_2 \in \varphi(x_2)$. We have to show $y_1 \wedge y_2 \in \varphi(x_1)$ and $y_1 \vee y_2 \in \varphi(x_2)$. This is obvious if $y_1 \le y_2$, hence we may suppose $y_1 > y_2$ since L is totally ordered. Since (x_1, y_1), $(x_2, y_2) \in \operatorname{graph}(\varphi)$ and φ is increasing we get $[x_1, x_2] \times [y_2, y_1] \subset \operatorname{graph}(\varphi)$, especially $y_1 \wedge y_2 = y_2 \in \varphi(x_1)$, $y_1 \vee y_2 = y_1 \in \varphi(x_2)$.

Now suppose that φ is $\sqsubseteq$-increasing. Consider (x_1, y_2), $(x_2, y_1) \in \operatorname{graph}(\varphi)$ with $x_1 \le x_2$, $y_1 \le y_2$ and take (x, y) in the rectangle $[x_1, x_2] \times [y_1, y_2]$. We have to show $(x, y) \in \operatorname{graph}(\varphi)$. We know $\varphi(x_1) \sqsubseteq \varphi(x) \sqsubseteq \varphi(x_2)$ since φ is $\sqsubseteq$-increasing. This entails $y_1 = y_2 \wedge y_1 \in \varphi(x_1)$ and similarly $y_2 \in \varphi(x_2)$. Hence $y \in \varphi(x_1)$, $\varphi(x_2)$ and, with an element $y_3 \in \varphi(x)$, $y \vee y_3$, $y_3 \wedge y \in \varphi(x)$, so $y \in \varphi(x)$. □

A function, perceived as a correspondence, is increasing iff it is an increasing function in the usual sense (for singletons $\sqsubseteq$ coincides with the ordering $\le$ on L). The results on increasing correspondences dualise in the obvious manner. So we can use freely **decreasing** (or **antitonic**) correspondences with their respective properties. As usual **monotone** means increasing or decreasing.

We investigate the lattice structure of the family $\mathcal{I}_L$.

Proposition 3.2 *Let $(L, \le)$ be a linearly ordered lattice. Then*

(i) *$(\mathcal{I}_L, \sqsubseteq)$ is a lattice, containing $(L, \le)$ as the sublattice of singletons;*

(ii) *$\{\mathbb{O}\}$ is the bottom and $\{\mathbb{I}\}$ the top of $(\mathcal{I}_L, \sqsubseteq)$ if $\mathbb{O}$ is the bottom and $\mathbb{I}$ the top of $(L, \le)$;*

(iii) *if $(L, \le)$ is complete, so is $(\mathcal{I}_L, \sqsubseteq)$;*

(iv) *$(\mathcal{I}_L, \sqsubseteq)$ is distributive. It is completely distributive if $(L, \le)$ is complete.*

Under the assumptions of Proposition 3.2 the lattice operations join and meet in $(\mathcal{I}_L, \sqsubseteq)$ will be denoted by $\sqcup$ and $\sqcap$. The proof of (i) shows

$$I_1 \sqcup I_2 = \{a_1 \vee a_2 \mid a_i \in I_i, i = 1, 2\}\,, \qquad (2)$$
$$I_1 \sqcap I_2 = \{a_1 \wedge a_2 \mid a_i \in I_i, i = 1, 2\}\,,$$

and similarly for any number of $\sqcup$ or $\sqcap$ (see (3)). For closed (and similarly for open) intervals these formulas look more familiar,

$$[a_1, b_1] \sqcup [a_2, b_2] = [a_1 \vee a_2, b_1 \vee b_2]\,, \quad [a_1, b_1] \sqcap [a_2, b_2] = [a_1 \wedge a_2, b_1 \wedge b_2].$$

Recall for (iv) that a linearly ordered complete lattice is completely distributive ([1] V.5). Proposition 3.2 can be generalised to arbitrary (distributive) lattices L if $\mathcal{I}_L$ is replaced by the set of closed nonempty intervals (cf. [21] 2.2).

Proof (i) Given I_1, $I_2 \in \mathcal{I}_L$, we have to show that a least upper bound (and similarly a greatest lower bound) of I_1 and I_2 does exist in $\mathcal{I}_L$. Our candidate is

$$J := \{a_1 \vee a_2 \mid a_i \in I_i, i = 1, 2\}\,.$$

First J is nonempty since the I_i are nonempty. Also J is a (preference) interval. To see this we show that the closed interval defined by any two points of J, say $a_1 \vee a_2 < b_1 \vee b_2$, is contained in J. Let $a_1 \vee a_2 < c < b_1 \vee b_2$, then for some i, $c < b_i$ and, for this i, also $a_i < c$. Hence $c \in I_i$. Let j be the complementary index to i, i.e. $\{j\} = \{1, 2\} \setminus \{i\}$, then we get $c = c \vee a_j \in J$ as desired.

We just verified that $J \in \mathcal{I}_L$ and have to show now that J is the least upper bound of I_1 and I_2, i.e. $K \sqsupseteq I_1,\ I_2,\ \ K \in \mathcal{I}_L$ imply $K \sqsupseteq J$. For $a_1 \vee a_2 \in J$ with $a_i \in I_i$ and $c \in K$ we know $(a_1 \vee a_2) \wedge c = (a_1 \wedge c) \vee (a_2 \wedge c) \in J$ and $(a_1 \vee a_2) \vee c = (a_1 \vee c) \vee (a_2 \vee c) \in K$. These are the conditions for $K \sqsupseteq J$.

Finally $a \mapsto \{a\} = [a, a]$ defines an isomorphism of $(L, \leq)$ to the sublattice of singletons in $\mathcal{I}_L$.

(ii) is obvious.

(iii) Let $\{I_i \mid i \in N\}$ be an arbitrary subset of $\mathcal{I}_L$. Since L is complete we know $\bigvee_{i \in N} a_i \in L$ for $a_i \in I_i$, $i \in N$. Now one verifies like for (i) that

$$\{\bigvee_{i \in N} a_i \mid a_i \in I_i, i \in N\} \tag{3}$$

is the least upper bound of the I_i, $i \in N$, in $\mathcal{I}_L$. Compared to (i) we now need complete distributivity of L, but this property holds for a complete linear lattice as we remarked above. The proof for the greatest lower bound runs dually.

(iv) Denoting the least upper bound (3) with $\bigsqcup_{i \in N} I_i$ and also applying the dual notation, we have to show

$$\bigsqcap_{k \in K} (\bigsqcup_{i \in N_k} I_{i,k}) = \bigsqcup_{\kappa \in \mathcal{K}} (\bigsqcap_{k \in K} I_{\kappa(k),k})$$

(and the dual equation) for $I_i \in \mathcal{I}_L$ first with a finite family K of finite index sets N_k, $k \in K$, then with an arbitrary family of arbitrary index sets. Here $\mathcal{K}$ denotes the set of functions $\kappa : K \to \bigcup_{k \in K} N_k$ (the union being disjoint) with $\kappa(k) \in N_k$. Using (2) and (3), this follows from (complete) distributivity of L,

$$\bigwedge_{k \in K} (\bigvee_{i \in N_k} a_{i,k}) = \bigvee_{\kappa \in \mathcal{K}} (\bigwedge_{k \in K} a_{\kappa(k),k}) \quad \text{for } a_{i,k} \in I_{i,k}\,.$$

□

Finally in this section we extend the ordering and lattice operations on $\mathcal{I}_L$ to $\mathcal{I}_L$-valued applications. For $\varphi, \psi : M \to \mathcal{I}_L$ we define $\varphi \sqsubseteq \psi$ iff $\varphi(x) \sqsubseteq \psi(x)$ for all $x \in M$. Similarly $\varphi \sqcup \psi$ and $\varphi \sqcap \psi$ are defined pointwise.

Proposition 3.3 *Let φ, $\psi : M \to \mathcal{I}_L$ be decreasing surjective correspondences from a complete linear lattice M to a complete linear lattice L. Then we have φ^{-1}, $\psi^{-1} : L \to \mathcal{I}_M$, these correspondences are decreasing and*

$$\varphi \sqsubseteq \psi \quad \textit{iff} \quad \varphi^{-1} \sqsubseteq \psi^{-1}\,.$$

Proof Since φ, ψ are surjective we know from (1) that φ^{-1}, $\psi^{-1} : L \to \mathcal{I}_M$ are decreasing. Now it is sufficient to prove the 'only if' part. Contrary to $\varphi^{-1} \sqsubseteq \psi^{-1}$ we assume that there is a point $y \in L$ so that $\varphi^{-1}(y) \not\sqsubseteq \psi^{-1}(y)$. Then either the intervals $\varphi^{-1}(y)$, $\psi^{-1}(y)$ are incomparable in $(\mathcal{I}_M, \sqsubseteq)$ or $\varphi^{-1}(y) \sqsupset \psi^{-1}(y)$. In both cases there exists a point $x \in \varphi^{-1}(y)$ with $\{x\} \sqsupset \psi^{-1}(y)$ or a point $x \in \psi^{-1}(y)$ with $\{x\} \sqsubset \varphi^{-1}(y)$. Both cases can be treated symmetrically, so we take the first one. Since ψ^{-1} is decreasing and $\{x\} \sqsupset \psi^{-1}(y) \neq \emptyset$ we know that graph$\,(\psi)$ cannot intersect the rectangle $[x, \mathbb{I}] \times [y, \mathbb{I}]$. Now, since $(x, y) \in$ graph$\,(\varphi)$, it is impossible that $\varphi(x) \sqsubseteq \psi(x)$. But this contradicts $\varphi \sqsubseteq \psi$. $\square$

4 Inner product of interval-valued functions

For interval-valued functions we will construct an ordinal analogue to the inner product of vectors. We investigate its properties mainly for monotone correspondences, since this will serve us to define the aggregation functionals in Section 7. The product will here be applied to saturate monotone correspondences, i.e. to fill the gaps in their domain.

Let L be a complete lattice and M a set. For φ, $\psi : M \to L$ we define the **(inner) product** as

$$\varphi *_M \psi := \bigvee_{x \in M} \varphi(x) \wedge \psi(x) \in L\,. \tag{4}$$

We write $\varphi * \psi$ if there is no ambiguity about the common domain M of φ and ψ. The name 'inner product' originates from the following observation. If $L = \mathbb{R}$ and M is finite, then the functions φ and ψ become vectors in $\mathbb{R}^{|M|}$ and the product (4) resembles the inner product of vectors with $\bigvee$ corresponding to $\sum$ and $\wedge$ to ordinary multiplication of numbers (see Section 2). With this interpretation the product behaves 'linear' in both factors as is shown by properties (iv) and (v) below together with (i).

Proposition 4.1 *Let L be a complete lattice and M a set. For $\varphi, \varphi_1, \varphi_2, \psi : M \to L$ and $a \in L$*

(i) $\varphi * \psi = \psi * \varphi$ *;*

(ii) the 'orthogonality' condition $\varphi * \psi = \mathbb{O}$ *holds iff* φ *and* ψ *have disjoint support;*

(iii) $\varphi_1 \leq \varphi_2$ *implies* $\varphi_1 * \psi \leq \varphi_2 * \psi$ *;*

(iv) $(\varphi_1 \vee \varphi_2) * \psi = (\varphi_1 * \psi) \vee (\varphi_2 * \psi)$ *if L is distributive;*

(v) $(a \wedge \varphi) * \psi = a \wedge (\varphi * \psi)$ *if L is completely distributive.*

Proof (i) The binary relation $*$ is commutative since $\sqcap$ is.

(ii) and (iii) are obvious.

(iv) Using distributivity of L we get $(\varphi_1 \vee \varphi_2) * \psi = \bigvee_{x\in M}(\varphi_1(x) \vee \varphi_2(x)) \wedge \psi(x) = \bigvee_{x\in M}(\varphi_1(x) \wedge \psi(x)) \vee (\varphi_2(x) \wedge \psi(x)) = (\varphi_1 * \psi) \vee (\varphi_2 * \psi)$.

(v) $(a\wedge\varphi)*\psi = \bigvee_{x\in M}(a\wedge\varphi(x))\wedge\psi(x) = \bigvee_{x\in M} a\wedge(\varphi(x)\wedge\psi(x)) = a\wedge(\varphi*\psi)$. For the last equality we applied complete distributivity of L. □

Now we confine ourselves to monotone correspondences and suppose that L is a complete linear lattice and that M is a linearly ordered set with bottom $\mathbb{O}$ and top $\mathbb{I}$. Then, by Proposition 3.2, $\mathcal{I}_L$ has all the properties required in Proposition 4.1 for L.

Example 4.1 For $a \in M$ let ϵ_a denote the indicator function $\mathbb{I}_{[a,\mathbb{I}]}$ of the interval $[a, \mathbb{I}] \subset M$. Then $\epsilon_a * \psi = \psi(a)$ for any decreasing correspondence $\psi : M \to \mathcal{I}_L$ from M to L. Thus, for monotone functions and in the analogy with the inner product of vectors, ϵ_a plays the role of the unit vector for 'coordinate' a. □

In general, the domain D of a decreasing correspondence $\psi : M \to 2^L$ from M to L is a proper subset of M, $D \subsetneq M$. We define the **saturation** $\widetilde{\psi}$ of ψ as

$$\widetilde{\psi}(x) := \epsilon_x *_D \psi\,, \quad x \in M,\ \text{where } D = \mathrm{dom}\,(\psi)\,.$$

$\widetilde{\psi}$ has domain M, is interval-valued, i.e. $\widetilde{\psi} : M \to \mathcal{I}_L$, and, by Example 4.1, $\widetilde{\psi}|_D = \psi$.

Proposition 4.2 *Let L be a complete linear lattice and M a linearly ordered set with top and bottom. The saturation $\widetilde{\psi}$ of a decreasing correspondence ψ from M to L is decreasing, too.*

Proof For $x_1 \le x_2$ we know $\epsilon_{x_1} \ge \epsilon_{x_2}$, hence by Proposition 4.1 (iii) $\widetilde{\psi}(x_1) = \epsilon_{x_1} *_D \psi \sqsupseteq \epsilon_{x_2} *_D \psi = \widetilde{\psi}(x_2)$. Now apply Proposition 3.1. □

If ψ is sharply decreasing, then $\widetilde{\psi}$ is not sharply decreasing, in general. So we define the **sharp saturation** of a decreasing correspondence ψ as

$$\widehat{\psi}(x) := \bigvee_{y\in\widetilde{\psi}(x)} y \quad \text{for } x \in M \setminus \mathrm{dom}\,(\psi)$$

and $\widehat{\psi}(x) := \widetilde{\psi}(x) = \psi(x)$ for $x \in \mathrm{dom}\,(\psi)$. Obviously, $\widehat{\psi} \sqsupseteq \widetilde{\psi}$.

Proposition 4.3 *Let L be a complete linear lattice and M a linearly ordered set with top and bottom.*

(i) The sharp saturation $\widehat{\psi}$ of a (sharply) decreasing correspondence ψ from M to L is (sharply) decreasing, too.

(ii) Given decreasing functions φ, $\psi : L \to M$ then $\varphi \le \psi$ implies $\widehat{\varphi^{-1}} \sqsubseteq \widehat{\psi^{-1}}$.

Simple examples show that (ii) does not hold for the saturations $\widetilde{\varphi^{-1}}$, $\widetilde{\psi^{-1}}$. We supposed φ, ψ to be functions only for simplicity.

Proof (i) Let $x_1 \leq x_2$. We have to show $\widehat{\psi}(x_1) \sqsupseteq \widehat{\psi}(x_2)$ and distinguish two cases. If $x_2 \in \mathrm{dom}\,(\psi)$ we know $\widehat{\psi}(x_1) \sqsupseteq \widetilde{\psi}(x_1) \sqsupseteq \widetilde{\psi}(x_2) = \widehat{\psi}(x_2)$ from Proposition 4.2. In the second case, $x_2 \notin \mathrm{dom}\,(\psi)$, we are done if also $x_1 \notin \mathrm{dom}\,(\psi)$. For the other case it is sufficient to prove $y_1 \geq \bar{y}_2$ for all $y_1 \in \psi(x_1)$ where $\{\bar{y}_2\} = \widehat{\psi}(x_2)$. Suppose the contrary, i.e. $y_1 < \bar{y}_2$ for some $y_1 \in \psi(x_1)$. Now the definition of $\widetilde{\psi}(x_2)$ together with the expression (3) for an arbitrary join of intervals imply the existence of an $y_3 \in \psi(x_3)$ with $x_3 \in \mathrm{dom}\,(\psi)$, $x_3 > x_2$ and $y_1 < y_3 \leq y_2$. But monotonicity of ψ would then imply that the rectangle $[x_1, x_3] \times [y_1, y_3]$ is contained in graph (ψ), contradicting $x_2 \notin \mathrm{dom}\,(\psi)$.

If, in addition, ψ is sharply decreasing, i.e. graph (ψ) does not contain any non-degenerate rectangle, then the same holds for $\widehat{\psi}$ since $\widehat{\psi}(x)$ is a singleton for any $x \notin \mathrm{dom}\,(\psi)$.

(ii) Setting $D := \mathrm{dom}\,(\varphi^{-1})$, $E := \mathrm{dom}\,(\psi^{-1})$ we have, for $y \in M$,

$$\widetilde{\varphi^{-1}}(y) = \epsilon_y *_D \varphi^{-1} = \bigsqcup_{y \leq u \in D} \varphi^{-1}(u)\,, \quad \widetilde{\psi^{-1}}(y) = \epsilon_y *_E \psi^{-1} = \bigsqcup_{y \leq v \in E} \psi^{-1}(v)\,.$$

For proving $\widehat{\varphi^{-1}}(y) \sqsubseteq \widehat{\psi^{-1}}(y)$ we have to distinguish the four cases that D or E contains y or not.

If $y \notin D$ and $y \notin E$ the least upper bounds of the intervals $\widetilde{\varphi^{-1}}(y)$, $\widetilde{\psi^{-1}}(y)$ have to be compared. It is sufficient for $\widehat{\varphi^{-1}}(y) \leq \widehat{\psi^{-1}}(y)$ to find for any point $u \in D$, $u \geq y$ and any point $x \in \varphi^{-1}(u)$ a point $v \in E$, $v \geq y$ such that $x \in \psi^{-1}(v)$. By assumption $u \in \varphi(x) \leq \psi(x)$. Then $v = \psi(x)$ will do the job.

If $y \in D$ and $y \notin E$ the same argument works since $\widehat{\psi^{-1}}(y) \sqsupseteq \widetilde{\psi^{-1}}(y)$ is still a singleton.

Next let $y \notin D$ and $y \in E$. Since $\widehat{\varphi^{-1}}(y)$ is a singleton it is sufficient for $\widehat{\varphi^{-1}}(y) \sqsubseteq \widehat{\psi^{-1}}(y)$ to show $\widehat{\varphi^{-1}}(y) \leq x$ for any $x \in \widehat{\psi^{-1}}(y) = \psi^{-1}(y)$. Since $(x, y) \notin$ graph (φ) we know $y = \psi(x) > \varphi(x)$. φ being decreasing its graph cannot intersect the rectangle $]x, \mathbb{I}] \times]\varphi(x), \mathbb{I}]$. Hence $\widetilde{\varphi^{-1}}(y) \sqsubseteq [\mathbb{O}, x]$ and this implies $\widehat{\varphi^{-1}}(y) \leq x$ as requested.

Finally let $y \in D$ and $y \in E$. For proving $\widehat{\varphi^{-1}}(y) \sqsubseteq \widehat{\psi^{-1}}(y)$ we apply Lemma 4.4. Like above for any $x \in \widehat{\varphi^{-1}}(y) = \varphi^{-1}(y)$ there exists $v \in E$, $v \geq y$ such that $x \in \psi^{-1}(v) = \widehat{\psi^{-1}}(v)$. Since ψ^{-1} is decreasing $\widehat{\psi^{-1}}(v) \sqsubseteq \widehat{\psi^{-1}}(y)$ by (i), hence there exists $x' \geq x$, $x' \in \widehat{\psi^{-1}}(y)$. According to Lemma 4.4 it remains to find for any $x' \in \widehat{\psi^{-1}}(y)$ a point $x \leq x'$, $x \in \widehat{\varphi^{-1}}(y)$. Since $\varphi(x') \leq \psi(x') = y$ we may take $x = x'$ if $\varphi(x') = \psi(x')$. If $\varphi(x') < \psi(x')$ the graph of φ does not intersect the rectangle $]x', \mathbb{I}] \times]\varphi(x'), \mathbb{I}]$, hence any point $x \in \varphi^{-1}(y) = \widehat{\varphi^{-1}}(y)$ will do the job. $\square$

Lemma 4.4 *Let L be a linearly ordered lattice and I_1, $I_2 \in \mathcal{I}_L$. $I_1 \sqsubseteq I_2$ holds iff for each $a_1 \in I_1$ there exists $a_2 \in I_2$ such that $a_1 \leq a_2$ and for each $b_2 \in I_2$ there exists $b_1 \in I_1$ such that $b_1 \leq b_2$.*

Proof First suppose $I_1 \sqsubseteq I_2$. Given $a_1 \in I_1$ take any $a \in I_2$. Then $a_1 \vee a \in I_2$ so that $a_2 := a_1 \vee a$ has the desired properties. The other condition derives similarly. For sufficiency we have to show $y_1 \wedge y_2 \in I_1$ and $y_1 \vee y_2 \in I_2$ for arbitrary $y_1 \in I_1$, $y_2 \in I_2$. By assumption there exists $b_1 \in I_1$ such that $b_1 \leq y_2$. Then $y_1 \wedge b_1 \leq y_1 \wedge y_2 \leq y_1$ and since I_1 is an interval containing $y_1 \wedge b_1$, y_1 we conclude $y_1 \wedge y_2 \in I_1$. The other condition proves analogously. □

For the sake of completeness we mention that a **dual product** can be defined as $\varphi *' \psi := \bigsqcap_{x \in M} \varphi(x) \sqcup \psi(x)$ and that the two products are related as follows.

Proposition 4.5 *Let M be a totally ordered set and L a complete linear lattice. If $\varphi : M \to \mathcal{I}_L$ is increasing and $\psi : M \to \mathcal{I}_L$ decreasing, then*

$$\varphi * \psi \sqsubseteq \varphi *' \psi \,.$$

Proof We are done if we show

$$\varphi(x_1) \sqcap \psi(x_1) \sqsubseteq \varphi(x_2) \sqcup \psi(x_2) \quad \text{for all } x_1, x_2 \in M. \tag{5}$$

First suppose $x_1 \leq x_2$. Using that φ is increasing we get $\varphi(x_1) \sqcap \psi(x_1) \sqsubseteq \varphi(x_1) \sqsubseteq \varphi(x_2) \sqsubseteq \varphi(x_2) \sqcup \psi(x_2)$. Similarly for $x_1 \geq x_2$ we get $\varphi(x_1) \sqcap \psi(x_1) \sqsubseteq \psi(x_1) \sqsubseteq \psi(x_2) \sqsubseteq \varphi(x_2) \sqcup \psi(x_2)$. Since M is totally ordered, the inequality (5) is proved for all $x_1, x_2 \in M$. □

5 Lattice-valued measures

Probability measures are monotone and assume only nonnegative values. We maintain this view in our ordinal context. In the cardinal theory of monotone measures there is a hierarchy of important subclasses: supermodular measures, totally monotone measures (belief functions), lower chain measures, necessity measures, (σ-)additive measures and the hierarchy of the respective dualisations. In the purely ordinal context among these only the chain measures and necessity (resp. possibility) measures survive. A continuity property will also be defined in our ordinal environment.

Let M denote a complete linear lattice with bottom $\mathbb{O}$ and top $\mathbb{I}$. M will be the scale of the measure to be defined. Throughout the paper, Ω denotes a nonempty set and $\mathcal{S} \subset 2^\Omega$ a family of subsets containing Ω and the empty set, $\emptyset, \Omega \in \mathcal{S}$. A M-valued set function,

$$\mu : \mathcal{S} \to M$$

is called a **measure**, if $\mu(\emptyset) = \mathbb{O}$, $\mu(\Omega) = \mathbb{I}$ and it is increasing, i.e.

$$A \subset B \quad \text{implies} \quad \mu(A) \leq \mu(B)\,.$$

Example 5.1 The cardinal case $M := [0, 1] \subset \mathbb{R}$ has already been studied extensively in many different contexts. So μ has many names in this case, (Choquet) capacity, non-additive or monotone measure, fuzzy measure, just to mention the most important ones. □

The **inner extension** of a measure μ is

$$\mu_*(A) := \bigvee_{\substack{B\in\mathcal{S}\\ B\subset A}} \mu(B) = \mu *_{\mathcal{S}} \zeta(\,\cdot\,, A)\,, \quad A \in 2^\Omega\,.$$

Here ζ denotes the zeta-function of the ordered set $(2^\Omega, \subset)$,

$$\zeta(B, A) := \begin{cases} \mathbb{I} & \text{if } A \supset B \\ \mathbb{O} & \text{else} \end{cases}\,, \quad A \in 2^\Omega\,.$$

[3] The outer extension is defined dually, $\mu^* := \mu *'_{\mathcal{S}} \zeta(A,\,\cdot\,)$. Since μ is increasing, so are μ_* and μ^*. For any increasing extension ν of μ to 2^Ω

$$\mu_* \leq \nu \leq \mu^*. \tag{6}$$

A measure $\mu : \mathcal{S} \to M$ is called a **lower chain measure**, if there is a chain w.r.t. set inclusion $\mathcal{K} \subset \mathcal{S}$ with $\emptyset, \Omega \in \mathcal{K}$ such that

$$\mu = (\mu|\mathcal{K})_*|\mathcal{S}\,.$$

We refer to $\mathcal{K}$ as a **(defining) chain** for μ. For a lower chain measure μ on $\mathcal{S}$,

$$\mu(\bigcap_{A\in\mathcal{A}} A) = \bigwedge_{A\in\mathcal{A}} \mu(A) \tag{7}$$

for all finite set systems $\mathcal{A} \subset \mathcal{S}$ such that $\bigcap_{A\in\mathcal{A}} A \in \mathcal{S}$. The straightforward proof can be found in [2] or use Proposition 4.1 (iii) together with the fact that ζ has property (7) for the second variable.

If property (7) holds for arbitrary $\mathcal{A} \subset \mathcal{S}$ and $\mathcal{S}$ is closed under arbitrary intersection, then μ is called **minitive** or a **necessity measure**. Dually, one defines **upper chain measures** and **maxitive** or **possibility measures**.

Example 5.2 Like in the cardinal theory we call $u_K := \zeta(K,\,\cdot\,)$ the **unanimity game** for 'coalition' $K \subset \Omega$. It is a lower chain measure with defining chain $\mathcal{K} := \{\emptyset, K, \Omega\}$ and it is minitive. But it is not maxitive, in general. Since $u_K(K^c) = \mathbb{O}$ we get

$$\overline{u}_K(A) := (u_K|\{\emptyset, K^c, \Omega\})^*(A) = \begin{cases} \mathbb{I} & \text{if } A \cap K \neq \emptyset \\ \mathbb{O} & \text{if } A \subset K^c \end{cases}\,, \quad A \in 2^\Omega\,.$$

and this is an upper chain measure. The unanimity game $u_{\{\omega\}}$ for the singleton $K = \{\omega\}$, often called Dirac measure at point ω, simultaneously is a lower and upper chain measure. □

Proposition 5.1 *Any minitive measure is a lower chain measure. Also, any maxitive measure is an upper chain measure.*

[3] In analogy with the cardianl theory one could regard μ, extended with value $\mathbb{O}$ from $\mathcal{S}$ to 2^Ω, as an ordinal Möbius transform of μ_*. It is not unique since also μ_* is a Möbius transform of $\mu_* = (\mu_*)_*$. For details see [11].

Proof Let $\mathcal{K}$ be the chain consisting of the sets $K_x := \bigcap\{B \mid B \in \mathcal{S},\, \mu(B) \geq x\}$, $x \in M$ and $\emptyset$, Ω. We have to prove $(\mu|\mathcal{K})_*|\mathcal{S} \geq \mu$ since the reversed inequality holds by (6). Let $A \in \mathcal{S}$ be arbitrary. With $x = \mu(A)$ we get $K_{\mu(A)} \subset A$ and by minitivity

$$\begin{aligned}(\mu|\mathcal{K})_*(A) &= \bigvee_{\substack{K\in\mathcal{S}\\K\subset A}} \mu(K)\\ &\geq \mu(K_{\mu(A)}) = \bigwedge_{\substack{B\in\mathcal{S}\\ \mu(B)\geq\mu(A)}} \mu(B)\\ &= \mu(A)\,.\end{aligned}$$

The proof for the maxitive measures runs similarly. □

The converse of Proposition 5.1 does not hold since, in contrast to a chain measure, a minitive (maxitive) measure has some continuity property: A minitive (maxitive) measure is continuous w.r.t. the lower (upper) topology of the complete lattices 2^Ω and M (see [15] III 1.2). But, of course, the class of minitive (maxitive) measures coincides with the class of lower (upper) chain measures if Ω is finite.

6 The quantile correspondence of a lattice-valued function

Here we do the first steps for aggregating lattice valued functions f on Ω w.r.t. a monotone lattice-valued measure in introducing the distribution function of f and the saturation of its inverse, the quantile correspondence. All this is done in close analogy to probability theory.

Let $\mu : 2^\Omega \to M$ be a measure, M and L linear lattices and $f : \Omega \to L$ a function. Like in probability theory the upper level sets $\{f \geq x\} := \{\omega \in \Omega \mid f(\omega) \geq x\}$ and $\{f > x\} := \{\omega \in \Omega \mid f(\omega) > x\}$ of f for level $x \in L$ will play an important role. Clearly the family of all upper level sets of f forms a chain. We denote it with $\mathcal{K}_f \subset 2^\Omega$. A family $F \subset L^\Omega$ of functions is called **comonotonic** if $\bigcup_{f\in F} \mathcal{K}_f$ forms again a chain. For other characterisations of comonotonicity see [3].

The **distribution function** $G_{\mu,f} : L \to M$ of f is defined as

$$G_{\mu,f}(x) := \mu(f \geq x)\,, \quad x \in L\,.$$

Obviously, $G_{\mu,f}$ is a decreasing function. Since, in general, $G_{\mu,f}$ is not surjective, the domain of $G^{-1}_{\mu,f}$ can be a proper subset of M. We extend it by means of the sharp saturation $\widehat{G^{-1}_{\mu,f}}$ defined in Section 4.[4] In analogy with probability theory,

[4] In many practical applications the scale L is finite. Then one can avoid the saturation in replacing $G_{\mu,f}$ by the following sharply decreasing interval-valued correspondence, which

we define for $p \in M$ the p**-quantile** of f w.r.t. μ as the interval

$$Q_{\mu,f}(p) := \widehat{G^{-1}_{\mu,f}}(p) \in \mathcal{I}_L .$$

The **quantile correspondence** $Q_{\mu,f}$ is sharply decreasing in the variable $p \in M$ (Proposition 4.3 (i)). If M has the additional structure of a reflection lattice with fixed point p_0 then $Q_{\mu,f}(p_0)$ is called the **median** of f w.r.t. μ (cf. Example 7.3 below).

Proposition 6.1 *Let $\mu : 2^\Omega \to M$ be a measure, M and L complete linear lattices and $f,\, g \in L^\Omega$ functions. Then*

$$\begin{aligned} G_{\mu,f\vee g} &\geq G_{\mu,f} \vee G_{\mu,g}\,, \quad G_{\mu,f\wedge g} \leq G_{\mu,f} \wedge G_{\mu,g} \\ Q_{\mu,f\vee g} &\sqsupseteq Q_{\mu,f} \sqcup Q_{\mu,g}\,, \quad Q_{\mu,f\wedge g} \sqsubseteq Q_{\mu,f} \sqcap Q_{\mu,g} \end{aligned}$$

and equality holds if $f,\, g$ are comonotonic. If μ is an upper (lower) chain measure then equality holds, too, in the formula for $f \vee g$ ($f \wedge g$, respectively).

Proof We first get $G_{\mu,f\vee g}(x) \geq G_{\mu,f}(x) \vee G_{\mu,g}(x)$ applying the monotone measure μ on the sets

$$\{f \vee g \geq x\} = \{f \geq x \text{ or } g \geq x\} = \{f \geq x\} \cup \{g \geq x\} \supset \{f \geq x\}\,,\ \{g \geq x\}\,.$$

If $f,\, g$ are comonotonic we have $\{f \geq x\} \supset \{g \geq x\}$ or the converse so that we get an equality. With an upper chain measure we can apply the dual of (7), so that we get equality, too. The assertion with $\wedge$ proves similarly.

The result for the distribution correspondences translates to the corresponding result for the quantile correspondences by means of Proposition 4.3 (ii): $G_{\mu,f\vee g} \geq G_{\mu,f}$ implies $Q_{\mu,f\vee g} = \widehat{G^{-1}_{\mu,f\vee g}} \sqsupseteq \widehat{G^{-1}_{\mu,f}} = Q_{\mu,f}$ and similarly with g in place of f on the right hand sides. Both relations together imply the result. □

The partial order on the set of distribution functions or on the set of quantile correspondences induces a partial order on L^Ω, often called stochastic dominance. In the next section we investigate extensions of this order to total orders.

7 Fan-Sugeno functionals

This is the main part of the article. The former results are applied to define the class of Fan-Sugeno functionals for lattice valued functions w.r.t. a lattice valued monotone measure and to derive the essential properties.

Here L and M are complete linear lattices. Let $\mu : 2^\Omega \to M$ be an M-valued measure on a set Ω and $\ell : M \to L$ an increasing function. ℓ relates the scale of the measure μ to the scale of the functions $f \in L^\Omega$, hence we call it the

is already surjective: $x \mapsto]\mu(f > x), \mu(f \geq x)]$ for $x \in L \setminus \{\mathbb{I}\}$ (here read $]a,a] = \{a\}$) and $\mathbb{I} \mapsto [\mathbb{O}, \mu(f \geq \mathbb{I})]$. Similarly, in the classical context with $L = \mathbb{R}$, $M = [0,1] \subset \mathbb{R}$ and μ continuous from above and below, one has to close the intervals above in order that the distribution correspondence becomes surjective.

commensurability function. The interval-valued **Fan-Sugeno functional** $S_{\mu,\ell} : L^\Omega \to \mathcal{I}_L$ is defined by means of the inner product $*$ of Section 4 as

$$S_{\mu,\ell}(f) := \ell * Q_{\mu,f} \,.$$

Always $S_{\mu,\ell}(f)$ is a nonempty interval. Often a single value is preferred, then the least upper bound is the right one. We define

$$\overline{S}_{\mu,\ell}(f) := \bigvee_{x \in S_{\mu,\ell}(f)} x \,.$$

If $L = M$ and ℓ is the identity mapping id_M on M, i.e. $\ell(p) = p$, $p \in M$, then we write S_μ or $\overline{S}_\mu$ for short. The name, we attribute to these functionals, deduces from the following special cases.

Example 7.1 Let $M = [0,1] \subset \mathbb{R}$, $L = [0,\infty]$, $\ell(x) = x$ for $x \in M$ and μ a probability measure on a σ-algebra. Since we have defined the Fan-Sugeno functional only for ordinal measures on the total power set 2^Ω we first extend μ, say to the inner extension μ_*. Now, for two real random variables f, g on Ω, the number $\overline{S}_{\mu_*,\ell}(|f-g|)$ is the distance $\|f-g\|_0$ of f and g in the Ky Fan metric of the space $L_0(\mu)$ of measurable functions ([8], see also theorems 9.2.2 and 9.2.3 in [7]). Convergence in probability is convergence in this metric. □

Example 7.2 Let $L = M = [0,1]$, μ a fuzzy measure on 2^Ω (Example 5.1) and $f : \Omega \to [0,1]$. Then

$$S_\mu(f) = \bigsqcup_{x \in [0,1]} \{x\} \sqcap Q_{\mu,f}(x) \,, \quad \overline{S}_\mu(f) = \bigvee_{x \in [0,1]} x \wedge G_{\mu,f}(x) \,.$$

The functional $\overline{S}_\mu(f)$ is the Sugeno integral of f w.r.t. μ ([23]). □

Another special case of our general functional is even better and longer known than the examples above.

Example 7.3 Let $p \in M$ and $\ell := \epsilon_p$ (see Example 4.1), then $S_{\mu,\epsilon_p}(f) = \epsilon_p * Q_{\mu,f} = Q_{\mu,f}(p)$, the p-quantile of f. The classical case is $M = [0,1] \subset \mathbb{R}$, $L = \mathbb{R}$, where $S_{\mu,\epsilon_p}(f)$ is the $(1-p)$-quantile of f in the usual terminology. The difference to our present terminology results from the fact that we are employing the decreasing distribution function whereas classically one employs the increasing one. Of special importance is the case $p = \frac{1}{2}$, then $S_{\mu,\epsilon_p}(f)$ is the median of f, which, in applications of probability theory, is the second important location parameter after the expected value. □

Proposition 7.1 *Let L, M be complete linear lattices. Let $\lambda, \mu : 2^\Omega \to M$ be M-valued measures on a set Ω and $k, \ell : M \to L$ commensurability functions. The Fan-Sugeno functional has the following properties where $f, g \in L^\Omega$, $a \in L$:*

(i) $\ell(\mu(A)) = \overline{S}_{\mu,\ell}(\mathbb{I}_A)$ for $A \subset \Omega$, especially μ can be reconstructed from $\overline{S}_\mu$;

(ii) $f \leq g$ *implies* $S_{\mu,\ell}(f) \sqsubseteq S_{\mu,\ell}(g)$;

(iii) If $\ell(\mathbb{O}) = \mathbb{O}$*, then* $S_{\mu,\ell}(a \wedge f) = \{a\} \sqcap S_{\mu,\ell}(f)$;

(iv) $S_{\mu,\ell}(f \vee g) \sqsupseteq S_{\mu,\ell}(f) \sqcup S_{\mu,\ell}(g)$
and equality holds if μ *is an upper chain measure;*

(v) Comonotonic maxitivity: if f, g *are comonotonic, then*
$S_{\mu,\ell}(f \vee g) = S_{\mu,\ell}(f) \sqcup S_{\mu,\ell}(g)$ *;*

(vi) $\lambda \leq \mu$ *and* $k \leq \ell$ *imply* $S_{\lambda,k}(f) \sqsubseteq S_{\mu,\ell}(f)$ *.*

Proof (i) Since $Q_{\mu,\mathbb{I}_A}(\mu(A)) =]\mathbb{O}, \mathbb{I}]$ and $Q_{\mu,\mathbb{I}_A}(p) = \mathbb{O}$ for $p > \mu(A)$ and $\mathbb{I}$ for $p < \mu(A)$ we see that $\ell(\mu(A))$ is the top of $\ell * Q_{\mu,\mathbb{I}_A} = \bigsqcup_{p \in M} \ell(p) \sqcap Q_{\mu,\mathbb{I}_A}(p)$.

(ii) From $f \leq g$ one easily derives $G_{\mu,f} \leq G_{\mu,g}$. Then by Proposition 4.3 (ii) $Q_{\mu,f} \sqsubseteq Q_{\mu,g}$ and by Proposition 4.1 (iii) the result follows.

(iii) We know $Q_{\mu,a \wedge f} = Q_{\mu,a} \sqcap Q_{\mu,f}$ from Proposition 6.1 since a and f are comonotonic. We have $Q_{\mu,a}(p) = a$, except for $p = \mathbb{O}$ and $p = \mathbb{I}$, where $Q_{\mu,a}(\mathbb{O}) =]a, \mathbb{I}]$ and $Q_{\mu,a}(\mathbb{I}) = [\mathbb{O}, a]$. Now observe that $\ell(\mathbb{O}) \sqcap Q_{\mu,a}(\mathbb{O}) = \mathbb{O} = \ell(\mathbb{O}) \sqcap \{a\}$ since $\ell(\mathbb{O}) = \mathbb{O}$, and $Q_{\mu,f}(\mathbb{I}) \sqcap Q_{\mu,a}(\mathbb{I}) = Q_{\mu,f}(\mathbb{I}) \sqcap \{a\}$ since $Q_{\mu,f}(\mathbb{I})$ is an interval with bottom $\mathbb{O}$. This proves that $(Q_{\mu,a} \sqcap Q_{\mu,f}) * \ell = (a \sqcap Q_{\mu,f}) * \ell$. We obtain the desired result by applying Proposition 4.1 (v).

(iv) We know $Q_{\mu,f \vee g} \sqsupseteq Q_{\mu,f} \sqcup Q_{\mu,g}$ (Proposition 6.1) and this relation is maintained if we $*$-multiply with ℓ (Proposition 4.1 (iii)). Finally, the result follows by distributivity of $*$ with $\sqcup$ (Proposition 4.1 (iv)).

(v) The proof runs like in (iv).

(vi) From $\lambda \leq \mu$ one easily derives $G_{\lambda,f} \leq G_{\mu,f}$. Then by Proposition 4.3 (ii) $Q_{\lambda,f} \sqsubseteq Q_{\mu,f}$ and, applying Proposition 4.1 (iii) twice, the result follows. □

We mention some further properties. The transformation rule proves like for the Choquet integral (see [3]) and the Fan-Sugeno operators are compatible with increasing transformations of L and M.

For better understanding some of these properties we again employ the analogy of $\vee$ or $\sqcup$ with the sum of real numbers and of $\wedge$ or $\sqcap$ with the product. Properties (iii) and (iv) tell us that the Fan-Sugeno functional is a 'linear' operator for upper chain measures. So, in the ordinal context, the upper chain measures play the role of probability measures.

The dual of properties (iv) and (v) for $f \wedge g$ cannot be proved here since $*$ is distributive with $\sqcup$, but not with $\sqcap$. These properties for $f \wedge g$ can be obtained with the dual functional, defined with the dual product $*'$ (see proposition 4.5). In case of the Sugeno integral (Example 7.2) $\overline{S}_\mu$ coincides with its dual (see e.g. [20]). We illustrate this duality by an example.

Example 7.4 Let $L = M = [0, 1]$, ℓ the identity mapping and μ, f so that

$$G_{\mu,f}(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq .2 \\ .5 & \text{for } .2 < x \leq .6 \\ 0 & \text{for } .6 < x \leq 1 \end{cases}$$

Then $S_\mu(f) =].2, .5]$, whereas the dual functional had the value $[.5, .6]$. □

For $\{0,1\}$-valued measures our functional coincides with the Choquet integral. We demonstrate this fact for unanimity games and their conjugate in the next example.

Example 7.5 Let $M = \{\mathbb{O}, \mathbb{I}\}$, $\ell(\mathbb{O}) = \mathbb{O}$, $\ell(\mathbb{I}) = \mathbb{I}$ then $S_{\mu,\ell}(f) = (\mathbb{O} \sqcap Q_{\mu,f}(\mathbb{O})) \sqcup (\mathbb{I} \sqcap Q_{\mu,f}(\mathbb{I})) = Q_{\mu,f}(\mathbb{I})$. Especially for $\mu = u_K$ (Example 5.2) we get $\overline{S}_{u_K,\ell}(f) = \bigvee\{x \mid \{f \geq x\} \supset K\} = \bigvee\{x \mid \bigwedge_{\omega\in K} f(\omega) \geq x\} = \bigwedge_{\omega \in K} f(\omega)$. Similarly $\overline{S}_{\overline{u}_K,\ell}(f) = \bigvee_{\omega\in K} f(\omega)$. □

8 Fan-Sugeno functionals for R-valued functions

Now we consider functions, for which the range is not only a linear lattice but has the structure of a linear reflection lattice as introduced in Section 2. Like for the Choquet integral there are two ways to extend the functional of the last section to functions, taking positive and negative values. One way results in a functional that is symmetric w.r.t. the reflection at $\mathbb{O}$, the other one will be an asymmetric functional. Both functionals can be applied at least to situations where the respective Choquet integrals can be applied.

Let R be a linear reflection lattice with positive part $L = L_+$. For a function $f : \Omega \to R$ we define the positive part f^+ and the negative part f^- by

$$f^+(\omega) := f(\omega) \vee \mathbb{O}\,, \quad f^- := (-f)^+$$

and one easily sees that f is the difference of f^+ and f^-,

$$f = f^+ \bigtriangledown (-f^-)\,.$$

By means of this representation we define the symmetric extension of the Fan-Sugeno functional $S_{\mu,\ell} : L^\Omega \to \mathcal{I}_{L_+}$ with a commensurability function $\ell : M \to L_+$ to the **symmetric Fan-Sugeno functional** $SS_{\mu,\ell} : R^\Omega \to \mathcal{R}$ as the pseudo-difference

$$SS_{\mu,\ell}(f) := S_{\mu,\ell}(f^+) \bigtriangledown (-S_{\mu,\ell}(f^-))\,. \tag{8}$$

Here the pseudo-addition $\bigtriangledown$ operates on intervals, not points of R, which remains to be defined. Clearly the disjoint union $\mathcal{R}$ of $\mathcal{I}_{L_-}$ and $\mathcal{I}_{L_+}$ with $\{-\mathbb{O}\} \in \mathcal{I}_{L_-}$ identified with $\{\mathbb{O}\} \in \mathcal{I}_{L_+}$ forms a non-linear reflection lattice so that the pseudo-addition $\bigtriangledown$ as defined in Section 2 applies to $\mathcal{R}$. On $\mathcal{I}_{L_+}$ the operation $\bigtriangledown$ coincides with $\sqcup$. A noticeable property of $SS_{\mu,\ell}(f)$ is that its value is $\mathbb{O}$ as soon as $S_{\mu,\ell}(f^+)$ and $S_{\mu,\ell}(f^-)$ are incomparable intervals in $\mathcal{I}_{L_+}$ (see footnote 1).

Using $(-f)^+ = f^-$, $(-f)^- = f^+$ and $a \bigtriangledown (-b) = -(b \bigtriangledown (-a))$ (Proposition 2.1 (iii)) one easily checks the symmetry property

$$SS_{\mu,\ell}(-f) = -SS_{\mu,\ell}(f)\,.$$

It is easy to check that defining $\overline{SS}_{\mu,\ell}$ by replacing in (8) $S_{\mu,\ell}$ by $\overline{S}_{\mu,\ell}$, then the symmetry property still holds. Also properties (i) and (ii) of Proposition 7.1 still hold for $SS_{\mu,\ell}$ and $\overline{SS}_{\mu,\ell}$ (see Proposition 2.1 (vi)).

Human behaviour with respect to gains and losses seem not to be symmetric (see e.g. [18]). So it might be useful to apply different commensurability functions for the positive and negative parts. Let $k, \ell : M \to L_+$ be increasing functions. One may generalise (8) to $SS_{\mu,k,\ell}(f) := S_{\mu,\ell}(f^+) \bigtriangledown (-S_{\mu,k}(f^-))$.

Now we need another pair of increasing commensurability functions, $\ell_+ : M \to L_+$ and $\ell_- : M \to L_-$ for defining the **asymmetric Fan-Sugeno functional** $AS_{\mu,\ell_-,\ell_+} : R^\Omega \to \mathcal{R}$,

$$AS_{\mu,\ell_-,\ell_+}(f) := S_{\mu,\ell_-}(f) \bigtriangledown S_{\mu,\ell_+}(f)\,.$$

Like before we have to check if the operation on the right hand side is well defined. This is the case since ℓ_- has values in L_- so that $S_{\mu,\ell_-}(f) \in \mathcal{I}_{L_-}$ and similar for the other term. Comparing with (8), here we have the same function in the two expressions on the right hand side but different commensurability functions. Also $-\ell_-$ and ℓ_+ cannot be compared without an additional structure of their common domain L_+ (to be a reflection lattice, for example) since $-\ell_-$ is decreasing and ℓ_+ increasing.

The asymmetric Fan-Sugeno functional is asymmetric in the following sense:

$$AS_{\mu,\ell_-,\ell_+}(-f) = -AS^*_{\mu,-\ell_+,-\ell_-}(f).$$

Here the upper $*$ denotes the **conjugate**[5] **Fan-Sugeno functional**, which is performed with the increasing distribution function and the decreasing commensurability correspondences $-\ell_+$, $-\ell_-$. We leave the details for further research. For the moment we can say that properties (i), (ii) and (vi) of Proposition 7.1 still hold for $AS_{\overline{\mu},\ell_+,\ell_-}$.

9 Ordinal metrics and norms

In the spirit of the classical Example 7.1 we define the **ordinal** (μ, ℓ)**-distance** of R-valued functions f, g. Since $\mathrm{dist}(f(\omega), g(\omega)) = |f(\omega) \bigtriangledown (-g(\omega))|$ is an L-valued function on Ω, L being the positive part L_+ of R, we can define

$$\mathrm{dist}_{\mu,\ell}(f, g) := \overline{S}_{\mu,\ell}(|f \bigtriangledown (-g)|)\,, \quad f, g \in R^\Omega\,.$$

Like at the end of Section 2 the usual properties of a distance hold with a restriction for the triangle inequality,

$$\begin{aligned}
\mathrm{dist}_{\mu,\ell}(f,g) &\geq \mathbb{O} \quad \text{and} \quad \mathrm{dist}_{\mu,\ell}(f,g) = \mathbb{O} \ \text{ if } \ f = g\,,\\
\mathrm{dist}_{\mu,\ell}(f,g) &= \mathrm{dist}_{\mu,\ell}(g,f)\,,\\
\mathrm{dist}_{\mu,\ell}(f,h) &\leq \mathrm{dist}_{\mu,\ell}(f,g) \vee \mathrm{dist}_{\mu,\ell}(g,h) \quad \text{if } \mu \text{ is an upper chain measure.}
\end{aligned}$$

[5] If the scale M of μ had the additional structure of an order reversing bijection existing on M, then the **conjugate** $\overline{\mu}$ could be defined like for $\mathbb{R}$-valued monotone measures and we would get $AS^*_{\mu,-\ell_+,-\ell_-} = AS_{\overline{\mu},\ell_+,\ell_-}$ (see [5]).

Let us prove the triangle inequality. Using the triangle inequality of dist, we have for all $\omega \in \Omega$:

$$|f(\omega) \bigtriangledown (-h(\omega))| \leq |f(\omega) \bigtriangledown (-g(\omega))| \vee |g(\omega) \bigtriangledown (-h(\omega))|$$

which entails for $x \in L$

$$\{|f \bigtriangledown (-h)| \geq x\} \subset \{|f \bigtriangledown (-g)| \geq x\} \cup \{|g \bigtriangledown (-h)| \geq x\}.$$

By monotonicity of μ and the dual of (7) we get:

$$G_{\mu,|f \bigtriangledown (-h)|} \leq G_{\mu,|f \bigtriangledown (-g)|} \vee G_{\mu,|g \bigtriangledown (-h)|}.$$

Then by Proposition 4.3 (ii) this relation translates to the corresponding quantile correspondences and is maintained if we $*$-multiply with ℓ (Proposition 4.1 (iii)). Finally, the result follows by distributivity of $*$ with $\sqcup$ (Proposition 4.1 (iv)).

We define the **ordinal** (μ, ℓ)**-norm**

$$\|f\|_{\mu,\ell} := \mathrm{dist}_{\mu,\ell}(f, \mathbb{O})\,.$$

It has (with $\wedge$ interpreted as multiplication) the homogeneity property of usual norms,

$$\|a \wedge f\|_{\mu,\ell} = |a| \wedge \|f\|_{\mu,\ell} \quad \text{for } a \in R\,,$$

which proves with Proposition 7.1 (iii). One also derives the triangle inequality

$$\|f \bigtriangledown g\|_{\mu,\ell} \leq \|f\|_{\mu,\ell} \vee \|g\|_{\mu,\ell} \quad \text{if } \mu \text{ is an upper chain measure.}$$

Important special cases with $L = M$ and $\ell = \mathrm{id}$ are the ordinal Ky-Fan norm w.r.t. μ

$$\|f\|_{\mathbb{O}} := \|f\|_{\mu,\mathrm{id}}$$

and the μ-essential supremum (cf. [3] Chapter 9)

$$\|f\|_\infty := \|f\|_{\mathrm{sign}(\mu),\mathrm{id}}\,.$$

f is called a μ-**nullfunction** if $\|f\|_\infty = \mathbb{O}$.

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