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Abstract

We consider a deconvolution model for 3D periodic flows. We show the existence of a global attractor for the model.

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1 Introduction

This note is concerned by the deconvolution model of order N introduced in [6] (model (2.7) below) for 3D periodic flows. This model takes inspiration in the class of the so called α-models (see in [2] and [4] and references inside) and also in the class of ADM models (see in [7]). We are interested by the question of the existence of a global attractor for this model.

The question of attractors has already been considered for the alpha model (see [1]), corresponding to the case $N = 0$. We prove in this work the existence of an attractor for each N (see Theorem 3.1).

In order to make the paper self contained, we describe carrefully how is constructed the deconvolution model. Next, we recall basic notions on the attractors, notions that can be founded in the book of R. Temam (see [8]). Finally we prove the existence of the attractor. The question of its dimension is under progress.

2 The Deconvolution model

2.1 Function Spaces

for $s \in \mathbb{R}$, let us define the space function

$$\mathbf{H}_s = \left\{ \mathbf{w} = \sum_{\mathbf{k}} \widehat{\mathbf{w}} e^{i\mathbf{k}\cdot\mathbf{x}},\ \nabla\cdot\mathbf{w} = 0,\ \widehat{\mathbf{w}}(\mathbf{0}) = \mathbf{0},\quad \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}(\mathbf{k},t)|^2 < \infty \right\}. \tag{2.1}$$

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We define the $\mathbf{H}_s$ norms by

$$||\mathbf{w}||_s^2 = \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}(\mathbf{k},t)|^2, \tag{2.2}$$

where of course $||\mathbf{w}||_0^2 = ||\mathbf{w}||^2$. It can be shown that when s is an integer, $||\mathbf{w}||_s^2 = ||\nabla^s \mathbf{w}||^2$ (see [3]).

We denote by P_L The Helmholtz-Leray orthogonal projection of $(L^2)^3$ onto $\mathbf{H}_0$ and by A the Stokes operator defined by $A = -P_L\triangle$ on $D(A) = \mathbf{H}_0 \cap (H^2)^3$. We note that in the space-periodic case $A\mathbf{w} = -\triangle\mathbf{w}$ for all $\mathbf{w} \in D(A)$.
The operator A^{-1} is a sef-adjoint positive definite compact operator from $\mathbf{H}_s$ onto $\mathbf{H}_s$, for $s = 1$ and $s = 2$ (see [5]). We denote λ_1 the smallest eigenvalue of A.
We introduce the trilinear form b defined by

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{\mathbf{i},\mathbf{j}} \int_\Omega u_i \partial_i v_j w_j dx. \tag{2.3}$$

wherever the integrals make sense. Note that $b(\mathbf{u},\mathbf{w},\mathbf{w}) = 0$ when $\nabla \cdot \mathbf{u} = 0$.

2.2 The Filter and the deconvolution process

Let $\mathbf{w} \in \mathbf{H}_0$ and $\overline{\mathbf{w}} \in \mathbf{H}_1$ be the unique solution to the following Stokes problem with periodic boundary conditions:

$$-\delta^2 \triangle \overline{\mathbf{w}} + \overline{\mathbf{w}} + \nabla r = \mathbf{w} \quad \text{in } \mathbb{R}^3, \quad \nabla \cdot \overline{\mathbf{w}} = 0, \quad \int_\Omega \overline{\mathbf{w}} = \mathbf{0}. \tag{2.4}$$

We denote the filtering operation by G so that $\overline{\mathbf{w}} = G\mathbf{w}$. Writing $\mathbf{w}(\mathbf{x},t) = \sum_{\mathbf{k}} \widehat{\mathbf{w}}(\mathbf{k},t) e^{-i\mathbf{k}\cdot\mathbf{x}}$, it is easily seen that $\nabla r = 0$ and $\overline{\mathbf{w}}(\mathbf{x},t) = \sum_{\mathbf{k}} \dfrac{\widehat{\mathbf{w}}(\mathbf{k},t)}{1+\delta^2|\mathbf{k}|^2} e^{-i\mathbf{k}\cdot\mathbf{x}}$.
Then writing $\overline{\mathbf{w}} = G(\mathbf{w})$, we see that in the corresponding spaces of the type $\mathbf{H}_s$, the transfer function of G, denoted by $\widehat{G}$, is the function $\widehat{G}(\mathbf{k}) = \frac{1}{1+\delta^2|\mathbf{k}|^2}$, and we also can write on the $\mathbf{H}_s$ type spaces

$$-\delta^2 \triangle \overline{\mathbf{w}} + \overline{\mathbf{w}} = \mathbf{w} \quad \text{in } \mathbb{R}^3, \quad \nabla \cdot \overline{\mathbf{w}} = 0, \quad \int_\Omega \overline{\mathbf{w}} = \mathbf{0}. \tag{2.5}$$

The procedure of deconvolution by the Van Citter approximation is described in [6]. This yields the operator $D_N\mathbf{w} = \sum_{n=0}^{N} (I-G)^n \mathbf{w}$.

Definition 2.1 *The truncation operator $H_N : \mathbf{H}_s \to \mathbf{H}_s$ is defined by $H_N\mathbf{w} := D_N\overline{\mathbf{w}} = (D_N \circ G)\mathbf{w}$.* ■

Note that, for any $s \geqslant 0$ we have the following proprieties (see [6]) :

$$\|H_N\mathbf{w}\|_s \leqslant \|\mathbf{w}\|_s, \quad \|H_N\mathbf{w}\|_{s+2} \leqslant C(\delta,N)\|\mathbf{w}\|_s. \tag{2.6}$$

2.3 The model

Let $\mathbf{u}_0 \in \mathbf{H}_0$, $f \in \mathbf{H}_{-1}$. For $\delta > 0$, let the averaging be defined by (2.4). The problem we consider is the following: for a fixed $T > 0$, find $(\mathbf{w}, q)$

$$(2.7) \quad \begin{cases} \mathbf{w} \in L^2([0,T], \mathbf{H}_1) \cap L^\infty([0,T], \mathbf{H}_0), \quad \partial_t \mathbf{w} \in L^2([0,T], \mathbf{H}_{-1}) \\ q \in L^2([0,T], L^2_{\text{per},0}), \\ \partial_t \mathbf{w} + (H_N(\mathbf{w}) \cdot \nabla)\, \mathbf{w} - \nu \triangle \mathbf{w} + \nabla q = H_N(\mathbf{f}) \quad \text{ in } \mathcal{D}'([0,T] \times \mathbb{R}^3), \\ \mathbf{w}(\mathbf{x}, 0) = H_N(\mathbf{u}_0) = \mathbf{w}_0. \end{cases}$$

where $L^2_{\text{per},0}$ denotes the scalar fields in $L^2_{loc}(\mathbb{R}^3)$, 2π-periodic with zero mean value. We prove in [6] the following result.

Theorem 2.1 *Problem* (2.7) *admits a unique solution* $(\mathbf{w}, q)$, $\mathbf{w} \in L^\infty([0,T], \mathbf{H}_1) \cap L^2([0,T], \mathbf{H}_2)$, *and the following energy equality holds:*

$$(2.8) \quad \frac{1}{2} \|\mathbf{w}(t)\|^2 + \nu \int_0^t \int_\Omega |\nabla \mathbf{w}|^2 d\mathbf{x} dt' = \frac{1}{2} \|H_N(\mathbf{u}_0)\|^2 + \int_0^t \int_\Omega H_N(\mathbf{f}).\mathbf{w}\, d\mathbf{x} dt'. \quad \blacksquare$$

3 Main result

3.1 Recall of basic notions about attractors

We denote by $\mathbf{w}(t, \cdot) = S(t)(\mathbf{w}_0)$ the (unique) solution of system (2.7) at time t. We recall the definitions of a global attractor and an absorbing set (see in [8]).

Definition 3.1 *We say that* $\mathcal{A} \subset \mathbf{H}_0$ *is a global attractor for the dynamical system* (2.7) *if and only if*

(P1) $\mathcal{A}$ *is compact in the space* $\mathbf{H}_0$,

(P2) $\forall\, t \in \mathbb{R}$, $S(t)(\mathcal{A}) \subset \mathcal{A}$,

(P3) For every bounded subset $B \subset \mathbf{H}_0$, $\rho(S(t)(B), \mathcal{A})$ *goes to zero when* t *goes to infinity, where* $\rho(S(t)(B), \mathcal{A}) = \sup_{v \in B} \inf_{u \in \mathcal{A}} ||u - v||$. $\blacksquare$

Definition 3.2 *1. A set* $A \subset \mathbf{H}_0$ *is an absorbing set if and only if for every bounded subset* $B \subset \mathbf{H}_0$ *there exists* $t_1 > 0$ *such that for all* $t \geq t_1$ *one has* $S(t)(B) \subset A$. *2. We say that the semi group* $S(t)$ *is uniformly compact if and only if for every bounded subset* $B \subset \mathbf{H}_0$ *there exists* $t_2 = t_2(B)$ *such that* $\overline{\bigcup_{t \geq t_2} S(t)(B)}$ *is compact.*

3. We denote by $\omega(A)$ *the set* $\omega(A) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)(A)}$. $\blacksquare$

Proposition 3.1 *Assume that there exists an absorbing bounded set* A *and that the semi group* $S(t)$ *is uniformly compact, then* $\mathcal{A} = \omega(A)$ *is the global attractor for the dynamical system defined by* $S(t)$.

see the proof in [8]).

3.2 Existence of a global attractor

We are now in order to state and prove the main result of this note.

Theorem 3.1 *The system* (2.7) *has a global attractor.* ■

Proof. Thanks to Proposition 3.1, it remains to prove that system (2.7) has an absorbing set and that $S(t)$ is uniformly compact, in the sense of definition 3.2. Both things are derived from basic estimates that we detail in the following.

Absorbing set in $\mathbf{H}_0$: We take the inner product of the first quation of system (2.7) with $\mathbf{w}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{w}\right\|^2 + b(H_N(\mathbf{w}),\mathbf{w},\mathbf{w}) + \nu||\mathbf{w}||_1^2 = (H_N(\mathbf{f}),\mathbf{w}). \tag{3.1}$$

Observing that $b(H_N(\mathbf{w}),\mathbf{w},\mathbf{w}) = 0$ due to $\nabla\cdot H_N(\mathbf{w}) = 0$, applying Young inequality, Poincare inequality $||\mathbf{w}|| \leqslant \lambda_1^{-\frac{1}{2}}||\mathbf{w}||_1$ and using (2.6) there remains

$$\frac{d}{dt}\left\|\mathbf{w}\right\|^2 + \nu\lambda_1||\mathbf{w}||^2 \leqslant \frac{1}{\nu\lambda_1}\left\|\mathbf{f}\right\|^2. \tag{3.2}$$

So, noting $\rho_0 = \frac{1}{\nu\lambda_1}\left\|\mathbf{f}\right\|$ and applying Gronwall lema we obtain

$$\left\|\mathbf{w}\right\|^2 \leqslant \left\|\mathbf{w}_0\right\|^2 e^{-\nu\lambda_1 t} + \rho_0^2(1-e^{-\nu\lambda_1 t}). \tag{3.3}$$

Considering $\mathbf{w}_0$ included in a ball $B(0,R)$ and choosing $\rho_0' > \rho_0$, the previous inequality implies that, for $t > T_0$,

$$\left\|\mathbf{w}(t)\right\|^2 < {\rho_0'}^2, \quad with \quad T_0 = \frac{1}{\nu\lambda_1} ln \frac{R^2}{{\rho_0'}^2 - {\rho_0}^2}. \tag{3.4}$$

Since each bounded set of $\mathbf{H}_0$ is included in a ball $B(0,R)$, one deduces that $B(0,\rho_0')$ is an absorbing set in $\mathbf{H}_0$.

More, as an alternative of (3.2) we may obtain

$$\frac{d}{dt}\left\|\mathbf{w}\right\|^2 + \nu||\mathbf{w}||_1^2 \leqslant \frac{1}{\nu\lambda_1}\left\|\mathbf{f}\right\|^2. \tag{3.5}$$

Integrating between t and $t+r$, we observe than, for $\mathbf{u}_0 \in B(0,R)$, $\rho_0' > \rho_0$ and $t > T_0$ $\left(with \quad T_0 = \frac{1}{\nu\lambda_1} ln \frac{R^2}{{\rho_0'}^2-{\rho_0}^2}\right)$:

$$\int_t^{t+r} \left\|\mathbf{w}(s)\right\|_1^2 ds \leqslant \frac{r}{\nu^2\lambda_1}\left\|\mathbf{f}\right\|^2 + \frac{{\rho_0'}^2}{\nu}. \tag{3.6}$$

Absorbing set in $\mathbf{H}_1$: We take now the inner product of the first equation of system (2.7) with $A\mathbf{w}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{w}\right\|_1^2 + b(H_N(\mathbf{w}),\mathbf{w},A\mathbf{w}) + \nu||A\mathbf{w}||^2 = (H_N(\mathbf{f}),A\mathbf{w}), \tag{3.7}$$

leading to

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{w}\right\|_1^2 + +\nu||A\mathbf{w}||^2 \leqslant \frac{1}{\nu}\left\|H_N(\mathbf{f})\right\|^2 + \frac{\nu}{4}\left\|A\mathbf{w}\right\|^2 + |b(H_N(\mathbf{w}),\mathbf{w},A\mathbf{w})|, \tag{3.8}$$

The trilinear form b satisfies the folowing inequality (see in [6]) :

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leqslant c'\left\|\mathbf{u}\right\|^{1/4}\left\|\mathbf{u}\right\|_1^{3/4}\left\|\mathbf{v}\right\|_1^{1/4}\left\|A\mathbf{v}\right\|^{3/4}\left\|\mathbf{w}\right\|. \tag{3.9}$$

Therefore, one has

$$|b(H_N(\mathbf{w}),\mathbf{w},A\mathbf{w})| \leqslant c'\left\|H_N(\mathbf{w})\right\|^{1/4}\left\|H_N(\mathbf{w})\right\|_1^{3/4}\left\|\mathbf{w}\right\|_1^{1/4}\left\|A\mathbf{w}\right\|^{7/4}. \tag{3.10}$$

Using (2.6) we have $\left\|H_N(\mathbf{w})\right\|_1 \leqslant \left\|H_N(\mathbf{w})\right\|_2 \leqslant C(\delta,N)\left\|\mathbf{w}\right\|$ and using (2.6) :

$$|b(H_N(\mathbf{w}),\mathbf{w},A\mathbf{w})| \leqslant C'(\delta,N)\left\|\mathbf{w}\right\|\left\|\mathbf{w}\right\|_1^{1/4}\left\|A\mathbf{w}\right\|^{7/4}. \tag{3.11}$$

By Young inequality we obtain

$$|b(H_N(\mathbf{w}),\mathbf{w},A\mathbf{w})| \leqslant \frac{\nu}{4}\left\|A\mathbf{w}\right\|^2 + \frac{C_1(\delta,N)}{2}\left\|\mathbf{w}\right\|^8\left\|\mathbf{w}\right\|_1^2, \tag{3.12}$$

thus

$$\frac{d}{dt}\left\|\mathbf{w}\right\|_1^2 + \nu||A\mathbf{w}||^2 \leqslant \frac{2}{\nu}\left\|H_N(\mathbf{f})\right\|^2 + C_1(\delta,N)\left\|\mathbf{w}\right\|^8\left\|\mathbf{w}\right\|_1^2 \tag{3.13}$$

We now use a Gronwall type proposition (see the proof in [8]):

Proposition 3.2 *Assume that y, g and h are positive, localy integrable functions on $]t_0,+\infty[$, and that for $t \geqslant t_0$,*

$$\frac{dy}{dt} \leqslant gy + h, \quad \int_t^{t+r} y(s)ds \leqslant k_1, \quad \int_t^{t+r} g(s)ds, \leqslant k_2, \quad \int_t^{t+r} h(s)ds \leqslant k_3,$$

where r, k_1, k_2, k_3 are four positive constants, then

$$y(t+r) \leqslant \left(\frac{k_1}{r} + k_3\right)e^{k_2}, \quad \forall t \geqslant t_0. \qquad \blacksquare$$

We can now finish the proof. Thanks to (3.4) and (3.6), using this lemma with $y = \left\|\mathbf{w}\right\|_1^2$, $\quad g = C_1(\delta,N)\left\|\mathbf{w}\right\|^8$ and $h = \frac{2}{\nu}\left\|H_N(\mathbf{f})\right\|^2$, we obtain,

$$\left\|\mathbf{w}(t)\right\|_1^2 \leqslant \left(\frac{k_1}{r} + k_3\right)e^{k_2}, \quad \forall t \geqslant T_0 + r, \tag{3.14}$$

with $\quad k_1 = \frac{r}{\nu^2\lambda_1}\left\|\mathbf{f}\right\|^2 + \frac{1}{\nu}{\rho_0'}^2, \quad k_2 = C_1(\delta,N){\rho_0'}^8, \quad k_3 = \frac{2r}{\nu}\left\|\mathbf{f}\right\|^2$.
Thus, after a time $T_1 = T_1\left(\left\|\mathbf{w}_0\right\|,\left\|\mathbf{f}\right\|,\nu\right)$, $\mathbf{w}$ is included in a ball or radius $R = R\left(\left\|\mathbf{f}\right\|,\nu,\delta,N\right)$. One deduces that there exists an absorbing set in $\mathbf{H}_1$.
Let B be a bounded set in $\mathbf{H}_1$. Estimate (3.14) implies that $\bigcup_{t\geq T_0+r} S(t)B$ is a bounded set in $\mathbf{H}_1$ wich is compactly imbeded in $\mathbf{H}_0$, so $S(t)$ is uniformly compact. Estimate (3.14) also implies the existence of an absorbing bonded set since k_1, k_2 and k_3 are independant of $\mathbf{w}_0$. Thanks to (3.1), this achieves the proof of the theorem. $\blacksquare$

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