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HILBERT DOMAINS QUASI-ISOMETRIC TO NORMED VECTOR SPACES

BRUNO COLBOIS AND PATRICK VEROVIC

ABSTRACT. We prove that a Hilbert domain which is quasi-isometric to a normed vector space is actually a convex polytope.

1. INTRODUCTION

A *Hilbert domain* in \mathbf{R}^m is a metric space $(\mathcal{C}, d_{\mathcal{C}})$, where \mathcal{C} is an *open bounded convex* set in \mathbf{R}^m and $d_{\mathcal{C}}$ is the distance function on \mathcal{C} — called the *Hilbert metric* — defined as follows.

Given two distinct points p and q in \mathcal{C} , let a and b be the intersection points of the straight line defined by p and q with $\partial\mathcal{C}$ so that $p = (1 - s)a + sb$ and $q = (1 - t)a + tb$ with $0 < s < t < 1$. Then

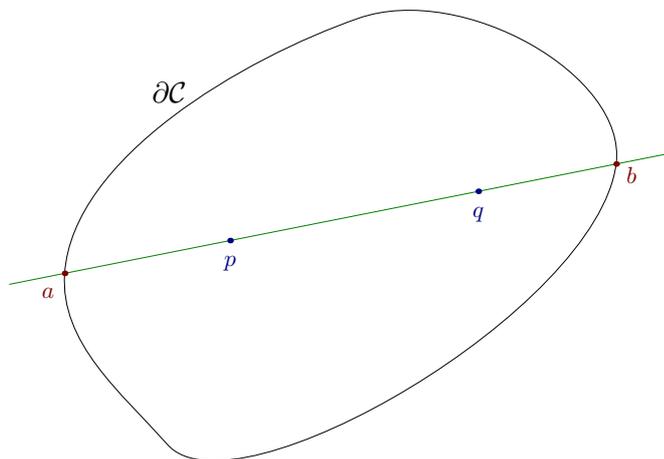
$$d_{\mathcal{C}}(p, q) = \frac{1}{2} \ln[a, p, q, b],$$

where

$$[a, p, q, b] = \frac{1 - s}{s} \times \frac{t}{1 - t} > 1$$

is the cross ratio of the 4-tuple of ordered collinear points (a, p, q, b) .

We complete the definition by setting $d_{\mathcal{C}}(p, p) = 0$.



The metric space $(\mathcal{C}, d_{\mathcal{C}})$ thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of \mathbf{R}^m and in which the affine open segments joining two points of the boundary $\partial\mathcal{C}$ are geodesics that are isometric to $(\mathbf{R}, |\cdot|)$.

For further information about Hilbert geometry, we refer to [4, 5, 9, 11] and the excellent introduction [15] by Socié-Méthou.

The two fundamental examples of Hilbert domains $(\mathcal{C}, d_{\mathcal{C}})$ in \mathbf{R}^m correspond to the case when \mathcal{C} is an ellipsoid, which gives the Klein model of m -dimensional hyperbolic geometry (see for

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example [15, first chapter]), and the case when the closure $\bar{\mathcal{C}}$ is a m -simplex for which there exists a norm $\|\cdot\|_{\mathcal{C}}$ on \mathbf{R}^m such that $(\mathcal{C}, d_{\mathcal{C}})$ is isometric to the normed vector space $(\mathbf{R}^m, \|\cdot\|_{\mathcal{C}})$ (see [8, pages 110–113] or [14, pages 22–23]).

Much has been done to study the similarities between Hilbert and hyperbolic geometries (see for example [7], [16] or [1]), but little literature deals with the question of knowing to what extent a Hilbert geometry is close to that of a normed vector space. So let us mention three results in this latter direction which are relevant for our present work.

Theorem 1.1 ([10], Theorem 2). *A Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in \mathbf{R}^m is isometric to a normed vector space if and only if \mathcal{C} is the interior of a m -simplex.*

Theorem 1.2 ([6], Theorem 3.1). *If \mathcal{C} is an open convex polygonal set in \mathbf{R}^2 , then $(\mathcal{C}, d_{\mathcal{C}})$ is Lipschitz equivalent to Euclidean plane.*

Theorem 1.3 ([2], Theorem 1.1. See also [17]). *If \mathcal{C} is an open set in \mathbf{R}^m whose closure $\bar{\mathcal{C}}$ is a convex polytope, then $(\mathcal{C}, d_{\mathcal{C}})$ is Lipschitz equivalent to Euclidean m -space.*

Recall that a convex *polytope* in \mathbf{R}^m (called a convex *polygon* when $m = 2$) is the convex hull of a finite set of points whose affine span is the whole space \mathbf{R}^m .

In light of these three results, it is natural to ask whether the converse of Theorem 1.3 — which generalizes Theorem 1.2 in higher dimensions — holds. In other words, if a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in \mathbf{R}^m is quasi-isometric to a normed vector space, what can be said about \mathcal{C} ? Here, by *quasi-isometric* we mean the following (see [3]):

Definition 1.1. Given real numbers $A \geq 1$ and $B \geq 0$, a metric space (S, d) is said to be (A, B) -quasi-isometric to a normed vector space $(V, \|\cdot\|)$ if and only if there exists a map $f : S \rightarrow V$ such that

$$\frac{1}{A}d(p, q) - B \leq \|f(p) - f(q)\| \leq Ad(p, q) + B$$

for all $p, q \in S$.

We can now state the result of this paper which asserts that the converse of Theorem 1.3 is actually true:

Theorem 1.4. *If a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in \mathbf{R}^m is (A, B) -quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geq 1$ and $B \geq 0$, then \mathcal{C} is the interior of a convex polytope.*

2. PROOF OF THEOREM 1.4

The proof of Theorem 1.4 is based on an idea developed by Förtsch and Karlsson in their paper [10].

It needs the following fact due to Karlsson and Noskov:

Theorem 2.1 ([12], Theorem 5.2). *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert domain in \mathbf{R}^m and $x, y \in \partial\mathcal{C}$ such that $[x, y] \not\subseteq \partial\mathcal{C}$. Then, given a point $p_0 \in \mathcal{C}$, there exists a constant $K(p_0, x, y) > 0$ such that for any sequences $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ in \mathcal{C} that converge respectively to x and y in \mathbf{R}^m one can find an integer $n_0 \in \mathbf{N}$ for which we have*

$$d_{\mathcal{C}}(x_n, y_n) \geq d_{\mathcal{C}}(x_n, p_0) + d_{\mathcal{C}}(y_n, p_0) - K(p_0, x, y)$$

for all $n \geq n_0$.

Now, here is the key result which gives the proof of Theorem 1.4:

Proposition 2.1. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert domain in \mathbf{R}^m which is (A, B) -quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geq 1$ and $B \geq 0$.*

Then, if $N = N(A, \|\cdot\|)$ denotes the maximum number of points in the ball $\{v \in V \mid \|v\| \leq 2A\}$ whose pairwise distances with respect to $\|\cdot\|$ are greater than or equal to $1/(2A)$, and if $X \subseteq \partial\mathcal{C}$ is such that $[x, y] \not\subseteq \partial\mathcal{C}$ for all $x, y \in X$ with $x \neq y$, we have

$$\text{card}(X) \leq N.$$

Proof.

Let $f : \mathcal{C} \rightarrow V$ such that

$$(2.1) \quad \frac{1}{A}d_{\mathcal{C}}(p, q) - B \leq \|f(p) - f(q)\| \leq Ad_{\mathcal{C}}(p, q) + B$$

for all $p, q \in \mathcal{C}$.

First of all, up to translations, we may assume that $0 \in \mathcal{C}$ and $f(0) = 0$.

Then suppose that there exists a subset X of the boundary $\partial\mathcal{C}$ such that $[x, y] \not\subseteq \partial\mathcal{C}$ for all $x, y \in X$ with $x \neq y$ and $\text{card}(X) \geq N + 1$. So, pick $N + 1$ distinct points x_1, \dots, x_{N+1} in X , and for each $k \in \{1, \dots, N + 1\}$, let $\gamma_k : [0, +\infty) \rightarrow \mathcal{C}$ be a geodesic of $(\mathcal{C}, d_{\mathcal{C}})$ that satisfies $\gamma_k(0) = 0$, $\lim_{t \rightarrow +\infty} \gamma_k(t) = x_k$ in \mathbf{R}^m and $d_{\mathcal{C}}(0, \gamma_k(t)) = t$ for all $t \geq 0$.

This implies that for all integers $n \geq 1$ and every $k \in \{1, \dots, N + 1\}$, we have

$$(2.2) \quad \left\| \frac{f(\gamma_k(n))}{n} \right\| \leq A + \frac{B}{n}$$

from the second inequality in Equation 2.1 with $p = \gamma_k(n)$ and $q = 0$.

On the other hand, Theorem 2.1 yields the existence of some integer $n_0 \geq 1$ such that

$$d_{\mathcal{C}}(\gamma_i(n), \gamma_j(n)) \geq 2n - K(0, x_i, x_j)$$

for all integers $n \geq n_0$ and every $i, j \in \{1, \dots, N + 1\}$ with $i \neq j$, and hence

$$(2.3) \quad \left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geq \frac{2}{A} - \frac{1}{n} \left(\frac{K(0, x_i, x_j)}{A} + B \right)$$

from the first inequality in Equation 2.1 with $p = \gamma_i(n)$ and $q = \gamma_j(n)$.

Now, fixing an integer $n \geq n_0 + AB + \max\{K(0, x_i, x_j) \mid i, j \in \{1, \dots, N + 1\}\}$, we get

$$\left\| \frac{f(\gamma_k(n))}{n} \right\| \leq 2A$$

for all $k \in \{1, \dots, N + 1\}$ by Equation 2.2 together with

$$\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geq \frac{1}{2A}$$

for all $i, j \in \{1, \dots, N + 1\}$ with $i \neq j$ by Equation 2.3.

But this contradicts the definition of $N = N(A, \|\cdot\|)$.

Therefore, Proposition 2.1 is proved. □

Remark. Given $v \in V$ such that $\|v\| = 2A$, we have $\|-v\| = 2A$ and $\|v - (-v)\| = 2\|v\| = 4A \geq 1/(2A)$, which shows that $N \geq 2$.

The second ingredient we will need for the proof of Theorem 1.4 is the following:

Proposition 2.2. *Let \mathcal{C} be an open bounded convex set in \mathbf{R}^2 .*

If there exists a non-empty finite subset Y of the boundary $\partial\mathcal{C}$ such that for every $x \in \partial\mathcal{C}$ one can find $y \in Y$ with $[x, y] \subseteq \partial\mathcal{C}$, then the closure $\bar{\mathcal{C}}$ is a convex polygon.

Proof.

Assume $0 \in \mathcal{C}$ and let us consider the continuous map $\pi : \mathbf{R} \rightarrow \partial\mathcal{C}$ which assigns to each $\theta \in \mathbf{R}$ the unique intersection point $\pi(\theta)$ of $\partial\mathcal{C}$ with the half-line $\mathbf{R}_+^*(\cos \theta, \sin \theta)$.

For each pair $(x_1, x_2) \in \partial\mathcal{C} \times \partial\mathcal{C}$, denote by $A(x_1, x_2) \subseteq \partial\mathcal{C}$ the arc segment defined by $A(x_1, x_2) := \pi([\theta_1, \theta_2])$, where θ_1 and θ_2 are the unique real numbers such that $\pi(\theta_1) = x_1$ and $\pi(\theta_2) = x_2$ with $\theta_1 \in [0, 2\pi)$ and $\theta_1 \leq \theta_2 < \theta_1 + 2\pi$.

Before proving Proposition 2.2, notice that adding a point of $\partial\mathcal{C}$ to Y does not change Y 's property at all, and therefore we may assume that $\text{card}(Y) \geq 2$.

So, write $Y = \{x_1, \dots, x_n\}$ with $x_1 = \pi(\theta_1), \dots, x_n = \pi(\theta_n)$, where $\theta_1 \in [0, 2\pi)$ and $\theta_1 < \dots < \theta_n < \theta_{n+1} := \theta_1 + 2\pi$, and let $x_{n+1} := \pi(\theta_{n+1}) = x_1$.

Fix $k \in \{1, \dots, n\}$ and pick an arbitrary $x \in A(x_k, x_{k+1}) \setminus \{x_k, x_{k+1}\}$.

By hypothesis, one can find $y \in Y$ with $[x, y] \subseteq \partial\mathcal{C}$.

Then the convex set \mathcal{C} is contained in one of the two open half-planes in \mathbf{R}^2 bounded by the line passing through the points x and y , and hence either $A(x, y) = [x, y]$, or $A(y, x) = [x, y]$.

Since $x_k \in A(y, x)$ and $x_{k+1} \in A(x, y)$, we then have $x_k \in [x, y]$ or $x_{k+1} \in [x, y]$, which yields $A(x_k, x) = [x_k, x]$ or $A(x, x_{k+1}) = [x, x_{k+1}]$.

Conclusion: $A(x_k, x_{k+1}) = S_k \cup S_{k+1}$, where $S_k := \{x \in A(x_k, x_{k+1}) \mid A(x_k, x) = [x_k, x]\}$ and $S_{k+1} := \{x \in A(x_k, x_{k+1}) \mid A(x, x_{k+1}) = [x, x_{k+1}]\}$.

Now, the set S_k (resp. S_{k+1}) satisfies $[x_k, x] \subseteq S_k$ (resp. $[x, x_{k+1}] \subseteq S_{k+1}$) whenever $x \in S_k$ (resp. $x \in S_{k+1}$).

So, if we consider $\alpha_0 := \max\{\theta \in [\theta_k, \theta_{k+1}] \mid A(x_k, \pi(\theta)) = [x_k, \pi(\theta)]\}$, we have $S_k = [x_k, \pi(\alpha_0)]$ and $S_{k+1} = [\pi(\alpha_0), x_{k+1}]$.

Hence, $A(x_k, x_{k+1})$ is the union of the two affine segments $[x_k, \pi(\alpha_0)]$ and $[\pi(\alpha_0), x_{k+1}]$.

Finally, since $\partial\mathcal{C} = \bigcup_{k=1}^n A(x_k, x_{k+1})$, this implies that $\partial\mathcal{C}$ is the union of $2n$ affine segments in \mathbf{R}^2 , and thus $\bar{\mathcal{C}}$ is a convex polygon. \square

Before proving Theorem 1.4, let us recall the following useful result, where a convex *polyhedron* in \mathbf{R}^m is the intersection of a finite number of closed half-spaces:

Theorem 2.2 ([13], Theorem 4.7). *Let P be a convex set in \mathbf{R}^m and $p \in \overset{\circ}{P}$.*

Then P is a convex polyhedron if and only if all its plane sections containing p are convex polyhedra.

Proof of Theorem 1.4.

Let $(\mathcal{C}, d_{\mathcal{C}})$ be a non-empty Hilbert domain in \mathbf{R}^m that is (A, B) -quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geq 1$ and $B \geq 0$.

According to Theorem 2.2, it suffices to prove Theorem 1.4 for $m = 2$ since any plane section of \mathcal{C} gives rise to a 2-dimensional Hilbert domain which is also (A, B) -quasi-isometric to $(V, \|\cdot\|)$.

So, let $m = 2$, and consider the set $\mathcal{E} := \{X \subseteq \partial\mathcal{C} \mid [x, y] \not\subseteq \partial\mathcal{C} \text{ for all } x, y \in X \text{ with } x \neq y\}$.

It is not empty since $\{x, y\} \in \mathcal{E}$ for some $x, y \in \partial\mathcal{C}$ with $x \neq y$ (indeed, \mathcal{C} is a non-empty open set in \mathbf{R}^2), which implies together with Proposition 2.1 that $n := \max\{\text{card}(X) \mid X \in \mathcal{E}\}$ does exist and satisfies $2 \leq n \leq N$ (recall that $N \geq 2$).

Then pick $Y \in \mathcal{E}$ such that $\text{card}(Y) = n$, write $Y = \{x_1, \dots, x_n\}$, and prove that for every $x \in \partial\mathcal{C}$ one can find $k \in \{1, \dots, n\}$ such that $[x, x_k] \subseteq \partial\mathcal{C}$.

Owing to Proposition 2.2, this will show that $\overline{\mathcal{C}}$ is a convex polygon.

So, suppose that there exists $x_0 \in \partial\mathcal{C}$ satisfying $[x_0, x_k] \not\subseteq \partial\mathcal{C}$ for all $k \in \{1, \dots, n\}$, and let us find a contradiction by considering $Z := Y \cup \{x_0\}$.

First, since $x_0 \neq x_k$ for all $k \in \{1, \dots, n\}$ (if not, we would get an index $k \in \{1, \dots, n\}$ such that $[x_0, x_k] = \{x_0\} \subseteq \partial\mathcal{C}$, which is false), we have $x_0 \notin Y$. Hence $\text{card}(Z) = n + 1$.

Next, since $Y \in \mathcal{E}$ and $[x_0, x_k] \not\subseteq \partial\mathcal{C}$ for all $k \in \{1, \dots, n\}$, we have $Z \in \mathcal{E}$.

Therefore, the assumption of the existence of x_0 yields a set $Z \in \mathcal{E}$ whose cardinality is greater than that of Y , which contradicts the very definition of Y .

Conclusion: $\overline{\mathcal{C}}$ is a convex polygon, and this proves Theorem 1.4. □

REFERENCES

- [1] BENOIST, Y. A survey on convex divisible sets. Lecture notes, International Conference and Instructional Workshop on Discrete Groups in Beijing, 2006.
- [2] BERNIG, A. Hilbert geometry of polytopes. Tech. rep., To appear in *Archiv der Mathematik*, 2008.
- [3] BURAGO, D., BURAGO, Y., AND IVANOV, S. *A course in metric geometry*. AMS, 2001.
- [4] BUSEMANN, H. *The geometry of geodesics*. Academic Press, 1955.
- [5] BUSEMANN, H., AND KELLY, P. *Projective geometry and projective metrics*. Academic Press, 1953.
- [6] COLBOIS, B., VERNICOS, C., AND VEROVIC, P. Hilbert geometry for convex polygonal domains. Tech. rep., University of Neuchâtel, 2008.
- [7] COLBOIS, B., AND VEROVIC, P. Hilbert geometry for strictly convex domains. *Geom. Dedicata* 105 (2004), 29–42.
- [8] DE LA HARPE, P. On Hilbert’s metric for simplices. *Lond. Math. Soc. Lect. Note Ser.* 1, 181 (1993), 97–119.
- [9] EGLOFF, D. Uniform Finsler Hadamard manifolds. *Ann. Inst. Henri Poincaré, Phys. Théor.* 66 (1997), 323–357.
- [10] FÖRTSCH, T., AND KARLSSON, A. Hilbert metrics and Minkowski norms. *J. Geom.* 83, 1-2 (2005), 22–31.
- [11] GOLDMAN, W. Projective geometry on manifolds. Lecture notes, University of Maryland, 1988.
- [12] KARLSSON, A., AND NOSKOV, G. The Hilbert metric and Gromov hyperbolicity. *Enseign. Math.* 48, 1-2 (2002), 73–89.
- [13] KLEE, V. Some characterizations of convex polyhedra. *Acta Math.* 102 (1959), 79–107.
- [14] NUSSBAUM, R. Hilbert’s projective metric and iterated nonlinear maps. *Mem. Am. Math. Soc.* 75, 391 (1988).
- [15] SOCIÉ-MÉTHOU, É. *Comportements asymptotiques et rigidités des géométries de Hilbert*. PhD thesis, University of Strasbourg, 2000.
- [16] VERNICOS, C. Introduction aux géométries de Hilbert. *Sémin. Théor. Spectr. Géom.* 23 (2005), 145–168.
- [17] VERNICOS, C. Lipschitz characterization of polytopal Hilbert geometries. Tech. rep., University of Maynooth, 2008.

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