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# HILBERT DOMAINS QUASI-ISOMETRIC TO NORMED VECTOR SPACES

BRUNO COLBOIS AND PATRICK VEROVIC

ABSTRACT. We prove that a Hilbert domain which is quasi-isometric to a normed vector space is actually a convex polytope.

## 1. INTRODUCTION

A *Hilbert domain* in  $\mathbf{R}^m$  is a metric space  $(\mathcal{C}, d_{\mathcal{C}})$ , where  $\mathcal{C}$  is an *open bounded convex* set in  $\mathbf{R}^m$  and  $d_{\mathcal{C}}$  is the distance function on  $\mathcal{C}$  — called the *Hilbert metric* — defined as follows.

Given two distinct points  $p$  and  $q$  in  $\mathcal{C}$ , let  $a$  and  $b$  be the intersection points of the straight line defined by  $p$  and  $q$  with  $\partial\mathcal{C}$  so that  $p = (1 - s)a + sb$  and  $q = (1 - t)a + tb$  with  $0 < s < t < 1$ . Then

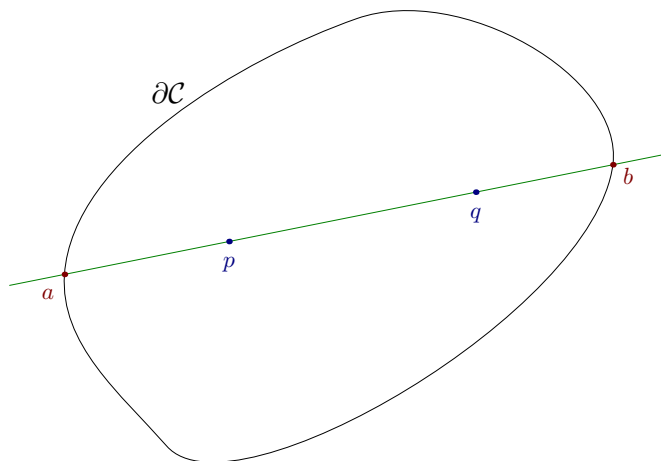
$$d_{\mathcal{C}}(p, q) = \frac{1}{2} \ln[a, p, q, b],$$

where

$$[a, p, q, b] = \frac{1 - s}{s} \times \frac{t}{1 - t} > 1$$

is the cross ratio of the 4-tuple of ordered collinear points  $(a, p, q, b)$ .

We complete the definition by setting  $d_{\mathcal{C}}(p, p) = 0$ .



The metric space  $(\mathcal{C}, d_{\mathcal{C}})$  thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of  $\mathbf{R}^m$  and in which the affine open segments joining two points of the boundary  $\partial\mathcal{C}$  are geodesics that are isometric to  $(\mathbf{R}, |\cdot|)$ .

For further information about Hilbert geometry, we refer to [4, 5, 9, 11] and the excellent introduction [15] by Socié-Méthou.

The two fundamental examples of Hilbert domains  $(\mathcal{C}, d_{\mathcal{C}})$  in  $\mathbf{R}^m$  correspond to the case when  $\mathcal{C}$  is an ellipsoid, which gives the Klein model of  $m$ -dimensional hyperbolic geometry (see for

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example [15, first chapter]), and the case when the closure  $\bar{\mathcal{C}}$  is a  $m$ -simplex for which there exists a norm  $\|\cdot\|_{\mathcal{C}}$  on  $\mathbf{R}^m$  such that  $(\mathcal{C}, d_{\mathcal{C}})$  is isometric to the normed vector space  $(\mathbf{R}^m, \|\cdot\|_{\mathcal{C}})$  (see [8, pages 110–113] or [14, pages 22–23]).

Much has been done to study the similarities between Hilbert and hyperbolic geometries (see for example [7], [16] or [1]), but little literature deals with the question of knowing to what extent a Hilbert geometry is close to that of a normed vector space. So let us mention three results in this latter direction which are relevant for our present work.

**Theorem 1.1** ([10], Theorem 2). *A Hilbert domain  $(\mathcal{C}, d_{\mathcal{C}})$  in  $\mathbf{R}^m$  is isometric to a normed vector space if and only if  $\mathcal{C}$  is the interior of a  $m$ -simplex.*

**Theorem 1.2** ([6], Theorem 3.1). *If  $\mathcal{C}$  is an open convex polygonal set in  $\mathbf{R}^2$ , then  $(\mathcal{C}, d_{\mathcal{C}})$  is Lipschitz equivalent to Euclidean plane.*

**Theorem 1.3** ([2], Theorem 1.1. See also [17]). *If  $\mathcal{C}$  is an open set in  $\mathbf{R}^m$  whose closure  $\bar{\mathcal{C}}$  is a convex polytope, then  $(\mathcal{C}, d_{\mathcal{C}})$  is Lipschitz equivalent to Euclidean  $m$ -space.*

Recall that a convex *polytope* in  $\mathbf{R}^m$  (called a convex *polygon* when  $m = 2$ ) is the convex hull of a finite set of points whose affine span is the whole space  $\mathbf{R}^m$ .

In light of these three results, it is natural to ask whether the converse of Theorem 1.3 — which generalizes Theorem 1.2 in higher dimensions — holds. In other words, if a Hilbert domain  $(\mathcal{C}, d_{\mathcal{C}})$  in  $\mathbf{R}^m$  is quasi-isometric to a normed vector space, what can be said about  $\mathcal{C}$ ? Here, by *quasi-isometric* we mean the following (see [3]):

**Definition 1.1.** Given real numbers  $A \geq 1$  and  $B \geq 0$ , a metric space  $(S, d)$  is said to be  $(A, B)$ -quasi-isometric to a normed vector space  $(V, \|\cdot\|)$  if and only if there exists a map  $f : S \rightarrow V$  such that

$$\frac{1}{A}d(p, q) - B \leq \|f(p) - f(q)\| \leq Ad(p, q) + B$$

for all  $p, q \in S$ .

We can now state the result of this paper which asserts that the converse of Theorem 1.3 is actually true:

**Theorem 1.4.** *If a Hilbert domain  $(\mathcal{C}, d_{\mathcal{C}})$  in  $\mathbf{R}^m$  is  $(A, B)$ -quasi-isometric to a normed vector space  $(V, \|\cdot\|)$  for some real constants  $A \geq 1$  and  $B \geq 0$ , then  $\mathcal{C}$  is the interior of a convex polytope.*

## 2. PROOF OF THEOREM 1.4

The proof of Theorem 1.4 is based on an idea developed by Förtsch and Karlsson in their paper [10].

It needs the following fact due to Karlsson and Noskov:

**Theorem 2.1** ([12], Theorem 5.2). *Let  $(\mathcal{C}, d_{\mathcal{C}})$  be a Hilbert domain in  $\mathbf{R}^m$  and  $x, y \in \partial\mathcal{C}$  such that  $[x, y] \not\subseteq \partial\mathcal{C}$ . Then, given a point  $p_0 \in \mathcal{C}$ , there exists a constant  $K(p_0, x, y) > 0$  such that for any sequences  $(x_n)_{n \in \mathbf{N}}$  and  $(y_n)_{n \in \mathbf{N}}$  in  $\mathcal{C}$  that converge respectively to  $x$  and  $y$  in  $\mathbf{R}^m$  one can find an integer  $n_0 \in \mathbf{N}$  for which we have*

$$d_{\mathcal{C}}(x_n, y_n) \geq d_{\mathcal{C}}(x_n, p_0) + d_{\mathcal{C}}(y_n, p_0) - K(p_0, x, y)$$

for all  $n \geq n_0$ .

Now, here is the key result which gives the proof of Theorem 1.4:

**Proposition 2.1.** *Let  $(\mathcal{C}, d_{\mathcal{C}})$  be a Hilbert domain in  $\mathbf{R}^m$  which is  $(A, B)$ -quasi-isometric to a normed vector space  $(V, \|\cdot\|)$  for some real constants  $A \geq 1$  and  $B \geq 0$ .*

*Then, if  $N = N(A, \|\cdot\|)$  denotes the maximum number of points in the ball  $\{v \in V \mid \|v\| \leq 2A\}$  whose pairwise distances with respect to  $\|\cdot\|$  are greater than or equal to  $1/(2A)$ , and if  $X \subseteq \partial\mathcal{C}$  is such that  $[x, y] \not\subseteq \partial\mathcal{C}$  for all  $x, y \in X$  with  $x \neq y$ , we have*

$$\text{card}(X) \leq N.$$

*Proof.*

Let  $f : \mathcal{C} \rightarrow V$  such that

$$(2.1) \quad \frac{1}{A}d_{\mathcal{C}}(p, q) - B \leq \|f(p) - f(q)\| \leq Ad_{\mathcal{C}}(p, q) + B$$

for all  $p, q \in \mathcal{C}$ .

First of all, up to translations, we may assume that  $0 \in \mathcal{C}$  and  $f(0) = 0$ .

Then suppose that there exists a subset  $X$  of the boundary  $\partial\mathcal{C}$  such that  $[x, y] \not\subseteq \partial\mathcal{C}$  for all  $x, y \in X$  with  $x \neq y$  and  $\text{card}(X) \geq N + 1$ . So, pick  $N + 1$  distinct points  $x_1, \dots, x_{N+1}$  in  $X$ , and for each  $k \in \{1, \dots, N + 1\}$ , let  $\gamma_k : [0, +\infty) \rightarrow \mathcal{C}$  be a geodesic of  $(\mathcal{C}, d_{\mathcal{C}})$  that satisfies  $\gamma_k(0) = 0$ ,  $\lim_{t \rightarrow +\infty} \gamma_k(t) = x_k$  in  $\mathbf{R}^m$  and  $d_{\mathcal{C}}(0, \gamma_k(t)) = t$  for all  $t \geq 0$ .

This implies that for all integers  $n \geq 1$  and every  $k \in \{1, \dots, N + 1\}$ , we have

$$(2.2) \quad \left\| \frac{f(\gamma_k(n))}{n} \right\| \leq A + \frac{B}{n}$$

from the second inequality in Equation 2.1 with  $p = \gamma_k(n)$  and  $q = 0$ .

On the other hand, Theorem 2.1 yields the existence of some integer  $n_0 \geq 1$  such that

$$d_{\mathcal{C}}(\gamma_i(n), \gamma_j(n)) \geq 2n - K(0, x_i, x_j)$$

for all integers  $n \geq n_0$  and every  $i, j \in \{1, \dots, N + 1\}$  with  $i \neq j$ , and hence

$$(2.3) \quad \left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geq \frac{2}{A} - \frac{1}{n} \left( \frac{K(0, x_i, x_j)}{A} + B \right)$$

from the first inequality in Equation 2.1 with  $p = \gamma_i(n)$  and  $q = \gamma_j(n)$ .

Now, fixing an integer  $n \geq n_0 + AB + \max\{K(0, x_i, x_j) \mid i, j \in \{1, \dots, N + 1\}\}$ , we get

$$\left\| \frac{f(\gamma_k(n))}{n} \right\| \leq 2A$$

for all  $k \in \{1, \dots, N + 1\}$  by Equation 2.2 together with

$$\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geq \frac{1}{2A}$$

for all  $i, j \in \{1, \dots, N + 1\}$  with  $i \neq j$  by Equation 2.3.

But this contradicts the definition of  $N = N(A, \|\cdot\|)$ .

Therefore, Proposition 2.1 is proved.  $\square$

**Remark.** Given  $v \in V$  such that  $\|v\| = 2A$ , we have  $\|-v\| = 2A$  and  $\|v - (-v)\| = 2\|v\| = 4A \geq 1/(2A)$ , which shows that  $N \geq 2$ .

The second ingredient we will need for the proof of Theorem 1.4 is the following:

**Proposition 2.2.** *Let  $\mathcal{C}$  be an open bounded convex set in  $\mathbf{R}^2$ .*

*If there exists a non-empty finite subset  $Y$  of the boundary  $\partial\mathcal{C}$  such that for every  $x \in \partial\mathcal{C}$  one can find  $y \in Y$  with  $[x, y] \subseteq \partial\mathcal{C}$ , then the closure  $\bar{\mathcal{C}}$  is a convex polygon.*

*Proof.*

Assume  $0 \in \mathcal{C}$  and let us consider the continuous map  $\pi : \mathbf{R} \rightarrow \partial\mathcal{C}$  which assigns to each  $\theta \in \mathbf{R}$  the unique intersection point  $\pi(\theta)$  of  $\partial\mathcal{C}$  with the half-line  $\mathbf{R}_+^*(\cos \theta, \sin \theta)$ .

For each pair  $(x_1, x_2) \in \partial\mathcal{C} \times \partial\mathcal{C}$ , denote by  $A(x_1, x_2) \subseteq \partial\mathcal{C}$  the arc segment defined by  $A(x_1, x_2) := \pi([\theta_1, \theta_2])$ , where  $\theta_1$  and  $\theta_2$  are the unique real numbers such that  $\pi(\theta_1) = x_1$  and  $\pi(\theta_2) = x_2$  with  $\theta_1 \in [0, 2\pi)$  and  $\theta_1 \leq \theta_2 < \theta_1 + 2\pi$ .

Before proving Proposition 2.2, notice that adding a point of  $\partial\mathcal{C}$  to  $Y$  does not change  $Y$ 's property at all, and therefore we may assume that  $\text{card}(Y) \geq 2$ .

So, write  $Y = \{x_1, \dots, x_n\}$  with  $x_1 = \pi(\theta_1), \dots, x_n = \pi(\theta_n)$ , where  $\theta_1 \in [0, 2\pi)$  and  $\theta_1 < \dots < \theta_n < \theta_{n+1} := \theta_1 + 2\pi$ , and let  $x_{n+1} := \pi(\theta_{n+1}) = x_1$ .

Fix  $k \in \{1, \dots, n\}$  and pick an arbitrary  $x \in A(x_k, x_{k+1}) \setminus \{x_k, x_{k+1}\}$ .

By hypothesis, one can find  $y \in Y$  with  $[x, y] \subseteq \partial\mathcal{C}$ .

Then the convex set  $\mathcal{C}$  is contained in one of the two open half-planes in  $\mathbf{R}^2$  bounded by the line passing through the points  $x$  and  $y$ , and hence either  $A(x, y) = [x, y]$ , or  $A(y, x) = [x, y]$ .

Since  $x_k \in A(y, x)$  and  $x_{k+1} \in A(x, y)$ , we then have  $x_k \in [x, y]$  or  $x_{k+1} \in [x, y]$ , which yields  $A(x_k, x) = [x_k, x]$  or  $A(x, x_{k+1}) = [x, x_{k+1}]$ .

Conclusion:  $A(x_k, x_{k+1}) = S_k \cup S_{k+1}$ , where  $S_k := \{x \in A(x_k, x_{k+1}) \mid A(x_k, x) = [x_k, x]\}$  and  $S_{k+1} := \{x \in A(x_k, x_{k+1}) \mid A(x, x_{k+1}) = [x, x_{k+1}]\}$ .

Now, the set  $S_k$  (resp.  $S_{k+1}$ ) satisfies  $[x_k, x] \subseteq S_k$  (resp.  $[x, x_{k+1}] \subseteq S_{k+1}$ ) whenever  $x \in S_k$  (resp.  $x \in S_{k+1}$ ).

So, if we consider  $\alpha_0 := \max\{\theta \in [\theta_k, \theta_{k+1}] \mid A(x_k, \pi(\theta)) = [x_k, \pi(\theta)]\}$ , we have  $S_k = [x_k, \pi(\alpha_0)]$  and  $S_{k+1} = [\pi(\alpha_0), x_{k+1}]$ .

Hence,  $A(x_k, x_{k+1})$  is the union of the two affine segments  $[x_k, \pi(\alpha_0)]$  and  $[\pi(\alpha_0), x_{k+1}]$ .

Finally, since  $\partial\mathcal{C} = \bigcup_{k=1}^n A(x_k, x_{k+1})$ , this implies that  $\partial\mathcal{C}$  is the union of  $2n$  affine segments in  $\mathbf{R}^2$ , and thus  $\bar{\mathcal{C}}$  is a convex polygon.  $\square$

Before proving Theorem 1.4, let us recall the following useful result, where a convex *polyhedron* in  $\mathbf{R}^m$  is the intersection of a finite number of closed half-spaces:

**Theorem 2.2** ([13], Theorem 4.7). *Let  $P$  be a convex set in  $\mathbf{R}^m$  and  $p \in \mathring{P}$ .*

*Then  $P$  is a convex polyhedron if and only if all its plane sections containing  $p$  are convex polyhedra.*

*Proof of Theorem 1.4.*

Let  $(\mathcal{C}, d_{\mathcal{C}})$  be a non-empty Hilbert domain in  $\mathbf{R}^m$  that is  $(A, B)$ -quasi-isometric to a normed vector space  $(V, \|\cdot\|)$  for some real constants  $A \geq 1$  and  $B \geq 0$ .

According to Theorem 2.2, it suffices to prove Theorem 1.4 for  $m = 2$  since any plane section of  $\mathcal{C}$  gives rise to a 2-dimensional Hilbert domain which is also  $(A, B)$ -quasi-isometric to  $(V, \|\cdot\|)$ .

So, let  $m = 2$ , and consider the set  $\mathcal{E} := \{X \subseteq \partial\mathcal{C} \mid [x, y] \not\subseteq \partial\mathcal{C} \text{ for all } x, y \in X \text{ with } x \neq y\}$ .

It is not empty since  $\{x, y\} \in \mathcal{E}$  for some  $x, y \in \partial\mathcal{C}$  with  $x \neq y$  (indeed,  $\mathcal{C}$  is a non-empty open set in  $\mathbf{R}^2$ ), which implies together with Proposition 2.1 that  $n := \max\{\text{card}(X) \mid X \in \mathcal{E}\}$  does exist and satisfies  $2 \leq n \leq N$  (recall that  $N \geq 2$ ).

Then pick  $Y \in \mathcal{E}$  such that  $\text{card}(Y) = n$ , write  $Y = \{x_1, \dots, x_n\}$ , and prove that for every  $x \in \partial\mathcal{C}$  one can find  $k \in \{1, \dots, n\}$  such that  $[x, x_k] \subseteq \partial\mathcal{C}$ .

Owing to Proposition 2.2, this will show that  $\overline{\mathcal{C}}$  is a convex polygon.

So, suppose that there exists  $x_0 \in \partial\mathcal{C}$  satisfying  $[x_0, x_k] \not\subseteq \partial\mathcal{C}$  for all  $k \in \{1, \dots, n\}$ , and let us find a contradiction by considering  $Z := Y \cup \{x_0\}$ .

First, since  $x_0 \neq x_k$  for all  $k \in \{1, \dots, n\}$  (if not, we would get an index  $k \in \{1, \dots, n\}$  such that  $[x_0, x_k] = \{x_0\} \subseteq \partial\mathcal{C}$ , which is false), we have  $x_0 \notin Y$ . Hence  $\text{card}(Z) = n + 1$ .

Next, since  $Y \in \mathcal{E}$  and  $[x_0, x_k] \not\subseteq \partial\mathcal{C}$  for all  $k \in \{1, \dots, n\}$ , we have  $Z \in \mathcal{E}$ .

Therefore, the assumption of the existence of  $x_0$  yields a set  $Z \in \mathcal{E}$  whose cardinality is greater than that of  $Y$ , which contradicts the very definition of  $Y$ .

Conclusion:  $\overline{\mathcal{C}}$  is a convex polygon, and this proves Theorem 1.4. □

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