Conditional Image Diffusion

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Abstract

In this paper, a theoretical framework for the conditional diffusion of digital images is presented. Different approaches have been proposed to solve this problem by extrapolating the idea of the anisotropic diffusion for a grey level images to vector-valued images. Then, the diffusion of each channel is conditioned to a direction which normally takes into account information from all channels. In our approach, the diffusion model assumes the *a priori* knowledge of the diffusion direction during all the process.

The consistency of the model is shown by proving the existence and uniqueness of solution for the proposed equation from the viscosity solutions theory. Also a numerical scheme adapted to this equation based on the neighborhood filter is proposed. Finally, we discuss several applications and we compare the corresponding numerical schemes for the proposed model.

1 Introduction

The most simple diffusion model is the heat equation

$$u_t = \Delta u$$

where Δu denotes the Laplacian of image u. The heat equation is linear and isotropic. However, isotropic diffusion is not well suited for natural images since it does not preserve the location and direction of edges which convey the main perceptual information of the image, the first works on anisotropic diffusion goes back to [16] and [19].

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Non linear PDE equations were introduced in order to favor diffusion directions adapted to the local geometry of the image. Perona and Malik [24] proposed the following equation,

$$\frac{\partial u}{\partial t} = \operatorname{div}(g(|Du|)\, Du), \tag{1}$$

where $g : I\!R \to I\!R$ is a decreasing function with $g(0) = 1$ and $\lim_{s\to\infty} g(s) = 0$. The equation can be decomposed as the sum of two directional diffusion terms

$$\frac{\partial u}{\partial t} = g(|Du|)u_{\xi\xi} + h(|Du|)u_{\eta\eta}, \tag{2}$$

where

$$u_{\xi\xi} = D^2 u\,(\xi, \xi)\,, \quad u_{\eta\eta} = D^2 u\,(\eta, \eta)\,,$$

respectively denote the second derivatives of u in the direction of $\eta = Du/|Du|$ and $\xi = Du^{\perp}/|Du|$. The diffusion term in the ξ direction, tangent to the level line passing through $\mathbf{x}$, is always positive. The diffusion term in the η direction, across the level line, can take positive or negative values depending on the magnitude of the gradient. This equation combines a filtering/deblurring strategy as already proposed by Gabor in the 60s [14]. The same objective can be attained by using reaction diffusion equations as proposed by Cottet et al. in [10].

A simpler non linear equation is the mean curvature motion [20]. The equation is written as

$$\frac{\partial u}{\partial t} = u_{\xi\xi}. \tag{3}$$

This equation diffuses only in the ξ direction preserving the main discontinuities. Other anisotropic diffusion equations have been more recently proposed [27, 7, 26].

The diffusion direction in the previous equations is defined implicitly by the equation itself. The aim of this work is to introduce a new theoretical framework in which the diffusion direction is given by a certain vector field $W(x, t)$ available a priori. The motivation for this work comes from the color image filtering equation presented in [9]. This equation conditions the color diffusion on the luminance

$$\frac{\partial u_i}{\partial t} = D^2 u_i \left(\frac{DI^{\perp}}{|DI|}, \frac{DI^{\perp}}{|DI|} \right), \quad i = 1, 2, 3, \tag{4}$$

where $\mathbf{u} : I\!R^2 \to I\!R^3$ is the color image, $\mathbf{u} = (u_1, u_2, u_3)$, and $I = \frac{u_1+u_2+u_3}{3}$. Because of the linearity of the expressions involved in (4), the luminance I

itself must satisfy a mean curvature equation

$$\frac{\partial I}{\partial t} = D^2 I \left(\frac{DI^{\perp}}{|DI|}, \frac{DI^{\perp}}{|DI|} \right). \tag{5}$$

Then, the luminance image can be previously filtered $I(t, x)$ and coded as *a priori* data inside a vector field $W(t, x) = DI(t, x)^{\perp}$. The equation (4) can then be rewritten as,

$$\frac{\partial u}{\partial t} = D^2 u \left(\frac{W(t, x)}{||W(t, x)||}, \frac{W(t, x)}{||W(t, x)||} \right). \tag{6}$$

This general diffusion model adapts to many applications in which the diffusion direction during all the process is known a priori. Cabral et al [6] used the same idea in order to filter noise images with fixed a priori vector fields. The obtained filtered noise reflects the vector field obtaining visually artistic images. This model is suitable for real applications where some additional information on the desired solution is available. For example, Caselles et al. [2] proposed to increase the resolution of the color channels of satellite images using the panchromatic component at the desired resolution. In [30], the authors proposed to restore color to damaged or old black and white images by diffusing the initial data in the orthogonal direction to the gray image gradient. The subjacent idea common to these applications is that the geometry of color images is contained in the luminance as proposed and tested in [8].

Conditional diffusion is not restricted to color or multivalued images. Recently, Almansa et al. [13] proposed to filter the disparity map of a stereo pair by a diffusion equation. The diffusion direction is computed on the original grey level image and not on the disparity image.

A similar constraint can be introduced when dealing with optical flow computation. The introduced requirement restricts the variation of the displacement vectors in directions with small or no variation of gray values. Different model have been proposed taking advantage of this technique Nagel et al. [21], Alvarez et al. [1] and Weickert et al. [28]

All these applications can be set under the same general framework given by equation (6) since the directional field is known at the beginning of the process. The aim of this work is to perform the theoretical study of this equation and discuss its possible applications. The theoretical results of this paper apply for time dependent vector fields even if in many application this field is taken to be constant in time.

The planning of the paper is as follows. In section 2 we introduce a general PDE equation for conditional diffusion. We prove the existence and uniqueness of solution in section 3. In section 4 we propose a numerical scheme adapted to this equation. Section 5 is devoted to the discussion and experimentation on the several applications of the model. Finally, section 6 exposes some conclusions.

2 Conditional diffusion model

Let $W(x,t)$ be a vector field giving the direction of the diffusion process. Then, the proposed equation reads

$$\frac{\partial u}{\partial t} = D^2 u\left(\frac{W(t,x)}{||W(t,x)||}, \frac{W(t,x)}{||W(t,x)||}\right). \tag{7}$$

Following the ideas in [17, 24] the above diffusion can be coupled with a diffusion term of decreasing magnitude in the orthogonal direction to the vector field, $W^{\perp}$. This second term avoids the singularity of the diffusion process when $W = 0$. Then, when there is no meaningful directional information, an isotropic regularization is performed.

The proposed final equation and a uniqueness result are stated in the next theorem.

Theorem 2.1 *Let $W : [0,\infty)\times I\!R^n \to I\!R^n$ be a continuous in time, Lipschitz continuous in the space variable and bounded vector field. Let $f : [0,\infty) \to [0,\infty)$ be a smooth decreasing function, $f \in C^1([0,+\infty))$, satisfying $f(0) = 1$, $f'_+(0) = 0$ and $\lim_{s\to+\infty} f(s) = 0$. Then the equation*

$$u_t = D^2 u\left(\frac{W}{||W||}, \frac{W}{||W||}\right) + f(||W||)D^2 u\left(\frac{W^{\perp}}{||W||}, \frac{W^{\perp}}{||W||}\right), \tag{8}$$

has a unique viscosity solution in $(0,T] \times \mathbb{R}^n$, given a continuous initial condition $u(0,x) = u_0(x)$.

The diffusion in the orthogonal direction decreases as the field magnitude increases. When this is small, both diffusion terms are combined leading to the heat equation. When this magnitude increases, the term in the orthogonal direction cancels and the diffusion is performed only in the field direction.

The result applies for multivalued functions $u = (u_1, \ldots, u_N)$. In that case, each component is diffused by the same field direction. This is the case for instance of color image filtering. The vector field $W \equiv W(t,x)$ is in general time dependent. It can be constant in time for several applications as we shall see in section 5.

3 Proof of the main result

The aim of this section is to prove the Theorem 2.1. It is divided into three subsections. In the first one, we present the assumptions that must satisfy the equation in order to have a comparison principle and the well-known result in the framework of viscosity solutions for degenerate parabolic equations [15]. In the second one we prove that the proposed equation (8) satisfies the above assumptions and then we have a comparison principle for equation (8) and, as a consequence, uniqueness of the solution is proved. Finally, in the last subsection, we prove the existence part of Theorem 2.1 by constructing a consistent and monotone scheme. Then, following the ideas of [4], the scheme converges to the solution of the equation.

3.1 A general uniqueness result

We adapt a comparison principle for second-order degenerate parabolic equations stated in [15]. We apply this result in order to prove the existence and uniqueness of equation (8) in Theorem 2.1.

Let T be any positive constant and consider a degenerate parabolic equation of the form

$$u_t = F(t,x,D^2u), \qquad \text{in} \qquad Q=(0,T]\times \mathbb{R}^n. \tag{9}$$

We list the assumptions on $F=F(t,x,X)$ which are necessary for the uniqueness result.

(F1) $F: Q\times S_n \to \mathbb{R}$ is continuous, where S_n is the space of real $n\times n$ symmetric matrices.

(F2) F is degenerate elliptic, i.e.,

$$F(t,x,X) \le F(t,x,X+Y), \qquad Y\ge 0.$$

(F3) For every $R>0$

$$c_R = \sup\{|F(t,x,X)|: \quad ||X||\le R, (t,x,X)\in Q\times S_n\} < \infty.$$

(F4) Suppose that

$$-\mu\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} \le \begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix} \le \nu\begin{pmatrix} I & -I\\ -I & I\end{pmatrix} + \omega\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} \tag{10}$$

with $\mu, \nu, \omega \geq 0$. Then it holds:

$$F(t, y, -Y) - F(t, x, X) \geq -m(\nu||x-y||^2) - m(2\omega)$$

with some modulus m independent of $t, x, y, X, Y, \mu, \nu, \omega$.

Next, we recall the definition of viscosity sub- and supersolution of (9) (see also [11]).

Definition 3.1 *An upper-semicontinuous function $u : Q \to I\!R$ is called a viscosity subsolution of (9) in Q if*

$$\tau \leq F(t, x, X) \quad \text{for all } (\tau, p, X) \in \mathcal{P}_Q^{2,+}u(t, x),\ (t, x) \in Q.$$

Similarly, a lower-semicontinuous function $u : Q \to I\!R$ is called a viscosity supersolution of (9) in Q if

$$\tau \geq F(t, x, X) \quad \text{for all } (\tau, p, X) \in \mathcal{P}_Q^{2,-}u(t, x),\ (t, x) \in Q.$$

Finally, a continuous function $u : Q \to I\!R$ is a viscosity solution of (9) in Q if it is both a viscosity subsolution and a viscosity supersolution of (9) in Q.

Here $\mathcal{P}_Q^{2,+}u(t, x)$ denotes the parabolic super 2-jet in Q, that is the set of $(\tau, p, X) \in I\!R \times I\!R^n \times S_n$ such that

$$u(s, y) \leq u(t, x) + \tau(s-t) + < p, y-x > + \frac{1}{2} < X(y-x), y-x > +$$
$$+ o(|s-t| + ||y-x||^2) \quad \text{as } (s, y) \to (t, x),$$

and $\mathcal{P}_Q^{2,-}u = -\mathcal{P}_Q^{2,+}(-u)$.

Then the comparison theorem for equation (9) is a particular adaptation of a more general kind of equations studied in [15].

Theorem 3.1 *Suppose that F of equation (9) satisfies (F1)-(F4). Let u and v be respectively, sub and super-solutions of (9). Assume that*

i) $u(t, x) \leq K(||x|| + 1)$, $v(t, x) \geq -K(||x|| + 1)$ *for some $K > 0$ independent of $(t, x) \in Q$.*

ii) $u(0, x) - v(0, y) \leq K||x-y||$ *for all $(x, y) \in I\!R^n \times I\!R^n$, for some $K > 0$ independent of (x, y).*

Then there is a modulus m such that

$$u(t, x) - v(t, y) \leq m(||x-y||) \quad \text{on } U = (0, T] \times I\!R^n \times I\!R^n.$$

In particular $u \leq v$ on Q.

The proof of this theorem is similar to the one of Theorem 4.2 in [15].

3.2 Proof of uniqueness of the solution for Theorem 2.1

Next, we apply the comparison principle, Theorem 3.1, to equation (8). This equation can be rewritten as

$$u_t = Tr(B(W)D^2u),$$

where, if we denote as $W = (w_1, w_2)$ then $B(W) \in \mathcal{S}_2$ and it writes as

$$B(W) = \begin{pmatrix} \frac{w_1^2 + f(||W||)w_2^2}{||W||^2} & \frac{w_1 w_2 (1 - f(||W||))}{||W||^2} \\ \\ \frac{w_1 w_2 (1 - f(||W||))}{||W||^2} & \frac{w_2^2 + f(||W||)w_1^2}{||W||^2} \end{pmatrix}. \qquad (11)$$

Note that $B(W)$ is not defined when $W = 0$ but due to the continuity of f we have that $\lim_{W \to 0} B(W) = I$. This fact implies that we can extend continuously B at points where $W = 0$ as $B(0) = I$. Therefore, if we write

$$A(t,x) = \begin{cases} B(W(t,x)) & \text{if } W(t,x) \neq 0, \\ I & \text{if } W(t,x) = 0, \end{cases} \qquad (12)$$

then equation (8) can be rewritten as

$$u_t = Tr(A(t,x)D^2u). \qquad (13)$$

This formulation, acting $B(W)$ as a diffusion tensor, can be found in [26] where the authors study a general framework that unifies a large number of methods for vector-valued image regularization.

Next result proves that $F(t,x,X) = Tr(A(t,x)D^2u)$ is under the hypothesis of Theorem 3.1 and by applying it, we obtain a uniqueness result for equation (13).

Theorem 3.2 *Suppose W is Lipschitz continuous in the space variable, continuous in time variable and bounded. Assume also that $f \in C^1([0, +\infty))$, is a decreasing function satisfying $f(0) = 1$, $f'_+(0) = 0$ and $\lim_{s \to +\infty} f(s) = 0$. Then, it holds*

(i) $A(t,x)$ defined by (12) is a positive and bounded matrix and $A(t,x)^{1/2}$ is Lipschitz in the space variable.

*(ii) $F(t,x,X) = Tr(A(t,x)D^2u)$ satisfies hypothesis **(F1)-(F4)** of Theorem 3.1.*

Proof: To prove part (i), note that W is bounded which implies $A(x,t) > \delta I > 0$. Therefore, if we prove that A is Lipschitz then it follows that $A^{1/2}$ is also Lipschitz continuous. From the hypothesis on W and f, it is easy to see that the matrix $B(W)$ defined above is differentiable with bounded differential, for all $W = (w_1, w_2)$. Therefore, if M is a bound of $DB(W)$ and L is the Lipschitz constant of $W(t,x)$, then we can apply the mean value theorem and we have, for any $(t,y),(t,x) \in Q$,

$$\begin{aligned}||A(t,y) - A(t,x)|| &= ||B(w_1(t,y), w_2(t,y)) - B(w_1(t,x), w_2(t,x))|| \\ &\leq ||DB(\psi_1, \psi_2)||||W(t,y) - W(t,x)|| \leq ML||y-x||,\end{aligned}$$

which proves that A is Lipschitz with Lipschitz constant ML.

To prove part (ii), note that **(F1)** and **(F2)** follow from the properties of A stated above.

To prove **(F3)**, note that if $||A|| \leq K$, then $|F(t,x,X)| \leq 2K||X||$ for all $(t,x) \in Q$. As a consequence, for every $R > 0$,

$$c_R = \sup\{|F(t,x,X)| : \ ||X|| \leq R, (t,x,X) \in Q \times \mathcal{S}^n\} = 2KR < \infty$$

Finally, for **(F4)**, we multiply the right side inequality in (10) by the nonnegative symmetric matrix (see also [11])

$$\begin{pmatrix} A(t,x) & A(t,y)^{1/2}A(t,x)^{1/2} \\ A(t,x)^{1/2}A(t,y)^{1/2} & A(t,y) \end{pmatrix}$$

and taking traces yields

$$\begin{aligned}Tr(A(t,x)X) + Tr(A(t,y)Y) &\leq \nu\, Tr((A(t,x)^{1/2} - A(t,y)^{1/2})^2) \\ &\quad +\omega\, Tr(A(t,x) + A(t,y)).\end{aligned}$$

By part (i), $A^{1/2}$ is Lipschitz continuous with constant $\bar{L}$, and then we have

$$\begin{aligned}F(t,y,-Y) - F(t,x,X) &= -Tr(A(t,y)Y) - Tr(A(t,x)X) \\ &\geq -2\nu\bar{L}^2||x-y||^2 - 4K\omega.\end{aligned}$$

If we choose $m(s) = as$ where $a = \max(2\bar{L}^2, 2K)$, then we have the desired inequality. $\square$

3.3 Proof of existence of the solution for Theorem 2.1

In this section, we will construct a numerical finite difference scheme for equation (13). Based on a work by Barles-Souganidis [4], the convergence to the viscosity solution of equation (13) is ensured if the scheme is monotone and consistent. In fact, the main requirement for convergence is that the scheme satisfies a discrete comparison principle (see also [23], [22] for the notion of finite difference scheme).

Due to the fact that equation (13) is degenerate and anisotropic, we consider a grid in space and orientation of mesh sizes denoted by dx and $d\theta$, respectively. Next, we give the definitions of consistency and monotonicity.

Definition 3.2 (**Consistency**) *A numerical scheme $\mathcal{F}_{dx,d\theta}$ is consistent if for every $\phi \in \mathcal{C}^2(Q)$, $(t,x) \in Q$, where $Q = (0,T) \times I\!R^2$, then*

$$\lim_{dx,d\theta \to 0} \mathcal{F}_{dx,d\theta}(\phi) = \phi_t - F(t,x,D^2\phi). \tag{14}$$

Definition 3.3 (**Monotonicity**) *If u, v are solutions of a scheme with boundary data f, g respectively, then $f \leq g$ on the boundary implies $u \leq v$ on the domain.*

The monotonicity is a discrete version of the comparison principle. For explicit schemes, this is satisfied if the solution map is a non decreasing function of the values at the previous step.

3.3.1 The scheme on a uniform grid

Equation (8) acts as the diffusion of the function u in two directions, one given by the vector field W and the other one by the orthogonal direction $W^\perp$. In each of one, the equation behaves like the one dimensional heat equation in this direction. Then, the numerical approach applying finite differences is based on this fact.

We consider a grid which depends on two parameters dx and $d\theta$. The time discretization will be uniform and a monotone scheme for equation (13) can be built, provided that a CFL condition is satisfied.

For a fixed time t and x_0 a reference grid point, denote $x_1, \ldots, x_m$ the x_0 neighbors and $u_i = u(x_i, t)$ the discrete values at x_i, $i = 1, \ldots, m$. We define the direction vectors $v_i = x_i - x_0$ and the distances $d_i = |x_i - x_0|$, $i = 1, \ldots, m$ and we denote as $\hat{v}_i = \frac{v_i}{d_i}$ the normalized vector, $i = 1, \ldots, m$. Define the local spatial resolution

$$dx = \max_{i=1}^{m} d_i,$$

and the local directional resolution

$$d\theta = \max_{v \in S^1} \min_{i=1}^{m} |v - \hat{v}_i|.$$

We choose the neighbors such that if v is a direction vector, then so is $-v$ and $v^\perp$. Then we have the following result

Proposition 3.1 *(*Consistency *) Let u be a $\mathcal{C}^2$ function in a neighborhood of x_0. Suppose $x_1, \dots x_m$ are the neighbors of x_0 such that if $\hat{v}_i$ is a direction vector, then so is $-\hat{v}_i$. Suppose also $\xi \in I\!R^2$ a given direction, $||\xi|| = 1$. Then there is a direction vector $\hat{v}_k$ such that*

$$u_{\xi\xi} = \frac{u_k + u_{k^*} - 2u_0}{d_k^2} + O(dx + d\theta), \tag{15}$$

where x_{k^*} denotes the symmetrical point to x_k with respect to x_0.

Proof: Let $\hat{v}_k$ be the direction vector closest to ξ, (resp. $\hat{v}_{k^*} = -\hat{v}_k$ the closest direction to $-\xi$), that is

$$\max_{1 \le i \le m} < \hat{v}_i, \xi > = < \hat{v}_k, \xi > .$$

By the Taylor expansion we have

$$u_i = u_0 + d_i < \hat{v}_i, Du(x_0) > + \frac{d_i^2}{2} D^2 u\,(\hat{v}_i, \hat{v}_i) + O(dx^3), \quad i = k, k^*.$$

Then, by taking into account that $d_k = d_{k^*}$, we get

$$\frac{u_k + u_{k^*} - 2u_0}{d_k^2} = \frac{1}{2} D^2 u\,(\hat{v}_k, \hat{v}_k) + \frac{1}{2} D^2 u\,(\hat{v}_{k^*}, \hat{v}_{k^*}) + O(dx). \tag{16}$$

By using $|\hat{v}_k - \xi| = O(d\theta)$ and $|\hat{v}_{k^*} + \xi| = O(d\theta)$, we have

$$\begin{array}{rcl} D^2 u\,(\hat{v}_k, \hat{v}_k) & = & D^2 u\,(\hat{v}_k - \xi, \hat{v}_k + \xi) + D^2 u\,(\xi, \xi) \\ & = & D^2 u\,(\xi, \xi) + O(d\theta), \\ D^2 u\,(\hat{v}_{k^*}, \hat{v}_{k^*}) & = & D^2 u\,(\hat{v}_{k^*} - \xi, \hat{v}_{k^*} + \xi) + D^2 u\,(\xi, \xi) \\ & = & D^2 u\,(\xi, \xi) + O(d\theta). \end{array} \tag{17}$$

Then by combining expressions (17) and (16), we have the desired result.

□

Theorem 3.3 *For equation (8), there exists a difference scheme which is consistent and monotone and it converges to the unique viscosity solution.*

Proof: Let (t, x_0) be a discrete point of the grid and suppose that $W(t, x_0) \neq 0$. We denote by $\xi = \frac{W}{||W||}$ and $\eta = \frac{W^\perp}{||W||}$ the normalized vector field and its orthogonal, respectively. Then, the operator associated to equation (8) writes as $F(t, x_0, D^2u) = u_{\xi\xi} + f(||W||)u_{\eta\eta}$.

By Proposition 3.1 there exist two orthogonal vectors u_k and u_l such that

$$u_{\xi\xi} = \frac{u_k + u_{k^*} - 2u_0}{d_k^2} + O(dx + d\theta),$$

$$u_{\eta\eta} = \frac{u_l + u_{l^*} - 2u_0}{d_l^2} + O(dx + d\theta),$$

where x_{k^*} and x_{l^*} denote the symmetrical points to x_k and x_l respectively with respect to x_0.

By using the explicit Euler discretization in time, we get the explicit discrete solution

$$\begin{aligned} u(t + dt, x_0) = (1 - 2\tfrac{dt}{d_k^2} - 2f(||W(t, x_0)||)\tfrac{dt}{d_l^2})u_0 + \\ \tfrac{dt}{d_k^2}(u_k + u_{k^*}) + f(||W(t, x_0)||)\tfrac{dt}{d_l^2}(u_l + u_{l^*}), \end{aligned} \tag{18}$$

where the time step is bounded by the nonlinear CFL condition $dt \leq \frac{dx'^2}{4}$, where $dx' = \min_{i=1}^m d_i$. Then Proposition (3.1) is applied to proof consistency of the method.

If $W(t, x_0) = 0$, then the operator F associated to equation (8) writes as the Laplacian Δu. In this case, there can be chosen any two orthogonal directions and apply the explicit Euler discretization (18) to obtain the consistency of the method with the same CFL condition.

To prove the monotonicity, we note that, in all cases, the discrete map (18) gives us $u(t + dt, x_0)$ as a convex combination of $u(t, x_0)$ and the value of two neighbors, which implies the monotonicity of the scheme.

Finally, the consistence and the monotonicity of the scheme imply its uniform convergence to the unique viscosity solution of equation (8) (see [4]). □

We note that the previous scheme allows only a finite number of smoothing directions. In order to extend this scheme, we propose an alternative one using neighborhood filters.

4 Numerical approximation

In the previous section we built a numerical scheme to proof the existence of solution to the proposed equation. This scheme involves the choice of a certain direction in which the second order derivatives are discretized. This choice can be noise dependent and therefore not accurate. For this reason, we introduce an extension based on neighborhood filters [18, 29] and a recent modification [5].

4.1 Local adaptive averages

Let u be a scalar function defined on a bounded domain $\Omega \subset \mathbb{R}^2$. Then the filtered value at $\mathbf{x}$ is defined as

$$NF_{h,\rho}u(\mathbf{x}) = \frac{1}{C(\mathbf{x})} \int_{B_\rho(\mathbf{x})} u(\mathbf{y}) e^{-\frac{|u(\mathbf{y})-u(\mathbf{x})|^2}{h^2}} d\mathbf{y}, \tag{19}$$

where only pixels inside $B_\rho(\mathbf{x})$ are averaged, h controls the grey level similarity and $C(\mathbf{x}) = \int_{B_\rho(\mathbf{x})} e^{-\frac{|u(\mathbf{y})-u(\mathbf{x})|^2}{h^2}} d\mathbf{y}$ is the normalization factor. This filter is usually called sigma [18] or neighborhood filter [29] and was introduced in the 80's for the restoration of digital images. In order to preserve the discontinuities of the image, only pixels with a similar gray level value to the one being restored are averaged.

The mathematical study of this filter was made in [5]. It is shown that the subjacent PDE of the neighborhood filter is equivalent to the Perona-Malik equation. The neighborhood filter suffers from a staircase effect, which was explained by the inverse diffusion term of its subjacent PDE. The authors also proposed a simple variant of the neighborhood filter replacing the average by a linear regression. This strategy amounts to find for every point $\mathbf{x} = (x_1, x_2)$ the plane locally approximating u in the following sense,

$$\arg \min_{a_0,a_1,a_2} \int_{B_\rho(\mathbf{x})} w(\mathbf{x}, \mathbf{y})(u(\mathbf{y}) - (a_2 y_2 + a_1 y_1 + a_0))^2 \, d\mathbf{y}, \tag{20}$$

where $w(\mathbf{x}, \mathbf{y}) = e^{-\frac{|u(\mathbf{y})-u(\mathbf{x})|^2}{h^2}}$ and then replacing $u(\mathbf{x})$ by the filtered value $a_2 x_2 + a_1 x_1 + a_0$. The weights used to define the minimization problem are the same as the ones used by the neighborhood filter. Thus, the points with a grey level value close to $u(\mathbf{x})$ will have a larger influence in the minimization process than those with a further grey level value. We denote the above linear regression correction as $LNF_{h,\rho}$. The subjacent PDE of this new filter is related to the mean curvature motion.

4.2 Extension for the proposed equation

The *LNF* can be extended following the model introduced in this paper. In fact, the neighborhood filter performs an average of the neighboring pixels, but under the condition that their color is close enough to the one of the pixels in restoration. In some sens, it is equivalent to impose some kind of constraint, the weights, in order to preserve the geometry of the image as the contours. Thinking about the vector-valued images, we can average pixels of the same channel u but the weights can be related to the geometry of another image I.

In other works, we can diffuse the image u conditionally to the geometry of any other image I. In that case the minimization process reads

$$\arg \min_{a_0,a_1,a_2} \int_{B_\rho(\mathbf{x})} w(\mathbf{x},\mathbf{y})(u(\mathbf{y}) - (a_2 y_2 + a_1 y_1 + a_0))^2 \, d\mathbf{y}, \tag{21}$$

where $w(\mathbf{x},\mathbf{y}) = e^{-\frac{|I(\mathbf{y})-I(\mathbf{x})|^2}{h^2}}$. Thus, the image is locally approximated by a plane with the weight distribution computed on the image I.

The next theorem shows that when ρ and h have the same order, the application of the linear regression neighborhood filter is equivalent to one step of the proposed model (8).

Theorem 4.1 *Suppose $u \in C^2(\Omega, R)$, $I \in C^2(\Omega, \mathbb{R})$, and let $\rho, h, \alpha > 0$ such that $\rho, h \to 0$ and $h = O(\rho^\alpha)$. Let $\tilde{f}$ be the continuous function defined as $\tilde{f}(0) = 1$,*

$$\tilde{f}(t) = \frac{3}{2t^2}\left(1 - \frac{2t\, e^{-t^2}}{E(t)}\right),$$

for $t \neq 0$, where $E(t) = 2\int_0^t e^{-s^2}\, ds$. Then, for $\boldsymbol{x} \in \Omega$

1. *If $\alpha < 1$,*

$$LNF_{h,\rho}u(\boldsymbol{x}) - u(\boldsymbol{x}) \simeq \frac{\triangle u(\boldsymbol{x})}{6}\rho^2,$$

where $\triangle u$ denotes the Laplacian of u.

2. *If $\alpha = 1$,*

$$LNF_{h,\rho}u(\boldsymbol{x}) - u(\boldsymbol{x}) \simeq \left[D^2u(\xi,\xi)(\boldsymbol{x}) + \tilde{f}(\frac{\rho}{h}|DI(\boldsymbol{x})|)D^2u(\eta,\eta)(\boldsymbol{x})\right]\frac{\rho^2}{6},$$

where $\xi = \frac{DI^\perp}{|DI|}$ and $\eta = \frac{DI}{|DI|}$ denote respectively the tangent and orthogonal directions to the level line passing through $\boldsymbol{x}$.

Proof: Let us suppose without loss of generality that $\mathbf{x} = 0$. In that case, it is easily seen that

$$LNF_{h,\rho}u(0) - u(0) = \frac{\det \tilde{A}}{\det A},$$

where

$$A = \begin{pmatrix} a(2,0) & a(1,1) & a(1,0) \\ a(1,1) & a(0,2) & a(0,1) \\ a(1,0) & a(0,1) & a(0,0) \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} a(2,0) & a(1,1) & \tilde{b}(1,0) \\ a(1,1) & a(0,2) & \tilde{b}(0,1) \\ a(1,0) & a(0,1) & \tilde{b}(0,0) \end{pmatrix}$$

and

$$a(\alpha_1, \alpha_2) = \int_{B_\rho(0)} t_1^{\alpha_1} t_2^{\alpha_2} w(t_1, t_2)\, dt_1 dt_2,$$

$$\tilde{b}(\alpha_1, \alpha_2) = \int_{B_\rho(0)} t_1^{\alpha_1} t_2^{\alpha_2} w(t_1, t_2)\, [u(t_1, t_2) - u(0)]\, dt_1 dt_2.$$

We take the Taylor expansion of $I(\mathbf{t})$ in the orthogonal system given by $\xi = DI^{\perp}/|DI|$ and $\eta = DI/|DI|$,

$$I(\mathbf{t}) = I(0) + p\, t_1 + I_{\eta\eta} t_1^2 + I_{\xi\xi} t_2^2 + I_{\xi\eta} t_1 t_2 + O(|\mathbf{t}|^3),$$

where $p = |DI(0)|$ and if $p > 0$

$$I_{\xi\xi} = \frac{1}{2} D^2 I(\xi, \xi), \quad I_{\eta\eta} = \frac{1}{2} D^2 I(\eta, \eta), \quad I_{\xi\eta} = D^2 I(\xi, \eta).$$

We take the Taylor expansion of $u(\mathbf{t})$ in the same orthogonal reference,

$$u(\mathbf{t}) = u(0) + u_\eta t_1 + u_\xi t_2 + u_{\eta\eta} t_1^2 + u_{\xi\xi} t_2^2 + u_{\xi\eta} t_1 t_2 + O(|\mathbf{t}|^3),$$

where $u_\xi = \frac{\partial u}{\partial \xi}$, $u_\eta = \frac{\partial u}{\partial \eta}$ and

$$u_{\xi\xi} = \frac{1}{2} D^2 u(\xi, \xi), \quad u_{\eta\eta} = \frac{1}{2} D^2 u(\eta, \eta), \quad u_{\xi\eta} = D^2 u(\xi, \eta).$$

When $\alpha < 1$, we apply the usual Taylor expansion of the exponential function. The terms of lower order of matrices A and $\tilde{A}$ are in their diagonal and the quotient can be approximated by the lower terms of $\tilde{b}(0,0)/a(0,0)$. Therefore, the analysis of the difference reduces to the computation of the two terms,

$$a(0,0) \simeq \int_{B_\rho(0)} dt_1 dt_2 \simeq 4\rho^2$$

and

$$\tilde{b}(0,0) \simeq \int_{B_\rho(0)} (q_{\eta\eta} t_1^2 + q_{\xi\xi} t_2^2)\, dt_1 dt_2 = \frac{4\Delta u}{6}\rho^4.$$

This proves *(1)*.

When $\alpha = 1$, we cannot apply the above expansion and we decompose the weight function as

$$w(t_1,t_2) \simeq e^{-\frac{p^2 t_1^2}{h^2}} \left(1 - \frac{2pt_1}{h^2}(u_{\eta\eta} t_1^2 + u_{\xi\xi} t_2^2 + u_{\xi\eta} t_1 t_2)\right),$$

In this case, the terms of lower order of the matrices A and $\tilde{A}$ are the diagonal elements, $a(1,0)$, $a(0,1)$, $\tilde{b}(1,0)$ and $\tilde{b}(0,1)$. Then, the terms of lower order of the quotient are given by the lower terms of

$$\frac{\det \tilde{A}}{\det A} \simeq \frac{a(2,0)a(0,2)\tilde{b}(0,0) - a(1,0)a(0,2)\tilde{b}(1,0) - a(2,0)a(0,1)\tilde{b}(0,1)}{a(2,0)a(0,2)a(0,0)}. \tag{22}$$

Therefore, the analysis of the difference reduces to the computation of the terms,

$$a(0,0) \simeq \int_{B_\rho(0)} e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2,$$

$$a(0,1) \simeq -\frac{2p}{h^2} \int_{B_\rho(0)} I_{\xi\eta}\, t_1^2 t_2^2\, e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2,$$

$$a(1,0) \simeq -\frac{2p}{h^2} \int_{B_\rho(0)} (I_{\eta\eta} t_1^2 + I_{\xi\xi} t_2^2)\, t_1^2\, e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2,$$

$$a(0,2) \simeq \int_{B_\rho(0)} t_2^2\, e^{-\frac{p^2 t_2^2}{h^2}}\, dt_1 dt_2, \quad a(2,0) \simeq \int_{B_\rho(0)} t_1^2\, e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2,$$

$$\tilde{b}(0,1) \simeq u_\eta \int_{B_\rho(0)} t_2^2\, e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2, \quad \tilde{b}(1,0) \simeq u_\xi \int_{B_\rho(0)} t_1^2\, e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2,$$

$$\begin{aligned}\tilde{b}(0,0) \quad \simeq \quad & \int_{B_\rho(0)} (u_{\xi\xi} t_1^2 + u_{\eta\eta} t_2^2) e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2 - \frac{2p}{h^2}\, u_\eta \int_{B_\rho(0)} t_1^2 (I_{\eta\eta} t_1^2 + I_{\xi\xi} t_2^2) e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2 \\ & -\frac{2p}{h^2}\, u_\xi \int_{B_\rho(0)} I_{\xi\eta} t_1^2 t_2^2\, e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2.\end{aligned}$$

Now, replacing the terms in (22) by the previous estimates we get

$$\frac{1}{\int_{B_\rho(0)} e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2} \int_{B_\rho(0)} (u_{\eta\eta} t_1^2 + u_{\xi\xi} t_2^2)\, e^{-\frac{p^2 t_1^2}{h^2}}\, dt_1 dt_2.$$

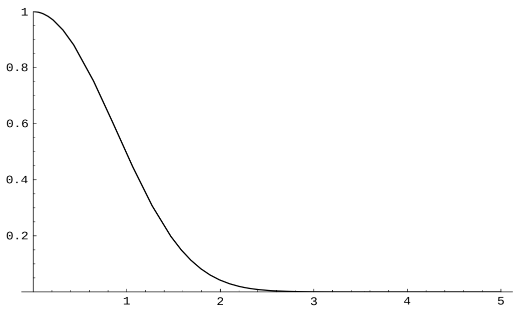

Figure 1: Weighting function of Theorem 4.1.

Computing the previous integrals and taking into account that $O(\rho) = O(h)$ we prove *(2)*. $\square$

The previous theorem tells us that when ρ and h have the same order, the subjacent PDE of this filter is equal to the proposed equation with $W = DI^{\perp}$. As displayed in Figure 1, the weighting function $\tilde{f}$ is positive and quickly decreasing to zero. It can be easily checked that it satisfies all the conditions of Theorem 2.1.

The iteration of this filter is not straightforward. As the filter is iterated the value of h must decrease in accordance. This is due to the decreasing of the gray level differences of pixels as the diffusion processes. For this reason, the above discretization will be only applied when the vector field W is not time dependent. In that case, the weight distribution is computed only once and maintained constant during the iteration process.

5 Discussion

We begin by discussing the application of the proposed equation (8) to color image filtering. The selected field W is equal to the orthogonal direction to the gradient of the luminance component as exposed in the introduction. The vector field is then obtained by diffusing the luminance according to the mean curvature equation, $W(t,x) = DI(t,x)^{\perp}$. This vector field is time dependent. If the vector field is taken constant in time then the sharpness of edges is lost due to the mismatch of color and luminance edges as time increases.

Figure 2 compares the application of the above model with the multivalued extension of the mean curvature motion presented by Sapiro et al [25]. In the mentioned work, the diffusion direction is computed as the direction

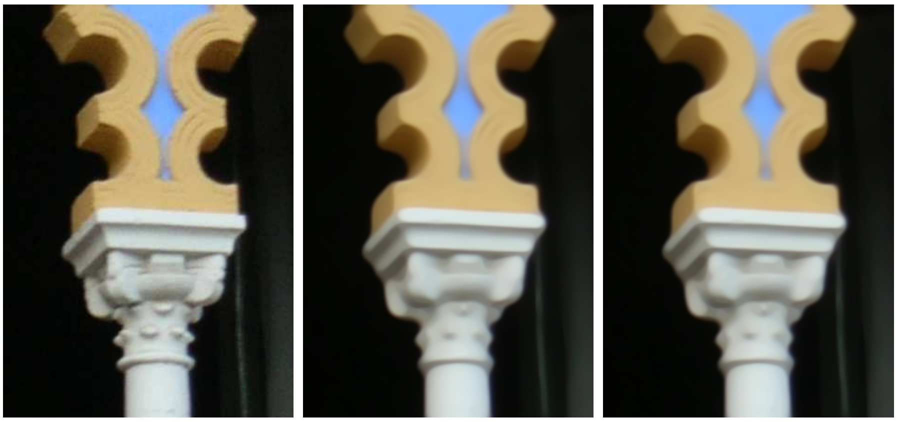

Figure 2: Color filtering comparison. Left: original image. Middle: filtered image by the extension of mean curvature motion [25]. Right: filtered image with the proposed equation. Both equations have been numerically implemented by a finite difference scheme. Even if the solutions are nearly identical, the difference is the theoretical well posedness of the presented model.

of minimum variation. This direction depends non linearly on the image at each time and therefore cannot be set under the conditional framework. Both solutions are nearly identical as displayed in the figure. The main difference of both models is the well posedness of the proposed equation while, as far as we know, there is no existence and uniqueness theorem for the equation proposed in [25].

For other applications is more suitable to take a constant in time vector field. This is the case of giving color to a grey level image. Some initial chromatic information is diffused according to the gradient of the grey level image, $W(x,t) = DI(x)^{\perp}$. Figure 3 displays a colorization experiment with equation (8). The initial chromatic information is used as boundary conditions.

Another application in which the direction field can be constant in time is the inpainting or filling-in problem. The objective is to fill in the missing information of a whole zone of the image in a non recognizable way. PDE approaches for filling-in try to diffuse the geometric information of the boundary inside the hole. A special case is when some information on the geometry inside the hole is available but not the grey level value. Common PDE models for image interpolation are not adapted to this case. Only in [3], the authors take this special case as a motivation for the final proposed functional. The final functional is presented in a general framework but it can be easily adapted to this possibility since the geometry and grey level variables are decoupled.

In Figure 4 we applied the proposed equation (8) to fill in the information on a colour image where only two of the three channels are damaged. The diffusion is only performed on damaged pixels. The orthogonal direction to the gradient of the non-damaged channel is used as the vector field W. This vector field is taken constant in time. We compare the finite difference scheme and the linear neighborhood one. The linear neighborhood scheme is able to continue better the edges as it is less diffusive than a finite difference scheme. This type of structures are the most difficult to fill in for PDE models, since finite difference schemes are rarely able to recover sharp edges. Because of the size of the removed zones, classical inpainting techniques (PDE and Efros-Leung based) are not able to interpolate separately each channel. Efros-Leung based interpolation techniques as [12] could be adapted to this problem in order to take advantage of the green channel but the main inconvenient of such a technique is the lack of a regularity condition assuring the non presence of artifacts.

6 Conclusions

In this work we have shown the feasibility and mathematical well posedness of a diffusion model with an a priori fixed directional field. Based on the theory of viscosity solutions we have proved the existence and uniqueness of the solution. We have also shown that a numerical scheme based on neighborhood filters is a better option to implement such a diffusion than a finite differences scheme. Experiments illustrate how this model can be applied to several image processing tasks including image filtering.

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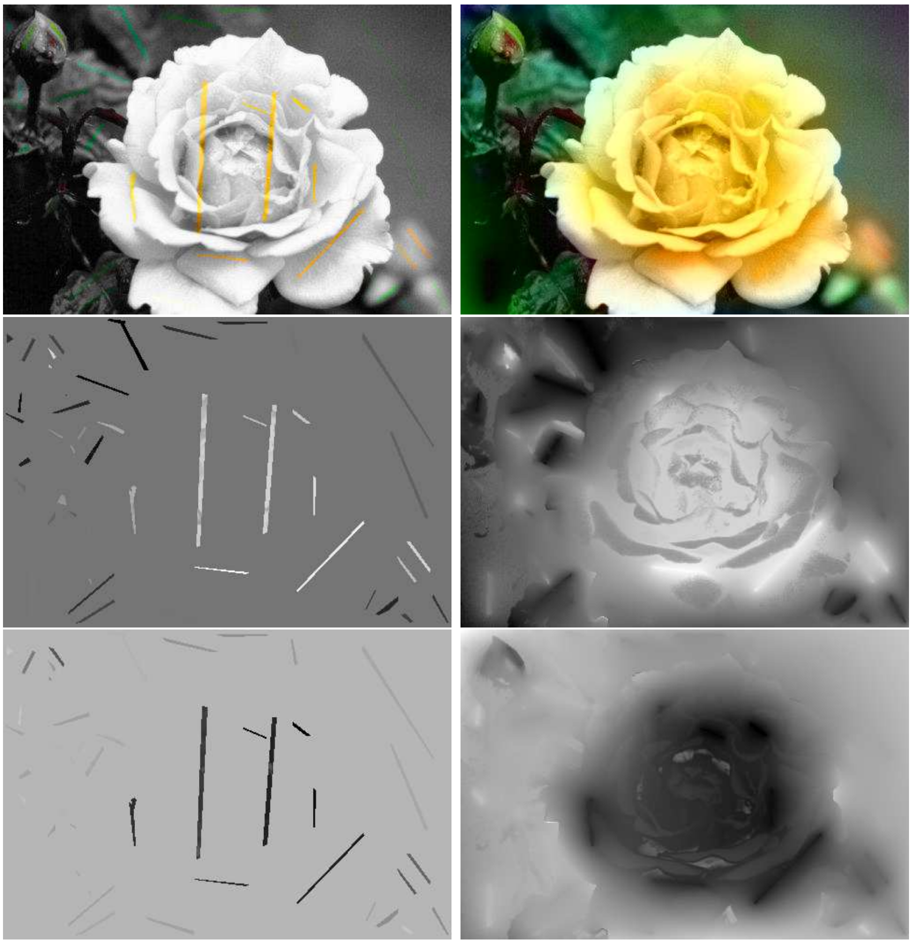

Figure 3: Colorization experiment using the conditional diffusion equation. Top Left: Input image with original luminance and initial data on the chromatic components. Top Right: Result image by applying the linear neighborhood scheme to the chromatic components using the initial chromatic data as boundary conditions. Middle left and bottom left: initial data on the two chromatic components. Middle right and bottom right: final interpolated components.

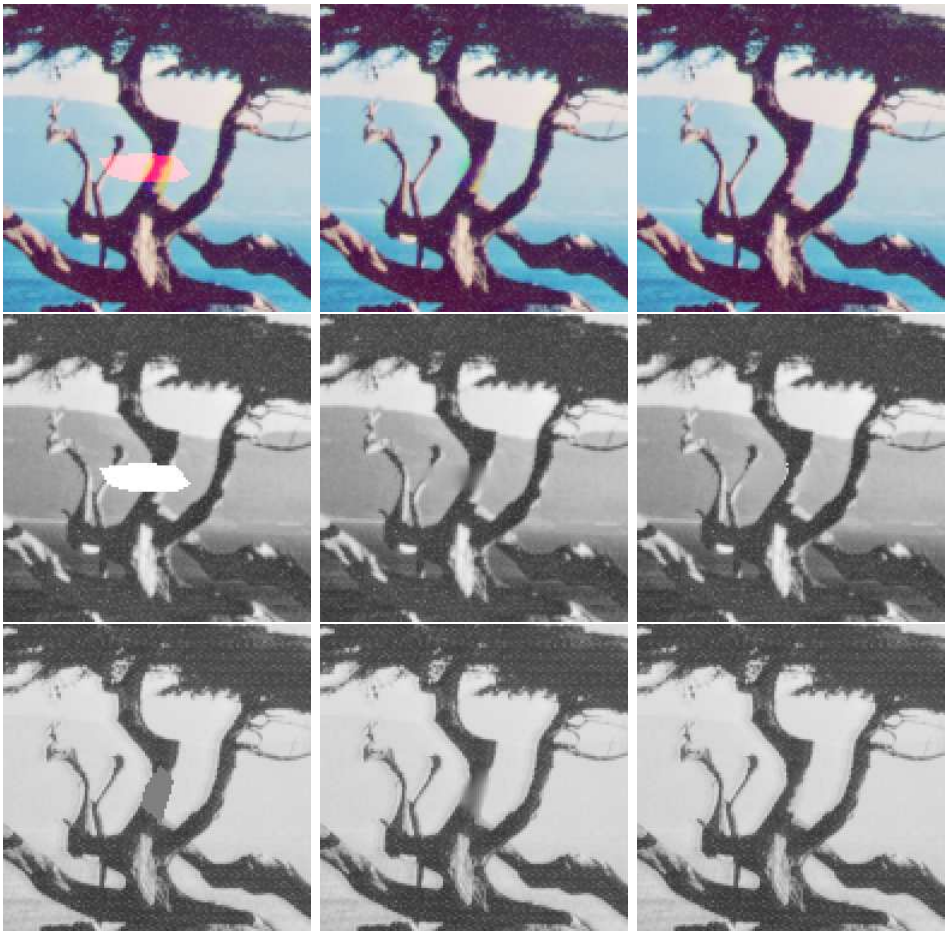

Figure 4: Inpainting example on a colour image where only two of the three channels are damaged. The tangent to the gradient direction of the non-damaged channel is used as the vector field W and maintained constant in time. Top: Original image, inpainted by the finite difference scheme and inpainted by the linear neighborhood scheme. Below: damaged channels and inpainted by the finite difference scheme and the linear neighborhood one. The linear neighborhood scheme is able to better recover the sharp edges as it is less diffusive than a finite difference scheme.