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Abstract

For estimating parameters in an unstable $AR(2)$ model, the paper proposes a sequential least squares estimate with a special stopping time defined by the trace of the observed Fisher information matrix. It is shown that the sequential LSE is asymptotically normally distributed in the stability region and on its boundary in contrast to the usual LSE, having six different types of asymptotic distributions on the boundary depending on the values of the unknown parameters. [1] [2]

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[2] *Key words*: Autoregressive process, least squares estimate, sequential estimation, asymptotic normality .

1 Introduction

Consider an autoregressive $AR(2)$ model

$$x_n = \theta_1 x_{n-1} + \theta_2 x_{n-2} + \varepsilon_n,\ n = 1, 2, \ldots, \tag{1.1}$$

where (x_n) is the observation, (ε_n) is a sequence of independent identically distributed (i.i.d.) random variables with $\mathbf{E}\varepsilon_1 = 0$ and $0 < \mathbf{E}\varepsilon_1^2 = \sigma^2 < \infty$, σ^2 is known, $x_0 = x_{-1} = 0$. The process (1.1) is assumed to can be unstable, that is, both roots of the characteristic polynomial

$$\mathcal{P}(z) = z^2 - \theta_1 z - \theta_2 \tag{1.2}$$

lie on or inside the unit circle. The model (1.1) is a particular case of unstable autoregressive processes $AR(p)$ which have been studied by many authors due to their applications in automatic control, identification and in modeling economic and financial time series (we refer the reader to Anderson (1971), Ahtola and Tiao (1987), Dickey and Fuller (1979), Chan and Wei (1988), Rao (1978) for details and futher references).

A commonly used estimate of parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ is the least squares estimate (LSE)

$$\boldsymbol{\theta}(n) = (\theta_1(n), \theta_2(n))' = M_n^{-1} \sum_{k=1}^{n} X_{k-1} x_k, \quad M_n = \sum_{k=1}^{n} X_{k-1} X_{k-1}', \tag{1.3}$$

where $X_k = (x_k, x_{k-1})'$; the prime denotes the transpose; M_n^{-1} denotes the inverse of matrix M_n if $\det M_n > 0$ and $M_n^{-1} = 0$ otherwise.

It is well known that

$$\sqrt{n}(\boldsymbol{\theta}(n) - \boldsymbol{\theta}) \stackrel{\mathcal{L}}{\Longrightarrow} \mathcal{N}(0, F),\ \text{as } n \to \infty,$$

for all $\boldsymbol{\theta} \in \Lambda$, where Λ is the stability region of process (1.1), that is,

$$\Lambda = \{\boldsymbol{\theta} = (\theta_1, \theta_2)' : -1 + \theta_2 < \theta_1 < 1 - \theta_2,\ |\theta_2| < 1\}, \tag{1.4}$$

$F = F(\boldsymbol{\theta})$ is a positive definite matrix (see, e.g., Anderson (1971), Th. 5.5.7), $\stackrel{\mathcal{L}}{\Longrightarrow}$ indicates convergence in law. If $\boldsymbol{\theta}$ belongs to the boundary $\partial\Lambda$ of the stability region Λ, the limiting distribution of LSE is no longer normal. Moreover, there is no one universal limiting distribution for all $\boldsymbol{\theta} \in \partial\Lambda$ and the corresponding set of limiting distributions numbers 6 different types depending on the values of roots z_1 and z_2 of the polynomial (1.2). Each limiting distribution of LSE on the boundary coincides with that of the ratio of certain Brownian functionals (we refer the reader to the paper of Chan and Wei (1988) for general results on the limiting distributions of the least squares estimates for unstable AR(p) processes and further details). For example, for conjugate complex roots $z_1 = e^{i\varphi}, z_2 = e^{-i\varphi}$ one has

$$n\cdot(\theta_1(n)-2\cos\varphi) \stackrel{\mathcal{L}}{\Longrightarrow} \frac{(W_1^2(1)-W_2^2(1))\sin\varphi + (W_1^2(1)+W_2^2(1)-2)\cos\varphi}{\int_0^1 [W_1^2(s)+W_2^2(s)]ds},$$

$$n\cdot(\theta_2(n)+1) \stackrel{\mathcal{L}}{\Longrightarrow} (2-W_1^2(1)-W_2^2(1))/\int_0^1 [W_1^2(s)+W_2^2(s)]ds\,,$$

where $(W_1(t), 0 \le t \le 1)$ and $(W_2(t), 0 \le t \le 1)$ are independent standard Brownian motion processes; if $\boldsymbol{\theta} = (2, -1)$, then (see Theorem 3.1.2 ibid)

$$\begin{pmatrix} n^2 & 0 \\ 0 & n \end{pmatrix} (\boldsymbol{\theta}(n) - \boldsymbol{\theta}) \stackrel{\mathcal{L}}{\Longrightarrow} G^{-1}\xi, \quad \xi = \begin{pmatrix} \int_0^1 Z(t)dW(t) \\ \int_0^1 W(t)dW(t) \end{pmatrix},$$

where

$$G = \begin{pmatrix} \int_0^1 W^2(t)dt & \int_0^1 W(t)Z(t)dt \\ \int_0^1 W(t)Z(t)dt & \int_0^1 Z^2(t)dt \end{pmatrix},\ Z(t) = \int_0^t W(s)ds\,.$$

It is well-known that a similar situation takes place in case of $AR(1)$ process

$$x_n = \theta x_{n-1} + \varepsilon_n, \tag{1.5}$$

for which the limiting distributions of the least squares estimate are not normal at the end-points $\theta = \pm 1$ of stability interval (-1,1) (see White (1958), Lai and Siegmund (1983)).

Lai and Siegmund (1983) for a first order non-explosive autoregressive process (1.5) proposed to use a sequential sampling scheme and proved that the sequential least squares estimate for θ with the stopping time based on the observed Fisher information is asymptotically normal uniformly in $\theta \in [-1, 1]$ in contrast with the ordinary LSE.

In this paper we develop a sequential sampling scheme for estimating parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ in model (1.1). We will use the sequential least squares estimate defined by the formula

$$\boldsymbol{\theta}(\tau(h)) = M_{\tau(h)}^{-1} \sum_{k=1}^{\tau(h)} X_{k-1} x_k, \tag{1.6}$$

where $\tau(h)$ is the stopping time for the threshold $h > 0$:

$$\tau(h) = \inf\{n \geq 1 : \sum_{k=1}^{n} (x_{k-1}^2 + x_{k-2}^2) \geq h\sigma^2\}, \ \inf\{\emptyset\} = +\infty. \tag{1.7}$$

This construction of sequential estimate is similar to that proposed in the paper of Lai and Siegmund for $AR(1)$ which is defined as

$$\hat{\theta}_{\tau(h)} = \left(\sum_{k=1}^{\tau(h)} x_{k-1}^2 \right)^{-1} \sum_{k=1}^{\tau(h)} x_{k-1} x_k \,, \tag{1.8}$$

$$\tau(h) = \inf\{n \geq 1 : \sum_{k=1}^{n} x_{k-1}^2 \geq h\sigma^2\}. \tag{1.9}$$

It should be noted, however, that the first factor in (1.6) is a random matrix and not a random variable, as in (1.8), and this makes additional difficulties.

For AR(1) the stopping time (1.9) turns the denominator in (1.8) practically into a constant $h\sigma^2$ and this allows to use the central limit theorem for martingales. In the case of $AR(2)$ the stopping time (1.7) enables one to control the inverse matrix $M_{\tau(h)}^{-1}$ in (1.6) only partially since it remains random. Nevertheless, we will see that such a change of time also enables one to improve the properties of the estimate (1.3).

In our paper (2006) we proved the following result.

Theorem 1.1. *Let $(\varepsilon_n)_{n\geq 1}$ in (1.1) be a sequence of i.i.d. random variables with $E\varepsilon_n = 0,\ 0 < E\varepsilon_n^2 = \sigma^2 < \infty$. Then, for any compact set $K \subset \Lambda_1$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \sup_{\boldsymbol{t}\in R^2} |\mathbf{P}_{\boldsymbol{\theta}} \left(M_{\tau(h)}^{1/2}(\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) \leq \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t}/\sigma)| = 0,$$

where $\Phi_2(\boldsymbol{t}) = \Phi(t_1)\Phi(t_2),\ \Phi$ is the standard normal distribution function,

$$\Lambda_1 = \{\boldsymbol{\theta} = (\theta_1, \theta_2)' :\ -1 + \theta_2 < \theta_1 < 1 - \theta_2,\ -1 \leq \theta_2 < 1\},\ \boldsymbol{t} = (t_1, t_2)'.$$

This theorem implies, in particular, that estimate (1.6) is asymptotically normal not only inside the stability region (1.4) but also on the part of its boundary $\{\boldsymbol{\theta} = (\theta_1, -1)' : -2 < \theta_1 < 2\}$ in contrast to the LSE (1.3).

The goal of this paper is to prove the asymptotic normality of the estimate (1.6),(1.7) in the whole region $[\Lambda]$ including its boundary $\partial\Lambda$.

Our main result (Theorem 3.1) claims that, as $h \to \infty$,

$$M_{\tau(h)}^{1/2}(\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) \stackrel{\mathcal{L}}{\Longrightarrow} \mathcal{N}(0, \sigma^2 I), \tag{1.10}$$

for any $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ inside the stability region Λ (1.4) and on its boundary $\partial\Lambda$, where I is the identity matrix. Thus the sequential estimate (1.6), (1.7) has a unique normal asymptotic distribution in the closure $[\Lambda]$ of the stability region (1.4). It will be observed that the normalizing factor $M_{\tau(h)}^{1/2}$ in the limit theorem (1.10) remains the same in the whole region $[\Lambda]$ in contrast to the case of the LSE (1.3), which has seven different limiting distributions in $[\Lambda]$ and in order to apply the limiting distributions one needs some knowledge about the location of unknown parameters (see Chan and Wei (1988)). The convergence of the sequential estimate (1.6), (1.7) to the normal distribution in (1.10) is not uniform in $\boldsymbol{\theta}$ for $\boldsymbol{\theta} \in [\Lambda]$. It can be explained by the fact that in the case, when the polynomial (1.2) has one root inside and the other on the unit circle, the rates of information provided by sample values x_n about the unknown parameters θ_1 and θ_2 may differ greatly.

Theorem 3.1 permits setting up tests of hypotheses about $\boldsymbol{\theta}$ and forming asymptotic confidence regions for $\boldsymbol{\theta}$ on the basis of standard normal distribution. Moreover, the asymptotic normality holds in $[\Lambda]$ for a broad class of the distributions of noises (ε_n).

The remainder of this paper is arranged as follows. Section 2 gives the asymptotic distribution of the stopping time (1.7) (Theorem 2.1) and some properties of the observed Fisher information matrix. In section 3 the asymptotic normality of sequential estimate (1.6) for unstable $AR(2)$ model is established (Theorem 3.1). Section 4 proposes the sequential estimation scheme for the case of unknown variance σ^2 in model (1.1). The appendix contains some technical results.

2 Properties of the stopping time $\tau(h)$ and the observed Fisher information matrix M_n.

In this section the attention is mainly focused on the case when the unknown parameter $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ belongs to the boundary $\partial\Lambda$ of the stability region (1.4). The boundary $\partial\Lambda$ includes three sides:

$$\Gamma_1 = \{\boldsymbol{\theta} : -\theta_1 + \theta_2 = 1, -2 < \theta_1 < 0\}, \Gamma_2 = \{\boldsymbol{\theta} : \theta_1 + \theta_2 = 1, 0 < \theta_1 < 2\},$$

$$\Gamma_3 = \{\boldsymbol{\theta} : -2 < \theta_1 < 2, \theta_2 = -1\} \tag{2.1}$$

and three apexes $(0, 1), (-2, -1), (2, -1)$. Denote

$$A = \begin{pmatrix} \theta_1 & \theta_2 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$W^{(n)}(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=0}^{[nt]} \varepsilon_i, \; W_1^{(n)}(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=0}^{[nt]} (-1)^i \varepsilon_i, \; 0 \le t \le 1, \tag{2.2}$$

and introduce the following functionals

$$\mathcal{J}_1(x;t)=\int_0^t x^2(s)ds,\ \mathcal{J}_2(x;t)=\int_0^t\left(\int_0^s x(u)du\right)^2 ds, \tag{2.3}$$

$$\mathcal{J}_3(x;y;t)=\int_0^t (x^2(s)+y^2(s))ds,\ \mathcal{J}_4(x;t)=\left(\int_0^t x(s)ds\right)^2.$$

Theorem 2.1. *Let $(\varepsilon_n)_{n\geq 1}$ in (1.1) be a sequence of i.i.d. random variables with $E\varepsilon_n=0, E\varepsilon_n^2=\sigma^2$ and $\tau(h)$ be defined by (1.7). Denote by a and b real roots of the polynomial (1.2), $-1\leq a<b\leq 1$. Then, for each $\boldsymbol{\theta}\in\Lambda$,*

$$\mathbf{P}_{\boldsymbol{\theta}}-\lim_{h\to\infty}\tau(h)/h=1/trF,\ F-AFA'=B. \tag{2.4}$$

Moreover, for each $\boldsymbol{\theta}\in\partial\Lambda$, as $h\to\infty$,

$$\frac{\tau(h)}{\psi(\boldsymbol{\theta},h)}\stackrel{\mathcal{L}}{\Longrightarrow}\begin{cases}\nu_1(W_1)=\inf\{t\geq 0:\mathcal{J}_1(W_1;t)\geq 1\}\text{ if }\boldsymbol{\theta}\in\Gamma_1,\\ \nu_2(W)=\inf\{t\geq 0:\mathcal{J}_1(W;t)\geq 1\}\text{ if }\boldsymbol{\theta}\in\Gamma_2,\\ \nu_3(W,W_1)=\inf\{t\geq 0:\mathcal{J}_3(W;W_1;t)\geq 1\}\text{ if }\boldsymbol{\theta}\in\Gamma_3\cup\{(0,1)\},\\ \nu_4(W)=\inf\{t\geq 0:\mathcal{J}_2(W;t)\geq 1\}\text{ if }\boldsymbol{\theta}=(2,-1),\\ \nu_5(W_1)=\inf\{t\geq 0:\mathcal{J}_2(W_1;t)\geq 1\}\text{ if }\boldsymbol{\theta}=(-2,-1),\end{cases} \tag{2.5}$$

where $\inf\{\emptyset\}=\infty$, Λ *is defined in (1.4),*

$$\psi(\boldsymbol{\theta},h)=\begin{cases}(1+b)\sqrt{h/2}\text{ if }\boldsymbol{\theta}\in\Gamma_1,\\ (1-a)\sqrt{h/2}\text{ if }\boldsymbol{\theta}\in\Gamma_2,\\ \sqrt{2h}\sin\varphi\text{ if }\boldsymbol{\theta}=(2\cos\varphi,-1)'\in\Gamma_3,\\ \sqrt{2h}\text{ if }\boldsymbol{\theta}=(0,1),\\ (h/2)^{1/4}\text{ if }\boldsymbol{\theta}\in\{(-2,-1),(2,-1)\},\end{cases} \tag{2.6}$$

$W(t), W_1(t)$ are independent standard Brownian motions.

Proof Assertion (2.4) easily follows from Lemma 3.12 in [6].

For $\boldsymbol{\theta}\in\partial\Lambda$ we decompose the original process (1.1) into two processes $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ using the transformation

$$QX_k=(u_k,v_k)', \tag{2.7}$$

where Q is a non-degenerate constant matrix of size 2×2 which will be chosen later depending on the values of $\boldsymbol{\theta}$. The limiting relation (2.5) for $\boldsymbol{\theta} \in \cup_{i=1}^{3} \Gamma_i$ has been proved in [7], Th 2.2. It remains to consider the apexes $(2,-1), (-2,-1), (0,1)$.

For $\boldsymbol{\theta} = (2,-1)$, putting in (2.7)

$$Q = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{2.8}$$

one obtains

$$v_k = \sum_{j=1}^{k} \varepsilon_j, \;\; u_k = \sum_{j=1}^{k} (x_j - x_{j-1}) = \sum_{j=1}^{k} v_j = \sum_{j=1}^{k} \sum_{i=1}^{j} \varepsilon_i \,,$$

$$\sum_{k=1}^{n} \|X_{k-1}\|^2 = \sum_{k=1}^{n} u_{k-1}^2 + \sum_{k=1}^{n} u_{k-2}^2 = 2 \sum_{k=1}^{n} u_{k-1}^2 - u_{n-1}^2 \,. \tag{2.9}$$

By the definition of $\tau(h)$ in (1.7), one gets

$$\mathbf{P}_{\boldsymbol{\theta}}\{\tau(h) \le t h^{1/4}\} = \mathbf{P}_{\boldsymbol{\theta}}\{ \sum_{k=1}^{[th^{1/4}]} \|X_{k-1}\|^2 \ge h\sigma^2\} \tag{2.10}$$

$$= \mathbf{P}_{\boldsymbol{\theta}}\{\frac{2}{h\sigma^2} \sum_{k=1}^{[th^{1/4}]} u_{k-1}^2 - \frac{1}{h\sigma^2} u_{[th^{1/4}]-1}^2 \ge 1\} \,.$$

Further we show (by the argument similar to that in the proof of Lemma 2.3 in the Appendix) that the sum

$$S_n(t) = \frac{1}{n^4 \sigma^2} \sum_{k=1}^{[nt]} u_{k-1}^2$$

satisfies the relation

$$S_n(t) = \mathcal{J}_2(W^{(n)}; t) + g^{(n)}(t),$$

where $g^{(n)}(t)$ is a random process such that, for any $\delta > 0$,

$$\lim_{n \to \infty} \mathbf{P}_{\boldsymbol{\theta}}(|g^{(n)}(t)| > \delta) = 0.$$

Now we check that

$$\lim_{n\to\infty} u_n^2/n^4 = 0 \quad \mathbf{P}_{\boldsymbol{\theta}} - \text{a.s.}\,. \tag{2.11}$$

By the Cauchy-Schwarz-Bunyakovskii inequality and the law of iterated logarithm we have

$$u_n^2/n^4 \le n^{-3}\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\varepsilon_i\right)^2,\ \sum_{k=1}^{n}\frac{1}{k^3}\left(\sum_{j=1}^{k}\varepsilon_i\right)^2 < \infty \quad \mathbf{P}_{\boldsymbol{\theta}} - \text{a.s.}\,.$$

These inequalities, in virtue of the Kronecker Lemma, imply (2.11).

From here and (2.10), (2.11), we obtain

$$\mathbf{P}_{\boldsymbol{\theta}}\left(\tau(h)/\psi(\boldsymbol{\theta},h) \le t\right) = \mathbf{P}_{\boldsymbol{\theta}}(\nu_{\boldsymbol{\theta}}^{(n)} \le t) + \beta_{\boldsymbol{\theta}}(h)\,,$$

where

$$\nu_{\boldsymbol{\theta}}^{(n)} = \inf\{t \ge 0 : \mathcal{J}_2(W^{(n)};t) \ge 1\},\ \lim_{h\to\infty}\beta_{\boldsymbol{\theta}}(h) = 0\,,$$

$W^{(n)}(t)$ is given in (2.2). This, by the functional Donsker theorem (see Billingsley (1968)), leads to (2.5) for $\boldsymbol{\theta} = (2,-1)$.

The case of the apexes $(0,1),(-2,-1)$ can be considered similarly with the use of Theorem 5.14 given in the Appendix. This completes the proof of Theorem 2.1. □

Now we will establish some properties of the observed Fisher information matrix M_n. Introduce the following subsets of the closed region $[\Lambda]$:

$$\Lambda_d = [\Lambda]\setminus\bigcup_{i=1}^{2} B_i,\ \Lambda_d = \Lambda_{d,1} + \Lambda_{d,2}, \tag{2.12}$$

where

$$\Lambda_{d,1} = \Lambda_d \cap V_d,\ \Lambda_{d,2} = \Lambda_d\setminus\Lambda_{d,1};$$

$$V_d = \left\{\boldsymbol{\theta} : -2 + \frac{d}{\sqrt{2}} \le \theta_1 \le 0,\ \frac{-\theta_1^2}{4} + \frac{d^2}{8} < \theta_2 \le 1 + \theta_1\right\}$$

$$\cup\left\{\boldsymbol{\theta} : 0 \le \theta_1 \le 2 - \frac{d}{\sqrt{2}},\ \frac{-\theta_1^2}{4} + \frac{d^2}{8} \le \theta_2 \le 1 - \theta_1\right\};$$

B_i are open balls of radius $d > 0$ centered at the apexes $(-2,-1),(2,-1)$.

In view of Theorem 1.1, it suffices to study the properties of M_n only for the parametric subset $\Lambda_{d,1}$ and the apexes $(-2,-1),(2,-1)$. In the case of $\Lambda_{d,1}$, one can use the transformation (2.7) with

$$Q = \begin{pmatrix} 1 & -b \\ 1 & -a \end{pmatrix}, \tag{2.13}$$

where $-1 \le a < b \le 1$. Substituting (2.7) and (2.13) in M_n (1.3) yields

$$M_n = Q^{-1} S_n (Q')^{-1} = Q^{-1} R_n^{-1} J_n R_n^{-1} (Q')^{-1}, \tag{2.14}$$

where

$$S_n = \begin{pmatrix} (u,u)_n & (u,v)_n \\ (u,v)_n & (v,v)_n \end{pmatrix}, \; R_n = \begin{pmatrix} (u,u)_n^{-1/2} & 0 \\ 0 & (v,v)_n^{-1/2} \end{pmatrix},$$

$$J_n = R_n S_n R_n = \begin{pmatrix} 1 & \xi_n \\ \xi_n & 1 \end{pmatrix}, \tag{2.15}$$

$$\xi_n = (u,u)_n^{-1/2} (v,v)_n^{-1/2} (u,v)_n \,, \; (u,v)_n = \sum_{k=1}^{n} u_{k-1} v_{k-1}. \tag{2.16}$$

Proposition 2.2. *Under conditions of Theorem 2.1, for any $d > 0, \delta > 0$,*

$$\lim_{h \to \infty} \sup_{\boldsymbol{\theta} \in \Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}} \left(\| J_{\tau(h)} - T(\theta_1, \theta_2) \| > \delta \right) = 0, \tag{2.17}$$

where

$$T(\theta_1, \theta_2) = \begin{pmatrix} 1 & r(a,b) \\ r(a,b) & 1 \end{pmatrix}, \; r(a,b) = \frac{\sqrt{1-a^2}\sqrt{1-b^2}}{1-ab}. \tag{2.18}$$

The proof of Proposition 2.2 is given in the Appendix.

Further we consider the asymptotic behaviour of the matrix J_n in the extreme cases when the process x_k is "most" unstable, that is, $\boldsymbol{\theta}$ coincides with one of the apexes $(-2,-1),(2,-1)$ of the parametric region $[\Lambda]$.

For $\boldsymbol{\theta} = (2, -1)$ we take the matrix Q from (2.8). This yields

$$u_k = \sum_{j=0}^{k} \sum_{i=0}^{j} \varepsilon_i, \ v_k = \sum_{j=0}^{k} \varepsilon_j, \ k \geq 1, \ u_0 = v_0 = \varepsilon_0 = 0. \tag{2.19}$$

For $\boldsymbol{\theta} = (-2, -1)$ we take

$$Q = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

This implies

$$u_k = (-1)^k \sum_{j=1}^{k} \sum_{i=1}^{j} (-1)^i \varepsilon_i, \ v_k = \sum_{j=1}^{k} (-1)^j \varepsilon_j.$$

Lemma 2.3. *Let ξ_n be given by (2.16) and $\boldsymbol{\theta} \in \{(-2, -1), (2, -1)\}$. Then*

$$\xi_n \stackrel{\mathcal{L}}{\Longrightarrow} \begin{cases} \varphi(W) \text{ if } \boldsymbol{\theta} = (2, -1), \\ \varphi(W_1) \text{ if } \boldsymbol{\theta} = (-2, -1), \end{cases} \text{ as } n \to \infty, \tag{2.20}$$

where

$$\varphi(W) = 2^{-1} \mathcal{J}_2^{-1/2}(W; 1) \mathcal{J}_1^{-1/2}(W; 1) \mathcal{J}_4(W; 1). \tag{2.21}$$

The proof of Lemma 2.3 is given in the Appendix.

3 Asymptotic normality.

It is known that the sequential least squares estimate (1.6),(1.7) is asymptotically normal just like the ordinary LSE for any value of $\boldsymbol{\theta}$ in the stability region Λ. Moreover, according to Theorem 1.1, this convergence of sequential LSE to normal law is uniform in $\boldsymbol{\theta}$ belonging to any compact set in Λ supplemented with the part of its boundary corresponding to complex roots of the polynomial (1.2). In this section, we will show that in contrast with the ordinary LSE (c.f. Chen and Wei (1988)), the sequential LSE is asymptotically normal also on the boundary $\partial\Lambda$ of the stability region Λ.

Theorem 3.1. *Suppose that in AR(2) model (1.1),* $(\varepsilon_n)_{n\geq 1}$ *is a sequence of i.i.d. random variables,* $E\varepsilon_n = 0$ *and* $0 < E\varepsilon_n^2 = \sigma^2 < \infty$. *Define* $\tau(h)$, $\boldsymbol{\theta}(\tau(h))$ *and* $M_{\tau(h)}$ *as in (1.6),(1.7) and (1.3). Then for any* $\boldsymbol{\theta} \in [\Lambda]$

$$\lim_{h\to\infty} \sup_{\boldsymbol{t}\in R^2} \left| \mathbf{P}_{\boldsymbol{\theta}} \left(M_{\tau(h)}^{1/2} (\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) \leq \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t}/\sigma) \right| = 0, \tag{3.1}$$

where $\Phi_2(\boldsymbol{t}) = \Phi(t_1)\Phi(t_2)$, $\boldsymbol{t} = (t_1, t_2)'$, Φ *is the standard normal distribution function;* $[\Lambda]$ *is the closure of the stability region (1.4).*

Proof of Theorem 3.1 In view of Theorem 1.1, we have to show (3.1) for $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2 \cup \{(0,1), (-2,-1), (2,-1)\}$. First we note that if $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2 \cup \{(0,1)\}$, the minimal and the maximal roots a and b of the polynomial (1.2) satisfy the inequalities $-1 \leq a < b \leq 1$. Therefore one can use the transformation (2.7),(2.13) to decompose the original process AR(2) (1.1) into two processes (u_k) and (v_k) which obey the equations

$$u_k = au_{k-1} + \varepsilon_k, \; v_k = bv_{k-1} + \varepsilon_k, \; u_0 = v_0 = 0. \tag{3.2}$$

Since the matrix Q in (2.13) is non-degenerate, one can represent the observed Fisher information matrix M_n in the form (2.14) to obtain

$$M_n^{1/2} = Q^{-1} R_n^{-1} J_n^{1/2}. \tag{3.3}$$

Substituting this matrix in the standardized deviation of the sequential estimate (1.6), one gets

$$M_{\tau(h)}^{1/2} (\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) = M_{\tau(h)}^{-1/2} \sum_{k=1}^{\tau(h)} X_{k-1}\varepsilon_k$$

$$= J_{\tau(h)}^{-1/2} R_{\tau(h)} \sum_{k=1}^{\tau(h)} Q X_{k-1}\varepsilon_k = J_{\tau(h)}^{-1/2} \boldsymbol{Z}_{\tau(h)},$$

where

$$\boldsymbol{Z}_n = \begin{pmatrix} (u,u)_n^{-1/2} \sum_{k=1}^n u_{k-1}\varepsilon_k \\ (v,v)_n^{-1/2} \sum_{k=1}^n v_{k-1}\varepsilon_k \end{pmatrix}. \tag{3.4}$$

Further we note that Proposition 2.2 implies that, for any $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Gamma_1\cup\Gamma_2\cup\{(0,1)\}} \mathbf{P}_{\boldsymbol{\theta}} \left(\|J_{\tau(h)}^{-1/2} - I\| > \delta \right) = 0\,. \tag{3.5}$$

Therefore in order to prove (3.1) for $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2 \cup \{(0,1)\}$ it suffices to establish the following result.

Proposition 3.2. *Let $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2 \cup \{(0,1)\}$. Then, for each constant vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)' \in R^2$ with $\|\boldsymbol{\lambda}\| = 1$, the random variable*

$$Y_h = \boldsymbol{\lambda}^{'} \boldsymbol{Z}_{\tau(h)}/\sigma \tag{3.6}$$

is asymptotically normal with mean 0 and unit variance, as $h \to \infty$, that is,

$$\lim_{h\to\infty} \sup_{t\in R} |\mathbf{P}_{\boldsymbol{\theta}}(Y_h \le t) - \Phi(t)| = 0.$$

The main difficulty in the analysis of Y_h is that the stopping time (1.7) enables one to control the sums $(u,u)_{\tau(h)},\ (v,v)_{\tau(h)}$ in the denominators of (3.6) only partially because one of them or both are random variables even in the asymptotics as $h \to \infty$.

The proof of Proposition 3.2 is given in the Appendix. The key idea of the proof is to replace Y_h by a more tractable random variable $\tilde{Y}_h$ equivalent to Y_h in distribution by making use of the Skorohod coupling theorem and then apply the Central Limit Theorem for martingales. The appendix contains also the proof of Theorem 3.1 for the case of $\boldsymbol{\theta} \in \{(-2,-1),(2,-1)\}$. This case is considered separately because the matrix J_n in (3.3) converges, according to Lemma 2.3, only in distribution. □

4 Asymptotic normaliy in the case of unknown variance.

In this section, we extend the sequential estimation scheme to model (1.1) with unknown variance. It is shown that the sequential least squares es-

timate modified to embrace this case remains asymptotically normal uniformly in $\boldsymbol{\theta}$ for any compact set in the region $\Lambda_1 = \Lambda \cup \Gamma_3$ (Th. 4.1) and it is asymptotically normal in the closure of the stability region $[\Lambda]$ (Th. 4.2).

Suppose that the variance σ^2 in (1.1) is unknown. A commonly used estimate for σ^2 in autoregression processes on the basis of observations $(x_1, \ldots, x_n)$ is defined as

$$\hat{\sigma}_n^2 = n^{-1} \sum_{k=1}^{n} (x_k - \boldsymbol{\theta}'(n) X_{k-1})^2 \,, \tag{4.1}$$

where $\boldsymbol{\theta}(n)$ is the least squares estimate of $\boldsymbol{\theta}$ defined in (1.3). Now we must modify the stopping time (1.7). At first sight, to this end one should replace σ^2 in (1.7) by $\hat{\sigma}_n^2$. However, we will use a different modification similar to that proposed by Lai and Siegmund for AR(1) model, which turns out to be more convenient in the theoretic studies. Define the sequential estimate as

$$\boldsymbol{\theta}(\hat{\tau}(h)) = M_{\hat{\tau}(h)}^{-1} \sum_{k=1}^{\hat{\tau}(h)} X_{k-1} x_k \,, \tag{4.2}$$

$$\hat{\tau}(h) = \inf\{n \geq 3 : \sum_{k=1}^{n} (x_{k-1}^2 + x_{k-2}^2) \geq h s_n^2\} \,, \tag{4.3}$$

where $s_n^2 = \hat{\sigma}_n^2 \vee \delta_n,\ \delta_n$ is a sequence of positive numbers with $\delta_n \to 0$.

The main results of this section are stated in the following theorems.

Theorem 4.1. *Let $(\varepsilon_n)_{n\geq 1}$ in (1.1) be a sequence of i.i.d. random variables, $E\varepsilon_n = 0,\ 0 < E\varepsilon_n^2 = \sigma^2 < \infty$. Then, for any compact set $K \subset \Lambda_1$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \sup_{\boldsymbol{t}\in R^2} |\mathbf{P}_{\boldsymbol{\theta}} \left(M_{\hat{\tau}(h)}^{1/2} (\boldsymbol{\theta}(\hat{\tau}(h)) - \boldsymbol{\theta}) / \hat{\sigma}_{\hat{\tau}(h)} \leq \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t})| = 0, \tag{4.4}$$

where $\Phi_2(\boldsymbol{t}) = \Phi(t_1)\Phi(t_2),\ \Phi$ is the standard normal distribution function,

$$\Lambda_1 = \{\boldsymbol{\theta} = (\theta_1, \theta_2)' :\ -1 + \theta_2 < \theta_1 < 1 - \theta_2,\ -1 \leq \theta_2 < 1\},\ \boldsymbol{t} = (t_1, t_2)' \,.$$

Theorem 4.2. *Let $(\varepsilon_n)_{n\geq 1}$ in (1.1) be a sequence of i.i.d. random variables, $E\varepsilon_n = 0,\ 0 < E\varepsilon_n^2 = \sigma^2 < \infty$. Then, for any $\boldsymbol{\theta} \in [\Lambda]$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{t}\in R^2} |\mathbf{P}_{\boldsymbol{\theta}} \left(M_{\hat{\tau}(h)}^{1/2}(\boldsymbol{\theta}(\hat{\tau}(h)) - \boldsymbol{\theta})/\hat{\sigma}_{\hat{\tau}(h)} \leq \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t})| = 0\,.$$

The proofs of Theorems 4.1- 4.2 proceed along the lines of those of Theorems 1.1 and 3.1 though they become more laborious because one needs to control the additional terms appearing as a result of the unknown variance. We will give only the proof of Theorems 4.1.

Proof of Theorems 4.1. Substituting (1.1) in (4.2) yields

$$M_{\hat{\tau}(h)}^{1/2}(\boldsymbol{\theta}(\hat{\tau}(h)) - \boldsymbol{\theta})/\hat{\sigma}_{\hat{\tau}(h)} = M_{\hat{\tau}(h)}^{-1/2} \sum_{k=1}^{\hat{\tau}(h)} X_{k-1}\varepsilon_k/\hat{\sigma}_{\hat{\tau}(h)}$$

$$= \left(M_{\hat{\tau}(h)}\hat{\sigma}_{\hat{\tau}(h)}^2/(\sigma^4 h/2) \right)^{-1/2} \sum_{k=1}^{\hat{\tau}(h)} X_{k-1}\varepsilon_k/(\sigma^2\sqrt{h/2})\,. \tag{4.5}$$

Further we need the following results.

Lemma 4.3. *Let $M_n, \hat{\tau}(h)$ be given by (1.3), (4.3). Then, for any compact set $K \subset \Lambda_1$ and $\delta > 0$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}} \left(\|M_{\hat{\tau}(h)}\hat{\sigma}_{\hat{\tau}(h)}^2/(\sigma^4 h/2) - L(\theta_1,\theta_2)\| > \delta \right) = 0\,, \tag{4.6}$$

where

$$L(\theta_1,\theta_2) = \begin{pmatrix} 1 & \theta_1/(1-\theta_2) \\ \theta_1/(1-\theta_2) & 1 \end{pmatrix}.$$

Lemma 4.4. *Under the assumptions of Theorem 1.1, for any compact set $K \subset \Lambda_1$ and for each constant vector $\boldsymbol{\lambda} = (\lambda_1,\lambda_2)'$ with $\|\boldsymbol{\lambda}\| = 1$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \sup_{t\in R} |\mathbf{P}_{\boldsymbol{\theta}}(Y_h \leq t) - \Phi(t)| = 0\,,$$

where

$$Y_h = \boldsymbol{\lambda}' L^{-1/2}(\theta_1,\theta_2) \sum_{k=1}^{\hat{\tau}(h)} X_{k-1}\varepsilon_k/(\sigma^2\sqrt{h/2})\,.$$

The proofs of these Lemmas are given below in this section.

Now we rewrite (4.5) as

$$M_{\hat\tau(h)}^{1/2}(\boldsymbol{\theta}(\hat\tau(h))-\boldsymbol{\theta})/\hat\sigma_{\hat\tau(h)} = \left(M_{\hat\tau(h)}\hat\sigma^2_{\hat\tau(h)}/(\sigma^4 h/2)\right)^{-1/2} L^{1/2}(\theta_1,\theta_2) \quad (4.7)$$

$$\times L^{-1/2}(\theta_1,\theta_2)\sum_{k=1}^{\hat\tau(h)} X_{k-1}\varepsilon_k/(\sigma^2\sqrt{h/2})\,.$$

According to Lemma 4.3 we have for each $\delta>0$

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in K}\mathbf{P}_{\boldsymbol{\theta}}\left(\|\left(M_{\hat\tau(h)}\hat\sigma^2_{\hat\tau(h)}/(\sigma^4 h/2)\right)^{-1/2} L^{1/2}(\theta_1,\theta_2)-I\|>\delta\right)=0\,.$$

From here and (4.7) by applying Lemma 4.4, we come to (4.4). This completes the proof of Theorem 4.1. □

In order to prove Lemmas 4.3, 4.4, we need the following result.

Proposition 4.5. *Let $\boldsymbol{\theta}(n)$ and $\hat\sigma_n^2$ be given by (1.3) and (4.1). Then, for any compact set $K\subset\Lambda_1$ and $\delta>0$,*

$$\lim_{m\to\infty}\sup_{\boldsymbol{\theta}\in K}\mathbf{P}_{\boldsymbol{\theta}}\left(\|\boldsymbol{\theta}(n)-\boldsymbol{\theta}\|>\delta \text{ for some } n\ge m\right)=0\,, \quad (4.8)$$

$$\lim_{m\to\infty}\sup_{\boldsymbol{\theta}\in K}\mathbf{P}_{\boldsymbol{\theta}}\left(|\hat\sigma_n^2-\sigma^2|>\delta \text{ for some } n\ge m\right)=0\,. \quad (4.9)$$

Proof. We have

$$\boldsymbol{\theta}(n)-\boldsymbol{\theta} = (M_n/(x,x)_n)^{-1}(x,x)_n^{-1}\sum_{k=1}^{n} X_{k-1}\varepsilon_k\,.$$

By Lemma 3.3 in [7], for any $\delta>0$ and any compact $K\subset\overset{\circ}{\Lambda}=[\Lambda]\setminus\{(0,1),(-2,-1),(2,-1)\}$,

$$\lim_{m\to\infty}\sup_{\boldsymbol{\theta}\in K}\mathbf{P}_{\boldsymbol{\theta}}\left(\|M_n/(x,x)_n-L(\theta_1,\theta_2)\|>\delta \text{ for some } n\ge m\right)=0\,. \quad (4.10)$$

Further it will be observed that, for any $0<C<\infty$ and compact set K, there exists a positive number Δ that, for all matrices $L(\theta_1,\theta_2)$ with $\boldsymbol{\theta}=(\theta_1,\theta_2)'\in K$ and B such that $\|B-L(\theta_1,\theta_2)\|<\Delta$, one has $\|B^{-1}\|\le C$.

Let C, B be such a pair. Then, for each $\boldsymbol{\theta} \in K$, we have the inclusions

$$(\|\boldsymbol{\theta}(n) - \boldsymbol{\theta}\| > \delta \text{ for some } n \geq m)$$

$$\subseteq \left(\|(M_n/(x,x)_n)^{-1}\|(x,x)_n^{-1}\| \sum_{k=1}^{n} X_{k-1}\varepsilon_k\| > \delta \text{ for some } n \geq m \right)$$

$$= (\bullet) \subseteq (\bullet, \|M_n/(x,x)_n - L(\theta_1,\theta_2)\| \leq \Delta \text{ for all } n \geq m)$$

$$\cup (\bullet, \|M_n/(x,x)_n - L(\theta_1,\theta_2)\| > \Delta \text{ for some } n \geq m)$$

$$\subset \left(C\,(x,x)_n^{-1}\| \sum_{k=1}^{n} X_{k-1}\varepsilon_k\| > \delta \text{ for some } n \geq m \right)$$

$$\cup (\|M_n/(x,x)_n - L(\theta_1,\theta_2)\| > \Delta \text{ for some } n \geq m)\ .$$

This yields

$$\mathbf{P}_{\boldsymbol{\theta}}(\|\boldsymbol{\theta}(n) - \boldsymbol{\theta}\| > \delta \text{ for some } n \geq m)$$

$$\leq \mathbf{P}_{\boldsymbol{\theta}} \left((x,x)_n^{-1}\| \sum_{k=1}^{n} X_{k-1}\varepsilon_k\| > \delta^{'} \text{ for some } n \geq m \right)$$

$$+\mathbf{P}_{\boldsymbol{\theta}}\,(\|M_n/(x,x)_n - L(\theta_1,\theta_2)\| > \Delta \text{ for some } n \geq m)\,,\ \delta^{'} = \delta/C\,.$$

By Lemmas 3.2,3.3 from [7], limiting $m \to \infty$, we come to (4.8).

Consider (4.9). Rewrite $\hat{\sigma}_n^2$ in (4.1) as

$$\hat{\sigma}_n^2 = n^{-1} \sum_{k=1}^{n} (\varepsilon_k + (\boldsymbol{\theta} - \boldsymbol{\theta}(n))^{'} X_{k-1})^2 = n^{-1} \sum_{k=1}^{n} \varepsilon_k^2$$

$$+2n^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}(n))^{'} \sum_{k=1}^{n} X_{k-1}\varepsilon_k + n^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}(n))^{'} \sum_{k=1}^{n} X_{k-1}X_{k-1}^{'}(\boldsymbol{\theta} - \boldsymbol{\theta}(n))\,.$$

Substituting here $\boldsymbol{\theta}(n)$ from (1.3) yields

$$\hat{\sigma}_n^2 - \sigma^2 = \left(n^{-1} \sum_{k=1}^{n} \varepsilon_k^2 - \sigma^2 \right) - 2n^{-1} \left(\sum_{k=1}^{n} X_{k-1}^{'}\varepsilon_k \right) M_n^{-1} \sum_{k=1}^{n} X_{k-1}\varepsilon_k$$

$$+n^{-1} \left(\sum_{k=1}^{n} X_{k-1}^{'}\varepsilon_k \right) M_n^{-1} M_n M_n^{-1} \sum_{k=1}^{n} X_{k-1}\varepsilon_k$$

$$= \left(n^{-1} \sum_{k=1}^{n} \varepsilon_k^2 - \sigma^2 \right) - n^{-1} \left(\sum_{k=1}^{n} X_{k-1}^{'}\varepsilon_k \right) M_n^{-1} \sum_{k=1}^{n} X_{k-1}\varepsilon_k$$

$$= \left(n^{-1} \sum_{k=1}^{n} \varepsilon_k^2 - \sigma^2 \right) - \frac{1}{n\,(x,x)_n} \left(\sum_{k=1}^{n} X'_{k-1} \varepsilon_k \right) (M_n/(x,x)_n)^{-1} \sum_{k=1}^{n} X_{k-1} \varepsilon_k \,.$$

The first term in the right-hand side of this equality converges to zero in virtue of the strong law of large numbers. Therefore, in order to prove (4.9), we have to verify that, for each $K \subset \Lambda_1$ and $\delta > 0$,

$$\lim_{m \to \infty} \sup_{\boldsymbol{\theta} \in K} \mathbf{P}_{\boldsymbol{\theta}} \left(\frac{1}{n} \left(\sum_{k=1}^{n} X'_{k-1} \varepsilon_k \right) M_n^{-1} \sum_{k=1}^{n} X_{k-1} \varepsilon_k > \delta \text{ for some } n \geq m \right) = 0$$

In view of Lemma 3.3 in [7], it is equivalent to the following limiting relations

$$\lim_{m \to \infty} \sup_{\boldsymbol{\theta} \in K} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \sum_{k=1}^{n} x_{k-1} \varepsilon_k \right|^2 > \delta n (x,x)_n \text{ for some } n \geq m \right) = 0 \,, \quad (4.11)$$

$$\lim_{m \to \infty} \sup_{\boldsymbol{\theta} \in K} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \sum_{k=1}^{n} x_{k-2} \varepsilon_k \right|^2 > \delta n (x,x)_n \text{ for some } n \geq m \right) = 0 \,. \quad (4.12)$$

To prove these relations we will make use of Lemma 2.2 from [10]. First we note that the matrix A defined in (2.2) possesses the property (see, [7]):

$$\sup_{\boldsymbol{\theta} \in K} \|A^n\| \leq \kappa, \;\; n = 1, 2, \ldots, \quad (4.13)$$

where κ is some positive number. This implies the following inequality

$$(x,x)_n \leq \kappa^2 \sum_{k=1}^{n} (\sum_{j=1}^{k} |\varepsilon_j|)^2 =: U_n \,. \quad (4.14)$$

Indeed, writing down (1.1) in the vector form

$$X_k = A X_{k-1} + \xi_k, \;\; \xi_k = (\varepsilon_k, 0)' \,,$$

and using the formula $X_k = \sum_{j=1}^{k} A^{k-j} \xi_j$, lead to the estimate

$$|x_k| \leq \|X_k\| \leq \kappa \sum_{j=1}^{k} |\varepsilon_j|$$

and, hence, to (4.14). By making use of the law of iterated logarithm and the Kronecker Lemma, one can show that U_n in (4.14) satisfies the following relation

$$U_n = o(n^4) \text{ a.s.} \quad (4.15)$$

Now let us prove, for example, (4.11). From the inequality under the sign of probability in (4.11), it follows that

$$\left|\sum_{k=1}^{n} x_{k-1}\varepsilon_k\right| > \delta^{1/2}(x,x)_n^{5/8}\left(n^4/(x,x)_n\right)^{1/8}$$

$$\geq \delta^{1/2}(x,x)_n^{5/8}\left(n^4/U_n\right)^{1/8}\,. \tag{4.16}$$

This enables us to obtain the following inclusions for $\Delta < \sigma^2$:

$$\left(|\sum_{k=1}^{n} x_{k-1}\varepsilon_k| > \delta^{1/2}n^{1/2}(x,x)_n^{1/2} \text{ for some } n \geq m\right)$$

$$\subseteq \left(\bullet,\, |n^{-1}\sum_{k=1}^{n}\varepsilon_k^2 - \sigma^2| \leq \Delta \text{ for all } n \geq m\right)$$

$$\cup \left(|n^{-1}\sum_{k=1}^{n}\varepsilon_k^2 - \sigma^2| > \Delta \text{ for some } n \geq m\right)$$

$$\subseteq \left(\bullet,\, n^{-1}\sum_{k=1}^{n}\varepsilon_k^2 > \sigma^2 - \Delta \text{ for all } n \geq m\right)$$

$$\cup \left(|n^{-1}\sum_{k=1}^{n}\varepsilon_k^2 - \sigma^2| > \Delta \text{ for some } n \geq m\right)$$

$$\subseteq \left(\bullet,\, n^{-1}\sum_{k=1}^{n}\varepsilon_k^2 > \sigma^2 - \Delta \text{ all } n \geq m, U_n/n^4 \leq 1 \text{ all } n \geq m\right)$$

$$\cup \left(|n^{-1}\sum_{k=1}^{n}\varepsilon_k^2 - \sigma^2| > \Delta \text{ for some } n \geq m\right) \cup \left(\frac{U_n}{n^4} > 1 \text{ for some } n \geq m\right).$$

From here one gets

$$\mathbf{P}_{\boldsymbol{\theta}}\left(\left|\sum_{k=1}^{n} x_{k-1}\varepsilon_k\right| > \delta^{1/2}n^{1/2}(x,x)_n^{1/2} \text{ for some } n \geq m\right)$$

$$\leq \mathbf{P}_{\boldsymbol{\theta}}\left(\left|\sum_{k=1}^{n} x_{k-1}\varepsilon_k\right| > \delta^{1/2}\left(4^{-1}n^{3/2}(\sigma^2-\Delta) \vee (x,x)_n^{5/8}\right) \text{ for some } n \geq m\right)$$

$$+\mathbf{P}_{\boldsymbol{\theta}}\left(|\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k^2 - \sigma^2| > \Delta \text{ for some } n \geq m\right) + \mathbf{P}_{\boldsymbol{\theta}}\left(\frac{U_n}{n^4} > 1 \text{ for some } n \geq m\right).$$

In order to come to (4.11), it remains to use Lemma 2.2 from [10], the strong law of large numbers and put $m \to \infty$. This completes the proof of Proposition 4.5. $\square$

Proof of Lemma 4.3. We start with the representation

$$\frac{M_{\hat\tau(h)}\hat\sigma^2_{\hat\tau(h)}}{\sigma^4 h/2} = \frac{M_{\hat\tau(h)}}{(x,x)_{\hat\tau(h)}} \cdot \frac{(x,x)_{\hat\tau(h)}}{2^{-1}\sum_{k=1}^{\hat\tau(h)} \|X_{k-1}\|^2} \cdot \frac{\sum_{k=1}^{\hat\tau(h)} \|X_{k-1}\|^2}{h s^2_{\hat\tau(h)}} \cdot \frac{s^2_{\hat\tau(h)}}{\sigma^2} \cdot \frac{\hat\sigma^2_{\hat\tau(h)}}{\sigma^2} \,.$$

It suffices to show, for any $\delta > 0$, the limiting relations

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}} \left(\|M_{\hat\tau(h)}/(x,x)_{\hat\tau(h)} - L(\theta_1,\theta_2)\| > \delta \right) = 0 \,, \tag{4.17}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| (x,x)_{\hat\tau(h)} \left(2^{-1} \sum_{k=1}^{\hat\tau(h)} \|X_{k-1}\|^2 \right)^{-1} - 1 \right| > \delta \right) = 0 \,, \tag{4.18}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \sum_{k=1}^{\hat\tau(h)} \|X_{k-1}\|^2 / (h s^2_{\hat\tau(h)}) - 1 \right| > \delta \right) = 0 \,, \tag{4.19}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}} \left(|s^2_{\hat\tau(h)}/\sigma^2 - 1| > \delta \right) = 0 \,, \tag{4.20}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}} \left(|\hat\sigma^2_{\hat\tau(h)}/\sigma^2 - 1| > \delta \right) = 0 \,. \tag{4.21}$$

Consider (4.17). We have

$$\mathbf{P}_{\boldsymbol{\theta}} \left(\|M_{\hat\tau(h)}/(x,x)_{\hat\tau(h)} - L(\theta_1,\theta_2)\| > \delta \right) \le \mathbf{P}_{\boldsymbol{\theta}}(\hat\tau(h) \le m) \tag{4.22}$$

$$+\mathbf{P}_{\boldsymbol{\theta}} \left(\|M_n/(x,x)_n - L(\theta_1,\theta_2)\| > \delta \text{ for some } n \ge m \right) \,.$$

In view of (4.10), we need to check only that, for each sufficiently large m,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}}(\hat\tau(h) \le m) = 0 \,. \tag{4.23}$$

Let m_0 be a number such that, for all $m \ge m_0$, the sequence (δ_m) satisfies the inequality $\delta_m \le \sigma^2/2$. By the definition of the stopping time $\hat\tau(h)$ in (4.3), it follows that

$$\mathbf{P}_{\boldsymbol{\theta}}(\hat\tau(h) \le m) = \mathbf{P}_{\boldsymbol{\theta}}(\sum_{k=1}^{m} \|X_{k-1}\|^2 \ge h s^2_m)$$

$$= \mathbf{P}_{\boldsymbol{\theta}}(\sum_{k=1}^{m} \|X_{k-1}\|^2 \geq h\delta_m,\ \delta_m \geq \hat{\sigma}_m^2) + \mathbf{P}_{\boldsymbol{\theta}}(\sum_{k=1}^{m} \|X_{k-1}\|^2 \geq h\hat{\sigma}_m^2,\ \delta_m < \hat{\sigma}_m^2)$$

$$\leq \mathbf{P}_{\boldsymbol{\theta}}(\hat{\sigma}_m^2 \leq \delta_m) + \mathbf{P}_{\boldsymbol{\theta}}\left(\sum_{k=1}^{m} \|X_{k-1}\|^2 \geq h\hat{\sigma}_m^2\right)$$

$$\leq \mathbf{P}_{\boldsymbol{\theta}}(|\hat{\sigma}_m^2 - \sigma^2| \geq \sigma^2/2) + \mathbf{P}_{\boldsymbol{\theta}}\left(\sum_{k=1}^{m} \|X_{k-1}\|^2 \geq h\hat{\sigma}_m^2\right). \tag{4.24}$$

Further we have

$$\mathbf{P}_{\boldsymbol{\theta}}\left(\sum_{k=1}^{m} \|X_{k-1}\|^2 \geq h\hat{\sigma}_m^2\right) = \mathbf{P}_{\boldsymbol{\theta}}(\bullet,\ |\hat{\sigma}_m^2 - \sigma^2| \leq \Delta) + \mathbf{P}_{\boldsymbol{\theta}}(\bullet,\ |\hat{\sigma}_m^2 - \sigma^2| > \Delta)$$

$$\leq \mathbf{P}_{\boldsymbol{\theta}}\left(\sum_{k=1}^{m} \|X_{k-1}\|^2 \geq h(\sigma^2 - \Delta)\right) + \mathbf{P}_{\boldsymbol{\theta}}(|\hat{\sigma}_m^2 - \sigma^2| > \Delta). \tag{4.25}$$

The inequalities (4.24),(4.25), in view of Proposition 4.5, imply (4.23). This leads to (4.17). To show (4.18) we use the identity

$$(x, x)_n = 2^{-1} \sum_{k=1}^{n} \|X_{k-1}\|^2 + x_{n-1}^2/2\,,$$

(4.23) and apply Lemma 3.1 from [7]. The relations (4.19)-(4.21) can be checked in a similar way. This completes the proof of Lemma 4.3. □

Proof of Lemma 4.4. We will use the argument similar to that in the proof of Proposition 2.1 in [10]. First we introduce a sequence $(\hat{x}_n)$ of truncated observations (x_n) defined as

$$\hat{x}_n = \begin{cases} x_n \text{ if } x_n^2 \leq \delta^2 h\,, \\ \delta\sqrt{h} \text{ if } x_n^2 > \delta^2 h,\ 0 < \delta < 1\,, \end{cases}$$

and the set

$$\hat{\Omega}_h = (x_n = \hat{x}_n \text{ for all } n < \hat{\tau}_h)\,.$$

Along the lines of the proof of Proposition 2.1, one can verify that

$$\lim_{h \to \infty} \sup_{\boldsymbol{\theta} \in K} \mathbf{P}_{\boldsymbol{\theta}}(\hat{\Omega}_h^c) = 0\,. \tag{4.26}$$

Further we introduce stopping times $\tilde{T}_h$ and T_h as

$$\tilde{T}_h = \inf\{n \geq 3 : \sum_{k=1}^{n} (\hat{x}_{k-1}^2 + \hat{x}_{k-2}^2) \geq h(\tilde{\sigma}_{n-1}^2 \vee \delta_n)\}\,,$$

$$T_h = \inf\{n \geq 3 : \sum_{k=1}^{n} (\hat{x}_{k-1}^2 + \hat{x}_{k-2}^2) \geq h(\sigma^2 \vee \delta_n)\}\,, \tag{4.27}$$

where

$$\tilde{\sigma}_{n-1}^2 = n^{-1} \sum_{k=1}^{n} (\hat{x}_{k-1} - \tilde{\theta}'(n)\hat{X}_{k-1})^2\,,$$

$$\tilde{\theta}(n) = \left(\sum_{k=1}^{n} \hat{X}_{k-1}\hat{X}_{k-1}' \right)^{-1} \sum_{k=1}^{n} \hat{X}_{k-1}\hat{x}_k\,.$$

On the set $\hat{\Omega}_h$, we have $\tilde{T}_h = \hat{\tau}(h),\ \tilde{\sigma}_n^2 = \hat{\sigma}^2$ and $\tilde{\theta}(n) = \theta(n)$.

Now we write down Y_h as

$$Y_h = \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{\tilde{T}_h} \hat{X}_{k-1}\varepsilon_k \mathbf{1}_{\hat{\Omega}_h} + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{\hat{\tau}(h)} X_{k-1}\varepsilon_k \mathbf{1}_{\hat{\Omega}_h^c}$$

$$= \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{\tilde{T}_h} \hat{X}_{k-1}\varepsilon_k + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\sum_{k=1}^{\hat{\tau}_h} X_{k-1}\varepsilon_k - \sum_{k=1}^{\tilde{T}_h} \hat{X}_{k-1}\varepsilon_k \right) \mathbf{1}_{\hat{\Omega}_h^c}$$

$$= \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{T_h} \hat{X}_{k-1}\varepsilon_k + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\sum_{k=1}^{\hat{\tau}_h} X_{k-1}\varepsilon_k - \sum_{k=1}^{\tilde{T}_h} \hat{X}_{k-1}\varepsilon_k \right) \mathbf{1}_{\hat{\Omega}_h^c}$$

$$+ \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\sum_{k=1}^{\tilde{T}_h} \hat{X}_{k-1}\varepsilon_k - \sum_{k=1}^{T_h} \hat{X}_{k-1}\varepsilon_k \right) = \tilde{Y}_h + r_h\,, \tag{4.28}$$

where $\mathbf{1}_A$ is the indicator of a set A, $K_{\boldsymbol{\theta}} = \sqrt{2}\boldsymbol{\lambda}' L^{-1/2}(\theta_1, \theta_2)$,

$$\tilde{Y}_h = \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k)\,, \tag{4.29}$$

$$r_h = \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\tilde{\varepsilon}}_k - \mathbf{E}\tilde{\tilde{\varepsilon}}_k) + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\hat{X}_{\tilde{T}_h - 1}\varepsilon_{\tilde{T}_h} - \hat{X}_{T_h - 1}\varepsilon_{T_h} \right) \tag{4.30}$$

$$+ \eta_h + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\sum_{k=1}^{\hat{\tau}_h} X_{k-1}\varepsilon_k - \sum_{k=1}^{\tilde{T}_h} \hat{X}_{k-1}\varepsilon_k \right) \mathbf{1}_{\hat{\Omega}_h^c}\,,$$

$$\eta_h = \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\sum_{k=1}^{\tilde{T}_h - 1} \hat{X}_{k-1}\varepsilon_k - \sum_{k=1}^{T_h - 1} \hat{X}_{k-1}\varepsilon_k \right) ,$$

$$\tilde{\varepsilon}_k = \varepsilon_k \mathbf{1}_{(|\varepsilon_k| \le 1/\sqrt{\delta})}, \ \tilde{\tilde{\varepsilon}}_k = \varepsilon_k \mathbf{1}_{(|\varepsilon_k| > 1/\sqrt{\delta})} .$$

Let us show that

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}}(|r_h| > \delta) = 0 , \tag{4.31}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \sup_{t\in R} \left| \mathbf{P}_{\boldsymbol{\theta}}(\tilde{Y}_h \le t) - \Phi(t) \right| = 0 . \tag{4.32}$$

The first term in the right-hand side of (4.30) can be estimated as

$$\mathbf{E}_{\boldsymbol{\theta}}(\frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\tilde{\varepsilon}}_k - \mathbf{E}\tilde{\tilde{\varepsilon}}_k))^2 \le \|K_{\boldsymbol{\theta}}/(\sigma^2\sqrt{h})\|^2 \mathbf{E}_{\boldsymbol{\theta}} \sum_{k=1}^{T_h} \|\hat{X}_{k-1}\|^2 \mathbf{E}(\tilde{\tilde{\varepsilon}}_k - \mathbf{E}\tilde{\tilde{\varepsilon}}_k)^2$$

$$\le \|K_{\boldsymbol{\theta}}/(\sigma^2\sqrt{h})\|^2 \mathbf{E}_{\boldsymbol{\theta}} \left(h(\sigma^2 \vee \delta_{T_h}) + h\delta^2 \right) \mathbf{D}(\tilde{\tilde{\varepsilon}}_1)$$

$$\le \|\frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}}\|^2 \left((\sigma^2 \vee \sup_{n\ge 1} \delta_n) + \delta^2 \right) \mathbf{D}(\tilde{\tilde{\varepsilon}}_1) .$$

From here, limiting $\delta \to 0$ we obtain

$$\lim_{\delta\to 0} \sup_{h>0} \sup_{\boldsymbol{\theta}\in K} \mathbf{E}_{\boldsymbol{\theta}} \left(\frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\tilde{\varepsilon}}_k - \mathbf{E}\tilde{\tilde{\varepsilon}}_k) \right)^2 = 0 . \tag{4.33}$$

The second term in (4.30) can be estimated as

$$\mathbf{E}_{\boldsymbol{\theta}} \left(\frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \hat{X}_{\tilde{T}_h - 1} \varepsilon_{\tilde{T}_h} \right)^2 \le \|\frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}}\|^2 \mathbf{E}_{\boldsymbol{\theta}} \|\hat{X}_{\tilde{T}_h - 1}\|^2 \mathbf{E}\varepsilon^2_{\tilde{T}_h}$$

$$\le \|K_{\boldsymbol{\theta}}/(\sigma^2\sqrt{h})\|^2 \delta^2 h \mathbf{E}\varepsilon^2_{\tilde{T}_h} = \|K_{\boldsymbol{\theta}/\sigma^2}\|^2 \delta^2 .$$

Therefore one gets

$$\lim_{\delta\to 0} \sup_{h>0} \sup_{\boldsymbol{\theta}\in K} \mathbf{E}_{\boldsymbol{\theta}} (\frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \hat{X}_{\tilde{T}_h - 1} \varepsilon_{\tilde{T}_h})^2 = 0 . \tag{4.34}$$

Further we note that

$$\mathbf{E}_{\boldsymbol{\theta}} \eta_h^2 \le \|K_{\boldsymbol{\theta}}/(\sigma^2\sqrt{h/2})\|^2 \mathbf{E}_{\boldsymbol{\theta}} t_h , \tag{4.35}$$

where

$$t_h = \left| \frac{1}{\sigma^2 h} \sum_{k=1}^{\tilde{T}_h - 1} \|\hat{X}_{k-1}\|^2 - \frac{1}{\sigma^2 h} \sum_{k=1}^{T_h - 1} \|\hat{X}_{k-1}\|^2 \right| .$$

Let us estimate t_h. If $T_h \geq \tilde{T}_h$, then

$$t_h \leq \frac{1}{\sigma^2 h} \sum_{k=1}^{T_h - 1} \|\hat{X}_{k-1}\|^2 \leq \frac{h(\sigma^2 \vee \delta_{T_h})}{\sigma^2 h} \leq (1 \vee (\sigma^{-2} \sup_{n \geq 1} \delta_n)) \,. \tag{4.36}$$

This estimate is also true in the case when $T_h < \tilde{T}_h$ because then $\tilde{\sigma}^2_{T_h - 1} < \sigma^2$ and, hence,

$$t_h \leq \frac{1}{\sigma^2 h} \sum_{k=1}^{\tilde{T}_h - 1} \|\hat{X}_{k-1}\|^2 \leq \frac{h}{\sigma^2 h} (\tilde{\sigma}^2_{\tilde{T}_h - 1} \vee \delta_{\tilde{T}_h}) \leq (1 \vee (\sigma^{-2} \sup_{n \geq 1} \delta_n)) \,.$$

Thus

$$t_h \leq \left(1 \vee (\sigma^{-2} \sup_{n \geq 1} \delta_n) \right) .$$

Now we will show that, for any $\Delta > 0$,

$$\lim_{h \to \infty} \sup_{\boldsymbol{\theta} \in K} \mathbf{P}_{\boldsymbol{\theta}}(t_h \geq \Delta) = 0 \,. \tag{4.37}$$

We have the following estimate

$$t_h \leq \left| \frac{1}{\sigma^2 h} \sum_{k=1}^{\tilde{T}_h - 1} \|\hat{X}_{k-1}\|^2 - \frac{1}{\sigma^2} (\tilde{\sigma}^2_{\tilde{T}_h - 1} \vee \delta_{\tilde{T}_h}) \right|$$

$$+ \left| \frac{1}{\sigma^2} (\tilde{\sigma}^2_{\tilde{T}_h - 1} \vee \delta_{\tilde{T}_h}) - (1 \vee \frac{\delta_{T_h}}{\sigma^2}) \right| + \left| (1 \vee \frac{\delta_{T_h}}{\sigma^2}) - \frac{1}{\sigma^2 h} \sum_{k=1}^{T_h - 1} \|\hat{X}_{k-1}\|^2 \right| .$$

From the definitions of $\tilde{T}_h$ and T_h, it follows that

$$\frac{1}{\sigma^2 h} \sum_{k=1}^{\tilde{T}_h - 1} \|\hat{X}_{k-1}\|^2 \leq \frac{h}{\sigma^2 h} (\tilde{\sigma}^2_{\tilde{T}_h - 1} \vee \delta_{\tilde{T}_h}) \leq \frac{1}{\sigma^2 h} \sum_{k=1}^{\tilde{T}_h} \|\hat{X}_{k-1}\|^2 \,,$$

$$\frac{1}{\sigma^2 h} \sum_{k=1}^{T_h - 1} \|\hat{X}_{k-1}\|^2 \leq \frac{h}{\sigma^2 h} (\sigma^2 \vee \delta_{T_h}) \leq \frac{1}{\sigma^2 h} \sum_{k=1}^{T_h} \|\hat{X}_{k-1}\|^2 \,.$$

Therefore

$$0 \leq \frac{1}{\sigma^2} (\tilde{\sigma}^2_{\tilde{T}_h - 1} \vee \delta_{\tilde{T}_h}) - \frac{1}{\sigma^2 h} \sum_{k=1}^{\tilde{T}_h - 1} \|\hat{X}_{k-1}\|^2 \leq \frac{1}{\sigma^2 h} \|\hat{X}_{\tilde{T}_h - 1}\|^2 \,,$$

$$0 \leq (1 \vee \sigma^{-2} \delta_{T_h}) - \frac{1}{\sigma^2 h} \sum_{k=1}^{T_h - 1} \|\hat{X}_{k-1}\|^2 \leq \frac{1}{\sigma^2 h} \|\hat{X}_{T_h - 1}\|^2$$

and this leads to the estimate

$$t_h \le (\sigma^2 h)^{-1}\|\hat{X}_{\tilde{T}_h-1}\|^2 + (\sigma^2 h)^{-1}\|\hat{X}_{T_h-1}\|^2 + \zeta_h\,,$$

where $\zeta_h = \left|\sigma^{-2}(\tilde{\sigma}^2_{\tilde{T}_h-1} \vee \delta_{\tilde{T}_h}) - (1 \vee (\sigma^{-2}\delta_{T_h}))\right|$.

Now we have to verify that, for any $\Delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}}((\sigma^2 h)^{-1}\|\hat{X}_{\tilde{T}_h-1}\|^2 > \Delta) = 0\,. \tag{4.38}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}}((\sigma^2 h)^{-1}\|\hat{X}_{T_h-1}\|^2 > \Delta) = 0\,, \tag{4.39}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}}(\zeta_h > \Delta) = 0\,. \tag{4.40}$$

For any $\Delta > 0$ and $0 < \tilde{\Delta} < 1$, one has the inclusions

$$\begin{aligned}
\left((\sigma^2 h)^{-1}\|\hat{X}_{\tilde{T}_h-1}\|^2 > \Delta\right) &\subset \left((\sigma^2 h)^{-1}\|\hat{X}_{\tilde{T}_h-1}\|^2 > \Delta, \hat{\Omega}_h\right) \cup \hat{\Omega}_h^c \\
&\subset \left((\sigma^2 h)^{-1}\|X_{\hat{\tau}(h)-1}\|^2 > \Delta, \hat{\Omega}_h\right) \cup \hat{\Omega}_h^c \\
\subset (\hat{\tau}(h) \le m) \cup \left((\sigma^2 h)^{-1}\|X_{\hat{\tau}(h)-1}\|^2 > \Delta, \hat{\Omega}_h, \hat{\tau}(h) > m\right) &\cup \hat{\Omega}_h^c \\
&\subset (\hat{\tau}(h) \le m) \cup \left(\|\sigma^2\, s^{-2}_{\hat{\tau}(h)} - 1\| > \tilde{\Delta}\right) \cup \hat{\Omega}_h^c \\
\cup \left(\|X_{\hat{\tau}(h)-1}\|^2 > \Delta(1-\tilde{\Delta}) \sum_{k=1}^{\hat{\tau}(h)} \|X_{k-1}\|^2, \hat{\tau}(h) > m\right) &\\
&\subset (\hat{\tau}(h) \le m) \cup \left(\|\sigma^2\, s^{-2}_{\hat{\tau}(h)} - 1\| > \tilde{\Delta}\right) \cup \hat{\Omega}_h^c \\
\cup(\|X_{n-1}\|^2 > \Delta(1-\tilde{\Delta}) \sum_{k=1}^{n} \|X_{k-1}\|^2 \text{ for some } n > m)\,.&
\end{aligned}$$

This yields the inequality

$$\begin{aligned}
\mathbf{P}_{\boldsymbol{\theta}}\left((\sigma^2 h)^{-1}\|\tilde{X}_{\tilde{T}_h-1}\|^2 > \Delta\right) &\le \mathbf{P}_{\boldsymbol{\theta}}(\hat{\tau}(h) \le m) + \mathbf{P}_{\boldsymbol{\theta}}\left(\|\sigma^2\, s^{-2}_{\hat{\tau}(h)} - 1\| > \tilde{\Delta}\right) \\
+\mathbf{P}_{\boldsymbol{\theta}}(\hat{\Omega}_h^c) + \mathbf{P}_{\boldsymbol{\theta}}(\|X_{n-1}\|^2 > \Delta(1-\tilde{\Delta}) &\sum_{k=1}^{n} \|X_{k-1}\|^2 \text{ for some } n > m)\,.
\end{aligned}$$

From here, (4.9),(4.23),(4.26) and Lemma 3.1 from [7], we come to (4.38). In a similar way one can check (4.39).

Further, for sufficiently large m, we have the inclusions

$$(\zeta_h > \Delta) = (\zeta_h > \Delta, \tilde{T}_h \le m, T_h \le m) \cup (\zeta_h > \Delta, \tilde{T}_h > m \text{ or } T_h > m)$$

$$\subset (T_h \le m) \cup (\zeta_h > \Delta, \tilde{T}_h > m, T_h > m) \cup (\zeta_h > \Delta, \tilde{T}_h > m, T_h \le m)$$

$$\cup(\zeta_h > \Delta, \tilde{T}_h \le m, T_h > m) \subset (T_h \le m) \cup (\tilde{T}_h \le m)$$

$$\cup(\zeta_h > \Delta, \tilde{T}_h > m, T_h > m) \subset (T_h \le m) \cup (\tilde{T}_h \le m) \cup (|\tilde{\sigma}^2_{\tilde{T}_h - 1}\sigma^{-2} - 1| > \Delta)\,.$$

This implies

$$\mathbf{P}_{\boldsymbol{\theta}}(\zeta_h > \Delta) \le \mathbf{P}_{\boldsymbol{\theta}}(T_h \le m) + \mathbf{P}_{\boldsymbol{\theta}}(\tilde{T}_h \le m) + \mathbf{P}_{\boldsymbol{\theta}}\left(|\tilde{\sigma}^2_{\tilde{T}_h - 1}\sigma^{-2} - 1| > \Delta\right)\,. \tag{4.41}$$

From the definition of T_h in (4.27), it follows that for sufficiently large m

$$\mathbf{P}_{\boldsymbol{\theta}}(T_h \le m) = \mathbf{P}_{\boldsymbol{\theta}}(\sum_{i=1}^{m} \|\hat{X}_{i-1}\|^2 \ge h(\sigma^2 \vee \delta_m)) \le \mathbf{P}_{\boldsymbol{\theta}}(\sum_{i=1}^{m} \|X_{i-1}\|^2 \ge h\sigma^2)$$

and, hence,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}}(T_h \le m) = 0\,. \tag{4.42}$$

Let us estimate the second term in the right-hand side of (4.41). In view of (4.27), one has the inclusion

$$(\tilde{T}_h \le m) = \left(\sum_{i=1}^{m} \|\hat{X}_{i-1}\|^2 \ge h(\tilde{\sigma}^2_{m-1} \vee \delta_m)\right) \subset \left(\sum_{i=1}^{m} \|\hat{X}_{i-1}\|^2 \ge h\delta_m\right)\,.$$

Thus

$$\mathbf{P}_{\boldsymbol{\theta}}(\tilde{T}_h \le m) \le \mathbf{P}_{\boldsymbol{\theta}}\left(\sum_{i=1}^{m} \|\hat{X}_{i-1}\|^2 \ge h\delta_m\right)$$

and one gets

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \mathbf{P}_{\boldsymbol{\theta}}(\tilde{T}_h \le m) = 0\,. \tag{4.43}$$

It remains to consider the last term in the right-hand side of (4.41). We have the following inclusions

$$\left(|\tilde{\sigma}^2_{\tilde{T}_h - 1}\sigma^{-2} - 1| > \Delta\right) \subset (\tilde{T}_h \le m) \cup \left(|\tilde{\sigma}^2_{\tilde{T}_h - 1}\sigma^{-2} - 1| > \Delta, \tilde{T}_h > m, \hat{\Omega}_h\right)$$

$$\cup\hat{\Omega}_h^c \subset \left(|\tilde{\sigma}^2_{\tilde{T}_h-1}\sigma^{-2}-1|>\Delta, \hat{\tau}(h)>m\right)\cup(\tilde{T}_h\le m)\cup\hat{\Omega}_h^c$$

$$\subset(\tilde{T}_h\le m)\cup\left(|\tilde{\sigma}^2_n\sigma^{-2}-1|>\Delta \text{ for some } n\ge m-1\right)\cup\hat{\Omega}_h^c\,.$$

This yields

$$\mathbf{P}_{\boldsymbol{\theta}}\left(|\tilde{\sigma}^2_{\tilde{T}_h-1}\sigma^{-2}-1|>\Delta\right)\le\mathbf{P}_{\boldsymbol{\theta}}(\tilde{T}_h\le m)+\mathbf{P}_{\boldsymbol{\theta}}(\hat{\Omega}_h^c)$$

$$+\mathbf{P}_{\boldsymbol{\theta}}\left(|\tilde{\sigma}^2_n-\sigma^2|>\sigma^2\Delta \text{ for some } n\ge m-1\right)\,.$$

From here, (4.9), (4.26), and (4.43) we come to (4.40). Taking into account (4.38)-(4.40) we have proved (4.37). Combining (4.35), (4.36) and (4.37) yields

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in K}\mathbf{E}_{\boldsymbol{\theta}}\eta_h^2=0\,. \tag{4.44}$$

From (4.30), (4.33), (4.34) and (4.44) we derive (4.31).

In order to complete the proof of Lemma 4.4, it remains to show (4.32).

Define the set

$$\tilde{\Omega}_h=(x_n=\hat{x}_n \text{ for all } n<T_h)\,.$$

By the definition of T_h in (4.27) one gets

$$\mathbf{P}_{\boldsymbol{\theta}}(\tilde{\Omega}_h^c)\le\sum_{k=1}^{m}\mathbf{P}_{\boldsymbol{\theta}}(x^2_{k-1}>\delta^2 h)$$

$$+\mathbf{P}_{\boldsymbol{\theta}}\left(T_h>m, x_n\ne\hat{x}_n \text{ for some } m\le n<T_h\right)$$

$$\le\sum_{k=1}^{m}\mathbf{P}_{\boldsymbol{\theta}}(x^2_{k-1}>\delta^2 h)+\mathbf{P}_{\boldsymbol{\theta}}\left(x_n^2\ge\delta^2\sum_{i=1}^{n}(\hat{x}^2_{i-1}+\hat{x}^2_{i-2}) \text{ for some } n\ge m\right)\,.$$

It can be proved that for the unstable model (1.1)

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in K}\mathbf{P}_{\boldsymbol{\theta}}\left(x_n^2\ge\delta^2\sum_{i=1}^{n}(\hat{x}^2_{i-1}+\hat{x}^2_{i-2}) \text{ for some } n\ge m\right)=0\,.$$

Therefore

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in K}\mathbf{P}_{\boldsymbol{\theta}}(\tilde{\Omega}_h^c)=0\,.$$

Let

$$T_0(h) = \inf\{n \geq 3 : \sum_{i=1}^{n} (x_{i-1}^2 + x_{i-2}^2) \geq h(\sigma^2 \vee \delta_n)\}\,.$$

Since, $T_h = T_0(h)$ on the set $\tilde{\Omega}_h$, we rewrite $\tilde{Y}_h$ defined in (4.29) as

$$\begin{aligned}
\tilde{Y}_h &= \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k)\mathbf{1}_{\tilde{\Omega}_h} + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k)\mathbf{1}_{\tilde{\Omega}_h^c} \\
&= \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k) - \sum_{k=1}^{T_0(h)} X_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k) \right) \mathbf{1}_{\tilde{\Omega}_h^c} \\
&\quad + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sum_{k=1}^{T_0(h)} X_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k) = \frac{1}{\sqrt{h}} \sum_{k=1}^{T_0(h)} g_{k-1} \frac{(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k)}{\sqrt{\mathbf{D}(\tilde{\varepsilon}_k)}} \\
&\quad + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k) - \sum_{k=1}^{T_0(h)} X_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k) \right) \mathbf{1}_{\tilde{\Omega}_h^c}\,,
\end{aligned}$$

where

$$g_{k-1} = \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \sqrt{\mathbf{D}(\tilde{\varepsilon}_k)} X_{k-1}\,.$$

Further we introduce the stopping time

$$\tau_0 = \tau_0(h) = \inf\{n \geq 3 : \sum_{k=1}^{n} g_{k-1}^2 \geq h\}$$

and represent $\tilde{Y}_h$ as

$$\tilde{Y}_h = \frac{1}{\sqrt{h}} \sum_{k=1}^{\tau_0(h)} g_{k-1} \frac{(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k)}{\sqrt{\mathbf{D}(\tilde{\varepsilon}_k)}} + \eta(h) + \Delta(h)\,,$$

where $\Delta(h) = \Delta_1(h) + \Delta_2(h)$,

$$\Delta_1(h) = h^{-1/2} g_{T_0(h)-1} \varepsilon_{T_0(h)}\,, \Delta_2(h) = -h^{-1/2} g_{\tau_0(h)-1} \varepsilon_{\tau_0(h)}\,,$$

$$\begin{aligned}
\eta(h) &= \frac{1}{\sqrt{h}} \sum_{k=1}^{T_0(h)-1} g_{k-1} \frac{(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k)}{\sqrt{\mathbf{D}(\tilde{\varepsilon}_k)}} - \frac{1}{\sqrt{h}} \sum_{k=1}^{\tau_0(h)-1} g_{k-1} \frac{(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k)}{\sqrt{\mathbf{D}(\tilde{\varepsilon}_k)}} \\
&\quad + \frac{K_{\boldsymbol{\theta}}}{\sigma^2\sqrt{h}} \left(\sum_{k=1}^{T_h} \hat{X}_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k) - \sum_{k=1}^{T_0(h)} X_{k-1}(\tilde{\varepsilon}_k - \mathbf{E}\tilde{\varepsilon}_k) \right) \mathbf{1}_{\tilde{\Omega}_h^c}\,.
\end{aligned}$$

Further analysis of $\tilde{Y}_h$ proceeds along the lines of Lemma 3.7 in [7] and is omitted. This completes the proof of Lemma 4.4. □

5 Auxiliary propositions.

This section contains the proofs of some results used in this paper.

1. Proof of Proposition 2.2. First we will prove the following three Lemmas.

Lemma 5.1. *Under conditions of Theorem 2.1, for each* $m = 1, 2, \dots$ *and for any* $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in[\Lambda]} \mathbf{P}_{\boldsymbol{\theta}}(\tau(h) < m) = 0, \ \lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in[\Lambda]} \mathbf{P}_{\boldsymbol{\theta}}(1/\tau(h) > \delta) = 0. \tag{5.1}$$

Proof. From the definition of stopping time $\tau(h)$ in (1.7) and (4.14), it follows that

$$(\tau(h) < m) = \left(\sum_{k=1}^{m}(x_{k-1}^2 + x_{k-2}^2) > h\sigma^2\right) \subset \left(U_m > h\sigma^2\right).$$

This implies (5.1). Hence Lemma 5.1. □

Lemma 5.2. *Under conditions of Theorem 2.1, for any* $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in[\Lambda]} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \frac{1}{\tau(h)} \sum_{k=1}^{\tau(h)} \varepsilon_k^2 - \sigma^2 \right| > \delta \right) = 0\,.$$

Proof. One has

$$\mathbf{P}_{\boldsymbol{\theta}} \left(|\frac{1}{\tau(h)} \sum_{k=1}^{\tau(h)} \varepsilon_k^2 - \sigma^2| > \delta \right) \le \mathbf{P}_{\boldsymbol{\theta}}(\tau(h) < m)$$

$$+\mathbf{P}_{\boldsymbol{\theta}} \left(|\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^2 - \sigma^2| > \delta \text{ for some } n \ge m \right).$$

Applying Lemma 5.1 and the strong law of large number one comes to the desired result. Hence Lemma 5.2. □

Lemma 5.3. *Let* $(u_k)_{k\ge 0}$ *and* $(v_k)_{k\ge 0}$ *be the processes defined in (3.2). Then, for each* $d > 0$ *and any* $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}} \left(|\tau(h)(u,u)_{\tau(h)}^{-1} - (1-a^2)/\sigma^2| > \delta \right) = 0, \tag{5.2}$$

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}}\mathbf{P}_{\boldsymbol{\theta}}\left(|\tau(h)(v,v)^{-1}_{\tau(h)}-(1-b^2)/\sigma^2|>\delta\right)=0.$$

Proof. Since these relations are similar, we verify only (5.2). First we show that, for each $d>0$ and any $\delta>0$,

$$\lim_{h\to\infty}\sup_{\substack{\boldsymbol{\theta}\in\Lambda_{d,1},\\ -\theta_1+\theta_2<1}}\mathbf{P}_{\boldsymbol{\theta}}\left(|\tau(h)(u,u)^{-1}_{\tau(h)}-(1-a^2)/\sigma^2|>\delta\right)=0. \tag{5.3}$$

Squaring both sides of the first equation in (3.2) and summing give

$$(1-a^2)\sum_{j=1}^{\tau(h)}u_{j-1}^2=u_0^2-u_{\tau(h)}^2+2a\sum_{j=1}^{\tau(h)}u_{j-1}\varepsilon_j+\sum_{j=1}^{\tau(h)}\varepsilon_j^2.$$

By making use of this equality one obtains

$$\begin{aligned}&|\tau(h)(u,u)^{-1}_{\tau(h)}-(1-a^2)/\sigma^2|\\ &\le\frac{u^2_{\tau(h)}}{\sigma^2(u,u)_{\tau(h)}}+\frac{2|\sum_{k=1}^{\tau(h)}u_{k-1}\varepsilon_k|}{\sigma^2(u,u)_{\tau(h)}}+\frac{|\sum_{k=1}^{\tau(h)}(\varepsilon_k^2-\sigma^2)|}{\sigma^2(u,u)_{\tau(h)}}\\ &\le\frac{u^2_{\tau(h)}}{\sigma^2(u,u)_{\tau(h)}}+\frac{2|\sum_{k=1}^{\tau(h)}u_{k-1}\varepsilon_k|}{\sigma^2(u,u)_{\tau(h)}}+\frac{|\sum_{k=1}^{\tau(h)}(\varepsilon_k^2-\sigma^2)|}{(\sigma^2/4)\sum_{k=1}^{\tau(h)-1}\varepsilon_k^2}.\end{aligned} \tag{5.4}$$

The last inequality follows from the estimate

$$\sum_{k=1}^{n-1}\varepsilon_k^2\le 4\sum_{k=1}^{n}u_{k-1}^2.$$

By Lemma 5.2, we have to show that, for each $d>0$ and any $\delta>0$,

$$\lim_{h\to\infty}\sup_{\substack{\boldsymbol{\theta}\in\Lambda_{d,1},\\ -\theta_1+\theta_2<1}}\mathbf{P}_{\boldsymbol{\theta}}\left(u^2_{\tau(h)}(u,u)^{-1}_{\tau(h)}>\delta\right)=0, \tag{5.5}$$

$$\lim_{h\to\infty}\sup_{\substack{\boldsymbol{\theta}\in\Lambda_{d,1},\\ -\theta_1+\theta_2<1}}\mathbf{P}_{\boldsymbol{\theta}}\left(|\sum_{k=1}^{\tau(h)}u_{k-1}\varepsilon_k|(u,u)^{-1}_{\tau(h)}>\delta\right)=0. \tag{5.6}$$

We have

$$\begin{aligned}\mathbf{P}_{\boldsymbol{\theta}}\left(u^2_{\tau(h)}(u,u)^{-1}_{\tau(h)}>\delta\right)&\le\mathbf{P}_{\boldsymbol{\theta}}(\tau(h)<m)\\ &+\mathbf{P}_{\boldsymbol{\theta}}\left(u_n^2(u,u)_n^{-1}>\delta\text{ for some }n\ge m\right).\end{aligned} \tag{5.7}$$

It is known (see, [10]) that

$$\lim_{m\to\infty}\sup_{|a|\le 1}\mathbf{P}_{\boldsymbol{\theta}}\left(u_n^2\left(\sum_{k=1}^{n}u_{k-1}^2\right)^{-1}>\delta \text{ for some } n\ge m\right)=0. \tag{5.8}$$

Applying this and Lemma 5.1 in (5.7) yields (5.5). To prove (5.6) we use the representation

$$\left|\sum_{k=1}^{\tau(h)}u_{k-1}\varepsilon_k\right|/(u,u)_{\tau(h)}=\zeta_{\tau(h)}\max\left(\frac{\tau(h)}{(u,u)_{\tau(h)}},\left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{1/4}\frac{1}{\sqrt[4]{\tau(h)}}\right),$$

where

$$\zeta_n=\frac{1}{\max\left(n,(u,u)_n^{3/4}\right)}\left|\sum_{k=1}^{n}u_{k-1}\varepsilon_k\right|.$$

By Lemmas 5.1, 5.2 and applying the uniform law of large numbers for martingales (see [10]) we come to (5.6). Combining (5.4) - (5.5) and Lemma 5.2 one gets (5.3). It remains to show that, for each $d>0$ and $\delta>0$,

$$\lim_{h\to\infty}\sup_{\substack{\boldsymbol{\theta}\in\Lambda_{d,1},\\ -\theta_1+\theta_2=1}}\mathbf{P}_{\boldsymbol{\theta}}(|\tau(h)/(u,u)_{\tau(h)}-(1-a^2)/\sigma^2|>\delta)=0. \tag{5.9}$$

If $\theta_1+\theta_2=1$, then $a=-1$ and the process u_k in (3.2) satisfies the limiting relation (see, e.g., Lai and Wei (1983))

$$\liminf_{n\to\infty}\sum_{k=1}^{n}u_k^2/(n^2/loglogn)=\frac{\sigma^2}{4}\text{ a.s.} \tag{5.10}$$

By making use of the inequality

$$\mathbf{P}_{\boldsymbol{\theta}}\left(\frac{\tau(h)}{(u,u)_{\tau(h)}}>\delta\right)\le P_{\boldsymbol{\theta}}(\tau<m)+\mathbf{P}_{\boldsymbol{\theta}}\left(\frac{n}{(u,u)_n}>\delta\text{ for some }n\ge m\right)$$

and (5.10), we come to (5.9). This completes the proof of Lemma 5.3. □

Now we can prove Proposition 2.2. We have to show that, for each $d>0$ and any $\delta>0$,

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}}\mathbf{P}_{\boldsymbol{\theta}}\left(|\xi_{\tau(h)}-r(a,b)|>\delta\right)=0. \tag{5.11}$$

Denote

$$\eta_h^{(l)} = A_h \sum_{k=l}^{\tau(h)} u_{k-l} v_{k-l}, l = 1, \ldots, \tau(h),\ A_h = (u,u)_{\tau(h)}^{-1/2} (v,v)_{\tau(h)}^{-1/2}. \quad (5.12)$$

From equations (3.2), one gets

$$\sum_{k=l}^{\tau(h)} u_{k-l} v_{k-l} = \sum_{k=l+1}^{\tau(h)} \left(a u_{k-l-1} + \varepsilon_{k-l}\right) \left(b v_{k-l-1} + \varepsilon_{k-l}\right)$$

$$= ab \sum_{k=l+1}^{\tau(h)} u_{k-l-1} v_{k-l-1} + a \sum_{k=l+1}^{\tau(h)} u_{k-l-1} \varepsilon_{k-l} +$$

$$+ b \sum_{k=l+1}^{\tau(h)} v_{k-l-1} \varepsilon_{k-l} + \sum_{k=l+1}^{\tau(h)} \varepsilon_{k-l}^2,\ l = 1, \ldots, \tau(h) - 1.$$

Substituting this in (5.12) yields

$$\eta_h^{(l)} = ab \eta_h^{(l+1)} + z_{\tau(h)-l},\ 1 \le l < \tau(h),$$

where

$$z_{\tau(h)-l} = A_h \left(a \sum_{k=l+1}^{\tau(h)} u_{k-l-1} \varepsilon_{k-l} + b \sum_{k=l+1}^{\tau(h)} v_{k-l-1} \varepsilon_{k-l} + \sum_{k=l+1}^{\tau(h)} \varepsilon_{k-l}^2 \right). \quad (5.13)$$

Putting $\zeta_m = \eta_h^{(\tau(h)-m)}$ we come to the equation

$$\zeta_m = ab \zeta_{m-1} + z_m, 1 \le m < \tau(h), \zeta_0 = 0.$$

Solving this equation one finds

$$\xi_{\tau(h)} = \zeta_{\tau(h)-1} = \sum_{j=0}^{\tau(h)-2} (ab)^j z_{\tau(h)-1-j}.$$

Introducing the sums

$$S_m = \sum_{l=0}^{m} (ab)^l, m \ge 0, \quad (5.14)$$

one can rewrite this formula as follows

$$\xi_{\tau(h)} = z_{\tau(h)-1} + \sum_{j=1}^{\tau(h)-2} (ab)^j z_{\tau(h)-1-j} \quad (5.15)$$

$$= z_{\tau(h)-1} + \sum_{j=1}^{\tau(h)-2} (S_j - S_{j-1}) z_{\tau(h)-1-j}$$

$$= z_{\tau(h)-1} + \sum_{j=1}^{\tau(h)-2} S_j z_{\tau(h)-1-j} - \sum_{j=0}^{\tau(h)-3} S_j z_{\tau(h)-2-j}$$

$$= \sum_{j=0}^{\tau(h)-2} S_j z_{\tau(h)-1-j} - \sum_{j=0}^{\tau(h)-3} S_j z_{\tau(h)-2-j}$$

$$= S_{\tau(h)-2} z_1 + \sum_{j=0}^{\tau(h)-3} S_j (z_{\tau(h)-1-j} - z_{\tau(h)-2-j}).$$

By making use of (5.13) one can easily verify that

$$z_{\tau(h)-1-j} - z_{\tau(h)-2-j} = aA_h u_{\tau(h)-2-j} \varepsilon_{\tau(h)-j-1}$$

$$+ bA_h v_{\tau(h)-2-j} \varepsilon_{\tau(h)-j-1} + A_h \varepsilon^2_{\tau(h)-j-1}.$$

Substituting this in (5.15) we obtain

$$\xi_{\tau(h)} = \xi_h^{(1)} + \xi_h^{(2)} + \xi_h^{(3)}, \tag{5.16}$$

where

$$\xi_h^{(1)} = A_h \sum_{k=1}^{\tau(h)-1} S_{\tau(h)-1-k} \varepsilon_k^2, \; \xi_h^{(2)} = aA_h \sum_{k=2}^{\tau(h)-1} S_{\tau(h)-1-k} u_{k-1} \varepsilon_k, \tag{5.17}$$

$$\xi_h^{(3)} = bA_h \sum_{k=2}^{\tau(h)-1} S_{\tau(h)-1-k} v_{k-1} \varepsilon_k.$$

To show (5.11) we have to check that, for each $d > 0$ and $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}}(|\xi_h^{(1)} - r(a,b)| > \delta) = 0, \tag{5.18}$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}}(|\xi_h^{(i)}| > \delta) = 0, \; i = 2, 3. \tag{5.19}$$

First we will verify the equalities for some subsets of $\Lambda_{d,1}$: for any $q \in]0, 1[$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|\le q\}} \mathbf{P}_{\boldsymbol{\theta}}(|\xi_h^{(1)} - r(a,b)| > \delta) = 0, \tag{5.20}$$

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|\le q\}}\mathbf{P}_{\boldsymbol{\theta}}(|\xi_h^{(i)}|>\delta)=0,\ i=2,3. \tag{5.21}$$

Denoting $\lim_{n\to\infty}S_n=(1-ab)^{-1}=S^*$ we rewrite $\xi_h^{(1)}$ as

$$\xi_h^{(1)}=A_hS^*\sum_{k=1}^{\tau(h)-1}\varepsilon_k^2+W_h, \tag{5.22}$$

where

$$W_h=A_h\sum_{k=1}^{\tau(h)-1}(S_{\tau(h)-1-k}-S^*)\varepsilon_k^2. \tag{5.23}$$

By Lemmas 5.1, 5.3 one gets

$$\begin{aligned}A_hS^*\sum_{k=1}^{\tau(h)-1}\varepsilon_k^2&=S^*\left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{1/2}\left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{1/2}\frac{1}{\tau(h)}\sum_{k=1}^{\tau(h)-1}\varepsilon_k^2 \\ &=\sqrt{1-a^2}\sqrt{1-b^2}(1-ab)^{-1}+\alpha_h,\end{aligned} \tag{5.24}$$

where α_h satisfies, for $d>0, 0<q<1$, and $\delta>0$, the limiting relation

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|\le q\}}\mathbf{P}_{\boldsymbol{\theta}}(|\alpha_h|>\delta)=0\,. \tag{5.25}$$

For $|W_h|$, on the set $(\tau(h)>N+1)$, one has the following estimate

$$\begin{aligned}|W_h|&=\left|A_h\left(\sum_{k=1}^{\tau(h)-N-1}(S_{\tau(h)-1-k}-S^*)\varepsilon_k^2+\sum_{k=\tau(h)-N}^{\tau(h)-1}(S_{\tau(h)-1-k}-S^*)\varepsilon_k^2\right)\right| \\ &\le\max_{n\ge N}|S_n-S^*|A_h\sum_{k=1}^{\tau(h)-N-1}\varepsilon_k^2+\max_{n\ge1}|S_n-S^*|A_h\sum_{k=\tau(h)-N}^{\tau(h)-1}\varepsilon_k^2 \\ &\le\max_{n\ge N}|S_n-S^*|\left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{1/2}\left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{1/2}\frac{1}{\tau(h)}\sum_{k=1}^{\tau(h)-1}\varepsilon_k^2 \\ &\quad+\max_{n\ge1}|S_n-S^*|\left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{1/2}\left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{1/2} \\ &\quad\times\left(\frac{1}{\tau(h)}\sum_{k=1}^{\tau(h)-1}\varepsilon_k^2-\frac{1}{\tau(h)}\sum_{k=1}^{\tau(h)-N-1}\varepsilon_k^2\right).\end{aligned}$$

From here, in view of the inequalities,

$$\max_{n\geq N}|S_n - S^*| \leq q^{N+1}/(1-q),\ \max_{n\geq 1}|S_n - S^*| \leq q/(1-q),$$

by applying Lemmas 5.1- 5.3, we obtain

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|\leq q\}}\mathbf{P}_{\boldsymbol{\theta}}(|W_h|>\delta)=0.$$

This and (5.22)-(5.25) imply (5.20).

By a similar argument, one can show (5.21).

Thus we have verified all limiting relationships (5.20),(5.21), which give the asymptotic convergence of random variables $\xi_h^{(i)}$ on the parametric set $\Lambda_{d,1}$ with the additional condition $|ab| \leq q$. It remains to show that $\xi_h^{(i)}$ converges on the set $\Lambda_{d,1}$.

It will be observed that, by the definition of parametric set $\Lambda_{d,1}$ in (2.12), there exists a number $q^* \in (0,1)$ such that for all $q^* \leq q < 1$ the corresponding set $\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|\leq q\}$ contains all points of $\Lambda_{d,1}$ except for those lying in some vicinity of the apex $(0,1)$. On the other hand, function $r(a,b)$ in (5.18) vanishes when $|ab|$ approaches 1. Therefore, for a given $\delta>0$, there exists a number $\tilde{q}\geq q^*$ such that, for every $\boldsymbol{\theta}\in\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|\geq\tilde{q}\}$,

$$\sqrt{1-a^2}\sqrt{1-b^2}<\delta/3,$$

which implies

$$r(a,b)<\delta/3. \tag{5.26}$$

Consider $\xi_h^{(1)}$. Since $S_n \leq 1$ for negative ab, then, in view of Lemmas 5.1, 5.3, $|\xi_h^{(1)}|$ can be estimated as

$$|\xi_h^{(1)}| \leq \left(\frac{\tau(h)}{(u,u)_{\tau(h)}}\right)^{1/2}\left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{1/2}\frac{1}{\tau(h)}\sum_{k=1}^{\tau(h)-1}\varepsilon_k^2$$

$$=\sqrt{1-a^2}\sqrt{1-b^2}+\alpha_h,$$

where

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}}\mathbf{P}_{\boldsymbol{\theta}}(|\alpha_h|>\delta/3)=0.$$

From here and (5.26) one has

$$|\xi_h^{(1)}-r(a,b)|\le 2\delta/3+\alpha_h.$$

Therefore, for any $\Delta>0$, there exists a number h_0 such that for all $h\ge h_0$

$$\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|>\tilde{q}\}}\mathbf{P}_{\boldsymbol{\theta}}(|\xi_h^{(1)}-r(a,b)|>\delta)\le\Delta. \tag{5.27}$$

In view of (5.22), for a given Δ, there exists a number h_1 such that for all $h\ge h_1$

$$\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|\le\tilde{q}\}}\mathbf{P}_{\boldsymbol{\theta}}(|\xi_h^{(1)}-r(a,b)|>\delta)\le\Delta. \tag{5.28}$$

Combining (5.27) and (5.28) we come to (5.18).

To prove (5.19), we estimate $|\xi_h^{(2)}|$ for $\boldsymbol{\theta}\in\Lambda_{d,1}\cap\{\boldsymbol{\theta}:|ab|>\tilde{q}\}$ as

$$|\xi_h^{(2)}|\le\left(\sum_{k=2}^{\tau(h)-1}\varepsilon_k^2\right)^{1/2}\left(\sum_{k=2}^{\tau(h)-1}u_{k-1}^2\right)^{1/2}(u,u)_{\tau(h)}^{-1/2}(v,v)_{\tau(h)}^{-1/2}$$

$$\le\left(\frac{\tau(h)}{(v,v)_{\tau(h)}}\right)^{1/2}\left(\frac{1}{\tau(h)}\sum_{k=2}^{\tau(h)-1}\varepsilon_k^2\right)^{1/2}=\sqrt{1-b^2}+\alpha_h^{(1)},$$

where $\alpha_h^{(1)}$ satisfies

$$\lim_{h\to\infty}\sup_{\boldsymbol{\theta}\in\Lambda_{d,1}}\mathbf{P}_{\boldsymbol{\theta}}(|\alpha_h^{(1)}|>\delta/3)=0.$$

This enables us, by the same argument as in the case of $\xi_h^{(1)}$, to show (5.19). The case of $\xi_h^{(3)}$ can be studied by a similar way. This completes the proof of Proposition 2.2. □

2. Proof of Lemma 2.3. Consider in detail the case when $\boldsymbol{\theta}=(2,-1)$. Denote

$$f^{(n)}(t)=\frac{1}{n}\sum_{j=0}^{[nt-1]}W_{\frac{j}{n}}^{(n)},\ I_t(f)=\int_0^t f(s)ds, \tag{5.29}$$

where $W_t^{(n)}$ is given in (2.2). Then the nominator in (2.16) becomes

$$\sum_{k=1}^{n} u_{k-1}v_{k-1} = \sum_{k=1}^{n} \left(\sum_{j=0}^{k-1} \sum_{i=0}^{j} \varepsilon_i \right) \sum_{l=0}^{k-1} \varepsilon_l$$

$$= n^3 \sum_{k=1}^{n} \left(n^{-1} \sum_{j=0}^{k-1} W_{\frac{j}{n}}^{(n)} \right) W_{\frac{k-1}{n}}^{(n)} \frac{1}{n} = n^3 \sum_{k=1}^{n} f^{(n)}(\frac{k}{n}) W_{\frac{k-1}{n}}^{(n)} \frac{1}{n}. \tag{5.30}$$

It will be observed that

$$f^{(n)}(t) = I_{\frac{[nt-1]}{n}}(W^{(n)}) + r_n^{(1)}(t), \tag{5.31}$$

$$|r_n^{(1)}(t)| \leq \omega(W^{(n)}; [0,1]; 1/n), \tag{5.32}$$

where $\omega(f; E; \delta)$ denotes the oscilation of a function $f : E \to R$ of radius $\delta > 0$, that is

$$\omega(f; E; \delta) = \sup_{|x-y| \leq \delta, x, y \in E} |f(x) - f(y)|.$$

By (5.29), (5.32)

$$r_n^{(1)}(t) \leq \max_{1 \leq i \leq n} |\varepsilon_i| / \sqrt{n} \to 0 \text{ a.s.} \tag{5.33}$$

Substituting (5.31) in (5.30) yields

$$\sum_{k=1}^{n} u_{k-1}v_{k-1} = n^3 \sum_{k=1}^{n} I_{\frac{k-1}{n}}(W^{(n)}) W_{\frac{k-1}{n}}^{(n)} \frac{1}{n} + n^3 r_n^{(2)}, \tag{5.34}$$

where

$$r_n^{(2)} = \sum_{k=1}^{n} r_n^{(1)}(\frac{k}{n}) W_{\frac{k-1}{n}}^{(n)} \frac{1}{n}.$$

Note that in view of (5.33)

$$|r_n^{(2)}| \leq \max_{0 \leq t \leq 1} |W_t^{(n)}| \cdot \max_{1 \leq i \leq n} \frac{|\varepsilon_i|}{\sqrt{n}} = \max_{1 \leq k \leq n} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{k} \varepsilon_i \right| \max_{1 \leq i \leq n} \frac{|\varepsilon_i|}{\sqrt{n}}.$$

Show that, for any $\delta > 0$,

$$\lim_{n \to \infty} \mathbf{P}_{\boldsymbol{\theta}} \left(|r_n^{(2)}| > \delta \right) = 0. \tag{5.35}$$

By applying the Kolmogorov inequality one gets, for any $\delta > 0$ and $\Delta > 0$,

$$\mathbf{P}_{\boldsymbol{\theta}}\left(|r_n^{(2)}| > \delta\right) \le \mathbf{P}_{\boldsymbol{\theta}}\left(\max_{1\le i\le n}\frac{|\varepsilon_i|}{\sqrt{n}} > \Delta\right) + \mathbf{P}_{\boldsymbol{\theta}}\left(\Delta \max_{1\le k\le n}\frac{1}{\sqrt{n}}\left|\sum_{i=1}^{k}\varepsilon_i\right| > \delta\right)$$

$$\le \mathbf{P}_{\boldsymbol{\theta}}\left(\max_{1\le i\le n}\frac{|\varepsilon_i|}{\sqrt{n}} > \Delta\right) + \frac{\Delta^2}{\delta^2 n}\mathbf{E}_{\boldsymbol{\theta}}\sum_{i=1}^{n}\varepsilon_i^2 = \mathbf{P}_{\boldsymbol{\theta}}\left(\max_{1\le i\le n}\frac{|\varepsilon_i|}{\sqrt{n}} > \Delta\right) + \frac{\Delta^2\sigma^2}{\delta^2}.$$

This implies (5.35).

Now we rewrite (5.34) as

$$\sum_{k=1}^{n} u_{k-1}v_{k-1} = n^3\int_0^1 I_{\frac{[tn-1]}{n}}(W^{(n)})W^{(n)}_{\frac{[tn-1]}{n}}dt + n^3 r_n^{(3)} \tag{5.36}$$

$$= n^3\int_0^1 I_t(W^{(n)})W_t^{(n)}dt + n^3 r_n^{(3)} + n^3 r_n^{(4)},$$

where

$$|r_n^{(3)}| \le \omega\left(I_{\frac{[tn-1]}{n}}(W^{(n)})W^{(n)}_{\frac{[tn-1]}{n}}; [0,1]; \frac{1}{n}\right) \tag{5.37}$$

$$\le \max_{0\le t\le 1}\left|I_{\frac{[tn-1]}{n}}(W^{(n)})\right| \cdot \omega\left(W^{(n)}_{\frac{[tn-1]}{n}}; [0,1]; \frac{1}{n}\right)$$

$$+ \max_{0\le t\le 1}\left|W^{(n)}_{\frac{[tn-1]}{n}}\right| \cdot \omega\left(I_{\frac{[tn-1]}{n}}(W^{(n)}); [0,1]; \frac{1}{n}\right)$$

$$\le \max_{0\le t\le 1}\left|W_t^{(n)}\right| \cdot \omega\left(I_t(W^{(n)}); [0,1]; \frac{1}{n}\right)$$

$$+ \max_{0\le t\le 1}\left|W_t^{(n)}\right| \cdot \omega\left(W_t; [0,1]; \frac{1}{n}\right) = 2\max_{1\le k\le n}\frac{1}{\sqrt{n}}\left|\sum_{i=1}^{k}\varepsilon_i\right| \max_{1\le i\le n}\frac{|\varepsilon_i|}{\sqrt{n}};$$

$$r_n^{(4)} = \int_0^1 I_{\frac{[tn-1]}{n}}(W^{(n)})W^{(n)}_{\frac{[tn-1]}{n}}dt - \int_0^1 I_t(W^{(n)})W_t^{(n)}dt = A_n + B_n, \tag{5.38}$$

$$A_n = \int_0^1 \left(I_{\frac{[tn-1]}{n}}(W^{(n)}) - I_t(W^{(n)})\right) W^{(n)}_{\frac{[tn-1]}{n}}dt,$$

$$B_n = \int_0^1 I_t(W^{(n)})\left(W^{(n)}_{\frac{[tn-1]}{n}} - W_t^{(n)}\right)dt.$$

For A_n and B_n one has the estimates

$$|A_n| \le n^{-1}\max_{0\le t\le 1}|W_t^{(n)}|^2 = n^{-2}\max_{1\le k\le n}\left|\sum_{i=1}^{n}\varepsilon_i\right|^2;$$

$$|B_n| \le \max_{0\le t\le 1} |I_t(W^{(n)})| \cdot \max_{1\le i\le n} |\varepsilon_i|/\sqrt{n} \le \max_{0\le t\le 1} |W_t^{(n)}| \cdot \max_{1\le i\le n} |\varepsilon_i|/\sqrt{n}.$$

From here and (5.37) and (5.38), it follows that, for any $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}_{\boldsymbol{\theta}}(|r_n^{(i)}| > \delta) = 0,\ i = 3, 4. \tag{5.39}$$

Consider now the sums in the denominator of (2.16). By the same argument, one can show that

$$(u,u)_n = n^4 \int_0^1 I_t^2(W^{(n)})dt + n^4 r_n^{(5)}, (v,v)_n = n^2 \int_0^1 \left(W_t^{(n)}\right)^2 dt + n^2 r_n^{(6)}, \tag{5.40}$$

where $r_n^{(5)}$ and $r_n^{(6)}$ are such that, for any $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}_{\boldsymbol{\theta}}(|r_n^{(i)}| > \delta) = 0,\ i = 5, 6. \tag{5.41}$$

Substituting (5.34) and (5.40) in (2.16) yields

$$\xi_n = \varphi(W^{(n)}) + r_n, \tag{5.42}$$

where r_n, in view of (5.35), (5.39), (5.41), satisfies, for any $\delta > 0$, the limiting relation

$$\lim_{n\to\infty} \mathbf{P}_{\boldsymbol{\theta}}(|r_n| > \delta) = 0.$$

One can check that functional $\varphi(x)$ given by (2.21) is continuous everywhere in $C[0,1]$ except for the point $x(t) \equiv 0$. Since the Wiener measure of the set $D = \{x \equiv 0\}$ equals zero we can apply the Donsker theorem to this functional in (5.42). This leads to (2.20). It remains to verify that $0 \le \varphi(W) \le 1$. It is obvious that the function $\varphi(W)$ in (2.21) can be viewed as the inner product of the functions

$$x(t) = \mathcal{J}_2^{-1/2}(W;1) \int_0^t W(s)ds,\ y(t) = \mathcal{J}_1^{-1/2}(W;1)\, W(t).$$

The equality $\varphi(W) = 1$ is possible iff the functions $x(t)$ and $y(t)$ are linearly dependent, that is, $x(t) = Cy(t),\ 0 \le t \le 1$, for some constant C. However

this does not hold with probability one, because $x(t)$ is absolutely continuous and $y(t)$ is non-differentiable almost everywhere. Hence the case $\boldsymbol{\theta} = (2, -1)$. By similar argument, one can show (2.20) for $\boldsymbol{\theta} = (-2, -1)$. This completes the proof of Lemma 2.3. □

3. Additional properties of the sums $\sum_{k=1}^{\tau(h)} u_{k-1}^2$ and $\sum_{k=1}^{\tau(h)} v_{k-1}^2$. In addition to Lemma 5.3 we will need the following results.

Lemma 5.4. *For each $d > 0$ and $\delta > 0$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap(\boldsymbol{\theta}:\theta_1\le 0)} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \frac{h\sigma^2}{\sum_{k=1}^{\tau(h)} u_{k-1}^2} - \frac{2(1+ab)}{(1-ab)(1-b^2)} \right| > \delta \right) = 0, \quad (5.43)$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \frac{1}{\sum_{k=1}^{\tau(h)} v_{k-1}^2} - \frac{1-b^2}{\tau(h)} \right| > \delta \right) = 0, \quad (5.44)$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}\cap(\boldsymbol{\theta}:\theta_1>0)} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \frac{h\sigma^2}{\sum_{k=1}^{\tau(h)} v_{k-1}^2} - \frac{2(1+ab)}{(1-ab)(1-a^2)} \right| > \delta \right) = 0\,, \quad (5.45)$$

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}} \left(\left| \frac{1}{\sum_{k=1}^{\tau(h)} u_{k-1}^2} - \frac{1-a^2}{\tau(h)} \right| > \delta \right) = 0. \quad (5.46)$$

Proof of Lemma 5.4. Consider first (5.44) and (5.46). By Lemma 5.3

$$\frac{\tau(h)}{\sum_{k=1}^{\tau(h)} u_{k-1}^2} = 1 - a^2 + \alpha_1(h), \ \frac{\tau(h)}{\sum_{k=1}^{\tau(h)} v_{k-1}^2} = 1 - b^2 + \alpha_2(h),$$

where $\alpha_1(h)$ and $\alpha_2(h)$ satisfy, for any $\delta > 0$, the relation

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in\Lambda_{d,1}} P_{\boldsymbol{\theta}}(|\alpha_i(h)| > \delta) = 0, \ i = 1, 2. \quad (5.47)$$

Therefore

$$\frac{1}{\sum_{k=1}^{\tau(h)} u_{k-1}^2} - \frac{1-a^2}{\tau(h)} = \frac{\alpha_1(h)}{\tau(h)}, \ \frac{1}{\sum_{k=1}^{\tau(h)} v_{k-1}^2} - \frac{1-b^2}{\tau(h)} = \frac{\alpha_2(h)}{\tau(h)}.$$

These equalities and (5.47) imply (5.44), (5.46). Denote

$$t_h = h\sigma^2 \, (u,u)_{\tau(h)}^{-1} - 2(1+ab)/((1-ab)(1-b^2)).$$

By the definition of stopping time $\tau(h)$ in (1.7), one has

$$h\sigma^2 = \sum_{k=1}^{\tau(h)-1} \|X_{k-1}\|^2 + \alpha_h \|X_{\tau(h)-1}\|^2 = \sum_{k=1}^{\tau(h)}{}' \|X_{k-1}\|^2,$$

where the prime at the sum sign means that the last addend is taken with the correction factor α_h providing the validity of the left-hand side equality, $0 < \alpha_h \leq 1$.

This equality implies

$$\frac{h\sigma^2}{\sum_{k=1}^{\tau(h)} u_{k-1}^2} = \frac{1}{\sum_{k=1}^{\tau(h)} u_{k-1}^2} tr \sum_{k=1}^{\tau(h)}{}' X_{k-1} X_{k-1}'$$

$$= tr Q^{-1} \begin{pmatrix} 1 & (u,v)_{\tau(h)}/(u,u)_{\tau(h)} \\ (u,v)_{\tau(h)}/(u,u)_{\tau(h)} & (v,v)_{\tau(h)}/(u,u)_{\tau(h)} \end{pmatrix} (Q^{-1})'. \quad (5.48)$$

By Lemma 5.3

$$\frac{(v,v)_{\tau(h)}}{(u,u)_{\tau(h)}} = \frac{\tau(h)}{(u,u)_{\tau(h)}} \times \frac{(v,v)_{\tau(h)}}{\tau(h)} = [\frac{1-a^2}{\sigma^2} + \alpha_1(h)](v,v)_{\tau(h)}/\tau(h)\,. \quad (5.49)$$

Since, on the set $\Lambda_{d,1} \cap (\boldsymbol{\theta} : \theta_1 \leq 0)$, parameter b is bounded away from the end-points of the interval $(-1, 1)$, then, for any $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta} \in \Lambda_{d,1} \cap (\boldsymbol{\theta}: \theta_1 \leq 0)} \mathbf{P}_{\boldsymbol{\theta}} \left(|(\tau(h))^{-1} (v,v)_{\tau(h)} - \sigma^2 (1-b^2)^{-1}| > \delta \right) = 0.$$

From here and (5.49), it follows that

$$(v,v)_{\tau(h)}/(u,u)_{\tau(h)} = (1-a^2)/(1-b^2) + \alpha_3(h),$$

where $\alpha_3(h)$ satisfies the following relation : for any $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta} \in \Lambda_{d,1}} \mathbf{P}_{\boldsymbol{\theta}}(|\alpha_3(h)| > \delta) = 0. \quad (5.50)$$

In view of (5.49), Lemma 5.3 and Proposition 2.2, the cross-term in (5.48) can be written as

$$\frac{(u,v)_{\tau(h)}}{(u,u)_{\tau(h)}} = \left(\frac{(v,v)_{\tau(h)}}{(u,u)_{\tau(h)}} \right)^{1/2} \frac{(u,v)_{\tau(h)}}{(u,u)_{\tau(h)}^{1/2} (v,v)_{\tau(h)}^{1/2}} = \frac{1-a^2}{1-ab} + \alpha_4(h)\,, \quad (5.51)$$

where $\alpha_4(h)$, in view of Proposition 2.2, also possesses the property given in (5.50). Hence

$$\frac{h\sigma^2}{(u,u)_{\tau(h)}} = trQ^{-1}\begin{pmatrix} 1 & (1-a^2)/(1-ab) \\ (1-a^2)/(1-ab) & (1-a^2)/(1-b^2) \end{pmatrix}(Q^{-1})' + r_h, \tag{5.52}$$

where

$$r_h = trQ^{-1}\begin{pmatrix} 0 & \alpha_4(h) \\ \alpha_4(h) & \alpha_3(h) \end{pmatrix}(Q^{-1})'.$$

One can easily verify that

$$trQ^{-1}\begin{pmatrix} 1 & (1-a^2)/(1-ab) \\ (1-a^2)/(1-ab) & (1-a^2)/(1-b^2) \end{pmatrix}(Q^{-1})' = \frac{2(1+ab)}{(1-ab)(1-b^2)}.$$

From here and (5.52), taking into account (5.50), we come to the assertion of Lemma 5.4. □

4. The Skorohod coupling theorem. Proof of Proposition 3.2. By Theorem 2.1, on the boundary $\partial\Lambda$ of the stability region (1.4), the stopping time $\tau(h)$ (1.7) converges in distribution to some functional of one or two Brownian motions. In order to prove Proposition 3.2 we need to strengthen this convergence by applying the following result.

Theorem 5.5. *(extended Skorohod coupling; see Theorem 4.30 and Corollary 6.12 in [9].) Let $f, f_1, f_2, \ldots$ be measurable functions from a Borel space S to a Polish space T, and let $\eta, \eta_1, \eta_2, \ldots$ be random elements in S with $f_n(\eta_n) \xrightarrow{\mathcal{L}} f(\eta)$. Then there exists a probability space with some random elements $\tilde{\eta} \stackrel{\mathcal{L}}{=} \eta$ and $\tilde{\eta}_n \stackrel{\mathcal{L}}{=} \eta_n$, $n \in \mathbb{N}$, with $f_n(\tilde{\eta}_n) \to f(\tilde{\eta})$ a.s.*

Let $W = (W(t))_{t\geq 0}$ and $W_1 = (W_1(t))_{t\geq 0}$ be independent Brownian motions and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ be an sequence of i.i.d. random variables with $\mathbf{E}\varepsilon_1 = 0$ and $\mathbf{E}\varepsilon_1^2 = \sigma^2$, which does not depend on W, W_1. Random elements

$$\eta = (\varepsilon, W, W_1)$$

take on values in the space $S = \mathbf{R}^{\infty} \times \mathbf{C}(\mathbf{R}_+) \times \mathbf{C}(\mathbf{R}_+)$, where $\mathbf{C}(\mathbf{R}_+)$ is the set of all continuous functions on $\mathbf{R}_+ = [0, \infty)$. Define the metric on S by the formula

$$\rho(\eta^{'}, \eta^{''}) = \rho_1(\varepsilon^{'}, \varepsilon^{''}) + \rho_2(W^{'}, W^{''}) + \rho_3(W_1^{'}, W_1^{''}),$$

where

$$\rho_1(\varepsilon^{'}, \varepsilon^{''}) = \sum_{k \geq 1} 2^{-k} \frac{|\varepsilon_k^{'} - \varepsilon_k^{''}|}{1 + |\varepsilon_k^{'} - \varepsilon_k^{''}|},$$

$$\rho_i(x, y) = \sum_{k \geq 1} 2^{-k} \frac{\max_{1 \leq t \leq k} |x(t) - y(t)|}{1 + \max_{1 \leq t \leq k} |x(t) - y(t)|}, \ i = 2, 3.$$

Let $(S, \mathcal{B}(S), \mathbf{P}_\eta)$ be the corresponding Borel space with the distribution $\mathbf{P}_\eta$ induced by η, that is, $\mathbf{P}_\eta = \mathbf{P}_\varepsilon \times \mathbf{P}_W \times \mathbf{P}_{W_1}$.

Now we are ready to prove Proposition 3.2.

Assume that $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2$.

Consider only the case when $\boldsymbol{\theta} \in \Gamma_1$ (the case $\boldsymbol{\theta} \in \Gamma_2$ is similar). For $\boldsymbol{\theta} \in \Gamma_1$ the processes $(u_k)_{k \geq 0}$ and $(v_k)_{k \geq 0}$ are described by equations (3.2) with $a = -1$ and $|b| < 1$. Let us apply the Skorohod Theorem 5.5 to the functional

$$f_n(\eta) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\tau(h)} v_{k-1}^2, \ n = [h/2],$$

and put $\eta_n \equiv \eta$. By Lemma 5.3 and Theorem 2.1 we have

$$f_n(\eta) \xrightarrow{\mathcal{L}} \nu_1(W_1)\sigma^2/(1-b) = f(\eta).$$

By Theorem 5.5 there exists $\tilde{\eta} = (\tilde{\varepsilon}, \tilde{W}, \tilde{W}_1)$ such that $\tilde{\eta} = (\tilde{\varepsilon}, \tilde{W}, \tilde{W}_1) \stackrel{\mathcal{L}}{=} \eta = (\varepsilon, W, W_1)$ and

$$f_n(\tilde{\eta}) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}_{k-1}^2 \stackrel{\text{a.s.}}{\to} \frac{\nu_1(\tilde{W}_1)\sigma^2}{1-b} = f(\tilde{\eta}), \ n = [h/2]. \tag{5.53}$$

It should be noted that all the sequences $(\tilde{x}_k), (\tilde{u}_k), (\tilde{v}_k)$ and the stopping time $\tilde{\tau}$ are defined by formulae (1.1),(3.2) and (1.7) with a given $\boldsymbol{\theta} \in \Gamma_1$

replacing in them $\varepsilon = (\varepsilon_k)$ by $\tilde{\varepsilon} = (\tilde{\varepsilon}_k)$. Besides we define a counterpart $\tilde{Y}_h$ for Y_h in (3.6) by the formula

$$\tilde{Y}_h = \frac{\lambda_1}{\sigma\sqrt{(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{u}_{k-1}\tilde{\varepsilon}_k + \frac{\lambda_2}{\sigma\sqrt{(\tilde{v},\tilde{v})_{\tilde{\tau}(h)}}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}_{k-1}\tilde{\varepsilon}_k \,. \tag{5.54}$$

By the construction the distribution of the random variable $\tilde{Y}_h$ coincides with that of Y_h and therefore, for our purposes, it suffices to study its asymptotic distribution as $h \to \infty$.

In view of (5.53) and Lemma 5.4, we start the analysis of $\tilde{Y}_h$ by rewriting it as

$$\begin{aligned}\tilde{Y}_h &= \frac{\lambda_1}{\sigma\sqrt{h}}\sqrt{\frac{h}{(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{u}_{k-1}\tilde{\varepsilon}_k + \frac{\lambda_2}{\sigma(h/2)^{1/4}}\sqrt{\frac{\sqrt{h/2}}{(\tilde{v},\tilde{v})_{\tilde{\tau}(h)}}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}_{k-1}\tilde{\varepsilon}_k \\ &= \frac{1}{\sigma^2\sqrt{h}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{g}_{k-1}\tilde{\varepsilon}_k + r_1(h), \end{aligned} \tag{5.55}$$

where

$$\tilde{g}_{k-1} = \frac{\lambda_1\sqrt{2}}{1+b}\tilde{u}_{k-1} + \lambda_2\sqrt{\frac{1-b}{\nu_1(\tilde{W}_1)}}(2h)^{1/4}\tilde{v}_{k-1}\,, \tag{5.56}$$

$$r_1(h) = \frac{1}{\sqrt{h}} t_h^1 \sum_{k=1}^{\tilde{\tau}(h)} \tilde{u}_{k-1}\tilde{\varepsilon}_k + \left(\frac{1}{h}\right)^{1/4} t_h^2 \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}_{k-1}\tilde{\varepsilon}_k \,, \tag{5.57}$$

$$t_h^1 = \frac{\lambda_1}{\sigma^2}\left(\sqrt{h\sigma^2/(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}} - \frac{\sqrt{2}}{1+b}\right),$$

$$t_h^2 = \frac{2^{1/4}\lambda_2}{\sigma^2}\left(\sqrt{\sigma^2\sqrt{(h/2)}/(\tilde{v},\tilde{v})_{\tilde{\tau}(h)}} - \sqrt{(1-b)/\nu_1(\tilde{W}_1)}\right).$$

Let us show that, for any $\delta > 0$,

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}^{'}(|r_1(h)| > \delta) = 0\,, \tag{5.58}$$

where $P_{\boldsymbol{\theta}}^{'}$ is the distribution of the process $(\tilde{x}_k)$. We rewrite $r_1(h)$ as

$$r_1(h) = \frac{t_h^1}{\sqrt{h}} \sum_{k=1}^{\tilde{\tau}(h)-1} \tilde{u}_{k-1}\tilde{\varepsilon}_k + \frac{t_h^1}{\sqrt{h}}\tilde{u}_{\tilde{\tau}(h)-1}\tilde{\varepsilon}_{\tilde{\tau}(h)} + \left(\frac{1}{h}\right)^{1/4} t_h^2 \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}_{k-1}\tilde{\varepsilon}_k \,.$$

For any $\delta > 0$ and any $C > 0$, we have the estimate

$$\mathbf{P}_{\boldsymbol{\theta}}^{'}(|r_1(h)| > \delta) \leq \mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\left|\frac{t_h^1}{\sqrt{h}}\sum_{k=1}^{\tilde{\tau}(h)-1}\tilde{u}_{k-1}\tilde{\varepsilon}_k\right| > \frac{\delta}{3}\right)$$

$$+\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(|\frac{t_h^1}{\sqrt{h}}\tilde{u}_{\tilde{\tau}(h)-1}\tilde{\varepsilon}_{\tilde{\tau}(h)}| > \frac{\delta}{3}\right) + \mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{1}{h^{1/4}}|t_h^2\sum_{k=1}^{\tilde{\tau}(h)}\tilde{v}_{k-1}\tilde{\varepsilon}_k| > \frac{\delta}{3}\right)$$

$$\leq \mathbf{P}_{\boldsymbol{\theta}}^{'}(\frac{1}{\sqrt{h}}|\sum_{k=1}^{\tilde{\tau}(h)-1}\tilde{u}_{k-1}\tilde{\varepsilon}_k| > C) + \mathbf{P}_{\boldsymbol{\theta}}^{'}(|t_h^1|C > \frac{\delta}{3}) + \mathbf{P}_{\boldsymbol{\theta}}^{'}(\frac{|\tilde{u}_{\tilde{\tau}(h)-1}\tilde{\varepsilon}_{\tilde{\tau}(h)}|}{\sqrt{h}} > \sqrt{\frac{\delta}{3}})$$

$$+\mathbf{P}_{\boldsymbol{\theta}}^{'}(|t_h^1| > \sqrt{\delta/3}) + \mathbf{P}_{\boldsymbol{\theta}}^{'}(\frac{1}{h^{1/4}}|\sum_{k=1}^{\tilde{\tau}(h)}\tilde{v}_{k-1}\tilde{\varepsilon}_k| > C) + \mathbf{P}_{\boldsymbol{\theta}}^{'}(|t_h^2|C > \frac{\delta}{3})\,. \quad (5.59)$$

Now we will study the asymptotic behaviour of the summands in the right-hand side of (5.59).

Lemma 5.6. *For each* $\boldsymbol{\theta} \in \Gamma_1$,

$$\lim_{C\to\infty}\sup_{h>0}\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{1}{\sqrt{h}}|\sum_{k=1}^{\tilde{\tau}(h)-1}\tilde{u}_{k-1}\tilde{\varepsilon}_k| > C\right) = 0\,. \quad (5.60)$$

Lemma 5.7. *For each* $\delta > 0$,

$$\lim_{h\to\infty}\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(h^{-1/2}|\tilde{u}_{\tilde{\tau}(h)-1}\tilde{\varepsilon}_{\tilde{\tau}(h)}| > \delta\right) = 0\,. \quad (5.61)$$

Lemma 5.8. *For any* $0 < C < \infty$ *and* $a > 0$,

$$\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{1}{h^{1/4}}|\sum_{k=1}^{\tilde{\tau}(h)}\tilde{v}_{k-1}\tilde{\varepsilon}_k| \geq C\right) \leq \frac{a}{C} + \mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{1}{\sqrt{h}}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{v}_{k-1}^2 \geq a\right)\,. \quad (5.62)$$

Lemma 5.9. *For any* $a > 0$ *and* $\Delta > 0$,

$$\lim_{h\to\infty}\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{1}{\sqrt{h}}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{v}_{k-1}^2 \geq a\right) \leq \mathbf{P}_{\boldsymbol{\theta}}^{'}(\nu_1(\tilde{W}_1) \geq a^{'})\,, \quad (5.63)$$

where $a^{'} = a\sqrt{2}(1+b)^{-1}\left(\sigma^2(1-b^2)^{-1} + \Delta\right)^{-1}$.

Proof of Lemma 5.6. By definition of stopping time $\tau(h)$ in (1.7) one obtains

$$\mathbf{E}_{\boldsymbol{\theta}}^{'}\left(\frac{1}{\sqrt{h}}\sum_{k=1}^{\tilde{\tau}(h)-1}\tilde{u}_{k-1}\tilde{\varepsilon}_k\right)^2=\frac{\sigma^2}{h}\mathbf{E}_{\boldsymbol{\theta}}^{'}\left(\sum_{k=1}^{\tilde{\tau}(h)-1}\tilde{u}_{k-1}^2\right)$$

$$\leq\frac{\sigma^2}{h}\mathbf{E}_{\boldsymbol{\theta}}^{'}\left(\sum_{k=1}^{\tilde{\tau}(h)-1}(\tilde{u}_{k-1}^2+\tilde{v}_{k-1}^2)\right)\leq\frac{\sigma^2}{h}\|Q\|^2\mathbf{E}_{\boldsymbol{\theta}}^{'}\left(\sum_{k=1}^{\tilde{\tau}(h)-1}\|\tilde{X}_{k-1}\|^2\right)\leq\sigma^4\|Q\|^2\,,$$

where $\mathbf{E}_{\boldsymbol{\theta}}^{'}$ is the expectation with respect to $\mathbf{P}_{\boldsymbol{\theta}}^{'}$. This implies (5.60). Hence Lemma 5.6. $\square$

Proof of Lemma 5.7. One has

$$\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(|\tilde{u}_{\tilde{\tau}(h)-1}\tilde{\varepsilon}_{\tilde{\tau}(h)}|/\sqrt{h}>\delta\right)\leq\mathbf{P}_{\boldsymbol{\theta}}\left(\|Q\|(\sqrt{h})^{-1}\|\tilde{X}_{\tilde{\tau}(h)-1}\|\cdot|\tilde{\varepsilon}_{\tilde{\tau}(h)}|>\delta\right)$$

$$\leq\mathbf{P}_{\boldsymbol{\theta}}^{'}(\bullet,\ |\tilde{\varepsilon}_{\tilde{\tau}(h)}|\leq C)+\mathbf{P}_{\boldsymbol{\theta}}^{'}(\bullet,\ |\tilde{\varepsilon}_{\tilde{\tau}(h)}|>C)$$

$$\leq\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{\|Q\|}{\sqrt{h}}\|\tilde{X}_{\tilde{\tau}(h)-1}\|C>\delta\right)+C^{-2}E_{\boldsymbol{\theta}}^{'}\tilde{\varepsilon}_{\tilde{\tau}(h)}^2$$

$$\leq\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(h^{-1}\|Q\|^2\|\tilde{X}_{\tilde{\tau}(h)-1}\|^2C^2>\delta^2\right)+C^{-2}\sigma^2.$$

It remains to show that

$$\lim_{h\to\infty}\mathbf{P}_{\boldsymbol{\theta}}^{'}(h^{-1}\|\tilde{X}_{\tilde{\tau}(h)-1}\|^2>\tilde{\delta})=0,\tag{5.64}$$

where $\tilde{\delta}=\delta^2\|Q\|^{-2}C^{-2}$. We have

$$\mathbf{P}_{\boldsymbol{\theta}}^{'}(h^{-1}\|\tilde{X}_{\tilde{\tau}(h)-1}\|^2>\tilde{\delta})\leq\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\|\tilde{X}_{\tau(h)-1}\|^2\geq\frac{\tilde{\delta}}{\sigma^2}\sum_{k=1}^{\tilde{\tau}(h)-1}\|\tilde{X}_{k-1}\|^2\right)$$

$$\leq\mathbf{P}_{\boldsymbol{\theta}}^{'}(\tilde{\tau}(h)\leq m)+P_{\boldsymbol{\theta}}\left(\|\tilde{X}_n\|^2\geq\frac{\tilde{\delta}}{\sigma^2}\sum_{k=1}^{n}\|\tilde{X}_{k-1}\|^2\text{ for some }n\geq m\right).$$

In virtue of relation (3.3) in [7] and Lemma 5.1, we come to (5.64). This completes the proof of Lemma 5.7. $\square$

Further we need the following Lenglart inequality.

Lemma 5.10. *(See, [13] Ch VII, 3, Th4). Let $(\xi_n, \mathcal{F}_n)$ be non-negative adapted sequence of random variables and $(A_n, \mathcal{F}_n)$ be predictable increasing sequence which dominates (ξ_n) in the sense that, for any stopping time σ with respect to $(\mathcal{F}_n)$, one has $\mathbf{E}\xi_\sigma \leq \mathbf{E}A_\sigma$. Then, for any $\varepsilon > 0$ and $a > 0$,*

$$\mathbf{P}(\sup_{1\leq j\leq \sigma} \xi_j \geq \varepsilon) \leq \varepsilon^{-1}\mathbf{E}(A_\sigma \wedge a) + \mathbf{P}(A_\sigma \geq a).$$

Proof of Lemma 5.8. Denote

$$\xi_n = h^{-1/2}\left(\sum_{k=1}^{n} \tilde{v}_{k-1}\tilde{\varepsilon}_k\right)^2, \ n \geq 1, \ \xi_0 = 0; \tag{5.65}$$

$$A_n = h^{-1/2}\sum_{k=1}^{n} \tilde{v}_{k-1}^2, \ n \geq 1, \ A_0 = 0 \tag{5.66}$$

Let us introduce the filtration $(\mathcal{F}_n)_{n\geq 0}$ with

$$\mathcal{F}_0 = \sigma\{\nu(\tilde{W}_1)\}, \ \mathcal{F}_n = \sigma\{\nu(\tilde{W}_1), \tilde{\varepsilon}_1 \ldots, \tilde{\varepsilon}_n\}. \tag{5.67}$$

Note that for each stopping time σ with respect to this filtration, one has

$$\mathbf{E}'_{\boldsymbol{\theta}}\xi_\sigma \leq \mathbf{E}'_{\boldsymbol{\theta}}A_\sigma.$$

Therefore the processes (5.65)-(5.66) satisfy the conditions of Lemma 5.10. Applying this Lemma with $\sigma = \tilde{\tau}(h)$ yields (5.62):

$$\mathbf{P}'_{\boldsymbol{\theta}}(\xi_{\tilde{\tau}(h)} \geq C) \leq C^{-1}\mathbf{E}'_{\boldsymbol{\theta}}(A_{\tilde{\tau}(h)} \wedge a) + \mathbf{P}'_{\boldsymbol{\theta}}(A_{\tilde{\tau}(h)} \geq a)$$

$$\leq aC^{-1} + \mathbf{P}'_{\boldsymbol{\theta}}(A_{\tilde{\tau}(h)} \geq a).$$

Hence Lemma 5.8.

Proof of Lemma 5.9. For any $\Delta > 0$ one has

$$\mathbf{P}'_{\boldsymbol{\theta}}\left(h^{-1/2}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)} \geq a\right) = \mathbf{P}'_{\boldsymbol{\theta}}\left((\tilde{\tau}(h)/\sqrt{h})\tilde{\tau}(h)^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)} \geq a\right)$$

$$= \mathbf{P}'_{\boldsymbol{\theta}}\left(\bullet, |\tilde{\tau}(h)^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)} - \sigma^2/(1-b^2)| < \Delta\right)$$

$$+\mathbf{P}'_{\boldsymbol{\theta}}\left(\bullet, |\tilde{\tau}(h)^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)} - \sigma^2/(1-b^2)| \geq \Delta\right)$$

$$\leq \mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{\tilde{\tau}(h)}{\sqrt{h}}(\frac{\sigma^2}{1-b^2}+\Delta)\geq a\right)+\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\left|\frac{(\tilde{v},\tilde{v})_{\tilde{\tau}(h)}}{\tilde{\tau}(h)}-\frac{\sigma^2}{1-b^2}\right|\geq\Delta\right)$$

$$=\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{\tilde{\tau}(h)}{(1+b)\sqrt{h/2}}\frac{(1+b)}{\sqrt{2}}(\sigma^2(1-b^2)^{-1}+\Delta)\geq a\right)$$

$$+\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(|\tilde{\tau}(h)^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)}-\sigma^2/(1-b^2)|\geq\Delta\right).$$

From here, in virtue of Theorem 2.1 and Lemma 5.3, we comme to (5.63). Hence Lemma 5.9. □

Now we are ready to show (5.58). Limiting in (5.59) $h\to\infty$ and taking into account Lemma 5.4, Lemmas 5.6- 5.9 and (5.53) we obtain

$$\limsup_{h\to\infty}\mathbf{P}_{\boldsymbol{\theta}}^{'}(|r_1(h)|>\delta)\leq\sup_{h>0}\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\frac{1}{\sqrt{h}}\left|\sum_{k=1}^{\tilde{\tau}(h)-1}\tilde{u}_{k-1}^2\tilde{\varepsilon}_k\right|>C\right)$$

$$+a/C+\mathbf{P}_{\boldsymbol{\theta}}^{'}(\nu_1(\tilde{W}_1)\geq a^{'}).$$

In view of Lemma 5.6, limiting $C\to\infty$ and then $a\to\infty$, we come to (5.58).

So we have

$$\tilde{Y}_h=\frac{1}{\sigma^2\sqrt{h}}\sum_{k=1}^{\tilde{\tau}(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k+r_1(h),\tag{5.68}$$

where $r_1(h)$ satisfies (5.58) and

$$\tilde{g}_{k-1}=\frac{\lambda_1\sqrt{2}}{1+b}\tilde{u}_{k-1}+\frac{\lambda_2\sqrt{1-b}}{\sigma\sqrt{\nu_1(\tilde{W}_1)}}(2h)^{1/4}\tilde{v}_{k-1}.$$

For a given $h>0$ we define the random variable

$$\tau_0(h)=\inf\{n\geq1:\sum_{k=1}^{n}\tilde{g}_{k-1}^2\geq h\sigma^2\},\ \inf\{\emptyset\}=\infty,\tag{5.69}$$

which is a stopping time with respect to the filtration $(\mathcal{F}_n)$ in (5.67), and rewrite $\tilde{Y}_h$ from (5.68) as

$$\tilde{Y}_h=\frac{1}{\sigma^2\sqrt{h}}\sum_{k=1}^{\tau_0(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k+r_1(h)+r_2(h),\tag{5.70}$$

where

$$r_2(h)=\frac{1}{\sigma^2\sqrt{h}}\left(\sum_{k=1}^{\tilde{\tau}(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k-\sum_{k=1}^{\tau_0(h)}\tilde{g}_{k-1}\tilde{\varepsilon}_k\right).\tag{5.71}$$

Now we observe that the first term in the right-hand side of (5.70) is a martingale with respect to the filtration $(\mathcal{F}_n)$ in (5.67)) stopped at the time (5.69). According to the Theorem 2.1 from [10], it is asymptotically normal with mean 0 and unit variance as $h \to \infty$. Therefore to end the proof of Theorem 3.1 for $\boldsymbol{\theta} \in \Gamma_1$ it remains to prove that, for any $\delta > 0$,

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}^{'}(|r_2(h)| > \delta) = 0\,. \tag{5.72}$$

First we will establish the following results.

Lemma 5.11. *For each $\boldsymbol{\theta} \in \Gamma_1$ and any $\delta > 0$,*

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\tilde{g}^2_{\tau_0(h)-1} \Big/ \sum_{k=1}^{\tau_0(h)-1} \tilde{g}^2_{k-1} > \delta\right) = 0\,. \tag{5.73}$$

Lemma 5.12. *For each $\boldsymbol{\theta} \in \Gamma_1$ and any $\delta > 0$,*

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}^{'}(U_h > \delta) = 0\,,$$

where

$$U_h = \frac{1}{h} \sum_{k=\tilde{\tau}(h)\wedge\tau_0(h)+1}^{\tilde{\tau}(h)\vee\tau_0(h)} \tilde{g}^2_{k-1}\,.$$

Proof of Lemma 5.11. One has the inclusions, for any $\Delta > 0$,

$$\left(\tilde{g}^2_{\tau_0(h)-1} \Big/ \sum_{k=1}^{\tau_0(h)-1} \tilde{g}^2_{k-1} > \delta\right) \subseteq \Big(\|J_{\tau_0(h)-1} - I\| > \Delta\Big) \cup A\,, \tag{5.74}$$

where I is 2×2 identity matrix,

$$A = \left(\tilde{g}^2_{\tau_0(h)-1} \Big/ \sum_{k=1}^{\tau_0(h)-1} \tilde{g}^2_{k-1} > \delta,\ \|J_{\tau_0(h)-1} - I\| \le \Delta\right)\,.$$

By (5.56) one gets

$$\tilde{g}^2_n = (z_1\tilde{u}_n + z_2\tilde{v}_n)^2 \le 2z_1^2\tilde{u}_n^2 + 2z_2^2\tilde{v}_n^2, \tag{5.75}$$

$$z_1 = \lambda_1\sqrt{2}/(1+b),\ z_2 = \lambda_2\sqrt{1-b}(2h)^{1/4}/(\sigma\sqrt{\nu_1(\tilde{W}_1)});$$

$$\sum_{k=1}^{n}\tilde{g}_{k-1}^2=\sum_{k=1}^{n}\left(Z'(\tilde{u}_{k-1},\tilde{v}_{k-1})'\right)^2=Z'R_n^{-1}J_nR_n^{-1}Z$$
$$=Z'R_n^{-1}(J_n-I)R_n^{-1}Z+Z'R_n^{-2}Z,\ Z=(z_1,z_2)';$$
$$Z'R_n^{-2}Z=z_1^2\,(u,u)_n+z_2^2\,(v,v)_n. \tag{5.76}$$

From here it follows that

$$A\subset\left(\tilde{g}_{\tau_0(h)-1}^2>\delta Z'\tilde{R}_{\tau_0(h)-1}^{-2}Z(1-\|\tilde{J}_{\tau_0(h)-1}-I\|),\ \|\tilde{J}_{\tau_0(h)-1}-I\|\le\Delta\right)$$
$$\subset\left(\tilde{g}_{\tau_0(h)-1}^2>\delta(1-\Delta)Z'\tilde{R}_{\tau_0(h)-1}^{-2}Z\right)\subset(\tau_0(h)\le m)$$
$$\cup\left(\tilde{g}_n^2>\delta(1-\Delta)Z'\tilde{R}_n^{-2}Z\text{ for some }n\ge m\right)$$
$$\subset(\tau_0(h)\le m)\cup\left(2z_1^2\tilde{u}_n^2>\delta 2^{-1}(1-\Delta)Z'\tilde{R}_n^{-2}Z\text{ for some }n\ge m\right)$$
$$\cup\left(2z_2^2\tilde{v}_n^2>\delta 2^{-1}(1-\Delta)Z'\tilde{R}_n^{-2}Z\text{ for some }n\ge m\right)$$
$$\subset(\tau_0(h)\le m)\cup\left(2\tilde{u}_n^2>\delta 2^{-1}(1-\Delta)\,(\tilde{u},\tilde{u})_n\text{ for some }n\ge m\right)$$
$$\cup\left(2\tilde{v}_n^2>\delta 2^{-1}(1-\Delta)\,(\tilde{v},\tilde{u})_n\text{ for some }n\ge m\right). \tag{5.77}$$

Combining inclusions (5.74), (5.77) yields

$$\left(\tilde{g}_{\tau_0(h)-1}^2\Big/\sum_{k=1}^{\tau_0(h)-1}\tilde{g}_{k-1}^2>\delta\right)\subset\left(\|\tilde{J}_{\tau_0(h)-1}-I\|>\Delta\right)\cup(\tau_0(h)\le m)$$
$$\cup\left(\tilde{u}_n^2>\delta'(\tilde{u},\tilde{u})_n\text{ for some }n\ge m\right)$$
$$\cup\left(\tilde{v}_n^2>\delta'(\tilde{v},\tilde{v})_n\text{ for some }n\ge m\right),\ \delta'=\delta\,4^{-1}(1-\Delta).$$

This implies

$$\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\tilde{g}_{\tau_0(h)-1}^2\Big/\sum_{k=1}^{\tau_0(h)-1}\tilde{g}_{k-1}^2>\delta\right)\le\mathbf{P}_{\boldsymbol{\theta}}^{'}(\|\tilde{J}_{\tau_0(h)-1}-I\|>\Delta) \tag{5.78}$$
$$+\mathbf{P}_{\boldsymbol{\theta}}^{'}(\tau_0(h)\le m)+\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\tilde{u}_n^2>\delta'(\tilde{u},\tilde{u})_n\text{ for some }n\ge m\right)$$
$$+\mathbf{P}_{\boldsymbol{\theta}}^{'}\left(\tilde{v}_n^2>\delta'(\tilde{v},\tilde{v})_n\text{ for some }n\ge m\right).$$

By the same argument as in the proofs of Lemma 5.1 and Proposition 2.2, one can show that for every $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2$ and for each $m = 1, 2, \ldots$ and any $\Delta > 0$, respectively,

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}^{'}(\tau_0(h) \le m) = 0, \ \lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}^{'}(\|\tilde{J}_{\tau_0(h)-1} - I\| > \Delta) = 0.$$

The last two terms in (5.78) also converge to zero by the well-known property of $AR(1)-$processes with parameter in the interval $[-1, 1]$ (see, [10]). This completes the proof of Lemma 5.11. □

Proof of Lemma 5.12. Note that

$$U_h = h^{-1} |(\tilde{g}, \tilde{g})_{\tilde{\tau}(h)} - (\tilde{g}, \tilde{g})_{\tau_0(h)}| \, .$$

In view of (5.75) this quantity can be estimated as

$$U_h = h^{-1} \left| Z' \tilde{R}^{-1}_{\tilde{\tau}(h)} (\tilde{J}_n - I) \tilde{R}^{-1}_{\tilde{\tau}(h)} Z + Z' \tilde{R}^{-2}_{\tilde{\tau}(h)} Z - (\tilde{g}, \tilde{g})_{\tau_0(h)} \right| \tag{5.79}$$

$$\le h^{-1} \|\tilde{J}_n - I\| Z' \tilde{R}^{-2}_{\tilde{\tau}(h)} Z + |h^{-1} Z' \tilde{R}^{-2}_{\tilde{\tau}(h)} Z - 1| + \tilde{g}^2_{\tau_0(h)-1} / (\tilde{g}, \tilde{g})_{\tau_0(h)-1}.$$

Now we show that, for any $\delta > 0$,

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}(|h^{-1} Z' \tilde{R}^{-2}_{\tilde{\tau}(h)} Z - 1| > \delta) = 0. \tag{5.80}$$

Using (5.76) and taking into account that for $\boldsymbol{\theta} \in \Gamma_1$, $z_1 = \lambda_1 \sqrt{2}/(1 + b)$, $z_2 = \lambda_2 (2h)^{1/4} \sqrt{1-b}/(\sigma \sqrt{\nu_1(\tilde{W}_1)})$ we obtain

$$h^{-1} Z' \tilde{R}^{-2}_{\tilde{\tau}(h)} Z - 1 = z_1^2 \, h^{-1} (\tilde{u}, \tilde{u})_{\tilde{\tau}(h)} + z_2^2 \, h^{-1} (\tilde{v}, \tilde{v})_{\tilde{\tau}(h)} - 1$$

$$= \lambda_1^2 \left(\frac{2}{(1+b)^2 h} (\tilde{u}, \tilde{u})_{\tilde{\tau}(h)} - 1 \right) + \lambda_2^2 \left(\frac{(1-b)(2h)^{1/2}}{\sigma^2 \nu_1(\tilde{W}_1)} (\tilde{v}, \tilde{v})_{\tilde{\tau}(h)} - 1 \right).$$

This, in view of (5.43) gives (5.80).

Now by applying Proposition 2.2 and Lemma 5.11 to (5.79) we come to desired result. Hence Lemma 5.12. □

The case $\boldsymbol{\theta} = (0,1)$. Then $a = -1, b = 1$ and equations (3.2) yield

$$u_k = (-1)^k \sum_{j=1}^{k} (-1)^j \varepsilon_j,\ v_k = \sum_{j=1}^{k} \varepsilon_j\,.$$

By Corollary 5.15 one has

$$\sigma^{-2}\left(n^{-2}\,(u,u)_n,\ n^{-2}\,(v,v)_n\right) \xrightarrow{\mathcal{L}} (\mathcal{J}_1(W_1;1),\ \mathcal{J}_1(W;1))\,. \tag{5.81}$$

Introduce a sequence of functionals

$$f_n(\eta) = \left((\sigma n)^{-2}\,(u,u)_n,\ (\sigma n)^{-2}\,(v,v)_n,\ \tau(h)/\sqrt{2h}\right),\ n = [h]\,.$$

From the definition of $\tau(h)$, Theorem 2.1 and (5.81) it follows that

$$f_n(\eta) \xrightarrow{\mathcal{L}} (\mathcal{J}_1(W_1;1), \mathcal{J}_1(W;1), \nu_3(W,W_1)) = f(\eta)\,.$$

By Theorem 5.5 there exists $\tilde{\eta}$ such that $\tilde{\eta} \stackrel{\mathcal{L}}{=} \eta$ and

$$f_n(\tilde{\eta}) \xrightarrow{\text{a.s.}} f(\tilde{\eta}) = \left(\mathcal{J}_1(\tilde{W}_1;1), \mathcal{J}_1(\tilde{W};1), \nu_3(\tilde{W},\tilde{W}_1)\right)\,.$$

On the basis of this $\tilde{\eta}$ we define as before $(\tilde{x}_k), (\tilde{u}_k), (\tilde{v}_k)$ and $\tilde{Y}_h$. It should be noted that the ratio

$$t_n = (\tilde{u},\tilde{u})_n/(\tilde{v},\tilde{v})_n$$

satisfies the limiting relation

$$\lim_{h\to\infty} t_{\tilde{\tau}(h)} = \kappa,\ \kappa = \mathcal{J}_1(\tilde{W}_1;1)/\mathcal{J}_1(\tilde{W};1)\,. \tag{5.82}$$

Further by making use of the equality

$$\sum_{k=1}^{\tilde{\tau}(h)} \|\tilde{X}_{k-1}\|^2 = tr \sum_{k=1}^{\tilde{\tau}(h)} \tilde{X}_{k-1}\tilde{X}'_{k-1}$$

$$= trQ^{-1} \begin{pmatrix} (\tilde{u},\tilde{u})_{\tilde{\tau}(h)} & (\tilde{u},\tilde{v})_{\tilde{\tau}(h)} \\ (\tilde{u},\tilde{v})_{\tilde{\tau}(h)} & (\tilde{v},\tilde{v})_{\tilde{\tau}(h)} \end{pmatrix} (Q^{-1})'$$

one gets

$$\frac{\sum_{k=1}^{\tilde{\tau}(h)} \|\tilde{X}_{k-1}\|^2}{(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}} = trQ^{-1} \begin{pmatrix} 1 & t_{\tilde{\tau}(h)}^{-1/2}\xi_{\tilde{\tau}(h)} \\ t_{\tilde{\tau}(h)}^{-1/2}\xi_{\tilde{\tau}(h)} & t_{\tilde{\tau}(h)}^{-1} \end{pmatrix} (Q^{-1})',$$

where ξ_n is defined in (2.16). Since for any $\delta > 0$

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}^{'}(|\xi_{\tilde{\tau}(h)}|t_{\tilde{\tau}(h)}^{-1/2} > \delta) = 0,$$

from here it follows that

$$\mathbf{P}_{\boldsymbol{\theta}}^{'} - \lim_{h\to\infty} \frac{\sum_{k=1}^{\tilde{\tau}(h)} \|\tilde{X}_{k-1}\|^2}{(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}} = trQ^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1/\kappa \end{pmatrix} (Q^{-1})' = \frac{1}{2}\left(1 + \frac{1}{\kappa}\right).$$

On the other hand, by the definition of stopping time $\tau(h)$ in (1.7) and Proposition 2.2 one has

$$\mathbf{P}_{\boldsymbol{\theta}}^{'} - \lim_{h\to\infty} h^{-1} \sum_{k=1}^{\tilde{\tau}(h)} \|\tilde{X}_{k-1}\|^2 = 1.$$

As result,

$$\mathbf{P}_{\boldsymbol{\theta}}^{'} - \lim_{h\to\infty} h\left((\tilde{u},\tilde{u})_{\tilde{\tau}(h)}\right)^{-1} = 2^{-1}\left(1 + \kappa^{-1}\right). \tag{5.83}$$

This and (5.82) give

$$\mathbf{P}_{\boldsymbol{\theta}}^{'} - \lim_{h\to\infty} h\left((\tilde{v},\tilde{v})_{\tilde{\tau}(h)}\right)^{-1} = 2^{-1}(1 + \kappa). \tag{5.84}$$

Now we rewrite (5.55) as

$$\tilde{Y}_h = \frac{1}{\sigma^2\sqrt{h}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{g}_{k-1}\tilde{\varepsilon}_k + r_1(h), \tag{5.85}$$

where

$$\tilde{g}_{k-1} = \lambda_1\sqrt{(1+\kappa^{-1})/2}\,\tilde{u}_{k-1} + \lambda_2\sqrt{(1+\kappa)/2}\,\tilde{v}_{k-1},$$

$$r_1(h) = \frac{\lambda_1}{\sigma^2}\left((\tilde{u},\tilde{u})_{\tilde{\tau}(h)}^{-1/2} - \sqrt{(1+\kappa^{-1})/2}\right)\sum_{k=1}^{\tilde{\tau}(h)} \tilde{u}_{k-1}\tilde{\varepsilon}_k$$

$$+\frac{\lambda_2}{\sigma^2}\left((\tilde{v},\tilde{v})_{\tilde{\tau}(h)}^{-1/2} - \sqrt{(1+\kappa)/2}\right)\sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}_{k-1}\tilde{\varepsilon}_k.$$

Taking into account (5.83), (5.84), one can show along the lines of the proof of Proposition 3.2 that, for any $\delta > 0$,

$$\lim_{h\to\infty} \mathbf{P}_{\boldsymbol{\theta}}^{'}(|r_1(h)| > \delta) = 0.$$

Further analysis of (5.85) repeats the case of $\boldsymbol{\theta} \in \Gamma_1$ and is omitted.

This completes the proofs of Proposition 3.2 and Theorem 3.1 for $\boldsymbol{\theta} \in \Gamma_1 \cup \Gamma_2 \cup \{(0,1)\}$. □

5. Proof of Theorem 3.1 for the case of multiple roots. Assume that $\boldsymbol{\theta} = (2,-1)$ (the proof for the case $\boldsymbol{\theta} = (-2,-1)$ is similar). This corresponds to the multiple root of the polynomial (1.2): $a = b = 1$. Since matrix Q in (2.7) is degenerate, we use the matrix (2.19) to transform the original process $(x_k)_{k\geq 0}$ into two components $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$. This leads to the equations

$$u_k = x_k,\ v_k = x_k - x_{k-1}$$

with the solutions given by the formulae (2.19).

Now we introduce a sequence of functionals

$$f_n(\eta) = \left(\xi_n, (\sigma n)^{-2}\sum_{k=1}^{n} v_{k-1}^2, \tau(h)/(h/2)^4\right),\ n = [h]\,,$$

where ξ_n is defined in (2.16).

By Lemma 2.3 and Theorem 2.1 we have

$$f_n(\eta) \xrightarrow{\mathcal{L}} (\varphi(W), \mathcal{J}_1(W;1), \nu_4(W)) = f(\eta)\,.$$

By Theorem 5.5 there exists $\tilde{\eta}$ such that $\tilde{\eta} \stackrel{\mathcal{L}}{=} \eta$ and

$$f_n(\tilde{\eta}) \xrightarrow{\text{a.s.}} f(\tilde{\eta}) = \left(\varphi(\tilde{W}), \mathcal{J}_1(\tilde{W};1), \nu_4(\tilde{W})\right)\,. \tag{5.86}$$

On the basis of $\tilde{\eta}$ we define $(\tilde{x}_k), (\tilde{u}_k), (\tilde{v}_k)$ and $\tilde{Y}_h$. In view of (5.86) we have

$$\lim_{h\to\infty} J_{\tilde{\tau}(h)} = T_1 \text{ a.s.}, \tag{5.87}$$

where

$$J_n = \begin{pmatrix} 1 & \tilde{\xi}_n \\ \tilde{\xi}_n & 1 \end{pmatrix}, T_1 = \begin{pmatrix} 1 & \varphi(\tilde{W}) \\ \varphi(\tilde{W}) & 1 \end{pmatrix} \tag{5.88}$$

$$\tilde{\xi}_n = \left(\sum_{k=1}^{n} \tilde{u}_{k-1}^2\right)^{-1/2} \left(\sum_{k=1}^{n} \tilde{v}_{k-1}^2\right)^{-1/2} \sum_{k=1}^{n} \tilde{u}_{k-1}\tilde{v}_{k-1}\,. \tag{5.89}$$

Besides, we will need the relations

$$\lim_{h\to\infty} \frac{1}{h} \sum_{k=1}^{\tilde{\tau}(h)} \|\tilde{X}_{k-1}\|^2 = \lim_{h\to\infty} \frac{1}{h} \left(2(\tilde{u},\tilde{u})_{\tilde{\tau}(h)-1} - \tilde{u}^2_{\tilde{\tau}(h)-1}\right) = \sigma^2, \tag{5.90}$$

$$\begin{aligned} \lim_{h\to\infty} \frac{1}{\sigma^2 h^{1/2}} (\tilde{v},\tilde{v})_{\tilde{\tau}(h)} &= \lim_{h\to\infty} \frac{\tilde{\tau}^2(h)}{(h/2)^{1/2}} \frac{1}{\sigma^2\sqrt{2}\tilde{\tau}^2(h)} (\tilde{v},\tilde{v})_{\tilde{\tau}(h)} \\ &= 2^{-1/2}\mathcal{J}_1(\tilde{W};1)\nu_4^2(\tilde{W}) := \mu\,, \end{aligned} \tag{5.91}$$

which directly follow from (5.86).

Consider now the standardized deviation of the sequential estimate (1.6):

$$M_{\tau(h)}^{1/2}\,(\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) = M_{\tau(h)}^{-1/2} \sum_{k=1}^{\tau(h)} X_{k-1}\varepsilon_k\,.$$

Its distribution coincides with that of the vector

$$\tilde{M}_{\tilde{\tau}(h)}^{-1/2} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{X}_{k-1}\tilde{\varepsilon}_k$$

constructed from $(\tilde{x}_k), (\tilde{\varepsilon}_k)$. Representing the matrix

$$\tilde{M}_n = \sum_{k=1}^{n} \tilde{X}_{k-1}\tilde{X}_{k-1}^{'}$$

in the form (2.14) yields

$$\begin{aligned} \tilde{M}_{\tilde{\tau}(h)}^{-1/2} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{X}_{k-1}\tilde{\varepsilon}_k &= \tilde{J}_{\tilde{\tau}(h)}^{-1/2} \tilde{R}_{\tilde{\tau}(h)} \sum_{k=1}^{\tilde{\tau}(h)} Q\tilde{X}_{k-1}\tilde{\varepsilon}_k \\ &= \tilde{J}_{\tilde{\tau}(h)}^{-1/2} T_1^{1/2} T_1^{-1/2} \tilde{Z}_{\tilde{\tau}(h)}\,, \end{aligned}$$

where

$$\tilde{Z}_n = \begin{pmatrix} (\tilde{u},\tilde{u})_n^{-1/2} \sum_{k=1}^{n} \tilde{u}_{k-1}\tilde{\varepsilon}_k \\ (\tilde{v},\tilde{v})_n^{-1/2} \sum_{k=1}^{n} \tilde{v}_{k-1}\tilde{\varepsilon}_k \end{pmatrix}.$$

Taking into account (5.87), it suffices to establish the following result.

Lemma 5.13. *For each constant vector* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^{'}$ *with* $\|\boldsymbol{\lambda}\| = 1$, *the random variable*

$$\tilde{Y}_h = \boldsymbol{\lambda}^{'} T_1^{-1/2} \tilde{Z}_{\tilde{\tau}(h)}/\sigma$$

is asymptotically normal with mean 0 *and unit variance as* $h \to \infty$.

Proof of Lemma 5.13. Represent $\tilde{Y}_h$ as

$$\tilde{Y}_h = \frac{1}{\sigma\sqrt{h}} \sum_{k=1}^{\tilde{\tau}(h)} \tilde{g}_{k-1} \tilde{\varepsilon}_k + r_1(h)\,,$$

where

$$\tilde{g}_{k-1} = \boldsymbol{\lambda}^{'} T_1^{-1/2} \begin{pmatrix} \sqrt{2}\tilde{u}_{k-1} \\ (h^{1/4}\sqrt{\mu})^{-1}\tilde{v}_{k-1} \end{pmatrix}, \tag{5.92}$$

$$r_1(h) = \frac{\boldsymbol{\lambda}^{'} T_1^{-1/2}}{\sigma} \begin{pmatrix} \left((\tilde{u},\tilde{u})_{\tilde{\tau}(h)}^{-1/2} - \sqrt{2/h}\right) \sum_{k=1}^{\tilde{\tau}(h)} \tilde{u}_{k-1}\tilde{\varepsilon}_k \\ \left((\tilde{v},\tilde{v})_{\tilde{\tau}(h)}^{-1/2} - (h^{1/4}\sqrt{\mu})^{-1}\right) \sum_{k=1}^{\tilde{\tau}(h)} \tilde{v}_{k-1}\tilde{\varepsilon}_k \end{pmatrix}.$$

By an argument similar to that in the proof of Proposition 3.2, one can verify that $r_1(h)$ satisfies (5.58). Further analysis of $\tilde{Y}_h$ holds true.

Let us check only that

$$\lim_{h\to\infty} (\sigma^2 h)^{-1} (\tilde{g}, \tilde{g})_{\tilde{\tau}(h)} = 1\,.$$

Using (5.92) one obtains

$$(\sigma^2 h)^{-1} (\tilde{g}, \tilde{g})_{\tilde{\tau}(h)} = \boldsymbol{\lambda}^{'} T_1^{-1/2} \sigma^{-2}$$

$$\times \begin{pmatrix} (2/h)(\tilde{u},\tilde{u})_{\tilde{\tau}(h)} & h^{-3/4}\sqrt{2/\mu}\,(\tilde{u},\tilde{v})_{\tilde{\tau}(h)} \\ h^{-3/4}\sqrt{2/\mu}\,(\tilde{u},\tilde{v})_{\tilde{\tau}(h)} & (\mu\sqrt{h})^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)} \end{pmatrix} T_1^{-1/2}\boldsymbol{\lambda}\,.$$

Now using (5.91) we rewrite the cross term as

$$h^{-3/4}\sqrt{2/\mu}\,(\tilde{u},\tilde{v})_{\tilde{\tau}(h)} = h^{-3/4}\sqrt{2/\mu}\left((\tilde{u},\tilde{u})_{\tilde{\tau}(h)}\right)^{1/2}\left((\tilde{v},\tilde{v})_{\tilde{\tau}(h)}\right)^{1/2}\tilde{\xi}_{\tilde{\tau}(h)}$$

$$= \left((h/2)^{-1}(\tilde{u},\tilde{u})_{\tilde{\tau}(h)}\right)^{1/2}\left((\mu\sqrt{h})^{-1}(\tilde{v},\tilde{v})_{\tilde{\tau}(h)}\right)^{1/2}\tilde{\xi}_{\tilde{\tau}(h)}\,.$$

From here, (5.90),(5.91) and (5.87) it follows that

$$\lim_{h\to\infty} h^{-3/4}\sqrt{2/h}(\tilde{u},\tilde{v})_{\tilde{\tau}(h)} = \varphi(\tilde{W})\,.$$

Hence

$$\lim_{h\to\infty}(\sigma^2 h)^{-1}(\tilde{g},\tilde{g})_{\tilde{\tau}(h)} = \sigma^{-2}\boldsymbol{\lambda}' T_1^{-1/2} T_1 T_1^{-1/2}\boldsymbol{\lambda} = \boldsymbol{\lambda}'\boldsymbol{\lambda} = 1\,.$$

This completes the proof of Theorem 3.1 for $\boldsymbol{\theta} \in \{(-2,-1),(2,-1)\}$. □

Theorem 5.14. *Let* $W^{(n)} = (W^{(n)}(t))_{0\le t\le 1}$ *and* $W_1^{(n)} = (W_1^{(n)}(t))_{0\le t\le 1}$ *be defined by (2.2). Then for the random functions*

$$X_n = \Big(X_n(s,t) = (W^{(n)}(s), W_1^{(n)}(t)) :\ 0\le s\le 1, 0\le t\le 1\Big)$$

with values in the product of the Skorohod spaces $\mathcal{D}[0,1]\times\mathcal{D}[0,1]$*, one has*

$$X_n \xrightarrow{\mathcal{L}} (W, W_1)\,,$$

where W *and* W_1 *are independent standard Brownian motions.*

This result is a straightforward consequence of Theorem 3.3 in Helland (1982). This functional central limit theorem implies the following result.

Corollary 5.15. *Let* $u_k = (-1)^k\sum_{j=1}^k (-1)^j\varepsilon_j$, $v_k = \sum_{j=1}^k \varepsilon_j$. *Then*

$$\big((\sigma n)^{-2}(u,u)_n,\ (\sigma n)^{-2}(v,v)_n\big) \xrightarrow{\mathcal{L}} (\mathcal{J}_1(W_1;1),\ \mathcal{J}_1(W;1))\ . \qquad (5.93)$$

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