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PRESCRIBING GAUSS CURVATURE OF SURFACES IN 3-DIMENSIONAL SPACETIMES

APPLICATION TO THE MINKOWSKI PROBLEM IN THE MINKOWSKI SPACE

THIERRY BARBOT*, FRANÇOIS BÉGUIN†, AND ABDELGHANI ZEGHIB‡

ABSTRACT. We study the existence of surfaces with constant or prescribed Gauss curvature in certain Lorentzian spacetimes. We prove in particular that every (non-elementary) 3-dimensional maximal globally hyperbolic spatially compact spacetime with constant non-negative curvature is foliated by compact spacelike surfaces with constant Gauss curvature. In the constant negative curvature case, such a foliation exists outside the convex core. The existence of these foliations, together with a theorem of C. Gerhardt, yield several corollaries. For example, they allow to solve the Minkowski problem in Min_3 for datas that are invariant under the action of a co-compact Fuchsian group.

1. INTRODUCTION

K-hypersurfaces — i.e. hypersurfaces with constant Gauss curvature — have always played a central role in Riemannian geometry and geometric analysis. It seems however that, unlike CMC-hypersurfaces — i.e. hypersurfaces with constant mean curvature —, they have not attracted much attention in mathematical relativity. The main reason might be that they are rigid and rare in dimension 4 and higher. Nevertheless, there is no reason not to manipulate K-surfaces in 3-dimensional gravity. For example, one can consider the Einstein equation in the gauge defined by a K-surface.

In the present paper, we study the existence of K-surfaces on 3-dimensional maximal globally hyperbolic spatially compact vacuum spacetimes. Roughly speaking, we prove that each such spacetime admits a unique foliation by K-surfaces. This foliation will provide us with a canonical time function on the spacetime under consideration.

Let us start by some general introduction, first on the problematic of existence of privileged time functions on spacetimes, and then on the class of spacetimes we are interested in.

1.1. Geometric times on MGHC spacetimes.

MGHC spacetimes. Recall that a Lorentz manifold M is *globally hyperbolic* if it admits a *Cauchy hypersurface*, i.e. a spacelike hypersurface which intersects every inextendible causal curve at exactly one point. A classical result of R. Geroch states that the existence of a single Cauchy hypersurface implies the existence of a foliation by such hypersurfaces. More precisely, recall that a *time function* on a spacetime is a submersion $t : M \to \mathbb{R}$ strictly increasing along every future-oriented causal curves. Geroch result states that a globally hyperbolic spacetime admits a time function whose levels are Cauchy hypersurfaces.

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A globally hyperbolic spacetime is said to be *spatially compact* if its Cauchy hypersurfaces are compact. This is equivalent to having a time function which is a proper map, or equivalently, a time function with compact levels. A globally hyperbolic spacetime (M, g) is *maximal* if every isometric embedding of M in another globally hyperbolic spacetime of the same dimension is onto. For short, we shall write MGHC for "maximal globally hyperbolic spatially compact". MGHC spacetimes are the tamest Lorentz manifolds from the geometric analysis viewpoint. These spacetimes appear as cosmological models in mathematical Relativity (MGHC spacetimes satisfying the strong positivity energy condition are sometimes called *cosmological spacetimes*).

Geometric time functions. The simplest examples of MGHC spacetimes are metric products, i.e. spacetimes of the type $M = (\mathbb{R}, -dt^2) \oplus (\Sigma, h)$, where (Σ, h) is a compact Riemannian manifold.

Let (M, g) be a MGHC spacetime. From a topological viewpoint, M is always homeomorphic to the product $\mathbb{R} \times \Sigma$, where Σ is a Cauchy hypersurface of M. More precisely, a MGHC spacetime M can always be written as a topological product $\mathbb{R} \times \Sigma$, where the first projection $T : (t, x) \to t$ is a time function, and $(\{t\} \times \Sigma)_{t\in\mathbb{R}}$ is a foliation of M by spacelike hypersurfaces.

¿From a metrical viewpoint, (M, g) is in general far from being isometric to a direct product. In the (t, x) coordinates given by the topological splitting $M \simeq \mathbb{R} \times \Sigma$, the metric on M has the ADM form: $g = -N(t, x)dt^2 \oplus (\omega_t dt + h_t)$, where h_t is a one parameter family of Riemannian metrics on Σ, ω_t a one parameter family of 1-forms on Σ, $N(t, x)$ a function (called the lapse function), and X_t the dual (with respect to h_t) of $(1/2)\omega_t$ is the shift (non-autonomous) vector field. (One can roughly say at this stage that a Lorentz structure on a MGHC space with topology $\mathbb{R} \times \Sigma$, is a curve in $\mathcal{M}et(\Sigma)$, the space of Riemannian metrics of Σ up to isotopy).

The topological splitting $M \simeq \mathbb{R} \times \Sigma$, in particular the time function $T : (t, x) \to t$, are by no means unique. Nonetheless, it is natural and worthwhile to ask if there are privileged splittings (or time function) for a given MGHC spacetime? It is specially exciting to ask what remains from the couple of the orthogonal foliations after perturbation of a direct product metric $-dt^2 \oplus h$?

So the general question we are posing is to produce canonical geometric foliations by Cauchy hypersurfaces, or equivalently geometric time functions, which yield a kind of measurement of the default for (M, g) to be a metric product. Actually, asymptotic behaviour, singularities, shocks, and similar questions are meaningful only in a "natural" coordinates system (which does not create artificial singularities).

Rigid time functions. Let us first give examples of local geometric conditions on times:

- *Static time:* $\frac{\partial}{\partial t}$ is a Killing vector field and is orthogonal to $\mathcal{F}$. In this case $g = -N(x)dt^2 \oplus h_0$.
- *Static geodesic time:* that is $\frac{\partial}{\partial t}$ has furthermore geodesic trajectories. This characterize the direct product case, $g = -dt^2 \oplus h_0$.
- *Homothetically static-geodesic case:* M is a warped product $g = -dt^2 \oplus w(t)h_0$.

However, all these situations are very rigid and happen only in a highly symmetric case. Worse, sometimes, a local existence of such times does imply neither existence of a global one, nor its uniqueness. The reason of rigidity of the last times is that they correspond to solutions of a *system* of PDE describing

the extrinsic geometry of the leaves of the spatial foliation $\mathcal{F}$. However, the reasonable situation is that of a single (scalar) geometric PDE, since solving systems involve compatibility (roughly comparable to the integrability conditions in Frobenius Theorem).

F-time functions. One defines a (scalar) F-curvature for a hypersurface Σ, where F is is a real valued function on the space of symmetric n by n matrices invariant by orthogonal conjugacy: such a data is equivalent to a map $F : \mathbb{R}^n \to \mathbb{R}$ invariant by permutation of the coordinates (here the space M has dimension $1 + n$). The F-curvature of Σ is the function obtained by evaluating F on the eigenvalues of the second fundamental form of S.

The uniformization (or geometrisation) problem can be stated as follows: find a foliation $\mathcal{F}$ on M by Cauchy hypersurfaces each of which has a constant F-curvature. We will furthermore require as an additional condition that the F-curvature is increasing along causal curves, i.e. that the map associating to x in M the F-curvature of the leaf $\mathcal{F}_x$ through x is a time function (that we may call a *F-time function* or a *F-time*). Such functions are automatically unique (see §1.3).

In the particular case where F is the (arithmetic) mean of eigenvalues, the F-curvature is the *mean curvature*. We are going here to consider the case where F is the opposite of the product of eigenvalues ; in the particular case, the F-curvature is called the Gauss-Killing-Kronecker-Lipschitz-curvature or K-curvature.

Constant mean curvature versus constant K-curvature. A CMC-hypersurface is a hypersurface with a constant mean curvature. A K-hypersurface is a hypersurface with a constant K-curvature.

¿From a PDE point of view, since one takes a linear sum in the definition of the mean curvature, the equation defining CMC hypersurfaces is the simplest one (among all the equation defining hypersurfaces with constant F-curvature). It is quasi-linear. In all the other cases, the PDE is fully non-linear of Monge-Ampère type.

Following G. Darboux, CMC surfaces are very important in physics, and equally are K-surfaces in geometry![1] Despite the marriage of geometry and physics via the Theory of General Relativity (some decades after Darboux), K-hypersurfaces remain mostly ignored by physicists, who are still investigating CMC hypersurfaces. In fact one of the central and natural question in Relativity is whether a given class of spacetimes admits CMC foliations?

One explanation of this CMC success is that the Cauchy problem for Einstein equations (in vacuum) can be formulated as a hyperbolic-elliptic well posed PDE system, in a CMC gauge. Roughly speaking, one can incorporate the CMC condition in the ADM representation of the metric, and set a well posed system, that is if initial data satisfy “CMC constraints”, then, the system has a solution. This solution is a Ricci flat Lorentz manifold extending the given initial Riemannian manifold, together with a *canonical* CMC foliation on a neighbourhood of it. In other words, one has a kind of a CMC flow.

It is natural to try to extend this consideration to the K-curvature case, that is to write Einstein equation in a K-gauge. This does not seem easy because of the the non-linear coupling of equations.

[1]...“On peut dire que la courbure totale a plus d’importance en Géométrie ; comme elle ne dépend que de l’élément linéaire, elle intervient dans toutes les questions relatives à la déformation des surfaces. En Physique mathématique, au contraire, c’est la courbure moyenne qui paraît jouer le rôle prépondérant”, G. Darboux, “Leçons sur la théorie générale des surfaces”, Livre V, chapitre II.

One beautiful corollary of CMC gauges is the regularity of global CMC foliations. If one knows that the ambient Lorentz metric is (real) analytic, then the locally defined CMC foliation given by the CMC flow is analytic (by analyticity of solutions of analytic hyperbolic systems with analytic initial data). Therefore, if one knows that the spacetime has a CMC foliation, and moreover, any CMC hypersurface is a leaf of it, then this foliation must be analytic (since it is locally defined by the CMC flow). One achievement of the present article is to show existence of K-foliations as well as uniqueness of K-hypersurfaces, but we can not yet deduce its analyticity, since we were not able to treat Einstein equation in a K-gauge.

Other geometric time functions. A time function $T : (x, t) \mapsto t$ is *of Gauss type* if the metric has the form $-dt^2 \oplus g_t$. If a level $\{t\} \times \Sigma$ of T is given, then the others levels are obtained by pushing along orthogonal geodesics. The cosmological time (CT for short) enjoys this property. It will be one important tool in our study of K-times.

1.2. **MGHC spacetimes of constant curvature.**

In the present paper, we restrict ourselves to MGHC spacetimes of constant curvature. Such spacetimes have locally trivial geometry, being locally isometric to Minkowski space Min_3, de Sitter space dS_3 or anti-de Sitter space AdS_3. However, the topology and the global geometry of these spacetimes may be highly nontrivial. Actually, all the questions studied in the present paper concern the global geometry of the spacetimes under consideration.

Focussing our attention on spacetimes with constant curvature is clearly an important restriction. Nevertheless, let us recall that, from a physical viewpoint, every 3-dimensional solution of the vacuum Einstein equation has constant curvature. Actually, 3-dimensional spacetimes with constant curvature have received much attention in the last fifteen years because of the role they play in quantum gravity (see for instance [24]), and since the discovery of the so-called BTZ black holes models ([8]). Moreover, from a purely mathematical viewpoint, 3-dimensional MGHC spacetimes with constant curvature are the exact Lorentzian analogs of the Riemannian *hyperbolic ends*, which play a fundamental role in 3-dimensional topology. More formally, there exists a duality between hyperbolic ends and 3-dimensional MGHC spacetimes of positive constant curvature, which allows to translate some of our results on MGHC spacetimes into results on hyperbolic ends (see §14).

The theory of 3-dimensional MGHC spacetimes of constant curvature naturally splits into two cases: the spacetimes whose Cauchy surfaces have genus 0 or 1 are called *elementary*, whereas the spacetimes whose Cauchy surfaces have genus at least 2 are called *non-elementary*. As suggested by the terminology, the global geometry of elementary spacetimes is much easier to understand than those of non-elementary spacetimes. In some sense, elementary spacetimes can be considered as "particular" or "exact" solutions of the Einstein equation, for which all geometrical problems can be solved explicitely "by hand" (see e.g. [4, §10.2 and §10.3]). Nevertheless, since tools are quite different, we will restrict ourselves to non-elementary spacetimes.

All the spacetimes we shall consider in the sequel are not only time-orientable, but actually time-oriented. If M is a non-elementary 3-dimensional MGHC spacetime of constant curvature $\Lambda \geq 0$, then M is always either past complete (all past directed causal geodesic rays are complete), or future complete. Reversing the time-orientation if necessary, we will always assume we are in the second case. Note that

a non-elementary 3-dimensional MGHC spacetime of constant curvature $\Lambda < 0$ is always neither past nor future complete.

1.3. **Some definitions: K-surfaces, K-slicings, K-times.**

Let (M, g) be a (time-oriented) 3-dimensional spacetime. Given a spacelike surface Σ in M, the second fundamental form k_x^Σ of Σ at x is defined by $k_x^\Sigma(X, Y) = - < \nabla_X n, Y >$ where n is the future oriented unit normal vector of Σ. The *K-curvature* (or Gauss-Killing-Kronecker-Lipschitz curvature) of Σ at x is $\kappa^\Sigma(x) = -\lambda_1(x)\lambda_2(x)$, where $\lambda_1(x)$ and $\lambda_2(x)$ are the principal curvatures of Σ at x, i.e. the eigenvalues of the second fundamental form k_x^Σ. Note that, in the case where the spacetime M has constant (sectional) curvature Λ, the Gaussian (i.e. sectional) curvature of Σ is $R^\Sigma = \Lambda + \kappa^\Sigma$, and its scalar curvature is $2R^\Sigma$.

A *K-surface* in M is a spacelike surface with constant K-curvature. Observe that, in the case where M has constant curvature, being a K-surface is equivalent to having constant Gaussian curvature.

¿From now on, we assume (M, g) to be globally hyperbolic and spatially compact. We define a *K-slicing* of M as a foliation of M by compact K-surfaces. Note that a K-slicing of M is always a trivial foliation, all the leaves of which are Cauchy surfaces of M. Equivalently, the leaves of a K-slicing of M are always the level sets of a certain time function.

Remark 1.1. (*Slicing versus Foliation*). Our choose of the term "slicing" instead of the familiar one "foliation" (in the mathematical literature) is to emphasize the fact that we are dealing only with topologically trivial foliations, e.g. with all leaves compact. However, one could imagine to study general K-foliations with non-trivial dynamics. Although we will not investigate the question here, we guess that, in our context of MGHC spacetimes of constant curvature, any K-foliation is in fact a slicing. A similar question about CMC-foliations of the 3-dimensional Euclidean space is handled in [46].

A K-slicing need not be unique in general. In order to get some uniqueness, we need to define a more restrictive notion. A C^2 spacelike surface Σ in M is said to be *convex* (resp. *strictly convex*) if it has non-positive (resp. negative) principal curvatures. Similarly, Σ is *concave* or *strictly concave* if the principal curvatures are non-negative or positive. A *K-time* on M is a time function $\tau : M \to \mathbb{R}$ such that, for every $a \in \tau(M)$, the level set $\tau^{-1}(a)$ is a compact strictly convex K-surface with K-curvature equal to a. Of course, the level sets of any K-time define a K-slicing of M. Nevertheless, the K-slicings defined by K-times are quite specific since all the leaves have negative principal curvatures, and since the K-curvature of the leaves increases with time. Note that the range $\tau(M)$ of a K-time is always included in $(-\infty, 0)$.

The maximum principle implies that K-times are unique. More precisely, if M admits a K-time $\tau : M \to \mathbb{R}$, then, for every $a \in \tau(M)$, the only compact K-surface of K-curvature a in M is the level set $\tau^{-1}(a)$. In particular, the only K-slicing on M is the one defined by the level sets of τ (see §4.4).

1.4. **Purpose of the paper.**

In the present paper, we are going here to consider the following questions:

- the existence of K-time (or K-slicings) in 3-dimensional MGHC spacetimes of constant curvature,
- the existence of Cauchy surfaces of prescribed K-curvature in 3-dimensional MGHC spacetimes of constant curvature,

– the Minkowski problem in the 3-dimensional Minkowski space.

The results concerning the last two items are essentially application of the first one.

2. Statements of results

2.1. K-slicings of MGHC spacetimes with constant curvature.

The following is our main result:

Theorem 2.1. *Let M be a 3-dimensional non-elementary MGHC spacetime with constant curvature Λ. If $\Lambda \geq 0$, reversing the time orientation if necessary, we assume that M is future complete.*

- *If $\Lambda \geq 0$ (flat case or locally de Sitter case), then M admits a unique K-slicing. The leaves of this slicing are the level sets of a K-time ranging over $(-\infty, -\Lambda)$.*
- *If $\Lambda < 0$ (locally anti-de Sitter case), then M does not admit any global K-slicing, but each of the two connected component of the complement of the convex core[2] of M admits a unique K-slicing. The leaves of the K-slicing of the past of the convex core are the level sets of a K-time ranging over $(-\infty, 0)$. The leaves of the K-slicing of the future of the convex core are the level sets of a reverse K-time[3] ranging over $(-\infty, 0)$.*

Let us make a few comments on this result.

Gaussian curvature. Since the Gaussian curvature of a surface is $R = \Lambda + \kappa$, the Gaussian curvature of the leaves of the K-slicings provided by Theorem 2.1 varies in $(-\infty, 0)$ when $\Lambda \geq 0$, and in $(-\infty, \Lambda)$ when $\Lambda < 0$.

CMC times. It is our interest on CMC-times that led us to extend our attention to more general geometric times. Existence of CMC times on MGHC spacetimes of constant non-positive curvature and any dimension was proved in [3, 4]. For spacetimes locally modelled on the de Sitter space, there are some restrictions but not in dimension 3.

Regularity. The slicings provided by Theorem 2.1 are continuous. It follows from their uniqueness (i.e. they are canonical). Extra smoothness is not automatic (e.g. the cosmological time is $C^{1,1}$, but not C^2). Here, we can hope that our K-slicings are (real) analytic, and even more, they depend analytically on the spacetime (within the space of MGHC spacetimes of curvature Λ and fixed topology). All this depends on consideration of Einstein equations in a K-gauge.

Non-standard isometric immersions of $\mathbb{H}^2$ in Min_3. It was observed by Hano and Nomizu [38], that the hyperbolic plane $\mathbb{H}^2$ admits non standard isometric immersions in the Minkowski space Min_3 (i.e. different from the hyperbola, up to a Lorentz motion).

[2] About the notion of convex core, see §7.

[3] A reverse K-time on N is a function $\tau : N \to \mathbb{R}$ which is strictly decreasing along every future oriented causal curve, and such that, for every a, the level set $\tau^{-1}(a)$ is a compact locally strictly concave surface of constant K-curvature equal to a.

Theorem 2.1, applied in the flat case, yields a K-slicing with (exactly) one isometric copy of $\mathbb{H}^2$ in $\mathbb{R}^{1,2}$, invariant by the holonomy group $\mathbf{\Gamma} \subset SO(1,2) \ltimes \mathbb{R}^3$ (since the curvature varies from $-\infty$ to 0). This isometric immersion is different from the standard one as soon as $\mathbf{\Gamma} \not\subset SO(1,2) \ltimes \{0\}$.

Observe that, in higher dimension, $\mathbb{H}^n$ is rigid in Min_{n+1}. Indeed, the theory of rigidity of submanifolds in the Euclidean space extends straightforwardly to the Minkowski space. From this theory, a hypersurface with a second fundamental form of rank ≥ 3 is rigid. By Gauss equation, any isometric immersion of a hyperbolic space in a Minkowski space is non-degenerate, and hence has the required condition on the rank, whence $n \geq 3$.

Spacetimes as curves in the Teichmüller space. For the sake of simplicity, let us restrict ourselves here to the case $\Lambda = 0$, i.e. that of flat spaces. Denote by Σ the abstract topological surface homeomorphic to a Cauchy surface of M, and by $\mathcal{M}et(\Sigma)$ the space of Riemannian metrics on Σ up to isotopy. Any time function of M gives rise to a family (Σ_t, g_t) of Riemannian metrics on Σ (well-defined up to isotopy) parametrized by the given time. We have in particular, associated to our three favourite, K, CMC and CT times, three curves in $\mathcal{M}et(\Sigma)$:

$$\mathrm{Met}^K \text{ (resp. } \mathrm{Met}^{cmc} \text{ and } \mathrm{Met}^{ct}) : \ t \to g_t \in \mathcal{M}et(\Sigma)$$

Now, let $Teich(\Sigma)$ be the Teichmüller space of Σ, i.e. the space of hyperbolic metrics (of curvature-1) on Σ, up to isotopy. Then, we have associated curves:

$$\mathrm{Teich}^K \text{ (resp. } \mathrm{Teich}^{cmc} \text{ and } \mathrm{Teich}^{ct}) : \ t \to [g_t] \in Teich(\Sigma)$$

Thus, $\mathrm{Met}(t)$ is the Riemannian metric of the t-level, and $\mathrm{Teich}(t)$ is its underlying complex structure.

Let us note the following facts (but we cannot give here a complete overview):

(1) The curves Met^K, Met^{cmc}, Met^{ct}, Teich^K, Teich^{cmc}, Teich^{ct} encode the metric properties and the geometry of the spacetime M.
(2) These curves all coincide exactly when the spacetime (M, g) is static (i.e. its universal cover is the solid lightcone).
(3) In the vein of V. Moncrief's work on the "reduction of Einstein equations to the Teichmüller space" in a CMC gauge in dimension $2+1$ [48], one can show in general that all the above curves in $Teich(\Sigma)$ have canonical lift in $T^*(Teich(\Sigma))$ or alternatively $T(Teich(\Sigma))$. There, they define (semi-)flows. In the CT-case, this is nothing but the grafting [20].
(4) It is an essential question to study the behaviour of these curves in both $\mathcal{M}et(\Sigma)$ and $Teich(\Sigma)$ when the parameter tends to one or the other extremity of the existence interval. A delicate point is that there are many compactifications for $Teich(\Sigma)$ and also many notions of convergences for sequences in $\mathcal{M}et(\Sigma)$. We have for instance, equivariant Gromov topology, convergence of spectrum, where in general the limit is a real tree. For existing asymptotic study, see for instance [22] in the CT-case and [2] in the CMC-case, and a forthcoming paper.
(5) One advantage of the K-slicing is that $\mathrm{Teich}^K(t)$ is directly given by $\mathrm{Met}^K(t)$, that is we don't need to uniformize it since it has constant curvature t (by definition of the parameter). In this case, all notions of convergence in $\mathcal{M}et(\Sigma)$ (up to scaling) and in $Teich(\Sigma)$, coincide.

Affine foliations. It is interesting to compare our results (existence of CMC-slicings and K-slicings) with similar results in the context of (equi-)affine geometry, in particular with the beautiful theory of affine spheres (see [43]): for every hypersurface Σ in the affine space $\mathbb{R}^{n+1}$, endowed with a parallel

volume form, one can define its affine principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_n$. The hypersurface is an affine hypershere if all these affine principal curvatures are equal and constant along Σ. It is equivalent to require that all the affine normals either intersect at one point, or are mutually parallel. If $\lambda_1 = \lambda_2 = \ldots = \lambda_n < 0$, the affine sphere is hyperbolic. Calabi's conjecture states that:

(1) Every hyperbolic affine sphere which is complete for its "affine" metric is asymptotic to the boundary of a proper open convex cone,
(2) Conversely, for every $\lambda < 0$, every proper open convex contains a unique hyperbolic affine sphere of affine principal curvatures λ.

The first part has been solved by Cheng and Yau under the hypothesis that the affine sphere is complete for an Euclidean metric on $\mathbb{R}^{n+1}$, hypothesis removed later by Li. Most interesting for us is the second part, proved by Sasaki and Gigena. It shows that any proper open cone admits a natural foliation by hyperbolic affine spheres, which can be considered as hypersurfaces of (affinely) constant scalar curvature, or as well of constant mean curvature.

For more details, see [43, Chapter 2], where the authors also discuss the method of C. P. Wang for constructing all hyperbolic affine 2-spheres which admit the action of a discrete subgroup of the equiaffine group with compact quotient.

2.2. **Duality hyperbolic-de Sitter, K-slicings of hyperbolic ends.**

The duality between the de Sitter space dS_3 and the hyperbolic space $\mathbb{H}^3$ (see §14) will allow us to deduce the existence of K-surfaces in hyperbolic ends from the existence of K-surfaces in MGHC spacetimes of constant positive curvature. So we will be able to recover the following result of F. Labourie:

Theorem 2.2 ([40])**.** *A 3-dimensional hyperbolic end possesses a K-slicing with K-curvature ranging in* $(0, 1)$.

Some explanations on hyperbolic ends can be found in §14. Note that, in the Riemannian setting, the K-curvature of a surface is just the product $\lambda_1\lambda_2$ of the principal curvatures.

Remarks 2.3.

(1) It was observed by Mess that the dS_3-case of our main Theorem 2.1 can be deduced from Labourie's Theorem, using the duality between dS_3 and $\mathbb{H}^3$.
(2) Our proof here of Theorem 2.1 in the dS_3-case is completely different from Labourie's one for hyperbolic ends. Actually, we think that our method gives the natural framework for approaching many results of existence "geometric" surfaces or foliation, even in the hyperbolic setting.

2.3. **Surfaces with prescribed K-curvature.**

The barriers method (see §3.1) allows one to find a surface with constant K-curvature κ in between two surfaces with (non-constant) K-curvatures bounded respectively from below and above by κ. This is an important ingredient in our proof of Theorem 2.1. This method generalizes to the case where κ is not a constant any more, but rather a function (see [31] or section 15). Using this generalisation and Theorem 2.1, we will get the following existence result:

Theorem 2.4. *Let M be a 3-dimensional non-elementary MGHC spacetime with constant curvature Λ. If $\Lambda \geq 0$, reversing the time-orientation if necessary, we assume that M is future complete. Let f be a*

smooth function on M admitting a range contained in a compact interval $[a,b] \subset]-\infty\,,\,\min(0,-\Lambda)[$. Then, there is a Cauchy surface Σ is M with K-curvature $\kappa^\Sigma(x) = f(x)$ for every $x \in \Sigma$.

In particular:

Corollary 2.5. *Let M be as above. Let Σ_0 be the abstract topological surface homeomorphic to the Cauchy surfaces of M, and let f_0 be a smooth real-valued function on Σ_0, such that the range of f_0 is a contained in a compact interval $[a,b] \subset]-\infty\,,\,\min(0,-\Lambda)[$. Then, there exists an embedding of $\phi : \Sigma_0 \hookrightarrow M$ such that $\Sigma = \phi(\Sigma_0)$ is a Cauchy surface in M, and the K-curvature of Σ at $\phi(x)$ is equal to $f_0(x)$ for every $x \in \Sigma_0$.*

2.4. **The Minkowski problem.**

Let us recall the classical formulation of the Minkowski problem (see for instance [25, 33, 40, 49]). If S is a closed convex smooth surface in the Euclidean space $\mathbb{R}^3$, then its Gauss map $\nu_S : S \to \mathbb{S}^2$ is a diffeomorphism, and one can consider the map $K^S := \kappa^S \circ \nu_S^{-1} : \mathbb{S}^2 \to (0,+\infty)$, where κ^S is the K-curvature of S. The Minkowski problem consists in characterizing the functions on the sphere $\mathbb{S}^2$ which have the form K^S for some surface S.

This problem can be transposed in the Lorentzian setting, by replacing the Euclidean space $\mathbb{R}^3$ by the Minkowski space Min_3, requiring the surface S to be spacelike (but not compact!), and replacing the sphere $\mathbb{S}^2$ by the hyperbolic space $\mathbb{H}^2$. Using Theorem 2.1 and a barrier theorem of Gerhardt ([31]), we are able to solve the Minkowski problem in Min_3 in the particular case where the function K^S is invariant under a co-compact Fuchsian group $\Gamma \subset \mathrm{SO}(1,2) = \mathrm{Isom}(\mathbb{H}^2)$:

Theorem 2.6. *Let Γ be a co-compact Fuchsian subgroup of $SO(1,2)$, and $f : \mathbb{H}^2 \to (-\infty,0)$ be a Γ-invariant smooth function. Then there exists a strictly convex spacelike surface S in Min_3 such that $f = \kappa^S \circ (\nu^S)^{-1}$, where $\nu^S : S \to \mathbb{H}^2$ is the Gauss map of S and $\kappa^S : S \to (0,+\infty)$ is the K-curvature (= Gauss curvature) of S. Moreover, if $\mathbf{\Gamma}$ is a subgroup of $SO(1,2) \ltimes \mathbb{R}^3 = \mathrm{Isom}(\mathrm{Min}_3)$ which projects bijectively on Γ, then there exists a unique such (convex) surface S which is $\mathbf{\Gamma}$-invariant.*

A few comments on the Minkowski problem in Min_3:

(1) There is no uniqueness in general for the Minkowski problem in Min_3. For example, for $f = -1$, any isometric copy of $\mathbb{H}^2$ in Min_3 is a solution of the Minkowski problem. The uniqueness in Theorem 2.6 strongly relies on the convexity and $\mathbf{\Gamma}$-invariance hypotheses.
(2) Despite the obvious and natural interest of the Minkowski problem in the Lorentzian setting, there seems to be very little work on this problem. We know essentially two substantial contributions: [44] by A. M. Li and [37] by B. Guan, H-Y Jian and R. Schoen. In the last paper, the authors proved a result similar to ours (equally in dimension $2+1$), with the invariance condition replaced by prescribing the asymptotic behaviour at infinity.
(3) There are generalisations of the Minkowski problem in other directions, still in a Riemannian setting, but for ambient spaces different from $\mathbb{R}^n$. The point here is to define a substitute of the Gauss map. This is indeed possible for universal spaces of constant curvature $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$. One can quote here recent works by C. Gerhardt which solve the corresponding Minkowski problem (for convex bodies)[32, 33], and a previous work of F. Labourie on an equivariant Minkowski problem for surfaces in the hyperbolic space $\mathbb{H}^3$ [40]. Finally, the Minkowski problem admits generalizations to other curvatures (see for instance [36, 53]).

2.5. **Related works.**

The general mathematical framework unifying our contributions here is that of prescribing (extrinsic) curvatures of hypersurfaces in M. Such a hypersurface N is sometimes called of Weingarten type, i.e. it is an F-hypersurface in the sense that: $F(\overrightarrow{\lambda}(x)) = F(\lambda_1(x), \dots, \lambda_n(x)) = f(x)$, where f is a given function on N, and F is a function of the principal curvatures. For instance, one asks the following general questions about them:
– A Dirichlet problem,
– A Dirichlet problem at infinity, also said "entire hypersurfaces problem" (i.e. existence of complete hypersurfaces with a prescribed asymptotic behaviour),
– Existence of F-slicings. This requires M to be topologically trivial, or that the slicing is defined on an end of M,
– A natural generalisation consists in considering curvature functions $F(\overrightarrow{\lambda}, \nu)$, i.e F also depends on a normal vector ν.
– Finally, regarding any of these questions, one specifies whether M is Riemannian or Lorentzian.

As said previously, there is a wide literature on the CMC case (also called the H-curvature). Let us quote some recent achievements concerning the K-curvature:
- The Dirichlet problem for the K-curvature in the Minkowski space was studied in particular by P. Delanoë [27] and B. Guan [35].
- Entire K-hypersurfaces were studied by B. Guan, H-Y Jian and R. Schoen [37]. (The corresponding CMC case was considered by A. Treibergs [56], and that on the "finite" Dirichlet problem, by Bartnik and Simon [14]).
- In [15, 57] P. Bayard and afterwards J. Urbas solved the Dirichlet problem for the scalar curvature in the Minkowski space of any dimension. Remark that for our dimension $2+1$, the scalar an K-curvatures coincide. Also [16] solved the entire hypersurface problem for the scalar curvature.
- C Gerhardt and O. Schnürer [30, 54] proved criteria for the solvability of the Dirichlet problem on general Lorentz manifolds, and for various curvature functions,
- Existence of K-slicings of hyperbolic ends in 3 dimension was proved by F. Labourie (see §2.2), and generalised by G. Smith for higher dimensional quasi-Fuchsian ends, where the K-curvature is replaced by a variant, governed by the "special Lagrangian" equation [55],
- Recently, R. Mazzeo and F. Pacard [45] considered manifolds that are merely asymptotically hyperbolic and admitting a conformal compactification. They prove existence of various geometric foliations near their boundary.

3. Ingredients of the proofs

Before going into the details, we want to the main ingredients of the proofs of our results. Let (M, g) be a non-elementary 3-dimensional MGHC spacetime with constant curvature Λ. If $\Lambda \geq 0$, reversing the time-orientation if necessary, we can (and we do) assume that M is future complete. To prove our main Theorem 2.1, we have to construct a K-slicing on M (or on the complement of the convex core of M if $\Lambda < 0$). Here are the main ingredients of this construction:

3.1. **Existence of barriers.**

A very general principle states that the existence of a surface with prescribed curvature should follow from the existence of so-called *barriers*. In our context, the suitable result was proved by C. Gerhardt:

Definition 3.1. For $\kappa \in \mathbb{R}$, a *pair of κ-barriers* is a pair of disjoint strictly convex Cauchy surfaces Σ^-, Σ^+ in M such that:

a. Σ^- is in the past of Σ^+,
b. the K-curvature of Σ^- is bounded from above by κ,
c. the K-curvature of Σ^+ is bounded from below by κ.

Theorem 3.2 (Gerhardt, [30][4]). *Given a real number $\kappa < -\Lambda$, if (M, g) admits a pair of κ-barriers (Σ^-, Σ^+), then (M, g) admits a strictly convex Cauchy surface Σ with constant K-curvature κ. Moreover, the Cauchy surface Σ is in the future of Σ^- and in the past of Σ^+.*

Remark 3.3. The sign convention of Gerhardt for the K-curvature is the opposite of ours (this is the reason why the main result of [30] asserts that one can find a Cauchy surface with constant Gauss curvature k for any $k > 0$ provided that there is a pair of barriers).

Also note Gerhardt uses the *past* directed unit normal vector to define the principal curvatures; it follows that a surface which is called convex in [30, 31, 33] is called concave here, and vice-versa. This is not a problem since all the result are valid both for convex and concave surfaces (one needs to be careful with the definition of the barriers).

We will prove the following result:

Theorem 3.4. *Assume Λ is non-negative*[5]*. There exists $\epsilon > 0$ such that there exists a pair of κ-barriers for every $\kappa \in (-\Lambda - \epsilon, -\Lambda)$.*

This result will be obtained by using previously known results on the existence of locally strictly convex Cauchy surfaces in flat spacetimes, and some estimates on the behaviour of the K-curvature when one pushes such a Cauchy surface along the orthogonal geodesics. Theorem 3.4 together with Gerhardt's theorem show the existence of a convex Cauchy surface Σ_κ with constant K-curvature κ in M for every $\kappa \in (-\Lambda - \epsilon, -\Lambda)$. Using classical arguments and some informations on our barriers, we will prove that the family of Cauchy surfaces $(\Sigma_\kappa)_{\kappa \in (-\Lambda-\epsilon, -\Lambda)}$ is a foliation a neighbourhood of the future end of M. This will provide us with a "local K-slicing" on a neighbourhood of the future end of M (in the case $\Lambda \geq 0$).

3.2. **Systole and distance to the past singularity.**

Let τ be the *cosmological time* on M (see §4.1 for more details). A major (and the most original) ingredient of our proof of Theorem 2.1 is the following result, which relates the systole of a Cauchy surface Σ in M and the "distance from Σ to the past singularity of M":

Theorem 3.5. *For every $\epsilon > 0$, there exists a constant $\alpha > 0$ such that, for any Cauchy surface Σ, if $\inf \tau_{|\Sigma}$ is smaller than α then the systole of Σ is smaller than ϵ.*

[4]The main result of [30] is stronger than the result stated above (see §15).

[5]See§11 for some statements in the case $\Lambda < 0$.

This statement should be understood as follows: *if the systole of a Cauchy surface Σ in M is not too small, then no point of Σ is close to the past singularity of M.* We will use this to prove that a sequence $(\Sigma_n)_{n\in\mathbb{N}}$ of Cauchy surfaces with K-curvature bounded away $-\infty$ remains far from the initial singularity (see §3.4).

The proof of Theorem 3.5 strongly relies on some fine knowledge of the geometry of MGHC spacetimes with constant curvature. We think that this Theorem 3.5 is not only a crucial step in the proof of our main result, but is also interesting in its own right. It can be used in various type of situations to prove that a sequence of Cauchy surfaces is relatively compact.

3.3. **Decreasing sequences of convex Cauchy surfaces.**

Another ingredient of our proof of Theorem 2.1 is the fact that the limit of a decreasing sequence of convex Cauchy surfaces is always "spacelike". More precisely:

Theorem 3.6. *Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of convex Cauchy surfaces*[6] *in M. Assume that this sequence is decreasing (i.e. $I^+(\Sigma_{n+1}) \supset I^+(\Sigma_n)$ for every n), and that the set $\Omega = \bigcup_{n\geq 0} I^+(\Sigma_n)$ is not the whole spacetime M. Let $\Sigma_\infty = \partial\Omega$ (note that the Ω is a locally geodesically convex set, and thus Σ_∞ is a convex topological surface). Then all the support planes of Σ_∞ are spacelike.*

Note that Σ_∞ is not a Cauchy surface in general. An easy (but important) corollary of Theorem 3.6 is the fact that a decreasing sequence of convex Cauchy surfaces is always "uniformly spacelike". To make this precise, let us denote by $T^{-1}M$ the set of all couples (x, P) where x is a point of M and P is a totally geodesic spacelike plane passing through x. The set $T^{-1}M$ is naturally identified with the subset of the tangent bundle TM of M made of all couples (x, v) where v is a future-directed tangent vector of norm -1 (the identification is of course through the correspondance between a totally geodesic spacelike plane and its future-directed unit normal vector). This endows $T^{-1}M$ with a natural topology. Clearly, $T^{-1}M$ is not compact (a sequence of spacelike planes may converge to a null plane!).

Definition 3.7. A sequence of spacelike surfaces $(\Sigma_n)_{n\in\mathbb{N}}$ is *uniformly spacelike* if, for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \Sigma_n$:

- either the sequence $(x_n)_{n\in\mathbb{N}}$ escapes from any compact subset of M,
- or the sequence $(x_n, P_n)_{n\in\mathbb{N}}$, where P_n is the tangent plane of the surface Σ_n at x_n, stays in a compact subset of $T^{-1}M$.

Corollary 3.8. *Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of convex Cauchy surfaces in M. Assume that this sequence is decreasing, and that the set $\Omega = \bigcup_{n\geq 0} I^+(\Sigma_n)$ is not the whole spacetime M. Then $(\Sigma_n)_{n\in\mathbb{N}}$ is uniformly spacelike.*

3.4. **Sequences of convex Cauchy surfaces with constant K-curvature.**

Using the two results discussed above, we will get a quite precise description of the possible asymptotic behaviour of a decreasing sequence of convex Cauchy surfaces with constant K-curvature:

Theorem 3.9. *Let $(\Sigma_n)_{(n\in\mathbb{N})}$ be a sequence of strictly convex Cauchy surfaces in M, such that, for every $n \in \mathbb{N}$, the surface Σ_n has constant K-curvature κ_n. Assume that this sequence is bounded away from the future end of M (i.e. the cosmological time is bounded from above on $\bigcup_n \Sigma_n$).*

[6] or generalized Cauchy surfaces, see §4.2.

If $\kappa_n \to \kappa$ with $-\infty < \kappa < \min(0, -\Lambda)$, then the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is precompact in the C^∞ topology. In particular, (Σ_n) is bounded away from the past singularity of M (there exists a Cauchy surface Σ such that $\Sigma_n \subset I^+(\Sigma)$ for all n), and there exists a subsequence of Σ_n converging to a smooth surface Σ_∞ with constant K-curvature κ.

On the contrary, if $\kappa_n \to -\infty$, then Σ_n is covering, that is *the union over $n \in \mathbb{N}$ of the sets $I^+(\Sigma_n)$ equals the whole spacetime M.*

The proof of this theorem involves several ingredients :

- an easy lemma which provides a uniform upper bound for the diameter of the Σ_n's,
- Margulis Lemma which tells that a uniform upper bound for the diameter of the Σ_n's yields some uniform lower bound for the systole of the Σ_n's (in the case where the sequence $(\kappa_n)_{n\in\mathbb{N}}$ is bounded from below),
- Theorem 3.5 which implies that the Σ_n's remain far from the initial singularity (in the case where he sequence $(\kappa_n)_{n\in\mathbb{N}}$ is bounded from below),
- Theorem 3.6 which roughly says that any limit point Σ_∞ of the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is uniformly spacelike,
- a result of Schlenker which says that a C^0 limit of convex spacelike surfaces either is a smooth surface or contains a complete geodesic ray,
- the geometric description of non-elementary MGHC spacetimes of constant curvature which entails the fact that such spacetimes never contain any complete geodesic ray.

3.5. **Perturbation of Cauchy surfaces with constant K-curvature.**

A classical calculation yields:

Proposition 3.10. *Assume that (M, g) admits a strictly convex Cauchy surface Σ with constant K-curvature $\kappa < -\Lambda$. Then, one can find a strictly convex Cauchy surface Σ^- in the past of Σ such that the K-curvature is strictly bounded from above by κ (i.e. such that (Σ^-, Σ) is a pair of κ'-barriers for every $\kappa' < \kappa$ such that κ' close enough to κ).*

The surface Σ^- can be obtained simply by pushing slightly Σ along the geodesics orthogonal to Σ. Proposition 3.10 and Theorem 3.2 show that, if M admits a convex Cauchy surface Σ with constant K-curvature $\kappa < -\Lambda$, then some neighbourhood of Σ in the past of Σ is foliated by convex Cauchy surfaces with constant K-curvature. Together with Theorem 3.9, this will allow us to prove that any local K-slicing on M can be extended towards the past untill it reaches the past singularity of M (i.e. until one gets a K-slicing on a neighbourhood of the past end of M). Since, in the case $\Lambda \geq 0$ (flat and de Sitter case), we have already explained how to get a local K-slicing on a neighbourhood of the future end of M (see §3.1), this will complete the proof of Theorem 2.1 in the flat case and in the de Sitter case.

3.6. **Duality between convex and concave surfaces in the anti-de Sitter space.**

All the arguments developped above in the case $\Lambda \geq 0$ (flat and de Sitter case) apply in the case $\Lambda < 0$ (anti-de Sitter case), except for one: if $\Lambda < 0$, the M is neither future complete nor past complete, and pushing along orthogonal geodesics does not provide, as explained in §3.1, a family of κ-barriers foliating a nieghbourhood of the future end of M.

So, in the anti-de Sitter case, one needs an alternative argument to complete the proof of Theorem 2.1. The proof goes as follows (see §13): from [13] we know that the spacetime contains a maximal surface Cauchy surface Σ. The set of points at Lorentzian distance $\pi/4$ from Σ is an union of two Cauchy surfaces $\Sigma^{\pi/4}$, $\Sigma^{-\pi/4}$, one in the future of Σ, the other in the past of Σ. These two surfaces have both constant K-curvature -1, and are thus κ-barriers. As discussed just above, we cannot prove that the future of $\Sigma^{-\pi/4}$ is foliated by K-surfaces. However, the arguments used in the flat and de Sitter cases ensure that the past $J^-(\Sigma^{-\pi/4})$ admits a K-time ranging over $(-\infty, -1)$. Similarly, the future $J^+(\Sigma^{\pi/4})$ of $\Sigma^{\pi/4}$ admits a reverse K-time. Now the key point is that there is a natural duality between convex and concave spacelike surfaces in AdS_3, preserving the property of having constant K-curvature. Therefore, the dual of the K-slicing of $J^+(\Sigma^{\pi/4})$ is a K-slicing of of the future of $\Sigma^{-\pi/4}$ in the past of the convex core, extending the K-slicing of $J^-(\Sigma^{-\pi/4})$.

3.7. **Ingredients of the proofs of Theorems 2.2, 2.4 and 2.6.**

As already explained in the introduction, Theorem 2.2 is an easy corollary of the existence of K-slicing for non-elementary 3-dimensional MGHC spacetimes of de Sitter type, and of the duality between the de Sitter space dS_3 and the hyperbolic space $\mathbb{H}^3$. Theorems 2.2 and 2.4 will follow from a generalized version of the Gerhardt's barriers theorem cited above and from Theorem 2.1: the barriers needed to apply Gerhardt's Theorem will be certain leaves of the K-slicing provided by Theorem 2.1.

3.8. **Organisation of the paper.**

Section 4 is devoted to some general preliminaries on time functions, surfaces with contant curvature, etc. In Section, 5, 6 and 7, we recall the main geometrical properties of MGHC spacetimes with constant curvature. In Section 8, we establish the desired relations between the systole of a Cauchy surface and the distance from this Cauchy surface to the initial singularity (Theorem 3.5). In Section 9, we prove that the limit of a decreasing sequence of convex Cauchy surfaces is always uniformly spacelike (Theorem 3.6). In Section 10, we prove Theorem 3.9 concerning decreasing sequence of Cauchy surfaces with constant K-curvature. In Section 11, we explain how to construct barriers; in particular, we prove Theorem 3.4 and Proposition 3.10. In Section 12 and 13, we prove our main Theorem 2.1 concerning the existence of K-slicings of 3-dimensional non-elementary MGHC spacetimes of constant curvature. In Section 14, we explain the duality between dS_3 and $\mathbb{H}^3$, and how to deduce Theorem 2.2 from Theorem 2.1. In Section 15, we deduce Theorem 2.4 from Theorem 2.1 and Gerhardt's barrier theorem. In Section 16, we consider the Minkowski problem and prove Theorem 2.6.

4. Cosmological time, Cauchy surfaces, maximum principle

For all basic Lorentzian notions such as chronological orientation, past/future, causal past/future, achronal subsets, acausal subsets, edgeless achronal subsets, Cauchy surfaces, global hyperbolicity, and maximal globally hyperbolic space, we refer to [50, 17]. Recall that MGHC is the acronym for "maximal globally hyperbolic spatially compact".

4.1. **Cosmological time.**

In any spacetime (M, g), we can define the *cosmological time* (see [5]):

Definition 4.1. The cosmological time of a spacetime (M, g) is the function $\tau : M \to [0, +\infty]$ defined by

$$\tau(x) = \text{Sup}\{\text{Length}(\gamma) \mid \gamma \in \mathcal{R}^-(x)\},$$

where $\mathcal{R}^-(x)$ is the set of all past-oriented causal curves starting at x, and Length(γ) the Lorentzian length of the causal curve γ.

In general, this function may have a very bad behavior: for example, if (M, g) is Minkowski space or the de Sitter space, then $\tau(x) = +\infty$ for every x.

Definition 4.2. A spacetime (M, g) is said to have *regular cosmological time*, if

(1) M has *finite existence time*, i.e. $\tau(x) < +\infty$ for every x in M,
(2) for every past-oriented inextendible causal curve $\gamma : [0, +\infty) \to M$, $\lim_{t\to\infty} \tau(\gamma(t)) = 0$.

Regular spacetimes admit many interesting properties ([5, Theorem 1.2]): in particular, they are globally hyperbolic, and their cosmological time is a locally Lipschitz time function. A common and important feature of all non-elementary MGHC spacetimes of constant curvature is that up to reversal of the time orientation, they have regular cosmological time ([3, 4, 20]).

Remark 4.3. For every spacetime (M, g) one can also introduce the *reverse cosmological time* $\check{\tau} : M \to [0, +\infty]$ of a spacetime (M, g), defined by $\check{\tau}(x) = \text{Sup}\{\text{Length}(\gamma) \mid \gamma \in \mathcal{R}^+(x)\}$, where $\mathcal{R}^+(x)$ is the set of all future-oriented causal curves starting at x. In other words, $\check{\tau}$ is the cosmological time of the spacetime obtained by reversing the time-orientation of (M, g).

4.2. **Generalized Cauchy surfaces.**

Let (M, g) be a globally hyperbolic spacetime.

First of all, we recall how Geroch proved that M is diffeomorphic to a product $\Sigma \times \mathbb{R}$ (see [34]). Select a Cauchy-time $t : M \to \mathbb{R}$, i.e. a time function such that every level $\{t = Cte\}$ is a Cauchy surface. Let X be the vector field $-\nabla t$, i.e. minus the gradient of t, and let ϕ^t be the flow of X. Let finally Σ be the level set $\{t = 0\}$. Then, the map $F : \Sigma \times \mathbb{R} \to M$ defined by $F(x, t) = \phi^t(x)$ is the required diffeomorphism.

It follows then nearly immediately from the definitions that every achronal subset E of M is the image by F of the graph of a locally Lipschitz function $u : \Lambda \to \mathbb{R}$, where Λ is a subset of Σ:

$$E = \{(x, t) \in \Lambda \times \mathbb{R} \mid t = u(x)\}$$

If the achronal subset is edgeless then Λ is open and u is proper. Furthermore, if E is edgeless achronal and compact, then $\Lambda = \Sigma$. In particular, Σ is compact too (i.e. M is spatially compact), and every inextendible causal curve intersects E.

Following the spirit of [34], we thus can define:

Definition 4.4. A *generalized Cauchy surface* in a MGHC spacetime is a compact achronal edgeless subset.

Now, let us observe that, in the above discussion, X can be replaced by any future-oriented Lipschitz-regular timelike vector field on M. The only difference is that the associated flow ϕ^t might be non-complete, but this does not matter: applying the process above we obtain an (bilipschitz) identification

$F : \Omega \to M$ where Ω is an open domain in $\Sigma \times \mathbb{R}$ defined by:

$$\Omega = \{(x,t) \in \Sigma \times \mathbb{R} \mid u_-(x) < t < u_+(x)\}$$

where Σ is any Cauchy surface in M and u_- (respectively u_+) is upper semi-continuous (respectively lower semi-continuous). We still have at our disposal a nice description of compact achronal edgeless subsets, i.e. generalized Cauchy surfaces: they are graphs of locally Lipschitz functions $u : \Sigma \to \mathbb{R}$ satisfying $u_- < u < u_+$. In the sequel, we will use this later remark with X being the opposite of the gradient of a cosmological time function, and the surface Σ being the level set $\tau^{-1}(1)$.

4.3. **Convex surfaces.**

Let (M, g) be a time-oriented spacetime. Recall that a C^2 spacelike surface $\Sigma \subset M$ is said to be *convex* (resp. *strictly convex*) if its second fundamental form is a negative (resp. negative definite) quadratic form. It is not hard to prove that Σ is convex if and only if its future $I^+(\Sigma)$ is locally geodesically convex (every point in M has an arbitrarly small neighbourhood U such that $U \cap I^+(\Sigma)$ is geodesically convex). The advantage of the later characterization is that its extends to non-smooth hypersurfaces:

Definition 4.5. An achronal edgeless set $\Sigma \subset M$ (e.g. a generalized Cauchy surface) is said to be *convex* if its future $I^+(\Sigma)$ is locally geodesically convex.

The three model spacetimes Min_3, dS_3 and AdS_3 admits some locally projective models (usually called *Klein models*). These models allow to reformulate the definition of convexity in the case where (M, g) is a 3-dimensional spacetime with constant curvature. Indeed in this particular case, every point $p_0 \in M$ admits arbitrarly small *convex charts centered at* p_0, i.e. causally convex and geodesically convex neighborhood U_0 such that there is a diffeomorphism $\varphi_0 : U_0 \to V_0$ where V_0 is an open domain in the vector space $\mathbb{R}^3$ with coordinates (x, y, z) satisfying the following properties:

(1) $\varphi_0(p_0) = 0$,
(2) φ_0 maps nonparametrized geodesics in U_0 to affine segments in V_0,
(3) $\varphi^* \frac{\partial}{\partial x_0}$ is a future oriented timelike vector field,
(4) V_0 is the domain $\{x^2 + y^2 < 1, -1 < z < 1\}$,
(5) for every $t \in]-1, +1[$ the hypersurface $\varphi_0^{-1}(\{z = t\})$ is a totally geodesic spacelike surface acausal in U_0,
(6) the image by φ_0 of $I^+_{U_0}(p_0)$ (resp. $I^-_{U_0}(p_0)$) is the open domain $\{z > 0, z^2 > x^2 + y^2\}$ (resp. $\{z < 0, z^2 > x^2 + y^2\}$).

Moreover, for any spacelike Cauchy surface Σ,

(7) for any convex chart centered at $p_0 \in \Sigma$, the image under φ_0 of the intersection $\Sigma \cap U_0$ is the graph of a Lipschitz map $(x, y) \mapsto f(x, y)$,
(8) if Σ is convex (resp. concave) if and only if for any sufficiently small convex chart centered at a point p_0 of Σ the function f is concave (resp. convex).

Another important property of spacetimes with constant curvature is the abundance of totally geodesic subspaces. Let M be a 3-dimensional spacetime with constant curvature, and Σ be a spacelike surface in M. A *lower support plane* of Σ is a 2-dimensional (necessarily spacelike) totally geodesic space $P \subset M$ such that:

(1) P intersects Σ at sme point p;
(2) Σ is locally contained in the future of P; more precisely, there exists a convex neighbourhood U of p such that $\Sigma \cap U \subset J_U^+(P)$.

One defines similarly the notion of *upper support plane* (by replacing "future" by "past" in the above definition). Using convex charts, it is easy to see that a spacelike surface Σ is convex (resp. concave) if and only if it admits a lower (resp. upper) support plane at each of its points.

4.4. **Maximum principle and K-times.**

Let (M, g) be a (time-oriented) MGHC spacetime. Recall that a K-time on M is a time function $\tau : M \to \mathbb{R}$ such that every level set of κ is a hypersurface One of the most important features of the K-times is their uniqueness. This property is a consequence of the so-called *maximum principle*.

Proposition 4.6 (Maximum Principle)**.** *Let Σ_1, Σ_2 be two convex spacelike surfaces in M. Assume that Σ_1, Σ_1 meet at some point x, and that Σ_1 is in the future of Σ_1. Then, the principal curvatures of Σ_1 are greater than or equal to those of Σ_1.*

To prove this, one only needs to write the surfaces Σ_1, Σ_1 as the graphs of some functions f_1, f_2 over the exponential of their common tangent plane at x, and to compare the Hessians of f_1 and f_2 at x. See, e.g., [13, Lemma 2.3].

Corollary 4.7 (Uniqueness of K-times)**.** *Assume that M admits a K-time $\kappa : M \to \mathbb{R}$. Then, for every $a \in \tau(M)$, the level set $C_a := \kappa^{-1}(a)$ is the unique compact K-surface of K-curvature a in M. In particular, the unique K-slicing on M is the one defined by the level sets of κ.*

Proof. Let $\Sigma \subset M$ be a compact K-surface with K-curvature a_0. For every $a \in \kappa(M)$, let $C_a := \kappa^{-1}(a)$. Define

$$a^- := \inf_S \kappa, \quad a^+ = \sup_S \kappa$$

Since Σ is compact, Σ meets C_{a^-} and C_{a^+}. Moreover, Σ is in the future of C_{a^-}, and in the past of C_{a^+}.

Let x^- be a point where Σ meets C_{a^-}. The maximum principle implies that the principal curvatures Σ at x^- are smaller than or equal to those of C_{a^-}. By definition of a K-time, the surface C_{a^-} is strictly convex, i.e. has negative principal curvatures. It follows that the principal curvatures of Σ at x^- are negative. Moreover, the product of the principal curvatures of Σ (resp. C_{a^-}) at x^- is equal to $-a_0$ (resp. $-a^-$). It follows that $a_0 \leq a^- < 0$.

Now, we see that the principal curvature of Σ are negative at every point of Σ. Indeed, they are negative at x^-, and their product is everywhere equal to $-a_0 \neq 0$. Let x^+ be a point where Σ meets C_{a^+}. By definition of a^+ and since κ is a K-time, Σ is in the past of C_{a^+}. Hence, the maximum principle implies that the principal curvatures curvatures of Σ at x^+ are bigger than those of C_{a^+}. Now, recall that the principal curvatures of Σ and C_{a^+} are negative, and the product of principal curvatures of Σ (resp. C_{a^+}) is equal to a_0 (resp. a^+). It follows that $a_0 \geq a^+$.

So we have $a^+ \leq a \leq a^-$ and $a^- \leq a^+$. It follows that $a = a^- = a^+$. Since Σ is in the future of C_{a^-} and in the past of C_{a^+}, we obtain the equality $\Sigma = C_{a^-} = C_{a^+}$. □

Remark 4.8. The strict convexity of the level sets of κ is crucial in the above proof.

5. Geometry of 3-dimensional MGHC flat spacetimes

Before starting the proof of Theorem 2.1, we need to recall some general facts on the geometry 3-dimensional MGHC spacetimes with constant curvature. In particular, we need to describe the decomposition of every such spacetime into simple "building blocks". We start here by discussing the case of flat spacetimes.

5.1. Mess' description of non-elementary MGHC flat spacetimes.

In his celebrated preprint [47], Mess has proved that every MGHC flat spacetime can be obtained as a quotient of domain of the Minkowski space Min_3 by a discrete group of isometries. More precisely:

Definition 5.1. A *future complete regular domain* in Min_3 is a domain of the form

$$E = \bigcap_{P\in\Lambda} I^+(P)$$

where Λ is a set of lightlike affine planes in Min_3, and $I^+(P)$ denotes the future of the lightlike plane P. Such a domain E is called *non-elementary* if Λ contains three pairwise non-parallel planes. The notion of *(non-elementary) past complete regular domain* is defined analogously.

Observe that every future (resp. past) complete regular domain $E \subset \mathrm{Min}_3$ is a convex set in Min_3. Moreover, if E is non-elementary, then E is a *proper* convex set, i.e. E does not contain any entire affine line of Min_3.

Theorem 5.2 (Mess, [47])**.** *Every non-elementary 3-dimensional MGHC flat spacetime is isometric to the quotient $\mathbf{\Gamma}\setminus E$ of a non-elementary future or past complete regular domain $E\subset \mathrm{Min}_3$ by a torsion-free discrete subgroup $\mathbf{\Gamma}$ of $\mathrm{Isom}_0(\mathrm{Min}_3) = \mathrm{SO}_0(1,2)\ltimes\mathbb{R}^3$. Moreover, the linear part of $\mathbf{\Gamma}$ is a co-compact Fuchsian subgroup Γ of $\mathrm{SO}_0(1,2)$, and the projection $p:\mathbf{\Gamma}\to\Gamma$ is one-to-one.*

Conversely, for every torsion-free discrete subgroup $\mathbf{\Gamma}$ of $\mathrm{Isom}_0(\mathrm{Min}_3) = \mathrm{SO}_0(1,2)\ltimes\mathbb{R}^3$, such that the linear part Γ of $\mathbf{\Gamma}$ is a co-compact Fuchsian subgroup of $\mathrm{SO}_0(1,2)$, and such that the projection $p:\mathbf{\Gamma}\to\Gamma$ is one-to-one, there exists a non-elementary future (resp. past) complete regular domain $E\subset\mathrm{Min}_3$ such that $\mathbf{\Gamma}$ preserves E, acts properly discontinuously on E, and such that $\mathbf{\Gamma}\setminus E$ is a non-elementary 3-dimensional MGHC flat spacetime.

The index 0 in $\mathrm{Isom}_0(\mathrm{Min}_3)$ and $\mathrm{SO}_0(1,2)$ means "identity component of the Lie group".

Remark 5.3. This theorem entails the fact that every non-elementary 3-dimensional MGHC flat spacetime is either future complete (and past incomplete) or past complete (and future incomplete).

5.2. Cosmological time of a regular domain.

Let E be a non-elementary future complete regular domain in Min_3, and $\widetilde{\tau}: E\to(0,+\infty]$ its cosmological time. Recall that, for every $x\in E$, the cosmological time $\widetilde{\tau}(x)$ is defined by

$$\widetilde{\tau}(x) = \sup\{\text{length}(\gamma)\mid \gamma\in\mathcal{R}^-(x)\},$$

where $\mathcal{R}^-(x)$ is the set of all past-oriented causal curves in E starting at x. Using the mere definition of a regular domain, Bonsante has given a much more concrete description of $\widetilde{\tau}$:

Proposition and definition 5.4 (Bonsante, [23])**.** *Let ∂E be the topological boundary of the domain E in* Min_3.

1. For every $x \in E$, there exists a unique point $r(x) \in \partial E$ such that $\widetilde{\tau}(x)$ is the length of the geodesic segment $[x, r(x)]$ (in particular $\widetilde{\tau}(x) < +\infty$). The map $x \mapsto r(x)$ is called the retraction on the singularity *of the domain E.*

2. The set $T_E = \{r(x), x \in E\}$ is made of all points of ∂E where E admits at least two distinct support planes (or equivalently, admits spacelike support planes). This set T_E is called the past singularity *of E.*

3. For every $x \in E$, $r(x)$ is the unique point of ∂E such that the geodesic segment $[r(x), x]$ is orthogonal to a spacelike support plane of ∂E at $r(x)$.

Remark 5.5. The set T_E, equipped with the metric induced by the Lorentzian metric of Min_3, is a $\mathbb{R}$-tree.

Using Proposition 5.4, one can prove that the cosmological time $\widetilde{\tau}$ satisfies nice regularity properties:

Proposition 5.6 (Bonsante, [23])**.** *The cosmological time $\widetilde{\tau} : E \to (0, +\infty)$ is a $C^{1,1}$ function, that is $\widetilde{\tau}$ is a C^1 function, whose gradient $\nabla\tilde{\tau}$ is a (locally) Lipschitz vector field. The levels of $\tilde{\tau}$ are $C^{1,1}$ Cauchy surfaces in E.*

5.3. **Decomposition of a regular domain in building blocks.**

We consider as above a non-elementary future complete regular $E \subset \mathrm{Min}_3$. We will now describe a natural partition of E in simple "building blocks". We use the notations $\widetilde{\tau}$, r, T_E defined in the previous paragraph. Some proofs of all the facts stated below can be found in [23].

For every $p \in T_E$, we consider the set $E(p) := r^{-1}(p)$. By item 1 of Proposition 5.4, $\{E(p)\}_{p \in T_E}$ is a partition of E. For every $p \in T_E$, let $\Delta(p)$ be the set of all spacelike support planes of E at p. Item 3 of Proposition 5.4 implies that $E(p)$ is the union of all future directed geodesic rays starting at p that are orthogonal to an element of $\Delta(p)$. Identifying every spacelike affine plane with its future directed unit normal vector, we can see $\Delta(p)$ as a subset of $\mathbb{H}^2$. Moreover, since E is obtained as the intersection of the futures of some lightlike planes, $\Delta(p)$ is an *ideal* convex subset of $\mathbb{H}^2$, that is $E(p)$ is the convex hull in $\mathbb{H}^2$ of a subset of $\partial\mathbb{H}^2$. We distinguish two cases:

1) if p is not a vertex of the $\mathbb{R}$-tree T_E, then $\Delta(p)$ is a geodesic of $\mathbb{H}^2$, and $E(p)$ is isometric to $\mathbb{R} \times (0, +\infty)$ endowed with the metric $t^2\,\mathrm{d}\theta^2 - \mathrm{dt}^2$.
2) if p is a vertex of the $\mathbb{R}$-tree T_E, then $\Delta(p)$ is a ideal convex subset of $\mathbb{H}^2$ with non-empty interior, and $E(p)$ is isometric to $\Delta(p) \times (0, +\infty)$ endowed with the metric $t^2\,{\mathrm{ds}_{\mathrm{hyp}}}^2 - \mathrm{dt}^2$.

Moreover, we observe that:

3) if γ is an edge of length ℓ in the $\mathbb{R}$-tree T_E, then $E(\gamma) := \bigcup_{p \in \gamma} E(p)$ is isometric to $\mathbb{R} \times (-\ell/2, \ell/2) \times (0, +\infty)$ endowed with the Lorentzian metric $t^2\,\mathrm{d}\theta^2 + dx^2 - \mathrm{dt}^2$, where ℓ is the length of γ. The set $E(\gamma)$ is foliated by sets of the form $E(p)$ and is maximal for this property.

Fact and definitions 5.7. *The partition $E = \bigcup_{p \in T_E} E(p)$ is called the* canonical decomposition *of E. Each element of this decomposition is:*

1) *either a* thin cone *isometric to* $(\mathbb{R} \times (0, +\infty)\ ,\ t^2\,\mathrm{d}\theta^2 - \mathrm{dt}^2)$,

2) *or a* truncated solid cone *isometric to* $(\Delta \times (0,+\infty)\ ,\ t^2\,{\rm ds_{hyp}}^2 - {\rm dt}^2)$ *for some ideal convex set with non-empty interior* $\Delta \subset \mathbb{H}^2$.

Every maximal subset of E *foliated by thin cones of the canonical decomposition is isometric to* $(\mathbb{R} \times (-\ell/2,\ell/2) \times (0,+\infty)\ ,\ t^2\,{\rm d}\theta^2 + dx^2 - {\rm dt}^2)$ *for some* ℓ*, and is called a* Misner sector of width ℓ.

Remark 5.8. In general, T_E is not a simplicial tree. In particular, there might be some points in T_E that neither are vertices, nor belong to a non-trivial edge. Therefore, in the canonical decomposition of E, there might be some Minkowski thin cones that do not belong to any Misner sector.

For every $a \in (0,+\infty)$, let $\widetilde{C}_a := \widetilde{\tau}^{-1}(a)$ be the a-level set of the cosmological time $\widetilde{\tau}$. Recall that $\widetilde{C}_a$ is a Cauchy surface in E. The canonical decomposition of the domain E induces a decomposition of the surface $\widetilde{C}_a$. It is easy to see that the elements of this decomposition are the leaves of a lamination and the connected components of the complement of this lamination. More precisely:

Fact 5.9. *There is a lamination* $\widetilde{\mathcal{L}}_a$ *in the surface* $\widetilde{C}_a$ *such that:*

1) *The leaves of* $\widetilde{\mathcal{L}}_a$ *are the intersections of* $\widetilde{C}_a$ *with the thin Minkowski cones of the canonical decomposition of* E*,*
2) *The connected components of* $\widetilde{C}_a \setminus \widetilde{\mathcal{L}}_a$ *are the intersections of* $\widetilde{C}_a$ *with the truncated solid cones of the canonical decomposition of* E*.*

The lamination $\widetilde{\mathcal{L}}_a$ *supports a transverse measure defined as follows : for every absolutely continuous path* $c : [a,b] \to \mathbb{H}^2$ *transverse to* $\widetilde{\mathcal{L}_a}$ *in* $\widetilde{C}_a$*, the measure of* c *is the length of the path* $r \circ c$ *in the* $\mathbb{R}$*-tree* T_E*. Note that:*

3) *The intersection of* $\widetilde{C}_a$ *with a Misner sector of width* ℓ *associated with the canonical decomposition of* E *is a strip (homeomorphic to* $\mathbb{R} \times [-\ell/2,\ell/2]$*) in* $\widetilde{C}_a$ *foliated by leaves of the lamination* $\widetilde{\mathcal{L}}_a$*. The width of this strip, measured with respect to the transverse measure of* $\widetilde{\mathcal{L}}_a$*, is exactly* ℓ*.*

The canonical decomposition of the domain E is invariant under the gradient flow of the cosmological time $\widetilde{\tau}$. It follows that, for every $a, a' \in (0,+\infty)$, the time $a' - a$ of the gradient flow of the cosmological time $\widetilde{\tau}$ maps the lamination $\widetilde{\mathcal{L}}_{a'} \subset \widetilde{C}_{a'}$ on the lamination $\widetilde{\mathcal{L}}_a \subset \widetilde{C}_a$.

Now, we can consider the map $\widetilde{N} : E \to \mathbb{H}^2$ which associates to each point $x \in \widetilde{E}$ the opposite of the gradient of $\widetilde{\tau}$ at x. Note that, for every $a \in (0,+\infty)$ and every $x \in \widetilde{C}_a$, the vector $\widetilde{N}(x)$ is the future-directed unit normal vector of $\widetilde{C}_a$ at x. The image under $\widetilde{N}$ of a thin Minkowski cone $E(p)$ of the canonical decomposition of E can be naturally identified with the geodesic $\Delta(p) \subset \mathbb{H}^2$. If $E(p), E(p')$ are two different Minkowski cones, the corresponding geodesics in $\mathbb{H}^2$ are clearly disjoint. Therefore:

Fact 5.10. *The image under* $\widetilde{N}$ *of the union of the thin Minkowski cones of the canonical decomposition of* $\mathcal{L}$ *is a geodesic lamination* $\widetilde{\mathcal{L}}$ *in* $\mathbb{H}^2$*.*

For every $a \in (0,+\infty)$, the image under $\widetilde{N}$ of the lamination $\widetilde{\mathcal{L}}_a$ is the geodesic lamination $\widetilde{\mathcal{L}}$. Therefore, one can push the transverse measure of $\widetilde{\mathcal{L}}_a$ to get a transverse measure on $\mathcal{L}$. This transverse measure is independant of a.

Since $\widetilde{\mathcal{L}}$ is a geodesic lamination in a hyperbolic surface, the arcwise connected components of $\widetilde{\mathcal{L}}$ are its leaves. It follows that:

Fact 5.11. *The pre-image (for the map $\widetilde{N}$) of a leaf of $\widetilde{\mathcal{L}}$ is an arcwise connected component of the union of the thin Minkowski cones of the canonical decomposition of E. Therefore:*

1) *The pre-image of a leaf of $\widetilde{\mathcal{L}}$ which carries a zero weight is a thin Minkowski cone of the canonical decomposition of E which is not included in a Misner sector.*
2) *The pre-image of a connected component of $\mathbb{H}^2 \setminus \widetilde{\mathcal{L}}$ is a truncated solid cone of the canonical decomposition of E.*
3) *The pre-image of a leaf of $\widetilde{\mathcal{L}}$ which carries a weight $\ell > 0$ is a Misner sector of width ℓ associated with the canonical decomposition of E.*

Remark 5.12. The $\mathbb{R}$-tree T_E is the dual of the measured lamination $\widetilde{\mathcal{L}}$ (cf. Proposition 3.7.2 and equality (3.6) in [20]).

5.4. **Decomposition of a 3-dimensional MGHC flat spacetime.**

Let (M, g) be a non-elementary MGHC flat spacetime. By Theorem 5.2, reversing the time orientation if necessary, (M, g) is isometric to the quotient of a non-elementary future complete regular domain $E \subset \mathrm{Min}_3$ by a torsion-free discrete subgroup $\mathbf{\Gamma}$ of $\mathrm{SO}_0(1,2) \ltimes \mathbb{R}^3$.

Clearly, the cosmological time $\widetilde{\tau}$ of the domain E is $\mathbf{\Gamma}$-invariant, and the projection of $\widetilde{\tau}$ in $M \simeq \mathbf{\Gamma}\backslash E$ is the cosmological time of the spacetime M. Therefore, the cosmological time τ of M enjoys the same nice regularity properties as $\widetilde{\tau}$. In particular, Propositon 5.6 implies that τ is a $C^{1,1}$ function, and, for every $a \in (0, +\infty)$, the level set $C_a := \tau^{-1}(a)$ is a $C^{1,1}$ Cauchy surface in M (in particular, C_a is a compact surface).

The canonical decomposition of E is $\mathbf{\Gamma}$-invariant. Therefore, it induces a decomposition of the spacetime M. More precisely:

Fact and definitions 5.13. *The canonical decomposition of E induces a* canonical decomposition *of M. Each element of this decomposition is:*

1) *either a* thin block *isometric to $\mathbb{R} \times (0, +\infty)$ or $\mathbb{S}^1 \times (0, +\infty)$ endowed with the metric $t^2\, \mathrm{d}\theta^2 - \mathrm{dt}^2$,*
2) *or a* solid block *isometric to $\Sigma \times (0, +\infty)$ with the metric $t^2\, \mathrm{ds}^2_{hyp} - \mathrm{dt}^2$ where $(\Sigma, \mathrm{ds}^2_{hyp})$ is the interior of a hyperbolic surface with boundary.*

Every maximal subset of M foliated by thin blocks of the canonical decomposition is isometric to $\mathbb{S}^1 \times (-\ell/2, \ell/2) \times (0, +\infty)$ with the metric $\mathrm{t}^2 \mathrm{d}\theta^2 + \mathrm{dx}^2 - \mathrm{dt}^2$, and is called a Misner block *of width ℓ.*

Now, for every $a \in (0, +\infty)$, the measured lamination $\widetilde{\mathcal{L}}_a$ projects to a measured lamination $\mathcal{L}_a$ in the surface $C_a = \mathbf{\Gamma}\backslash\widetilde{C}_a = \tau^{-1}(a)$. Moreover:

1) The leaves of $\mathcal{L}_a$ are the intersections of C_a with the thin blocks of the canonical decomposition of M. More precisely, the closed leaves of $\mathcal{L}_a$ are the intersections of C_a with the thin blocks homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and the non-closed leaves of $\mathcal{L}_a$ are the intersections of C_a with the thin blocks homeomorphic to $\mathbb{R} \times \mathbb{R}$.
2) The connected components of $C_a \backslash \mathcal{L}_a$ are the intersections of C_a with the solid blocks of the canonical decomposition of M.

3) The intersection of C_a with a Misner block of width ℓ associated with the canonical decomposition of M is a annulus in M foliated by closed leaves of $\mathcal{L}_a$. The width of this annulus, measured with respect to the transverse measure of $\mathcal{L}_a$, is exactly ℓ.

Now, the map $\widetilde{N} : E \to \mathbb{H}^2$ is $\mathbf{\Gamma}$-equivariant, where $\mathbf{\Gamma}$ acts on $\mathbb{H}^2$ through its linear part $\Gamma \subset \mathrm{SO}_0(1,2)$. It follows that $\widetilde{N}$ induces a Lipschitz map $N : M \to C := \Gamma\backslash\mathbb{H}^2$, that the measured geodesic lamination $\widetilde{\mathcal{L}}$ induces a measured geodesic lamination $\mathcal{L}$ on the compact hyperbolic surface $C := \Gamma\backslash\mathbb{H}^2$, and that $\mathcal{L}$ is the image of the union of all the thin blocks in the canonical decomposition of M under the map N. Moreover:

1) The pre-image (for the map $N : M \to C$) of a leaf of $\mathcal{L}$ which carries a zero weight is a thin block of the canonical decomposition of M which is not included in a Misner block.
2) The pre-image of a connected component of $C \setminus \mathcal{L}$ is a solid block of the canonical decomposition of E.
3) The pre-image of a leaf of $\mathcal{L}$ which carries a weight $\ell > 0$ is a Misner block of width ℓ associated with the canonical decomposition of E.

Note that, since C is compact, a leaf of $\mathcal{L}$ which carries a non-zero weight is necessarily closed, and that there are only finitely many such leaves in $\mathcal{L}$. It follows that there are only finitely many Misner blocks associated with the canonical decomposition of M.

Remark 5.14. In [47] G. Mess observed that the pair $(C, \mathcal{L})$ parametrizes the set of 3-dimensional non-elementary MGHC flat spacetimes of prescribed topology. In [22, 23], F. Bonsante studied in detail this parametrization, discussed its extension to higher dimensions, and proved in particular that this parametrization is continuous at many respect.

6. Geometry of 3-dimensional MGHC dS_3-spacetimes

6.1. **The 3-dimensional de Sitter space.**

Consider the quadratic form $Q_{1,3} = -x_1^2 + x_2^2 + x_3^2 + x_4^2$ on $\mathbb{R}^4$. The *linear model* dS_3 of the 3-dimensional de Sitter space is the hyperboloid $\{Q_{1,3} = 1\}$ endowed with the Lorentzian metric induced by $Q_{1,3}$. This spacetime is time-orientable; we choose the time orientation for which the curve $t \mapsto (\sinh t, \cosh t, 0, 0)$ is future-oriented. Moreover, dS_3 is globally hyperbolic and that the coordinate x_1 is a time function on dS_3. The group of orientation and time orientation preserving isometries of dS_3 is the group $\mathrm{O}_0(1,3)$.

It is sometimes convenient to consider the *Klein model* $\mathbb{DS}_3$ of the de Sitter space. By definition, $\mathbb{DS}_3$ is the image of dS_3 by the radial projection $\pi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{S}^3$ (endowed with the push-forward of the Lorentzian metric of dS_3). The projection $\pi : \mathrm{dS}_3 \to \mathbb{DS}_3$ is a diffeomorphism. The boundary of $\mathbb{DS}_3$ in $\mathbb{S}^3$ is the image under π of the cone $\{Q_{1,3} = 0\}$. This boundary is the disjoint union of two round 2-spheres: $\mathbb{S}^2_- := \pi(\{Q_{1,3} = 0\} \cap \{x_1 < 0\})$ and $\mathbb{S}^2_+ := \pi(\{Q_{1,3} = 0\} \cap \{x_1 > 0\})$. Note that every future oriented inextendible causal curve $\gamma \subset \mathbb{DS}_3$ "goes from $\mathbb{S}^2_-$ to $\mathbb{S}^2_+$" (more precisely: the α-limit and ω-limit sets of γ are single points lying respectively in $\mathbb{S}^2_-$ and $\mathbb{S}^2_+$).

Now denote by H^3_- (resp. H^3_+) the connected components of the hyperboloid $\{Q_{1,3} = -1\}$ contained respectively in the half-space $\{x_1 < 0\}$ (resp. $\{x_1 > 0\}$), endowed with the Riemannian metric induced by $Q_{1,3}$. Note that H^3_- and H^3_+ are two copies of the 3-dimensional hyperbolic space. The Klein

models of H^3_- (resp. H^3_+) is the image $\mathbb{H}^3_-$ (resp. $\mathbb{H}^3_+$) under the projection π. Observe that $\mathbb{S}^3 = \mathbb{H}^3_- \sqcup \mathbb{S}^2_- \sqcup \mathbb{DS}_3 \sqcup \mathbb{S}^2_+ \sqcup \mathbb{H}^3_+$, and that $\mathbb{S}^2_-$ (resp. $\mathbb{S}^2_+$) is the topological boundary of $\mathbb{H}^3_-$ (resp. $\mathbb{H}^3_+$) in $\mathbb{S}^3$.

The group $\mathrm{O}_0(1,3)$ is simultaneously the isometry group of the Lorentzian space $\mathbb{DS}_3$, the isometry group of the hyperbolic spaces $\mathbb{H}^3_-$ and $\mathbb{H}^3_+$, and the conformal group of the spheres $\mathbb{S}^2_-$ and $\mathbb{S}^2_+$.

A key ingredient in the sequel is the fact that the de Sitter space $\mathbb{DS}_3$ can be thought of as the space of (non-trivial open) round balls in $\mathbb{S}^2_+$. Indeed, for every point $x \in \mathbb{DS}_3$, let us denote by $\partial^+ I^+(x)$ the set of the future endpoints in $\mathbb{S}^2_+$ of all the future oriented timelike geodesic rays starting at x. Then, for every $x \in \mathbb{DS}_3$, the set $\partial^+ I^+(x)$ is an open round ball in $\mathbb{S}^2_+$. One can easily check that the map associating to x the round ball $\partial^+ I^+(x)$ establishes a one-to-one correspondance between the points in $\mathbb{DS}_3$ and the (non-trivial open) round balls in $\mathbb{S}^2_+$. Of course, there is a similar identification between the points of $\mathbb{DS}_3$ and the round balls in $\mathbb{S}^2_-$.

6.2. **Scannell's description of MGHC dS_3 spacetimes.**

We now sketch the correspondance between MGHC dS_3 spacetimes and compact Möbius surfaces. For more details (and for proofs of the facts stated below), we refer to [51] or [4].

Let us consider a simply connected Möbius surface S (i.e. a surface endowed with a $(\mathrm{O}_0(1,3), \mathbb{S}^2)$-structure). We will construct a simply connected spacetime $\mathcal{B}^+_0(S)$ associated to S and locally isometric to dS_3. For this purpose, we consider a developping map $d : S \to \mathbb{S}^2_+$ (such a map d does exists since S is a Möbius surface).

An (open) *round ball* U in S is an open domain in S such that the developing map d to U is one-to-one in restriction to U, and such that $d(U)$ is an open round ball in $\mathbb{S}^2_+$. A round ball $U \subset S$ is said to be *proper* if the image under d of the closure $\overline{U}$ of U in S is the closure of $d(U)$ in $\mathbb{S}^2_+$.

We will denote by $\mathcal{B}_0(S)$ the set of proper round balls. The set $\mathcal{B}_0(S)$ is naturally ordered by the inclusion. For every element U of $\mathcal{B}_0(S)$, we denote by $W(U)$ the subset of $\mathcal{B}_0(S)$ made of the proper round balls U' such that $\overline{U'} \subset U$. Given two elements U, V of $\mathcal{B}_0(S)$ such that $\overline{U} \subset V$, we denote by $W(U,V)$ the set of all proper round balls U' in S such that $\overline{U} \subset U'$ and $\overline{U'} \subset V$. The sets $W(U,V)$ generate a topology on $\mathcal{B}_0(S)$ that we call the *Alexandrov topology*. It can be proved that the set $\mathcal{B}_0(S)$, equipped with the Alexandrov topology, is a manifold.

We already observed that the de Sitter space $\mathbb{DS}_3$, as a set, is canonically identified with the space $\mathcal{B}_0(\mathbb{S}^2_+)$ of all open round balls in the sphere $\mathbb{S}^2_+$. Actually, the identification between $\mathbb{DS}_3$ and $\mathcal{B}_0(\mathbb{S}^2_+)$ is an homeomorphim, once $\mathcal{B}_0(\mathbb{S}^2_+)$ is endowed with the Alexandrov topology.

The map $\mathcal{D}^+ : \mathcal{B}_0(S) \to \mathbb{DS}_3$, composition of the developing map $d : \mathcal{B}_0(S) \to \mathcal{B}_0(\mathbb{S}^2)$ and the identification of $\mathcal{B}_0(\mathbb{S}^2)$ with $\mathbb{DS}_3$, is a local homeomorphism. Hence, we can consider the pull-back by $\mathcal{D}^+$ of the de Sitter metric on $\mathcal{B}_0(S)$. We will denote by $\mathcal{B}^+_0(S)$ the manifold $\mathcal{B}_0(S)$ equipped with the pull-back by $\mathcal{D}^+$ of the de Sitter metric. By construction, $\mathcal{B}^+_0(S)$ is a spacetime locally isometric to $\mathbb{DS}_3$.

Now suppose that the simply connected Möbius surface S considered above is the universal cover of a compact Möbius surface Σ. Denote by Γ the fundamental group of Σ. Then the developping map $d : S \to \mathbb{S}^2_+$ is Γ-equivariant: there exists a representation $\rho : \Gamma \to \mathrm{O}_0(1,3)$ such that $d \circ \gamma = \rho(\gamma) \circ d$ for every $\gamma \in \Gamma$. It follows that $\rho(\Gamma)$ acts by isometries on the spacetime $\mathcal{B}^+_0(S)$. This action is free, and properly discontinuous. So, we can consider the spacetime $M^+(\Sigma) := \rho(\Gamma) \backslash \mathcal{B}^+_0(S)$. Up to isometry,

this spacetime does not depend on the choice of the developing map d. The important result for us is the following:

Theorem 6.1 (Scannell)**.** *Every MGHC* dS_3*-spacetime is past or future complete. Every future complete MGHC* dS_3*-spacetime* M *is isometric to the spacetime* $M^+(\Sigma)$ *for some compact Möbius surface* Σ*.*

Remark 6.2. This correspondance between Möbius surfaces and MGH dS_3-spacetimes is an equivalence of category. In particular, any isometric map $\mathcal{B}_0^+(S_1) \to \mathcal{B}_0^+(S_2)$ induces a conformal projective map $S_1 \to S_2$.

6.3. **Canonical decomposition of MGHC** dS_3**-spacetimes.**

Let M be a future complete non-elementary MGHC dS_3-spacetime. Analysing the cosmological time τ of M, and following the same train of thoughts as in the flat case (see §5.3 and 5.4), one gets:

Fact and definitions 6.3. *The spacetime* M *admits a* canonical decomposition*. Each element of this decomposition is:*

1) *either a* thin block *isometric to* $\mathbb{R} \times (0, +\infty)$ *or* $\mathbb{S}^1 \times (0, +\infty)$ *endowed with the metric* $\sinh(t)^2\,\mathrm{dx}^2 - \mathrm{dt}^2$*,*
2) *or a* solid block *isometric to* $\Sigma \times (0, +\infty)$ *endowed with the Lorentzian metric* $\sinh(t)^2\,\mathrm{ds}^2_{hyp} - \mathrm{dt}^2$ *where* $(\Sigma, \mathrm{ds}^2_{hyp})$ *is the interior of a hyperbolic surface with boundary.*

Every maximal subset of M *foliated by thin blocks of the canonical decomposition is isometric to* $\mathbb{S}^1 \times (-\ell/2, \ell/2) \times (0, +\infty)$ *endowed with the metric* $\cosh(t)^2\,\mathrm{d}\theta^2 + \sinh(t)^2\,\mathrm{dx}^2 - \mathrm{dt}^2$*, and is called a* Misner block *of width* ℓ*.*

Fact 6.4. *For every* $a \in (0, \pi/2)$*, there exists a measured lamination* $\mathcal{L}_a$ *in the surface* $C_a = \tau^{-1}(a)$*, such that:*

1) *The leaves of* $\mathcal{L}_a$ *are the intersections of* C_a *with the thin blocks of the canonical decomposition of* M*.*
2) *The connected components of* $C_a \setminus \mathcal{L}_a$ *are the intersections of* C_a *with the solid blocks of the canonical decomposition of* M*.*
3) *The intersection of* C_a *with a Misner block associated with the canonical decomposition of* M *is a annulus in* C_a *foliated by closed leaves of* $\mathcal{L}_a$*.*

Fact 6.5. *The gradient of the cosmological time of* M *defines a Lipschitz map* $N : M \to C_\infty$ *where* C_∞ *is a hyperbolic surface. The image under* N *of the union of the thin blocks of the canonical decomposition of* N *is a geodesic lamination* $\mathcal{L}_\infty$ *on* C_∞*. The transverse measures of the laminations* $\mathcal{L}_a$ *induce a transverse measure on* $\mathcal{L}_\infty$*. Moreover:*

1) *The pre-image (for* N*) of a non-weighted leaf of* $\mathcal{L}_\infty$ *is a thin block of the canonical decomposition of* M *which is not included in a Misner block.*
2) *The pre-image of a connected component of* $C_\infty \setminus \mathcal{L}_\infty$ *is a solid block of the canonical decomposition of* M*.*
3) *The pre-image of a leaf of* $\mathcal{L}_\infty$ *which carries a weight* $\ell > 0$ *is a Misner block of width* ℓ *associated with the canonical decomposition of* M*.*

7. Geometry of 3-dimensional MGHC AdS_3-spacetimes

7.1. The 3-dimensional anti-de Sitter space.

Consider the quadratic form $Q_{2,2} = -x_1^2 - x_2^2 + x_3^2 + x_4^2$ on $\mathbb{R}^4$. The 3-dimensional anti-de Sitter space AdS_3 is the quadric $\{Q_{2,2} = -1\}$, equipped with the Lorentzian metric induced by $Q_{2,2}$. We choose the time-orientation for which the timelike curve $\theta \to (\cos\theta, \sin\theta, 0, 0)$ is future-oriented. The group of isometries of AdS_3 preserving the orientation and the time-orientation is the connected component $\mathrm{O}_0(2,2)$ of the identity in $\mathrm{O}(2,2)$.

Observe that the vector space of 2 by 2 matrices $\mathfrak{gl}(2,\mathbb{R})$ endowed with the quadratic form $-\det$ is isometric to $(\mathbb{R}^4, Q_{2,2})$. Hence AdS_3 is naturally identified with $\mathrm{SL}(2,\mathbb{R})$ endowed with its Killing form. The actions of $\mathrm{SL}(2,\mathbb{R})$ on itself by left and right translations preserves this Lorentzian metric and commutes each other. Hence we have a natural isomorphism $\mathrm{O}_0(2,2) \approx (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/I$ where $I = \{(Id, Id), (-Id, -Id)\}$.

We will also consider the *Klein model* $\mathbb{ADS}_3$ of the anti-de Sitter space. By definition, $\mathbb{ADS}_3$ is the image of AdS_3 under the radial projection $\pi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{S}^3$. The projection $\pi : \mathrm{AdS}_3 \to \mathbb{ADS}_3$ is a diffeomorphism.

We denote by $\mathcal{Q}$ the boundary of $\mathbb{ADS}_3$ in $\mathbb{S}^3$. The projection of $\mathcal{Q}$ in $\mathbb{RP}^3 = \mathbb{S}^3/\{\mathrm{Id}, -\mathrm{Id}\}$ is a one-sheeted hyperboloid, which notoriously admits two transverse rulings by projective lines. This provides an identification between $\mathcal{Q}/\{\mathrm{Id}, -\mathrm{Id}\}$ and $\mathbb{RP}^1 \times \mathbb{RP}^1$ (equivalently, between $\mathcal{Q}$ and a two fold cover of $\mathbb{RP}^1 \times \mathbb{RP}^1$). The isometric action of $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/\mathcal{I}$ on $\mathbb{ADS}_3$ extends to the boundary $\mathcal{Q}$. The identification between $\mathcal{Q}/\{\mathrm{Id}, -\mathrm{Id}\}$ and $\mathbb{RP}^1 \times \mathbb{RP}^1$ can be chosen so that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{Q}/\{\mathrm{Id}, -\mathrm{Id}\}$ coordinate by coordinate:

$$(\gamma_L, \gamma_R).(\theta_L, \theta_R) = (\gamma_L\theta_L, \gamma_R\theta_R)$$

7.2. Mess' description of non-elementary MGHC AdS_3 spacetimes.

We briefly recall the correspondance between non-elementary MGHC AdS_3 spacetimes and pairs of Fuchsian representations of a surface group. For more details, see [47] or [13, 20, 10].

Let Γ be the fundamental group of a closed surface of genus $g \geq 2$, and consider a pair of Fuchsian representations $\rho_L, \rho_R : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$. Two such representations are always topologically conjugate: there is a (unique) homeomorphism $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ such that:

$$\forall \gamma \in \Gamma, \quad \rho_R(\gamma) \circ f = f \circ \rho_L(\gamma)$$

The graph of f is a closed topological circle $\bar{\Lambda}_\rho$ embedded in $\mathcal{Q}/\{\mathrm{Id}, -\mathrm{Id}\} \approx \mathbb{RP}^1 \times \mathbb{RP}^1$ which is invariant under the action of Γ defined by $\bar{\rho} = (\rho_L, \rho_R)$. Moreover, this topological circle $\bar{\Lambda}_\rho \subset \mathcal{Q}/\{\mathrm{Id}, -\mathrm{Id}\} \subset \mathbb{RP}^3$ is contained in an affine chart of $\mathbb{RP}^3$.

The preimage in $\mathcal{Q}$ of $\bar{\Lambda}$ is the union of two acausal topological circles Λ_ρ^-, Λ_ρ^+, one image of the other by the antipodal map. For each of them one can define the *invisible domain* $E(\Lambda_\rho^\pm)$: it is the set of all points $p \in \mathbb{ADS}_3$ such that there is no causal curve γ starting at p and ending at a point $x \in \Lambda_\rho^\pm$, such that $\gamma \setminus \{x\}$ is contained and in an affine chart, and such that γ is causal. The intersection between the quadric $\mathcal{Q}$ and the closure of $E(\Lambda_\rho^\pm)$ in $\mathbb{S}^3$ is exactly the topological circle $\Lambda_\rho^\pm$, and $E(\Lambda_\rho^\pm) \setminus \Lambda_\rho^\pm$ is contained in $\mathbb{ADS}_3$. Moreover, $E(\Lambda_\rho^+)$ and $E(\Lambda_\rho^-)$ are disjoint one from the other.

The representation $\bar{\rho}$ lifts to a faithfull representation $\rho : \Gamma \to (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/\mathcal{I}$ preserving Λ_ρ^- and Λ_ρ^+, hence also the invisible domains $E(\Lambda_\rho^\pm)$. These actions are free and properly discontinuous. The quotient space $M(\rho) = \rho(\Gamma)\backslash E(\Lambda_\rho^+)$ is a MGHC spacetime with closed Cauchy surfaces of genus g. It turns out that every non-elementary MGHC AdS_3-spacetime can be obtained in this way:

Theorem 7.1 (Mess)**.** *For $g \geq 2$, let Γ_g be the fundamental group of the compact surface of genus g. Then every MGHC AdS_3-spacetime with Cauchy surface of genus g is isometric to $M(\rho) = \rho(\Gamma)\backslash E(\Lambda_\rho^+)$ for some Fuchsian representation $\rho = (\rho_L, \rho_R) : \Gamma_g \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.*

Remark 7.2. One may consider the quotient $\rho(\Gamma)\backslash E(\Lambda_\rho^-)$ instead of $\rho(\Gamma)\backslash E(\Lambda_\rho^+)$, but it would lead to the same result up to isometry since $E(\Lambda_\rho^-)$ is the image of $E(\Lambda_\rho^+)$ under the antipodal map.

One can also define the *convex hull* $\mathrm{Conv}(\Lambda_\rho^+)$: it is simply the convex hull in the usual meaning in any affine chart U of $\mathbb{S}^3$ containing Λ_ρ^+ (it does not depend on the choice of U). It turns out that $\mathrm{Conv}(\Lambda_\rho^+) \setminus \Lambda_\rho^+$ is contained in $\mathbb{ADS}_3$. The complement of Λ_ρ^+ in the convex hull $\mathrm{Conv}(\Lambda_\rho^+)$ is contained in $E(\Lambda_\rho^+)$. The projection of $\mathrm{Conv}(\Lambda_\rho^+) \setminus \Lambda_\rho^+$ in $M(\rho)$ is a closed region $\mathcal{C}(\rho)$, diffeomorphic to $S \times [-1,+1]$, that we call *convex core* of the spacetime $M(\rho)$. The complementary of $\mathcal{C}(\rho)$ has two connected components: the past and the future of $\mathcal{C}(\rho)$, denoted respectively by $M_-(\rho)$ and $M_+(\rho)$, called respectively *past tight region* and *future tight region* of $M(\rho)$.

The regions $\mathcal{C}(\rho)$, $M_-(\rho)$ and $M_+(\rho)$ can also be defined using the cosmological time and the reverse cosmological time of $M(\rho)$:

- The spacetime $M(\rho)$ has a regular cosmological time $\tau : M(\rho) \to (0,\pi)$, as well as a regular reverse cosmological time $\check{\tau} : M(\rho) \to (0,\pi)$.
- The convex core $\mathcal{C}(M)$ corresponds to the set $\{\tau \leq \pi/2,\ \check{\tau} \leq \pi/2\}$. The past tight region $M_-(\rho)$ corresponds to the set $\{\tau \leq \pi/2\}$. The future tight region $M_+(\rho)$ corresponds to the set $\{\check{\tau} \leq \pi/2\}$.
- The cosmological time τ (resp. the reverse cosmological time $\check{\tau}$) is a $C^{1,1}$ function in restriction to $M_-(\rho)$ (resp. $M_+(\rho)$).

7.3. **Canonical decomposition of MGHC AdS_3-spacetimes.**

We have seen in the two previous sections that every non-elementary MGHC flat or dS_3 spacetime admits a canonical decomposition in so-called thin blocks and solid blocks. A similar decomposition exists in the AdS_3-case, except that it does not concern the whole spacetime but only its past (or future) tight region.

Let $M = M(\rho)$ be a non-elementary MGHC AdS_3-spacetime, and M_- be the past tight region of M. Analysing the cosmological time of M, and following the same train of thoughts as in the flat case, one gets:

Fact and definitions 7.3. *The past tight region M_- admits a* canonical decomposition. *Each element of this decomposition is:*

1) *either a* thin block *isometric to $\mathbb{R} \times (0,+\infty)$ or $\mathbb{S}^1 \times (0,+\infty)$ endowed with the metric $\sin(t)^2\,\mathrm{dx}^2 - \mathrm{dt}^2$,*
2) *or a* solid block *isometric to $\Sigma \times (0,+\infty)$ endowed with the Lorentzian metric $\sin(t)^2\,\mathrm{ds}^2_{hyp} - \mathrm{dt}^2$ where $(\Sigma, \mathrm{ds}^2_{hyp})$ is the interior of a hyperbolic surface with boundary.*

Every maximal subset of M_- foliated by thin blocks of the canonical decomposition is isometric to $\mathbb{S}^1 \times (-\ell/2, \ell/2) \times (0, +\infty)$ endowed with the metric $\cos(t)^2\,\mathrm{d}\theta^2 + \sin(t)^2\,\mathrm{dx}^2 - \mathrm{dt}^2$, and is called a Misner block *of width ℓ.*

Fact 7.4. *For every $a \in (0, \pi/2)$, there exists a measured lamination $\mathcal{L}_a$ in the surface $C_a = \tau^{-1}(a)$, such that:*

1) *The leaves of $\mathcal{L}_a$ are the intersections of C_a with the thin blocks of the canonical decomposition of M_-.*
2) *The connected components of $C_a \backslash \mathcal{L}_a$ are the intersections of C_a with the solid blocks of the canonical decomposition of M_-.*
3) *The intersection of C_a with a Misner block associated with the canonical decomposition of M_- is a annulus in M_- foliated by closed leaves of $\mathcal{L}_a$.*

Fact 7.5. *The gradient of the cosmological time of M defines a Lipschitz map $N : M_- \to C_\infty^-$ where is a hyperbolic surface. The image under N of the union of the thin blocks of the canonical decomposition of N is a geodesic lamination $\mathcal{L}_\infty^-$ on C_∞^-. The transverse measures of the laminations $\mathcal{L}_a$ induce a transverse measure on $\mathcal{L}_\infty^-$. Moreover:*

1) *The pre-image (for N) of a non-weighted leaf of $\mathcal{L}_\infty^-$ is a thin block of the canonical decomposition of M_- which is not included in a Misner block.*
2) *The pre-image of a connected component of $C \setminus \mathcal{L}_\infty^-$ is a solid block of the canonical decomposition of M_-.*
3) *The pre-image of a leaf of $\mathcal{L}_\infty^-$ which carries a weight $\ell > 0$ is a Misner block of width ℓ associated with the canonical decomposition of M_-.*

Remark 7.6. The hyperbolic surface C_∞^- admits a (non-smooth) isometric embedding in M, as a pleated surface. This pleated surface is actually the topological boundary of the past tight region M_- in M, i.e. the $(\pi/2)$-level set of the cosmological time τ. The pleating lamination is the measured geodesic lamination $\mathcal{L}_\infty^-$ defined above.

Remark 7.7. The future tight region M^+ of M also admits a canonical decomposition associated with a hyperbolic surface C_∞^+ and a measured geodesic lamination $\mathcal{L}_\infty^+$ which are in general different from the hyperbolic surface C_∞^- and the measured geodesic laminations $\mathcal{L}_\infty^-$.

Remark 7.8. Supports of measured geodesic laminations in hyperbolic surfaces have zero Lebesgue measure, but it is not true that the complement in $\mathcal{L}_a$ of the Misner blocks has zero Lebesgue measure. Actually, the area of C_a for the induced metric depends continuously on the measured lamination $\mathcal{L}_\infty^-$. In the case where $\mathcal{L}_a$ is simplicial, i.e. a simple closed geodesic of length L with a weight l, this area is the sum of the area of C_∞ and the area of the inserted annulus, i.e. alL. It follows that in general the area of C_a is $2\pi(2g-2) + al(\mathcal{L}_\infty)$, where $l(\mathcal{L}_\infty)$ is the *length of the measured lamination* (cf. § 7 of [21]). In particular, even if C_a has no Misner block, the lamination $\mathcal{L}_a$ has nonzero Lebesgue measure. This remark applies of course in the flat, de Sitter and anti-de Sitter cases as well.

8. Upper bound of the systole

The purpose of this section is to prove Theorem 3.5. All over the section, (M, g) is a MGHC spacetime of constant curvature Λ. Rescalling the metric if necessary, we may (and we do) assume that $\Lambda = 0$ (flat

case), $+1$ (dS_3 case) or -1 (AdS_3 case). In order to treat simultaneously the three possible cases, we define M_0 to be:

- the whole spacetime M in the flat and in the dS_3 case,
- the past tight region of M in the anti-de Sitter case.

We use the notations defined in the preceeding section. In particular, we denote by $\tau : M_0 \to (0, +\infty)$ the cosmological time of M_0. We denote by N the opposite of the gradient of τ, and by $(\phi^t)_{t\in\mathbb{R}}$ the flow associated to N. We denote by $\tau_{\max}$ the supremum of τ over M_0 (recall that $\tau_{\max} = +\infty$ if $\Lambda = 0$ or $+1$, and $\tau_{\max} = \pi/2$ if $\Lambda = -1$). For every $a \in (0, \tau_{\max})$, we denote by C_a the a-level of τ. As explained in §4.2, there is a Lipschitz parametrization of M_0 obtained simply by pushing the surface C_1 along the flow ϕ^t. More precisely, the map

$$\begin{array}{rccc} \Phi \ : & C_1 \times (0, \tau_{\max}) & \to & M_0 \\ & (x, t) & \mapsto & \phi^{t-1}(x) \end{array}$$

is a global bi-Lipschitz homeomorphism. Recall that the cosmological time τ defines a measured lamination $\mathcal{L}_1$ on the surface C_1. The lamination $\mathcal{L}_1$ induces a decomposition of M_0 as a disjoint union of "buildings blocks", each of which being :

- a solid block $\Phi(U \times (0, \tau_{\max}))$ associated to a connected component U of $C_1 \setminus \mathcal{L}_1$,
- or a thin block $\Phi(\gamma \times (0, \tau_{\max}))$ associated to a non-closed leaf γ of $\mathcal{L}_1$,
- or a Misner block $\Phi(A \times (0, \tau_{\max}))$ associated to a maximal annulus $A \subset C_1$ foliated by closed leaves of $\mathcal{L}_1$ (such a Misner block can itself be decomposed as a disjoint union of thin blocks associated to the closed leaves of $\mathcal{L}_1$ foliating A).

8.1. **The expanding character of the cosmological flow.**

Here, we consider a Lipschitz curve $c : [a, b] \to M_0$ which is spacelike (i.e. the velocity $c'(s)$ is almost everywhere spacelike). We denote by $c_1 : [a, b] \to M_0$ the projection of curve c on the Cauchy surface C_1 along the flow ϕ^t. Observe that

$$c_1(s) = \phi^{1-\tau(c(s))}(c(s)). \tag{1}$$

It follows that c_1 is a Lipschitz curve. In particular, the length of the spacelike curve c_1 is well defined.

Proposition 8.1. *Assume that the curve c is contained in the past of the Cauchy surface C_1 (i.e. the restriction of the cosmological time τ to the curve c is bounded from above by 1). Then:*

(i) *the length of c is smaller than or equal to the length of c_1,*
(ii) *the cosmological time variation $|\tau(c(b)) - \tau(c(a))|$ is smaller than or equal to the length of c_1.*

Proof. For $s \in [a, b]$, let $t(s) = \tau(c(s))$, so that $c(s) = \phi^{t(s)-1}(c_1(s))$. Since the curve c is spacelike, one has, for almost every $s \in [a, b]$,

$$0 \leq |c'(s)|^2$$

On the other hand, the equality $c(s) = \phi^{t(s)}(c_1(s))$ yields almost everywhere:

$$\begin{aligned} |c'(s)|^2 &= \left| t'(s)N(c(s)) + d\left(\phi^{t(s)-1}\right)(c_1(s)).c_1'(s)\right|^2 \\ &= \left| d\left(\phi^{t(s)-1}\right)(c_1(s)).c_1'(s)\right|^2 + t'(s)^2|N(c(s))|^2 \\ &= \left| d\left(\phi^{t(s)-1}\right)(c_1(s)).c_1'(s)\right|^2 - t'(s)^2 \end{aligned}$$

Indeed, $d\left(\phi^{t(s)-1}\right)(c_1(s)).c_1'(s)$ is tangent to the level $C_{t(s)}$, which is orthogonal to $N = -\nabla\tau$, and $|N|^2 = -1$. In particular, for almost every s, one has

$$|c'(s)|^2 \leq \left| d\left(\phi^{t(s)-1}\right)(c_1(s)).c_1'(s)\right|^2. \tag{2}$$

Moreover, the map $\phi^{t-1} : C_1 \to C_t$ is contracting for every $t < 1$ (see [23, Lemma 7.4]) [7]. It follows that, for almost every s, one has

$$\left| d\left(\phi^{t(s)-1}\right)(c_1(s)).c_1'(s)\right|^2 \leq |c_1'(s)|^2. \tag{3}$$

Putting together inequalities (2) and (3), one gets that $|c'(s)| \leq |c_1'(s)|$ for almost every s. Integrating over $[a, b]$, one gets that the length of c is smaller than or equal to the length of c_1.

¿From the above inequalities, one gets, for almost every $s \in [a, b]$

$$t'(s)^2 \leq \left| d\left(\phi^{t(s)-1}\right)(c_1(s))(c_1'(s))\right|^2 \leq |c_1'(s))|^2.$$

Integrating over the interval $[a, b]$, one gets that $|\tau(c(b)) - \tau(c(a))| = |t(b) - t(a)|$ is smaller than or equal to the length of c_1, as wanted. □

8.2. **Key estimates.**

As in the previous paragraph, we consider a spacelike Lipschitz curve $c : [a, b] \to M_0$, and we denote by c_1 the projection of c on the level set C_1 along the flow ϕ^t. There is one particular situation where Proposition 8.1 can be dramatically improved : when the curve c is contained in a solid block or in a thin block.

Proposition 8.2. *Assume that c is contained in a solid block, or in a thin block. Denote by ℓ the length of c_1. Then, one has*

$$\nu_\Lambda^{-1}.\exp(-\ell) \leq \frac{\tau(c(b))}{\tau(c(a))} \leq \nu_\Lambda.\exp(\ell)$$

where ν_Λ is a constant (depending on Λ but not on M), and

$$\textit{length}(c) \leq \mu_\Lambda(\tau(c(a)), \ell)$$

where $\mu_\Lambda : (0, 1) \times (0, +\infty) \to (0, +\infty)$ is a universal function (depending on Λ but not on M) such that: for every $L > 0$, the map $\tau \mapsto \mu_\Lambda(\tau, L)$ is increasing and $\lim_{\tau\to 0} \mu_\Lambda(\tau, L) = 0$.

Remark 8.3. The last estimate should be interpreted as follows: *given the length of the projection of the curve c on the cosmological level set C_1, if the cosmological time takes a small value somewhere on c, then the length of c is small.*

[7] It does not follow directly from the fact that it is obviously true on $C_1 \setminus \mathcal{L}_1$ and in every Misner block in C_1, since the union of non-closed leaves in $\mathcal{L}_1$ may have non-zero Lebesgue measure (cf. Remark 7.8).

Proof of Proposition 8.2: static blocks. Assume that c is contained in a solid block. This block is of the form $M_0(U) := \Phi(U \times (0, \tau_{\max}))$ where U is a connected component of $C_1 \setminus \mathcal{L}_1$. If ds^2_{hyp} denotes the hyperbolic metric, then $(M_0(U), g)$ is locally isometric to $\mathbb{H}^2 \times (0, \tau_{\max})$ equipped with the metric:

- $g_0 = \tau^2\, \mathrm{ds}^2_{hyp} - d\tau^2$ in the flat case (see §5.4),
- $g_0 = \sinh^2(\tau)\, \mathrm{ds}^2_{hyp} - d\tau^2$ in the dS_3 case (see §6.3)
- $g_0 = \sin^2(\tau)\, \mathrm{ds}^2_{hyp} - \tau^2$ in the AdS_3 case (see §7.3)

The flat case. Let $T = \log(\tau)$. Then $(M_0(U), g)$ is locally isometric to $\mathbb{H}^2 \times (0, +\infty)$ equipped with the Lorentzian metric

$$g_0 = \exp(2T)(\mathrm{ds}^2_{hyp} - dT^2) = t^2(\mathrm{ds}^2_{hyp} - dT^2).$$

This metric is conformally equivalent to the metric $k = \mathrm{ds}^2_{hyp} - dT^2$. In particular, the curve c is space-like for the metric k. Therefore, the same arguments (for the metric k) as in the proof of Proposition 8.1 show that T-variation $|T(c(a) - T(c(b))|$ is smaller than or equal to the length of c_1 (note that the length of c_1 for the metric $g_0 = \exp(2T)(\mathrm{ds}^2_{hyp} - dT^2)$ and for the metric $k = \mathrm{ds}^2_{hyp} - dT^2$ are equal, since $T = 0$ on c_1). Since $T = \log(\tau)$, this yields the desired inequality :

$$\exp(-\ell) \leq \frac{\tau(c(b))}{\tau(c(a))} \leq \exp(\ell).$$

For any $s \in [a, b]$, if we replace b by s in the above arguments, one gets $\exp(-\ell) \leq \tau(c(s))/\tau(c(a)) \leq \exp(\ell)$. In particular,

$$\tau(c(s)) \leq \tau(c(a)) \exp(\ell).$$

Now, the same calculations as in the proof of Proposition 8.1 (for the metric k) show that $|c'(s)|^2_k \leq |c'_1(s)|^2_k$ where $|\cdot|_k$ denotes the norm of a vector for the metric k. Using $|c'(s)|^2_k = \tau(c(s))^2.|c'(s)|^2_{g_0}$ and $|c'_1(s)|^2_k = \tau(c_1(s))^2 |c'_1(s)|^2_{g_0} = |c'_1(s)|^2_{g_0}$, we get

$$|c'(s)|_{g_0} \leq \tau(c(s)) |c'_1(s)|_{g_0}.$$

Putting together the two last inequalities above, one obtains

$$\begin{aligned} \text{length}_g(c) = \text{length}_{g_0}(c) &= \int_a^b |c'(s)|_{g_0} ds \\ &\leq \tau(c(a)) \exp(\ell) \int_a^b |c'_1(s)|_{g_0} ds \\ &= \tau(c(a)) \ell \exp(\ell) \end{aligned}$$

We have thus proved Proposition 8.2 in the case $\Lambda = 0$, with $\nu_0 = 1$ and $\mu_0(\tau, l) = l \exp(l)\tau$.

The dS_3 *case.* Let $T = \log(\tanh(\frac{\tau}{2}))$ (note that $\tau \mapsto \log(\tanh(\frac{\tau}{2}))$ is the antiderivative of $\tau \mapsto \frac{1}{\sinh(\tau)}$). Then $(M(U), g)$ is locally isometric to $\mathbb{H}^2 \times]0, T_{\max}[$ endowed with the metric:

$$g_1 = \sinh(\tau)^2 (\mathrm{ds}^2_{hyp} - dT^2)$$

As in the flat case, the T-variation c is therefore at most ℓ. The inequality

$$\exp(-\ell) \leq \frac{\tanh(\tau(c(b))/2)}{\tanh(\tau(c(a))/2)} \leq \exp(\ell)$$

follows. This yields

$$\nu_1^{-1}\exp(-\ell) \leq \frac{\tau(c(b))}{\tau(c(a))} \leq \nu_1 \exp(\ell)$$

where $\nu_1 = \frac{e+1}{e-1}$ is the upper bound of $\frac{\tau}{\tanh(\tau/2)}$ on $[0,1]$. Now, by the same arguments as in the flat case, we get for every $s \in [a,b]$

$$\tanh(\tau(c(s)/2)) \leq \tanh(\tau(c(a)/2))\exp(\ell)$$

and

$$|c'(s)|_{g_1} \leq \sinh(\tau(c(s)))|c_1'(s)|_{g_1} = \frac{2\tanh(\tau(c(s))/2)}{1-\tanh^2(\tau(c(s))/2)}|c_1'(s)|_{g_1}.$$

Putting these two last inequalities together, and integrating over $[a,b]$, we obtain as wanted length$(c) \leq \mu_1(\tau(c(a)),\ell)$ for

$$\mu_1(\tau,l) = \frac{2l\exp(l)\tanh(\tau/2)}{1-\exp(2l)\tanh^2(\tau/2)}.$$

The AdS_3 *case.* Similar the dS_3 case, after replacing $\sinh$ and $\tanh$ by $\tan$ and $\sin$. The constant ν_{-1} is the supremum of $\frac{\tau}{\tan(\tau/2)}$ over $[0,\pi/2]$, and the universal map μ_{-1} is:

$$\mu_{-1}(\tau,l) = \frac{2l\exp(l)\tan(\tau/2)}{1+\exp(2l)\tan^2(\tau/2)}.$$

This completes the proof of Proposition 8.2 is the case where c is contained in a solid block. □

Proof of Proposition 8.2: thin blocks. Now, we assume that c is contained in a thin block $M_0(\gamma) := \Phi(\gamma \times (0,\tau_{\max}))$ where γ is leaf of $\mathcal{L}_1$. The block $(M_0(\gamma), g)$ is isometric to $\mathbb{R}\times(0,\tau_{\max})$ (if γ is not closed) or $\mathbb{S}^1 \times (0,\tau_{\max})$ (if γ is closed) equipped with the metric:

- $g_0 = \tau^2\,\mathrm{d}\theta^2 - d\tau^2$ in the flat case (see §5.4),
- $g_0 = \sinh^2(\tau)\,\mathrm{d}\theta^2 - d\tau^2$ in the dS_3 case (see §6.3)
- $g_0 = \sin^2(\tau)\,\mathrm{d}\theta^2 - d\tau^2$ in the AdS_3 case (see §7.3)

The end of the proof is exactly the same as in the case where c is contained in a solid block, replacing ds^2 by $\mathrm{d}\theta^2$. □

8.3. **Proof of Theorem 3.5.**

We consider a constant $\epsilon > 0$ and a spacelike Cauchy surface Σ in M_0, such that the systole of Σ bigger than or equal to ϵ (i.e. that any closed curve in Σ whose length is less than ϵ is homotopically trivial).

The discussion in §4.2 shows that we can write Σ as a graph over C_1. More precisely, there exists a map $t : C_1 \to (0,\tau_{\max})$ such that

$$\Sigma = \{\phi^{t(x)-1}(x) \mid x \in C_1\}.$$

Observe that this definition of the function t yields

$$\tau\left(\phi^{t(x)-1}(x)\right) = t(x)$$

for every $x \in C_1$. We will prove that the function $t : C_1 \to (0, \tau_{\max})$ is bounded from below by a constant $\alpha_\epsilon > 0$. Using the equality above, this will imply that the restriction of the cosmological time τ to the surface S is bounded from below by α_ϵ. Theorem 3.5 wil follow.

Remark 8.4. Assume that the systole of the cosmological level set C_1 is bigger than or equal to ϵ. Let $\hat{t}(x) = \min(t(x), 1)$. Then the systole of the surface $\widehat{\Sigma} := \{\phi^{\hat{t}(x)-1}(x) \mid x \in C_1\}$ where $\hat{t}(x) = \min(t(x), 1)$ is also greater than or equal to ϵ. And of course, if $\hat{t}$ is bounded from below by some constant α_ϵ, then t is also bounded from below by α_ϵ. Therefore, in the sequel, we can (and we will) assume than that t is bounded from above by 1. In other words, we will assume that the surface S is in the past of the cosmological level set C_1.

8.3.1. *Decomposition of the lamination $\mathcal{L}_1$.* We denote by $\mathcal{L}_1^c$ (resp. $\mathcal{L}_1^{nc}$) the union of the closed (resp. non-closed) leaves of the lamination $\mathcal{L}_1$. Since $\mathcal{L}^1$ is a measured lamination, $\mathcal{L}_1^c$ and $\mathcal{L}_1^{nc}$ are closed. Indeed, if $\mathcal{L}_1^{nc}$ were not closed, there would be a bunch of leaves of $\mathcal{L}_1^{nc}$ spiraling on some closed leaf γ of $\mathcal{L}_1^c$. But then, the measure of any small arc transverse to γ would be infinite. Contradiction.

Therefore, $\mathcal{L}_1^c$ and $\mathcal{L}_1^{nc}$ are sublaminations of $\mathcal{L}_1$. So we have a decomposition of $\mathcal{L}_1$ into two disjoint sublaminations $\mathcal{L}_1^c$ and $\mathcal{L}_1^{nc}$, the first one made only of closed leaves, the second one without closed leaves.

Observe that the support of $\mathcal{L}_1^c$ is exactly the intersection of the surface C_1 with the Misner blocks of M_0. In particular, this support of $\mathcal{L}_1^c$ is a finite union of disjoint annuli in C_1, each of these annuli being foliated by leaves of $\mathcal{L}_1^c$.

8.3.2. *Estimates of t on $\mathcal{L}_1^c$.*

Lemma 8.5. *There exists a constant $\eta^c > 0$ such that $t(x) \geq \eta^c$ for every x in the support of the lamination $\mathcal{L}_1^c$.*

Proof. Let us first recall that the support of the lamination $\mathcal{L}_1^c$ is exactly the intersection of the surface C_1 with the Misner blocks of M_0. The intersection of C_1 with a Misner block is an annulus foliated by leaves of $\mathcal{L}_1^c$, all of which have the same length. There are only finitely many Misner blocks in M_0. Therefore there are only finitely many possible lengths for the leaves of $\mathcal{L}_1^c$. We denote these lengths by $\ell_1, \ldots, \ell_k$.

Now, let $x \in C_1$ be a point in the support of the lamination $\mathcal{L}_1^c$. Denote by $\gamma_{1,x}$ the leaf of $\mathcal{L}_1^c$ containing x, and by γ_x be the projection of $\gamma_{1,x}$ on the surface Σ along the flow ϕ^t (of course, it follows that $\gamma_{1,x}$ is the projection of γ_x on the surface C_1). From Proposition 8.2, we get

$$\text{length}(\gamma_x) \leq \mu_\Lambda(t(x), \ell_i).$$

where ℓ_i is the length of the closed leaf $\gamma_{1,x}$. This closed leaf $\gamma_{1,x}$ is not homotopically trivial in C_1 (since the image of $\gamma_{1,x}$ under the Gauss map $N : C_1 \to C_\infty$ is a closed leaf of the geodesic lamination $\mathcal{L}_\infty$, i.e. a closed geodesic in the hyperbolic surface C_∞). Hence the closed curve γ_x is not homotopically trivial in Σ. So our assumption on the systole of Σ and the above estimate on the length of γ_x implies

$$\mu_\Lambda(t(x), \ell_i) \geq \epsilon.$$

It remains to choose a constant $\eta^c > 0$ such that $\mu_\Lambda(\tau, \ell_i) < \epsilon$ for every $\tau \leq \eta$ and every $i \in \{1, \dots, k\}$. The above inequality implies that $t(x) \geq \eta^c$ for every x in the support of $\mathcal{L}_1^c$. This completes the proof of Lemma 8.5. □

8.3.3. *Estimates of t on $\mathcal{L}_1^{nc}$.* We will prove the following:

Lemma 8.6. *There exists a constant $\eta^{nc} > 0$ such that $t(x) \geq \eta^{nc}$ for every $x \in \mathcal{L}_1^{nc}$.*

Proof. For every $x \in \mathcal{L}_1$, we denote by $\mathcal{L}_1(x)$ the leaf of $\mathcal{L}_1$ containing x. The idea of the proof is to construct, for every $x \in \mathcal{L}_1^{nc}$, a closed homotopicaly non-trivial loop containing x, by closing long segment of the leaf of $\mathcal{L}_1(x)$. Recall that a lamination $\mathcal{L}$ is *minimal* if every leaf of it is dense (in its support). We will use the following fact (see for example Proposition (iii) in [42]): *the lamination $\mathcal{L}_1^{nc}$ is a union of finitely many minimal sublaminations $\mathcal{L}_1^1, \dots, \mathcal{L}_1^k$.*

Let $i \in \{1, \dots, k\}$. Going to a double cover if necessary, we assume that the lamination $\mathcal{L}_1^i$ is orientable. Since $\mathcal{L}_1^i$ is minimal (and compact), there exists $\ell_i \in \mathbb{R}$ such that every $x \in \mathcal{L}_1^i$ is $(\ell_i, \frac{\epsilon}{2})$-*recurrent*,: there is a segment $\alpha_{1,x}$ starting at x in the leaf $\mathcal{L}_1(x)$, such that the length of $\alpha_{1,x}$ is at most ℓ_i, and such that the ends of $\alpha_{1,x}$ can be joined by a small arc $\beta_{1,x}$ transverse[8] to $\mathcal{L}_1$ of length at most $\frac{\epsilon}{2}$.

For any $x \in \mathcal{L}_1^i$, we consider the concatenation of the arcs $\alpha_{1,x}$ and $\beta_{1,x}$. This is a closed loop $\gamma_{1,x}$. This loop $\gamma_{1,x}$ can be perturbed to get a loop transverse to the lamination $\mathcal{L}_1$. So, by Poincaré-Bendixon Theorem, $\gamma_{1,x}$ is homotopically non trivial in C_1. We denote by γ_x the projection of the loop $\gamma_{1,x}$ on Σ along the flow ϕ^t. Of course, γ_x is the concatenation of the projections α_x and β_x of the arcs $\alpha_{1,x}$ and $\beta_{1,x}$.

Since $\alpha_{1,x}$ is contained in a thin geodesic cone, we can apply Proposition 8.2 to this arc. We get that the length of α_x is bounded from above by $\mu_\Lambda(t(x), \ell_x)$, where ℓ_x is the length of α_x. On the other hand, according to the Proposition 8.1 and Remark 8.4, the length of the arc β_x is less than the length of the arc $\beta_{1,x}$. In particular, it is less than $\epsilon/2$. So we get an upper bound for the length of the loop γ_x:

$$\text{length}(\gamma_x) \leq \sup_{\ell \leq \ell_i} \mu_\Lambda(t(x), \ell) + \epsilon/2.$$

Since γ_x is a non-homotopically trivial loop in Σ, and since the systole is bounded from below by ϵ, we get

$$\epsilon \leq \epsilon/2 + \sup_{\ell \leq \ell_i} \mu_\Lambda(t(x), \ell),$$

which leads to

$$\epsilon/2 \leq \sup_{\ell \leq \ell_i} \mu_\Lambda(t(x), \ell).$$

Now, let η^c be a positive number such that, if $\tau \leq \eta^c$, then

$$\sup_{\ell \leq \ell_i} \mu_\Lambda(\tau, \ell) < \epsilon/2$$

for every $i \in \{1, \dots, k\}$. The inequalities above imply that $t(x) \geq \eta^c$ for every $x \in \mathcal{L}_1^{nc}$. □

[8]By such, we mean that $\beta_{1,x}$ always intersects $\mathcal{L}_1$ with the same orientation.

8.3.4. *Estimates on $C_1 \setminus \mathcal{L}_1$.* Let δ be the diameter of the Cauchy surface C_1 (for the riemanian metric induced by g on C_1).

Lemma 8.7. *Let $\eta = \min(\eta^c, \eta^{nc})$, where η^c, η^{nc} are the positive numbers defined in Lemma 8.5 and 8.6. Then*

$$t(x) \geq \nu_\lambda \exp(\delta)\eta \quad \textit{for every} \quad x \in C_1 \setminus \mathcal{L}_1.$$

Proof. Any point $x \in C_1 \setminus \mathcal{L}_1$ can be joined to a point $y \in \mathcal{L}_1$ by a path c_1 of length at most δ. We can assume the whole of c_1 but y is contained in $C_1 \setminus \mathcal{L}_1$, that is the curve c_1 minus y is contained in a solid block. The first inequality in Proposition 8.2 yields:

$$t(x) \geq \nu_\Lambda \exp(\delta) t(y).$$

Since $y \in \mathcal{L}_1 = \mathcal{L}_1^c \cup \mathcal{L}_1^{nc}$, Lemmas 8.5 and 8.6 yield

$$t(y) \leq \eta.$$

Putting together the two inequalities above, we get $t(x) \geq \nu_\lambda \exp(\delta)\eta$, as wanted. □

8.3.5. *Conclusion.* Using Lemmas 8.5, 8.6 and 8.7 together with the decomposition

$$C_1 := \mathcal{L}_1^c \cup \mathcal{L}_1^{nc} \cup (C_1 \setminus \mathcal{L}_1),$$

we get a uniform positive lower bound for t on C_1. Since $\phi^{t(x)-1}(x)$ ranges over Σ when x ranges over C_1, and since $\tau\left(\phi^{t(x)-1}(x)\right) = t(x)$, this yields a uniform positive lower bound for the cosmological time τ on Σ. This completes the proof of Theorem 3.5. □

9. Sequences of convex Cauchy surfaces

The aim of this section is to prove Theorem 3.6 and Corollary 3.8.

Let us recall what are the objects and the hypotheses. We consider a 3-dimensional non-elementary MGHC spacetime (M, g) of constant curvature, and a sequence $(\Sigma_n)_{(n\in\mathbb{N})}$ of (locally) convex generalized Cauchy surfaces in M (see §4.2 and 4.3). We assume that this sequence is *decreasing* (i.e. $I^+(\Sigma_{n+1}) \supset I^+(\Sigma_n)$ for every n), and we assume that the set $\Omega = \bigcup_{n\in\mathbb{N}} I^+(\Sigma_n)$ is not the whole M.

We want to prove that every support plane of Ω is spacelike (Theorem 3.6), and that the Σ_n's are uniformly spacelike (Corollary 3.8; see definition 3.7).

Observe that Ω is a future domain, that is, for every x in Ω, we have $I^+(x) \subset \Omega$. It follows that the boundary $\Sigma_\infty = \partial\Omega$ is a closed (but non-compact in general) edgeless achronal subset (see [50, §14, Corollary 27]), hence a topological surface.

Lemma 9.1. *The set Σ_∞ is the set of limits in M of sequences $(p_n)_{n\in\mathbb{N}}$ with $p_n \in \Sigma_n$.*

Proof. Let x be an element of Σ_∞, and let $c :]-\infty, +\infty[\to M$ be an inextendible future timelike curve with $c(0) = p$. Since every Σ_n is a Cauchy surface, for every n there is a unique real number t_n such that $c(t_n)$ belongs to Σ_n. Then $c(]t_n, +\infty[)$ is contained in $I^+(\Sigma_n)$. Since p does not belong to Ω_n we have $t_n \geq 0$. Since the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is decreasing, the sequence $(t_n)_{n\in\mathbb{N}}$ is also decreasing and admits a limit $\bar{t}$. Then $c(\bar{t})$ belongs to Σ_∞: we obtain $\bar{t} = 0$ since Σ_∞ is achronal.

Conversely, if p is a limit of a sequence $(p_n)_{n\in\mathbb{N}}$ with $p_n \in \Sigma_n$, it belongs to the closure of Ω. If p belongs to Ω, then it belongs to some Ω_{n_0}. The same then would be true for all the p_n with n sufficiently big, but this is impossible for $n > n_0$. Hence, p belongs to the boundary $\Sigma_\infty = \partial\Omega$. □

Let $p \in \Sigma_\infty$, and let $\varphi : U \to V$ be a convex chart centered at p. By our discussion in §4.3, the surfaces $\varphi(\Sigma_n \cap U)$ are graphs of convex functions. Using Lemma 9.1, it follows that there is a convex map $(x, y) \to z = f_\infty(x, y)$ such that $\varphi(U \cap \Sigma_\infty)$ is the graph of f_∞. In other words, the topological surface Σ_∞ is convex.

Proof of Theorem 3.6. Since Σ_∞ is achronal, it is quite immediate that a support plane of Σ_∞ cannot contain a timelike curve. Hence, we must show that no support plane to Σ_∞ at p is lightlike.

Assume by contradiction that Σ_∞ admits a lightlike hyperplane H at p in a convex chart U centered at p. On one hand, $\Sigma_\infty \cap U$ is contained in $J^+_U(H)$. On the other hand, $\Sigma_\infty \cap U$ is contained in the causal past of the future $I^+_U(p)$. But $H \cap J^+_U(p) = H^\perp$ is a null segment, and hence this null segment is contained in Σ_∞.

More formally, we have proved that there is a future-oriented null geodesic segment $c : [0, 1] \to M$ with $c(0) = p$ such that every $c(t)$ belongs to Σ_∞. Let $\hat{c} : [0, T[\to M$ be the inextendible geodesic ray extending c in the future. Let T' be the $\sup$ of times t such that $\hat{c}(t)$ belongs to Σ_∞. Observe that $T' > 0$. Assume $T' < T$. Since Σ_∞ is closed, $q = \hat{c}(T')$ belongs to Σ_∞.

Since there is a null geodesic segment $[p, q]$ contained in Σ_∞, there is only one support plane to Σ_∞ at q: the null plane containing $\hat{c}(t)$ for $0 < t$ near T'. The argument above then implies that $\hat{c}(t)$ belongs to Σ_∞ for $t > T'$ and close enough to T'. This contradicts the definition of T'. Therefore, $T' = T$, that is, the whole future oriented null ray $\hat{c}$ is contained in Σ_∞.

Now, for every n, the point $p = c(0)$ is in the past of Σ_n, and Σ_n is a Cauchy surface, thus, the inextendible geodesic ray $\hat{c}$ intersects $I^+(\Sigma_n)$. Since $I^+(\Sigma_n) \subset I^+(\Sigma_\infty)$, it follows that the geodesic ray $\hat{c}$ intersects $I^+(\Sigma_\infty)$. This contradicts the fact that Σ_∞ is achronal and $\hat{c}(t)$ is contained in Σ_∞. □

Proof of Corollary 3.8. Assume that the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is not uniformly spacelike. Then one can find a sequence of points $(x_n)_{n\in\mathbb{N}}$, with $x_n \in \Sigma_n$, such that:

- extracting a subsequence if necessary, the sequence $(x_n)_{n\in\mathbb{N}}$ converges towards a point $x \in M$,
- if P_n is the unique support plane of Σ_n at x, then, extracting a subsequence if necessary, the sequence $(x_n, P_n)_{n\in\mathbb{N}}$ converges towards a couple (x, P) where P is a null plane at x.

By Lemma 9.1, the point x is on the topological surface Σ. For every n, since P_n is the support plane of Σ_n at x_n, the surface Σ_n is in the future of the totally geodesic plane P_n. Using again Lemma 9.1, it follows that Σ is in the future of the null plane P. Since $x \in \Sigma \cap P$, this implies that P is a support plane of Σ at x, contradicting Theorem 3.6.

As a consequence, the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is uniformly spacelike. □

10. Sequences of Cauchy surfaces of constant K-curvature

The present section is devoted to the proof of Theorem 3.9. Recall that we are considering a 3-dimensional non-elementary MGHC spacetime (M, g) of constant curvature Λ, and a decreasing sequence of Cauchy surfaces $(\Sigma_n)_{n\in\mathbb{N}}$ in M, such that, for every n, the surface Σ_n is locally strictly convex and has a constant K-curvature $\kappa_n < -\Lambda$ (note that this condition is automatically satisfied if $\Lambda \leq 0$ since locally strictly convex surfaces have negative K-curvature).

We will see later that $(\kappa_n)_{n\in\mathbb{N}}$ is automatically a decreasing sequence of real numbers (see Remark 12.3). Anyway, we do not really need this here: we only need to assume that

$$\kappa_n \to \kappa \in [-\infty, \min(-\Lambda, 0)) \quad \text{when} \quad n \to \infty.$$

If $\kappa > -\infty$, we have to prove that the sequence Σ_n converges to a smooth surface Σ_∞ of constant K-curvature κ. If $\kappa = -\infty$, we have to prove that $M = \bigcup_{n\in\mathbb{N}} I^+(\Sigma_n)$.

10.1. The $\kappa > -\infty$ case.

The main idea of the proof is to apply a result of Schlenker which describes the asymptotic behaviour of a sequence of immersions of a disc in a Lorentzian space of constant curvature, when the sequence of immersions itself does not converge, although the sequence of metrics obtained by the pulling-back the Lorentzian metric of constant curvature by the immersions does converge (see Theorem 10.5 below for a precise statement).

We consider an abstract compact surface Σ with the same genus as the Σ_n's. For every n, we denote by g_n the Riemannian metric induced on the Cauchy surface Σ_n by the Lorentzian metric g. We can write

$$g_n = |\kappa_n|^{\frac{1}{2}}.\hat{g}_n$$

where $\hat{g}_n$ is a metric on Σ_n with constant K-curvature -1. We denote by $[\hat{g}_n]$ the class of the metric $\hat{g}_n$ in the Teichmüller space Teich(S).

Lemma 10.1. *The sequence $([\hat{g}_n])_{n\in\mathbb{N}}$ is relatively compact in the Teichmüller space.*

Proof. Since $(\Sigma_n)_{n\in\mathbb{N}}$ is a decreasing sequence of Cauchy surfaces, all the Σ_n's are in the past of Σ_0. Therefore all the Σ_n's are in the past of a level set of the cosmological time of M, say the level set C_a. By Proposition 8.1, this implies that the sequence of the diameters of the (Σ_n, g_n)'s is bounded (by the diameter of the level set $\mathcal{C}_a$). Since $\kappa \neq 0$, deleting a finite number of elements of the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ if necessary, we can assume that the κ_n's are bounded away from 0. It follows that the sequence of the diameters of the $(\Sigma_n, \hat{g}_n)$'s is also bounded. By the collar neighbourhood lemma, this implies that the sequence of the systoles of the $(\Sigma_n, \hat{g}_n)$'s is bounded away from 0. It follows that the sequence $(\Sigma_n, [\hat{g}_n])_{n\in\mathbb{N}}$ is relatively compact in Teich(S). □

Lemma 10.1 can be reformulated as follows:

Corollary 10.2. *For every $n \in \mathbb{N}$, one can find a smooth embedding $f_n : \Sigma \hookrightarrow M$ such that $f_n(\Sigma) = \Sigma_n$ and such that the sequence $(f_n^*\hat{g}_n)_{n\in\mathbb{N}}$ is relatively compact in the space of the Riemannian metrics on Σ endowed with C^∞ topology.* □

Since the sequence $(\kappa_n)_{n\in\mathbb{N}}$ converges, the sequence $(f_n^* g_n)_{n\in\mathbb{N}}$ is also relatively compact in the space of the Riemannian metrics on S endowed with C^∞ topology. Extracting a subsequence if necessary, we get a sequence of embeddings (that we still denote by $(f_n)_{n\in\mathbb{N}}$) of Σ in M such that the sequence of metrics $(f_n^* g_n)_{n\in\mathbb{N}}$ converges in the C^∞ topology towards a metric g_∞ on Σ.

Now, let us consider a point x in Σ. For every n, let $y_n := f_n(x)$.

Lemma 10.3. *The sequence of points $(y_n)_{n\in\mathbb{N}} = (f_n(x))_{n\in\mathbb{N}}$ has a convergent subsequence.*

Proof. For every $n \in \mathbb{N}$, the point y_n belongs to the surface Σ_n. So, it is enough to prove that all the Σ_n's are contained in a compact region of M.

On the one hand, since the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is decreasing, all the Σ_n's are contained in the past of the Cauchy surface Σ_0. On the other hand, we have seen that the sequence of the systoles of the $(\Sigma_n, \hat{g}_n)$'s is bounded away from 0 (proof of Lemma 10.1). Since the sequence $(\kappa_n)_{n\in\mathbb{N}}$ is bounded away from 0 (deleting a finite number of surfaces Σ_n if necessary), it follows that the sequence of the systoles of the (Σ_n, g_n)'s is also bounded away from 0 . By Theorem 3.5, this implies that all the Σ_n's are contained in the future of some level set C_a of the cosmological time of M. As a consequence, all the Σ_n's are contained in the compact subset $I^+(C_a) \cap I^-(\Sigma_0)$ of M. □

Lemma 10.4. *The sequence of 1-jets $(j^1 f_n(x))_{n\in\mathbb{N}}$ has a convergent subsequence.*

Proof. By Lemma 10.3, up to extracting a subsequence, we may assume that the sequence of points $(y_n)_{n\in\mathbb{N}}$ converges towards a point $y \in M$. For every $n \in \mathbb{N}$, the positive definite quadratic form $q_n := (f_n^* g_n)_{|T_x\Sigma}$ is the pull back by $df_n(x)$ of the positive definite quadratic form $s_n := g_{n|T_{y_n}\Sigma_n}$. On the one hand, the sequence $(q_n)_{n\in\mathbb{N}}$ converges towards the positive definite quadratic form q on $g_{\infty|T_x\Sigma}$. On the other hand, Corollary 3.8 implies that, up to extracting a subsequence, the sequence of subspaces $T_{y_n}\Sigma_n$ converges towards a spacelike plane P in T_yM, and thus the sequence of quadratic forms $(s_n)_{n\in\mathbb{N}}$ converges towards the positive definite quadratic form $s = g_{|P}$. Since the space of linear maps leaving invariant a positive definite quadratic form is compact, it follows that, up to extracting a subsequence, the sequence of linear maps $df_n(x)$ converges towards a linear map $d : T_x\Sigma \to P$ such that q is the pull-back of s by d. □

Schlenker's result only concerns embeddings of discs in a simply connected Lorentzian space. So we need to lift everything to the universal covering. The universal covering of Σ is the 2-disc $\mathbb{D}$. The universal covering $\widetilde{M}$ of M is a convex open domain in $X = \mathrm{Min}_3$, dS_3 or $\widetilde{\mathrm{AdS}_3}$. Corollary 10.2 and Lemma 10.4 implies that, for every $n \in \mathbb{N}$, we can find a lift $\widetilde{\Sigma}_n$ of Σ_n in X, a lift $\widetilde{f}_n : \mathbb{D} \hookrightarrow X$ of f_n such that:

- the sequence of metrics $(\widetilde{f}_n^* \widetilde{g}_n)_{n\in\mathbb{N}}$ converges towards a Riemannian metric $\widetilde{g}_\infty$ on $\mathbb{D}$ in the C^∞ topology (where $\widetilde{g}_n$ is the Riemannian metric induced on $\widetilde{\Sigma}_n$ by the Lorentzian metric of X),
- for every $\widetilde{x} \in \mathbb{D}$ the sequence of 1-jets $(j^1 \widetilde{f}_n(\widetilde{x}))_{n\in\mathbb{N}}$ has a convergent subsequence.

Now, we argue by contradiction: *we assume that the sequence of surfaces Σ_n does not converge in the C^∞ topology towards a smooth surface S of constant curvature κ.* It follows in particular that the sequence of embeddings $(f_n)_{n\in\mathbb{N}}$ does not converge in the C^∞ topology. *A fortiori*, the sequence of embeddings $(\widetilde{f}_n)_{n\in\mathbb{N}}$ does not converge in the C^∞ topology. So we are under the assumption of Schlenkers result:

Theorem 10.5 (see [52], Théorème 5.6)**.** *Let $(\widetilde{f}_n)_n \in \mathbb{N} : \mathbb{D} \to X$ be a sequence of uniformly elliptic immersions*[9] *of a disc $\mathbb{D}$ in a simply connected Lorentzian spacetime of constant curvature $(X, \widetilde{g})$. On the one hand, assume that the metrics $\widetilde{f}_n^* \widetilde{g}$ converges in the C^∞ topology towards a Riemannian metric $\widetilde{g}_\infty$ on $\mathbb{D}$, and that there exists a point $x \in \mathbb{D}$ such that the sequence of 1-jets $(j^1 \widetilde{f}_n(x))_{n\in\mathbb{N}}$ converges. On the other hand, assume that the sequence $(\widetilde{f}_n)_{n\in\mathbb{N}}$ does not converge in the C^∞ topology in a neighbourhood of x. Then there exists a maximal geodesic γ of $(\mathbb{D}, \widetilde{g}_\infty)$ and a geodesic arc Γ of $(X, \widetilde{g})$ such that the sequence $(\widetilde{f}_{n|\gamma})_{n\in\mathbb{N}}$ converges towards an isometry $f_\infty : \gamma \to \Gamma$.* □

By this theorem, there exists a maximal geodesic γ of $(\mathbb{D}, \widetilde{g}_\infty)$ and a geodesic segment Γ in X, such that $f_{n|\gamma}$ converges towards an isometry from $f_\infty : \gamma \to \Gamma$. Since $\widetilde{g}_\infty$ is obtained by lifting a Riemannian metric on a compact surface, it is geodesically complete. In particular, the geodesic γ has infinite length. And, as a consequence, the geodesic arc Γ also has infinite length. But since $f_n(\Sigma) \subset \widetilde{M}$ for every n, the geodesic arc Γ must be contained in the closure of $\widetilde{M}$ in X. Morever, in the AdS_3-case, Γ must be contained in the closure of the complement of the lift of the convex core of M (since all the Σ_n's are contained in the complement of the convex core of M). This contradicts the following proposition:

Proposition 10.6. *Let (M, g) be a non-elementary 3-dimensional MGHC spacetime with constant curvature Λ. If $\Lambda \geq 0$ (flat case or locally de Sitter case), then there is no complete spacelike geodesic in M. If $\Lambda < 0$ (locally anti-de Sitter case), then every complete spacelike geodesic of M is contained in the convex core of M.*

Proof. In the flat case, we know that the spacetime is a quotient $\bar{\Gamma}\backslash E$ where E is a *proper* convex domain, i.e. contains no complete affine line (see [47]): the proposition follows.

In the locally anti-de Sitter case, we observe that any complete spacelike geodesic c in M lifts to a complete spacelike geodesic in $E(\Lambda_\rho^+) \subset \mathrm{AdS}_3$. Such a geodesic admits two extremities in $\mathcal{Q} = \partial\mathbb{ADS}_3$. But these two extremities must belong to the closure of $E(\Lambda_\rho^+)$ in $\mathbb{ADS}_3$. The intersection between this closure and $\mathcal{Q}$ is reduced to Λ_ρ^+: hence, the spacelike geodesic under consideration has extremities in Λ_ρ^+. It follows that c is contained in the convex core.

The remaining case is the locally de Sitter case. Let $c : \mathbb{R} \to M$ be the spacelike geodesic, parametrized by arc-length, and let $\tilde{c} : \mathbb{R} \to \widetilde{M}$ be its lifting in the universal covering. The composition $\tilde{\zeta} = \mathcal{D} \circ \tilde{c}$ with the developing map is a complete spacelike geodesic in dS_3. In particular, the image of $\tilde{\zeta}$ is contained in a 2-plane P of $\mathrm{dS}_3 \subset \mathbb{R}^{1,3}$.

Reversing the time-orientation if necessary, we can assume that (M, g) is future complete. Then $(\widetilde{M}, g)$ is isometric to the spacetime $\mathcal{B}_0(S)$ associated to a simply conncted Möbius surface S (naturally homeomorphic to the universal cover of the compact Σ) (see §6.2). It follows that the future of the image of $\tilde{c}$ in $\widetilde{M}$ is isometric to the universal covering $\widetilde{\Omega}(\Delta)$ - here the geodesic Δ of $\mathbb{H}^3$ is the intersection between $\mathbb{H}^3$ and the orthogonal $P^\perp$ in $\mathbb{R}^{1,3}$.

Now we observe that $\widetilde{\Omega}(\Delta)$ can also be described as $\mathcal{B}_0(\mathbb{CP}^1 - \{0, +\infty\})$ if $0, \infty$ denotes the extremities of Δ. In other words, we have an embedding $F : \mathcal{B}_0(\mathbb{C}^*) \to \mathcal{B}_0(S)$. According to Remark 6.2, we

[9] An immersion f of a $(n-1)$-dimensional manifold N in a n-dimensional Lorentzian space X of constant curvature Λ such that $f(N)$ is spacelike and has constant K-curvature $\kappa < -\Lambda$ is a typical example of *uniformly elliptic immersion*.

have a map $f : \mathbb{C}^* \to S$, inducing a holomorphic map $\bar{f} : \mathbb{C}^* \to \Sigma$. But since Σ has genus ≥ 2 such an holomorphic map must be constant: this is a contradiction. $\square$

The contradiction we have obtained shows that our assumption was absurd. Therefore, the sequence of Cauchy surfaces $(\Sigma_n)_{n\in\mathbb{N}}$ towards a surface Σ_∞ in the C^∞ topology. Clearly, this implies that Σ_∞ is a convex Cauchy surface with constant K-curvature $\kappa = \lim \kappa_n$. This completes the proof of Theorem 3.9 in the case where $\kappa_n \to \kappa > -\infty$.

10.2. **The $\kappa = -\infty$ case.**

We are left to consider the case where $\kappa_n \to -\infty$. In this case, we want to prove that the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is covering, i.e. that $\bigcup_n I^+(\Sigma_n) = M$.

We argue by contradiction: we suppose that the convex domain $\Omega := \bigcup_n I^+(\Sigma_n)$ is not the whole spacetime M. Then $\Sigma_\infty := \partial\Omega$ is non-empty. As discussed in the beginning of §9, Σ_∞ is a topological surface. Consider an open set with compact closure N in Σ_∞, and choose a local time function on a neighbourhood of N. This local time function allows to decompose some neighbourhood (with compact closure) W of N as a product $W = N \times I$ where I is an intervall in $\mathbb{R}$ and $\{x\} \times I$ is timelike for every $x \in N$. Using Lemma 9.1, we see that this allows to write, for every n large enough, the surface $\Sigma_n \cap W$ as a graph over N. By Corollary 3.8, this sequence of graphs is uniformly spacelike. Obviously, this implies that the area of the local surface $\Sigma_n \cap W$ does not tend to 0 when $n \to \infty$. In particular, the area of the Cauchy surface Σ_n does not tend to 0 when $n \to \infty$. This contradicts the Gauss-Bonnet formula: up to a multiplicative constant (depending on the genus of the Cauchy surfaces of M) the area of Σ_n equals $-\frac{1}{\Lambda+\kappa_n}$.

This completes the proof of Theorem 3.9.

11. Construction of barriers

This section is essentially devoted to the proof of Theorem 3.4 and Proposition 3.10. The proof of Theorem 3.4 can be divided into two steps:

- first, we show that every 3-dimensional future complete non-elementary MGHC spacetime with constant curvature $\Lambda \geq 0$ contains a convex Cauchy surface Σ with K-curvature strictly bounded from above by $-\Lambda$,
- then we push Σ along the geodesics orthogonal to Σ in order to get convex Cauchy surfaces whose K-curvature is arbitrarily close to $-\Lambda$.

Proposition 3.10 will be an easy consequence of our estimates on the behaviour of the principal curvatures when one pushes a surface along the orthogonal geodesics.

It is important at this stage to recall our conventions. Let Σ be a spacelike surface in (M, g). Recall that the second fundamental form of Σ is defined by $II(X, Y) = -\langle \nabla_X n, Y\rangle$ where n is the future oriented unit normal of Σ. The principal curvatures λ_1, λ_2 of Σ are the eigenvalues of the second fundamental form, and the K-curvature of Σ equals $-\lambda_1\lambda_2$. For example, with these conventions, if M is the Minkowski space $\mathbb{R}^{1,2}$ endowed with the lorentzian metric $-dt^2 + dx^2 + dy^2$, the surface $\{-t^2 + x^2 + y^2 = -c^2, t > 0\}$ has principal curvatures $\lambda = \mu = -c$.

11.1. **Existence of a Cauchy surface with controlled curvature.**

11.1.1. *The flat case.* We will prove the following result:

Proposition 11.1. *Every non-elementary* 3*-dimensional future complete flat MGHC spacetime contains a locally strictly convex Cauchy surface (in particular, a Cauchy surface with negative K-curvature).*

Equivalently, every non-elementary 3-dimensional flat MGHC spacetime contains a Cauchy surface with negative principal curvatures. In order to prove this Proposition 11.1, we will use the existence of constant mean curvature surfaces in flat MGHC spacetimes and the following proposition which is essentially due to A. Treibergs:

Proposition 11.2. *Let* (M, g) *be a non-elementary MGHC flat spacetime. Then, every constant mean curvature Cauchy surface in* M *has negative principal curvatures.*

Proof. Any Cauchy surface with constant mean curvature Σ in M lifts as a complete spacelike surface with constant mean curvature $S \simeq \widetilde{\Sigma}$ in the Minkowski space. It is shown in [56] (see also [25]) such a surface S is either *strictly convex* (i.e. has negative principal curvatures), or splits as a direct product of a curve by an affine line. The splitting case is excluded in our context, since we have assumed that our spacetime is non-elementary. The proposition follows. □

Proof of Proposition 11.1. In [1] and [3], it was shown that every future complete flat spacetime contains a Cauchy surface S with constant mean curvature -1 (see [1, 3]). Together with Proposition 11.2, this proves Proposition 11.1. □

11.1.2. *Transferring properties of CMC hypersurfaces from flat to* dS_3 *and* AdS_3 *spacetimes.* Now, we would like to get an analog of Proposition 11.1 in dS_3 and AdS_3-spacetimes. For this purpose, we will use a correspondance between CMC hypersurfaces in flat spacetimes, CMC hypersurface in dS spacetimes and CMC hypersurfaces in AdS spacetimes.

Proposition 11.3 (See e.g. [7])**.** *Let* Σ *be a compact surface,* η *be a Riemannian metric on* Σ *with scalar curvature* R_η*, and* h *be a quadratic differential on* Σ*. Let* Λ *and* H *be two real numbers. There exists an isometric embedding of* (Σ, η) *as a spacelike Cauchy surface with constant mean curvature* H *having* h *is the second fundamental form in* some *MGHC spacetime* (M, g) *with constant curvature* Λ *if and only if the two following conditions are satisfied:*

(1) *the quadratic differential* $h_0 = h - H.\eta$ *(i.e. the trace free part of* h*) is holomorphic;*
(2) *the Gauss equation* $R_\eta + \det_\eta(h_0) = \Lambda - H^2$ *is satisfied.*

Remark 11.4. If (Σ, η, h) satisfies conditions (1) and (2), then:

- one can take as a spacetime (M, g) the Cauchy development of the solution of Einstein equation with a cosmological constant Λ, in a CMC gauge, with initial Cauchy data (Σ, η, h),
- (S, η, h) is called a *CMC* (H, Λ)*-initial Cauchy data*.

The following immediate corollary of Proposition 11.3 will allow us to transfer properties of CMC hypersurfaces in flat spacetimes into properties of CMC hypersurfaces in dS_3 and AdS_3 spacetimes.

Corollary 11.5. *A CMC (H, Λ)-initial data $(\Sigma, \eta, h = h_0 + H\eta)$ gives rise to a CMC (H', Λ')-initial data of the form $(\Sigma, \eta, h' = h_0 + H'\eta)$ provided that*

$$\Lambda' - H'^2 = \Lambda - H^2.$$

Remark 11.6. Assume that $(\Sigma, \eta, h = h_0 + H\eta)$ is a CMC (H, Λ)-initial data and $(\Sigma, \eta, h' = h_0 + H'\eta)$ is a CMC (H', Λ')-initial data. Denote by λ, μ (resp. Λ', μ') the principal curvatures of Σ seen as a CMC (H, Λ)-initial Cauchy data (resp. as a CMC (H', Λ')-initial Cauchy data). Then

$$\lambda' = \lambda + (H' - H) \quad \text{and} \quad \mu' = \mu + (H' - H).$$

11.1.3. *The* dS_3 *case.* We will prove the following result:

Proposition 11.7. *Every non-elementary 3-dimensional future complete MGHC spacetime with constant curvature $\Lambda = +1$ contains a convex Cauchy surface with K-curvature strictly bounded from above by $-\Lambda$.*

Proof. Consider a non-elementary 3-dimensional future complete MGHC spacetime (M, g) with constant curvature $+1$. It was proved in [4] that (M, g) admits a Cauchy hypersurface Σ with constant mean curvature $-\sqrt{2}$. Denote by η and h the first and the second fundamental forms of Σ. Then $(\Sigma, \eta, h = h_0 - \sqrt{2}\eta)$ is a CMC $(-\sqrt{2}, 1)$-initial Cauchy data. So, by Corollary 11.5, $(\Sigma, \eta, h' = h_0 - \eta)$ is a CMC $(-1, 0)$-initial Cauchy data (i.e. (Σ, η) admits an isometric embedding in a flat MGHC spacetime with constant mean curvature -1 and second fundamental form h'). Denote by λ and μ (resp. λ' and μ') the principal curvatures of Σ seen as a CMC $(-\sqrt{2}, +1)$ initial Cauchy data (resp. as a CMC $(-1, 0)$ initial Cauchy data). According to Proposition 11.2, the principal curvatures λ' and μ' are negative. Using Remark 11.6, this implies that

$$\lambda < -\sqrt{2} + 1 \quad \text{and} \quad \mu < -\sqrt{2} + 1$$

(in particular, λ and μ are negative, so that Σ is convex in M.) Moreover, since $\lambda + \mu = 2H = -2\sqrt{2}$, the above inequality yields

$$-\sqrt{2} - 1 < \lambda < -\sqrt{2} + 1 \quad \text{and} \quad -\sqrt{2} - 1 < \mu < -\sqrt{2} + 1.$$

These bounds on λ, μ and the equality $\lambda + \mu = -2\sqrt{2}$ imply that

$$-\lambda.\mu < -1,$$

i.e. that the K-curvature of the Cauchy surface Σ in M is strictly bounded from above by $-\Lambda$, as wanted. □

11.1.4. *The* AdS_3 *case.* The arguments developped above are not sufficient to get a directly a convex Cauchy surface in every AdS_3 spacetime. Nevertheless, these argument provide us with a Cauchy surface with controlled principal curvatures:

Proposition 11.8 (See also Lemma 3.11 in [39])**.** *Every non-elementary 3-dimensional MGHC spacetime with constant curvature $\Lambda = -1$ contains a maximal Cauchy surface (i.e. a Cauchy surface with constant mean curvature 0). Moreover, the principal curvatures of this maximal surface stay within the interval $(-1, 1)$.*

Proof. Consider a 3-dimensional non-elementary MGHC spacetime (M, g) with constant curvature -1. It was proved in [13] that (M, g) admits a maximal Cauchy surface Σ. Denote by η and h the first and the second fundamental forms of Σ. Then $(\Sigma, \eta, h = h_0)$ is a CMC $(0, -1)$-initial Cauchy data. So, by Corollary 11.5, $(\Sigma, \eta, h' = h_0 - \eta)$ is a CMC $(-1, 0)$-initial Cauchy data. Denote by λ and μ (resp. λ' and μ') the principal curvatures of Σ seen as a CMC $(0, -1)$ initial Cauchy data (resp. as a CMC $(-1, 0)$ initial Cauchy data). According to Proposition 11.2, the principal curvatures λ', μ' are negative. Using Remark 11.6, this implies $\lambda < 1$ and $\mu < 1$. Since $H = -(\lambda + \mu) = 0$, it follows that $-1 < \lambda < 1$ and $-1 < \mu < 1$. □

Remark 11.9. In subsection 11.4, we will explain how to show that every non-elementary MGHC spacetime with constant curvature -1 contains a convex (and a concave) Cauchy surface.

11.2. **Pushing a Cauchy surface along the orthogonal geodesics.**

Let (M, g) be an (arbitrary) 3-dimensional spacetime. Let Σ be a spacelike surface in M. For every $x \in \Sigma$, denote by $n(x)$ the future oriented unit normal vector of Σ at x. For t small enough, the map

$$\begin{array}{rccl} \phi^t_\Sigma : & \Sigma & \to & M \\ & x & \mapsto & \exp_x(t.n(x)) \end{array}$$

is well-defined, and is an embedding. For such a t, the surface

$$\Sigma^t := \phi^t_\Sigma(\Sigma)$$

is obviously a compact spacelike surface, hence a Cauchy surface. We say that the Cauchy surface $\Sigma^t := \phi^t_\Sigma(\Sigma)$ is obtained by *pushing Σ along orthogonal geodesics for a time t*.

In our situtation, since the ambient curvature Λ is constant, one can compute explicitely the principal curvatures of the surface Σ^t:

Proposition 11.10. *Pick a point $x \in \Sigma$ and a real number t. We assume that the map ϕ^t_Σ is well-defined (i.e. that the geodesics that are orthogonal to Σ exists for a time at least t). We denote by λ, μ the principal curvature of Σ at x.*

1. The flat case. *Assume that (M, g) is flat (i.e. locally isometric to Min_3). If $\lambda.t \neq 1$ and $\mu.t \neq 1$ then ϕ^t_Σ is an embedding in the neighborhood of x, and the principal curvature of the surface Σ^t at the point $\phi_t(x)$ are*

$$\lambda_t = \frac{\lambda}{1 - \lambda t} \quad \textit{and} \quad \mu_t = \frac{\mu}{1 - \mu t}.$$

2. The dS_3-case. *Assume that (M, g) has positive constant curvature (i.e. is locally isometric to dS_3). If $\lambda \tanh(t) \neq 1$ and $\mu \tanh(t) \neq 1$ then ϕ^t_Σ is an embedding in the neighborhood of x, and the principal curvature of the surface Σ^t at the point $\phi_t(x)$ are*

$$\lambda_t = \frac{\lambda - \tanh(t)}{1 - \lambda \tanh(t)} \quad \textit{and} \quad \mu_t = \frac{\mu - \tanh(t)}{1 - \mu \tanh(t)}.$$

3. The AdS_3-case. *Assume that (M, g) has negative constant curvature (i.e. is locally isometric to AdS_3) If $\lambda \tan(t) \neq 1$ and $\mu \tan(t) \neq 1$ then ϕ^t_Σ is an embedding in the neighborhood of x, and the principal curvature of the surface Σ^t at the point $\phi^t_\Sigma(x)$ are*

$$\lambda_t = \frac{\lambda + \tan(t)}{1 - \lambda \tan(t)} \quad \textit{and} \quad \mu_t = \frac{\mu + \tan(t)}{1 - \mu \tan(t)}.$$

Proof. Straightforward computation. □

As an immediate corollary, we get:

Corollary 11.11. *Pick a point $x \in \Sigma$. Denote by H (resp. κ) the mean curvature (resp. the K-curvature) of the surface Σ at x. For t small enough, denote by κ_t the K-curvature of the surface Σ^t at $\phi^t_\Sigma(x)$. Then,*

$$\frac{\partial \kappa_t}{\partial t}_{|t=0} = 2(\kappa + \Lambda)H.$$

Remark 11.12. In particular, the K-curvature of Σ^t at $\phi^t_\Sigma(x)$ increases with t (for t close to 0) provided that the surface Σ is convex and has a K-curvature κ strictly bounded from above by $-\Lambda$.

11.3. **Proof of Theorem 3.4.**

We consider a 3-dimensional non-elementary MGHC spacetime (M, g) with constant K-curvature $\Lambda \geq 0 = 0$. Reversing the time orientation if necessary, we assume that (M, g) is future complete.

Proposition 11.13. *If Σ is a convex Cauchy surface in M whose K-curvature is strictly bounded from above by $-\Lambda$, then:*

(1) *for every $t \geq 0$, the map $\phi^t_\Sigma : \Sigma \to M$ is well-defined and is an embedding,*
(2) *for every $t \geq 0$, the Cauchy surface $\Sigma^t := \phi^t_\Sigma(\Sigma)$ is convex and has a K-curvature strictly bounded from above by $-\Lambda$,*
(3) *the K-curvature of Σ^t tends uniformly towards $-\Lambda$ when $t \to \infty$.*

Proof. The map ϕ^t_Σ is defined for every $t \geq 0$ because the spacetime was assumed to be future complete. The other assertions follow from Proposition 11.10. □

Proposition 11.1 (in the case $\Lambda = 0$) or Proposition 11.7 (in the case $\Lambda = +1$) provide us with a convex Cauchy surface Σ with K-curvature strictly bounded by $-\Lambda$. Choose $\epsilon > 0$ such that the K-curvature of Σ is bounded from above by $-\Lambda - \epsilon$. Let $\kappa \in (-\Lambda - \epsilon, \Lambda)$. By item 1 and 2 of Proposition 11.13, for every $t \geq 0$, the surface $\Sigma^t = \phi^t_\Sigma(\Sigma)$ is well-defined, convex, and has K-curvature strictly bounded from above by $-\Lambda$. Moreover, by item 3 of Proposition 11.13, for $t > 0$ large enough, the K-curvature of Σ_t is strictly bounded from below by κ. Therefore, for $t > 0$ large enough, (Σ, Σ_t) is a pair of κ-barriers. This completes the proof of Theorem 3.4.

11.4. **Convex and concave Cauchy surfaces in AdS_3 spacetimes.**

Proposition 11.14. *Let (M, g) be a non-elementary 3-dimensional MGHC spacetime of AdS_3 type (i.e. with negative constant curvature). Then M contains two Cauchy surfaces with constant K-curvature -1, one strictly convex and the other strictly concave.*

Proof. Let Γ be the fundamental group of the Cauchy surfaces of M. By Theorem 7.1, there exists a representations $\rho : \Gamma \to \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, a curve Λ^+_ρ in $\partial\, \mathbb{ADS}_3$ associated to ρ, and a open set $E(\Lambda^+_\rho)$ in $\mathbb{ADS}_3 \simeq \mathrm{AdS}_3$ such that $M \simeq \rho(\Gamma)\backslash E(\Lambda^+_\rho)$. We will need the following lemma:

Lemma 11.15. *Let Σ be a Cauchy surface in M, and S be a lift of Σ in $E(\Lambda^+_\rho)$. Let t_0 be a real number such that the map $\phi^t_S : S \to \mathrm{AdS}_3$ is an immersion for every t between 0 and t_0. Then $S^{t_0} = \phi^{t_0}_S(S)$ is contained in $E(\Lambda^+_\rho)$ and projects to a Cauchy surface Σ^{t_0} in M.*

Remark 11.16. *A posteriori*, the surface Σ^{t_0} can of course be obtained by pushing Σ along the orthogonal geodesics for a time t_0. Nevertheless, one cannot define Σ^{t_0} directly in M (without passing to the universal cover) since one does not know *a priori* that the geodesics of M that are orthogonal to Σ exist for a long enough time (recall that M is neither past nor future complete).

Let us postpone the proof of Lemma 11.15, and complete the proof of Proposition 11.14. According to Proposition 11.8, the spacetime M admits a maximal Cauchy surface Σ whose principal curvatures stay within the interval $(-1,1)$. Let S be a lift of Σ in $E(\Lambda_\rho^+)$. We denote by $\lambda(x), \mu(x)$ the principal curvatures of the surface S at x. Since $|\lambda(x)| < 1$ and $|\mu(x)| < 1$ for every $x \in \Sigma$, Proposition 11.10 implies that the map ϕ_S^t is an immersion for every $t \in [-\pi/4, 0]$. Therefore, Lemma 11.15 implies that $S^{-\pi/4}$ is embedded, contained in the domain $E(\Lambda_\rho^+)$, and projects to Cauchy surface $\Sigma^{-\pi/4}$ in M, which is obtained by pushing the surface Σ along orthogonal geodesics for a time $(-\pi/4)$. Moreover, Proposition 11.10 implies that, for every $x \in S$, the principal curvatures of the surface $S^{-\pi/4}$ at the point $\phi_S^{-\pi/4}(x)$ are

$$\lambda_{-\pi/4}(x) = \frac{\lambda(x)-1}{1+\lambda(x)} \quad \text{and} \quad \mu_{-\pi/4}(x) = \frac{\mu(x)-1}{1+\mu(x)}.$$

In particular, the Cauchy surface $S^{-\pi/4}$ is strictly convex and has constant K-curvature -1. It follows immediately that the Cauchy surface $\Sigma^{-\pi/4}$ is also stricty convex, and has constant K-curvature -1.

The same arguments show that the Cauchy surface $\Sigma^{\pi/4}$ is well-defined, strictly concave, and has constant K-curvature -1. □

We are left to prove Lemma 11.15. The key point is the following :

Sub-lemma 11.17. *Let $\widetilde{\Sigma}$ be the (abstract) universal cover of the surface Σ and $\phi : \widetilde{\Sigma} \to \mathrm{AdS}_3$ be a Γ-equivariant*[10] *immersion, such that $\phi(\widetilde{\Sigma})$ is spacelike. Then ϕ is a proper embedding, and $\phi(\widetilde{\Sigma})$ is contained in $E(\Lambda_\rho^+)$ or in $E(\Lambda_\rho^-)$.*

Proof of Sub-lemma 11.17. Since $\phi(\widetilde{\Sigma})$ is spacelike, the Lorentzian metric of AdS_3 induces a Riemannian metric on $\phi(\widetilde{\Sigma})$. Let us denote by g the pull-back (by ϕ) of this Riemannian metric on $\widetilde{\Sigma}$. Since ϕ is Γ-equivariant, this Riemannian metric g is Γ-invariant. And since Γ is co-compact, g is complete. Therefore, ϕ is a locally isometric immersion of the complete Riemannian surface $(\widetilde{\Sigma}, g)$ in AdS_3. By a lemma of Mess (see [47, Lemma 6]), this implies that ϕ is a proper embedding, and that $\phi(\widetilde{\Sigma})$ is achronal[11]. In particular, $\phi(\widetilde{\Sigma})$ is a closed achronal surface in $\mathrm{AdS}_3 \simeq \mathbb{ADS}_3$. As explained in §7.2, this implies that the boundary in $\mathbb{ADS}_3 \cup \partial\,\mathbb{ADS}_3$ is either the curve Λ_ρ^+ or the curve Λ_ρ^- (see [10, Theorem 10.13]). This implies that the Cauchy development $D(\phi(\widetilde{\Sigma}))$ is contained in $E(\Lambda_\rho^+)$ or $E(\Lambda_\rho^-)$ (see [13, Proposition 5.18]); in particular, the surface $\phi(\widetilde{\Sigma})$ is contained in $E(\Lambda_\rho^+)$ or $E(\Lambda_\rho^-)$. □

Proof of Lemma 11.15. Let $\widetilde{\Sigma}$ be the (abstract) universal cover of Σ, and choose a Γ-equivariant homeomorphism $\phi_0 : \widetilde{\Sigma} \to S$. For every t between 0 and t_0, the map $\phi_S^t : S \to S^t$ is $\rho(\Gamma)$-equivariant

[10] By such, we mean that, for every $\widetilde{x} \in \widetilde{\Sigma}$ and every $\gamma \in \Gamma$, one has $\phi(\gamma.x) = \rho(\gamma).\phi(\widetilde{x})$.

[11] Remember that this means that every timelike curve in $\widetilde{\mathrm{AdS}_3}$ cannot intersect a lift of $\phi(\widetilde{\Sigma})$ at two different points. We define this property in $\widetilde{\mathrm{AdS}_3}$ (rather than in AdS_3) because AdS_3 itself is not a causal space: any two points of AdS_3 can be joined by a timelike curve in AdS_3.

(since the group $\rho(\Gamma)$ acts by isometries of AdS_3). Therefore, the map $\phi_t := \phi_S^t \circ \phi_0 : \widetilde{\Sigma} \to \mathrm{AdS}_3$ is a Γ-equivariant immersion, and $S^t = \phi_t(\widetilde{\Sigma})$ is spacelike. By Lemma 11.17, it follows that S^t is properly embedded and contained in $E(\Lambda_\rho^+)$ or $E(\Lambda_\rho^-)$. Now, the open sets $E(\Lambda_\rho^+)$ and $E(\Lambda_\rho^-)$ are disjoint. The surface S^t depends continuously on t, and is contained in $E(\Lambda_\rho^+)$ for $t = 0$. Hence, S^t is contained in $E(\Lambda_\rho^+)$ for every t between 0 and t_0. In particular, S^{t_0} is contained in $E(\Lambda_\rho^+)$. So, we have proved that the map $\phi_{t_0} : \widetilde{\Sigma} \to \mathrm{AdS}_3$ is a $\rho(\Gamma)$-equivariant embedding, and that the spacelike surface $S^{t_0} = \phi_{t_0}(\widetilde{\Sigma})$ is contained in $E(\Lambda_\rho^+)$. It follows that S^{t_0} projects to a compact spacelike surface in $M \simeq \rho(\Gamma)\backslash E(\Lambda_\rho^+)$. To conclude the proof, we recall that every compact spacelike surface in a MGHC spacetime is a Cauchy surface. □

11.5. **Proof of Proposition 3.10.**

We consider a 3-dimensional non-elementary MGHC spacetime (M, g) with constant K-curvature Λ. We assume that (M, g) admits a Cauchy surface Σ which is strictly convex (i.e. has negative principal curvatures) and has constant K-curvature $\kappa < -\Lambda$. Note that, since Σ is strictly convex, the mean curvature of Σ is negative.

For t small enough, we define the map ϕ_Σ^t and the surface Σ^t as in Subsection 11.2. For $t < 0$, the Cauchy surface Σ^t is of course in the past of Σ. For t small enough, the principal curvatures of Σ^t are close to those of Σ. In particular, for t small enough, the Cauchy surface Σ^t is strictly convex. Moreover, Corollary 11.11 implies that, for $t < 0$ small enough, the K-curvature curvature of Σ^t is strictly bounded from above by κ (see also Remark 11.12). This completes the proof of Proposition 3.10. □

Remark 11.18. In the case here Λ is negative (AdS_3 spacetimes), there is an analog statement concerning concave surfaces:
Assume that (M, g) admits a stricly concave Cauchy surface Σ with constant K-curvature κ. Then, in the future of Σ (and arbitrarily close to Σ) one can find a strictly concave Cauchy surface whose K-curvature is strictly bounded from below by κ.

12. K-SLICINGS OF FLAT AND dS_3 SPACETIMES

We are now ready to prove the existence of K-slicings of non-elementary 3-dimensional MGHC spacetimes of constant curvature curvature. In the present §, we will only consider the cases of flat and locally de Sitter spacetimes (items 1 and 2 of Theorem 2.1). The case of locally anti-de Sitter spacetime (item 3 of Theorem 2.1) will be treated in §13.

All along this section, we consider a non-elementary 3-dimensional MGHC spacetime (M, g) of constant curvature curvature $\Lambda \geq 0$. Reversing time-orientation if necessary, we assume that (M, g) is future complete. As explained in section 3, we will first get a local K-slicing (i.e. a K-slicing of some open subset of M), and then extend this local K-slicing to a global one.

12.1. **Local K-slicings.**

Definition 12.1. A *local K-slicing* in M is a 1-parameter family $(\Sigma_\kappa)_{\kappa \in I}$ of Cauchy surfaces in M such that:

(1) the parameter set I is an interval included in $(-\infty, -\Lambda)$,

(2) for every $\kappa \in I$, the Cauchy surface Σ_κ is convex and has constant K-curvature equal to κ.

The set $U = \bigcup_{\kappa \in I} \Sigma_\kappa$ is called the *support* of the local K-slicing.

Note that, in the above definition, one does not require the Σ_κ's to be pairwise disjoint, or to depend continously on κ. Actually, these properties are automatically satisfied:

Lemma 12.2. *Every local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$ satisfies the following* monotonicity property*: for every $\kappa_1, \kappa_2 \in I$ such that $\kappa_1 < \kappa_2$, the Cauchy surface Σ_{κ_1} is strictly in the past of the Cauchy surface Σ_{κ_2}.*

In particular, if $(\Sigma_\kappa)_{\kappa \in I}$ is a local K-slicing, then the Σ_κ's are pairwise disjoint.

Proof of Lemma 12.2. Consider a local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$ in M, pick up two real numbers $\kappa_1, \kappa_2 \in I$ such that $\kappa_1 < \kappa_2$, and assume that Σ_{κ_1} is not strictly in the past of Σ_{κ_2}.

Since M is future complete, the map $\phi^t = \phi^t_{\Sigma_{\kappa_2}}$ does exists for every $t \geq 0$. Moreover, since Σ_{κ_2} is convex and $\Lambda = 0$ or 1, Proposition 11.13 implies that $\phi^t_{\Sigma_{\kappa_2}}$ is an embedding for every $t \geq 0$. So, for every $t \geq 0$, $\Sigma^t_{\kappa_2} := \phi^t_{\Sigma_{\kappa_2}}(\Sigma_{\kappa_2})$ is a smooth Cauchy surface. For $t > 0$ large enough, the Cauchy surface $\Sigma^t_{\kappa_2}$ is strictly in the future of the Cauchy surface Σ_{κ_1}. Therefore, we may consider the infimum t_{min} of all real numbers $t > 0$ such that $\Sigma^t_{\kappa_2}$ is in the future of Σ_{κ_1}. Clearly, the surfaces $\phi_t(\Sigma_{\kappa_2})$ and Σ_{κ_1} intersect at some point x (otherwise, t_{min} would not be minimal). From Proposition 11.13 and Remark 11.12, the K-curvature of $\Sigma_{\kappa_2})$ is strictly bounded from below by κ_2. Since $\kappa_1 < \kappa_2$, it follows in particular that the K-curvature of the surface $\Sigma^t_{\kappa_2}$ at x is strictly bigger than those of the surface Σ_{κ_1}. This contradicts the maximum principle (see §4.4) since the surface $\Sigma^t_{\kappa_2}$ is in the future of the surface Σ_{κ_1}. □

Remark 12.3. Actually, the argument of the above proof shows that: if Σ, Σ' are two convex Cauchy surfaces in M, whose K-curvatures are strictly bounded from above by $-\Lambda$, and such that the supremum of the K-curvature of Σ is smaller than the infimum of the K-curvature of Σ', then Σ is in the past of Σ'.

In particular, for every $\kappa < -\Lambda$, there exists at most one convex Cauchy surface with constant K-curvature κ.

Lemma 12.4. *If $(\Sigma_\kappa)_{\kappa \in I}$ is a local K-slicing in M, then the surface Σ_κ depends continuously on κ.*

Proof. Consider $\kappa_0 \in I$, and a neighbourhood V of the Cauchy surface Σ_{κ_0}. We have to prove that, for κ close enough to κ_0, the Cauchy surface Σ_κ is contained in V.

By definition of a local K-slicing, the Cauchy surface Σ_{κ_0} is convex, and has constant K-curvature κ. For t small enough, the Cauchy surface $\Sigma^t_{\kappa_0} = \phi^t_{\Sigma_{\kappa_0}}(\Sigma_{\kappa_0})$ is well defined, contained in V, convex, has K-curvature strictly bounded from above by $-\Lambda$. Moreover, by Remark 11.12, for $t > 0$ (resp. $t < 0$) small enough, the K-curvature Cauchy surface $\Sigma^t_{\kappa_0}$ is strictly bounded from below by κ . Therefore, there exists $\eta > 0$ and $\epsilon > 0$, such that the surfaces $\Sigma^{-\eta}_{\kappa_0}$ and $\Sigma^{\eta}_{\kappa_0}$ are contained in V, and such that $(\Sigma^{-\eta}_{\kappa_0}, \Sigma^{\eta}_{\kappa_0})$ is a pair of κ-barriers for every $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$. By Theorem 3.2, it follows that, for every $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$, the neighbourhood V contains a convex Cauchy surface with constant K-curvature κ. By the uniqueness explained in Remark 12.3, it follows that the Cauchy surface Σ_κ is contained in V for every $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$. □

Lemma 12.2 and 12.4 show that a local K-slicing is indeed a *slicing* (i.e. a *foliation*) of some open subset U of M. Moreover, Lemma 12.4 implies that the support U of a local K-slicing $(\Sigma_\kappa)_{\kappa\in I}$ has "no hole": for every $\kappa_1, \kappa_2 \in I$ such that $\kappa_1 < \kappa_2$, the intersection $I^+(\Sigma_{\kappa_1}) \cap I^-(\Sigma_{\kappa_2})$ is contained in U.

12.2. **Construction of a local K-slicing.**

Theorem 3.4 provides us with a non-empty open interval $I_0 :=]-\Lambda-\epsilon, -\Lambda[$ such that, for every $\kappa \in I_0$, there exists a pair of κ-barriers. Using these barriers and Theorem 3.2, we get obtain for every $\kappa \in I_0$ a convex Cauchy surface Σ_κ with constant K-curvature equal to κ. The family $(\Sigma_\kappa)_{\kappa\in I_0}$ is by definition a local K-slicing. We denote by U_0 the support of this local K-slicing.

Lemma 12.5. *The set U_0 is a neighbourhood of the future end of M, i.e. it contains the whole future of a Cauchy surface of M.*

Proof. Thanks to Lemmas 12.2 and 12.4, it is enough to prove that the Cauchy surface Σ_κ "escapes towards the future end of M" when $\kappa \to -\Lambda$: more precisely, if K is a compact subset of M, then K is in the past of the surface Σ_κ for every κ close enough to $-\Lambda$.

In order to prove this, we consider any convex Cauchy surface $\widehat{\Sigma}$ in M such that the curvature is strictly bounded from above by $-\Lambda$ (for example, one might take $\widehat{\Sigma}$ to be a leaf of our local K-slicing). The map $\phi^t_{\widehat{\Sigma}} : \widehat{\Sigma} \to M$ is well-defined and is an embedding for every $t \geq 0$ (item 1 of Proposition 11.13). For $t \geq 0$, denote by $\kappa_{max}(t)$ the supremum of the K-curvature of the surface $\widehat{\Sigma}^t = \phi^t_{\widehat{\Sigma}}(S)$. By item 2 of Proposition 11.13, one has $\kappa_{max}(t) < -\Lambda$ for every $t \geq 0$. By Remark 12.3, the surface Σ_κ is in the future of $\widehat{\Sigma}^t$ for every $\kappa > \kappa_{max}(t)$.

But of course, by definition of the surface $\widehat{\Sigma}^t$, it "escapes towards the future end of M" when $t \to \infty$. Therefore, the Cauchy surface Σ_κ escapes towards the future end of M when $\kappa \to -\Lambda$. □

12.3. **Getting a global K-slicing.**

We say that a local K-slicing $({\Sigma'}_\kappa)_{\kappa\in I'}$ *extends* a local K-slicing $(\Sigma_\kappa)_{\kappa\in I}$ if $I \subset I'$ and ${\Sigma'}_\kappa = \Sigma_\kappa$ for $\kappa \in I$. Our goal is to prove that our local K-slicing $(\Sigma_\kappa)_{\kappa\in I_0}$ can be extended to get K-slicing $(\Sigma_\kappa)_{\kappa\in]-\infty,-\Lambda[}$ whose support is the whole manifold M.

For this purpose, we consider a local K-slicing $(\Sigma_\kappa)_{\kappa\in I}$ which extends our initial local K-slicing $(\Sigma_\kappa)_{\kappa\in I_0}$, and is *maximal* among the local K-slicings extending $(\Sigma_\kappa)_{\kappa\in I_0}$. We denote by U the support of $(\Sigma_\kappa)_{\kappa\in I}$. We have to prove that $I = (-\infty, -\Lambda)$ and $U = M$.

Assume that $I \neq (-\infty, -\Lambda)$. Then, there exists $\alpha \in]-\infty, -\Lambda[$ such that $I =]\alpha, -\Lambda[$ or $I = [\alpha, -\Lambda[$. On the one hand, Theorem 3.9 tells us that the first possibility cannot occur. Indeed, if $I =]\alpha, -\Lambda[$, then there exists a convex Cauchy surface Σ_α with constant K-curvature α such that $I^+(\Sigma_\alpha) = U$. Clearly, $(\Sigma_\kappa)_{\kappa\in[\alpha,-\Lambda)}$ is a local K-slicing which extends the local K-slicing $(\Sigma_\kappa)_{\kappa\in I}$, contradicting the maximality of $(\Sigma_\kappa)_{\kappa\in I}$. On the other hand, Proposition 3.10 implies that the second possibility cannot occur either. Indeed, if $I = [\alpha, -\Lambda[$, then Proposition 3.10 provides us with a surface Σ such that (Σ, Σ_α) is a pair of κ barriers for every κ smaller than α and close enough to α. Using Theorem 3.2, we get a convex Cauchy surface Σ_κ with constant K-curvature κ for every $\kappa \in]\alpha', \alpha[$ for some $\alpha' < \alpha$. So, we get a local K-slicing $(\Sigma_\kappa)_{\kappa\in]\alpha',-\Lambda[}$ which extends $(\Sigma_\kappa)_{\kappa\in I}$, contradicting the maximality of $(\Sigma_\kappa)_{\kappa\in I}$.

So, we have proved that $I =]-\infty, -\Lambda[$. Using once again Theorem 3.9, we see that this implies that $U = M$. Henceforth, $(\Sigma_\kappa)_{\kappa\in I}$ is a global K-slicing of our spacetime M. This completes the proof of of Theorem 2.1 in the flat case and in the locally de Sitter case.

13. K-SLICINGS OF AdS_3-SPACETIME

The strategy used in §12 to get K-slicings on flat and dS_3-spacetimes does not fully apply in the AdS_3-case. The main problem is the lack of an analog of Proposition 11.13 in the AdS_3-setting[12]. To bypass this problem, we will develop an alternative argument based on the duality between convex and concave Cauchy surfaces in MGHC AdS_3-spacetimes.

13.1. **Duality in AdS_3-spacetimes.**

13.1.1. *Duality between points and totally geodesic planes.* We consider the quadratic form $Q_{2,2} = -x_1^2 - x_2^2 + x_3^2 + x_4^2$ on $\mathbb{R}^4$. Recall that AdS_3 is the quadric $\{Q_{2,2} = -1\}$ endowed with the Lorentzian metric induced by $Q_{2,2}$. For every set $A \subset \mathbb{R}^4$, we denote by $A^\perp$ the $Q_{2,2}$-orthogonal of A in $\mathbb{R}^4$.

Let x be a point in $\mathrm{AdS}_3 \subset \mathbb{R}^4$. Then $x^\perp$ is an hyperplane in $\mathbb{R}^4$, and the restriction of $Q_{2,2}$ to $x^\perp$ has signature $(-,+,+)$. It follows that $x^\perp \cap \mathrm{AdS}_3$ is the disjoint union of two spacelike totally geodesic planes in AdS_3. We denote these totally geodesic planes by $x^\flat$ and $x^\sharp$, in such a way that : if one starts at the point x, and follows a future-oriented (resp. past-oriented) timelike geodesic, then one crosses $x^\sharp$ before $x^\flat$ (resp. $x^\flat$ before $x^\sharp$)[13].

Conversely, let P be a spacelike totally geodesic plane in AdS_3. Then $P^\perp$ is a line in $\mathbb{R}^4$, and the restriction of $Q_{2,2}$ to $P^\perp$ is negative definite. It follows that $P^\perp$ intersects AdS_3 at two (antipodal) points. Every geodesic of AdS_3 orthogonal to P passes through these two points. We denote these two points by $P^\flat$ and $P^\sharp$ in such a way that : if one starts at a point of P, and follows a future-oriented (resp. past-oriented) timelike geodesic orthogonal to P, then one passes through $P^\sharp$ before passing through $P^\flat$ (resp. passes through $P^\flat$ before passing through $P^\sharp$)

Remark 13.1. It is easy to verify that every geodesic of AdS_3 which is orthogonal P passes through the two points $P^\flat$ and $P^\sharp$. Moreover, the length of a geodesic segment, orthogonal to P, and going from P to $P^\flat$ or $P^\sharp$ is exactly $\pi/2$. In other words, $P^\sharp$ (resp. $P^\flat$) is obtained by pushing P along the orthogonal geodesics for a time $\pi/2$ (resp. $-\pi/2$).

It clearly follows from the definitions above that, for every point $x \in \mathrm{AdS}_3$ and every spacelike totally geodesic plane $P \subset \mathrm{AdS}_3$, one has

$$(x^\flat)^\sharp = (x^\sharp)^\flat = x \quad \text{and} \quad (P^\flat)^\sharp = (P^\sharp)^\flat = P \tag{4}$$

[12]There can be no analog of Proposition 11.13 in the AdS_3-setting, simply because MGHC spacetimes of AdS_3-type are neither future complete, nor past complete.

[13]The rough idea is that "$x^\flat$ is in the past of x and $x^\sharp$ is in the future of x". Unfortunately, things are not so simple since AdS_3 is not a causal spacetime

13.1.2. *Duality between strictly convex and concave surfaces in* AdS_3. Let S be a (smooth) spacelike surface in AdS_3. For every $x \in S$, we denote by $P_{S,x}$ the totally geodesic plane in AdS_3 which is tangent to S at x, and we recall that $(P_{S,x})^\flat$ and $(P_{S,x})^\sharp$ are two antipodal points in AdS_3. Then we define the *past dual set* $S^\flat$ and the *future dual set* $S^\sharp$ of S as follows:

$$S^\flat = \{(P_{S,x})^\flat \mid x \in S\} \quad , \quad S^\sharp = \{(P_{S,x})^\sharp \mid x \in S\}.$$

Clearly, $S^\flat$ and $S^\sharp$ are two subsets of AdS_3, which are the image of each other under the antipodal map. Note that $S^\sharp$ and $S^\flat$ are subsets of AdS_3 not surfaces in general; for example, if S is a totally geodesic plane, then $S^\sharp$ (resp. $S^\flat$) is a single point.

Remark 13.2. A point y is in $S^\flat$ (resp. $S^\sharp$) if and only if the totally geodesic plane $y^\sharp$ (resp. $y^\flat$) is tangent to S at some point. This follows from the definitions of $S^\flat, S^\sharp$ and equalities (4).

Proposition 13.3. *If S is a strictly convex (resp. strictly concave) spacelike surface, then the sets $S^\sharp$ and $S^\flat$ are strictly concave (resp. strictly convex) spacelike surfaces.*

In order to prove this proposition, we introduce the set $\mathcal{P}_{\mathrm{AdS}_3}$ of all spacelike totally geodesic planes in AdS_3. Note that $\mathcal{P}_{\mathrm{AdS}_3}$ has a natural structure of three-dimensional manifold. For every spacelike surface S in AdS_3, we consider the map

$$\begin{array}{rccc} P_S : & S & \to & \mathcal{P}_{\mathrm{AdS}_3} \\ & x & \mapsto & P_{S,x} \end{array}$$

The following lemma is a simple remark, but plays a crucial role in the proof of Proposition 13.3 :

Lemma 13.4. *Given a spacelike surface S in AdS_3 and a point $x \in S$, the second fundamental form of S at x is non-degenerate if and only if the derivative at x of the map P_S has maximal rank (i.e. rank 2).*

Proof. Let $\mathrm{T}^{-1}\,\mathrm{AdS}_3$ be the set of all pairs (x, n) where $x \in \mathrm{AdS}_3$ and n is a future-oriented unit timelike vector in $\mathrm{T}_x\,\mathrm{AdS}_3$. Let $\mathrm{T}^{\mathcal{P}}\,\mathrm{AdS}_3$ be the set of all pairs (x, P) where $x \in \mathrm{AdS}_3$ and P is a spacelike totally geodesic plane containing x. For every $x \in \mathrm{AdS}_3$ and every spacelike totally geodesic plane P containing x, let $n_{P,x}$ be the future-oriented unit timelike vector orthogonal to P at x. The map $(x, P) \mapsto (x, n_{P,x})$ is clearly a diffeomorphism between $\mathrm{T}^{\mathcal{P}}\,\mathrm{AdS}_3$ and $\mathrm{T}^{-1}\,\mathrm{AdS}_3$. For every spacelike surface S, this diffeomorphism induces an identification between the derivative of the map P_S and the Weingarten operator of S. Therefore, for every $x \in S$, the derivative at x of P_S has maximal rank if and only if the Weingarten operator of S at x has maximal rank. □

Proof of Proposition 13.3. We assume S is a strictly convex spacelike surface in AdS_3. We will prove that $S^\sharp$ is a strictly concave spacelike surface. The same arguments apply for $S^\flat$ and/or in the case where S is strictly concave.

First step: $S^\sharp$ is an immersed surface. Since S is strictly convex, for every $x \in S$, the second fundamental form of S at x is non-degenerate. Together with Lemma 13.4, this shows that, for every $x \in S$, the derivative at x of the map P_S has maximal rank. Therefore, the map $P_S : x \mapsto P_{S,x}$ is an immersion of S in $\mathcal{P}_{\mathrm{AdS}_3}$. Moreover, the map $P \mapsto P^\sharp$ is obviously a local diffeomorphism between $\mathcal{P}_{\mathrm{AdS}_3}$ and AdS_3. Therefore the map

$$\begin{array}{ccc} S & \to & \mathrm{AdS}_3 \\ x & \mapsto & (P_{S,x})^\sharp \end{array}$$

is an immersion of S in AdS_3. The range of this map is the set $S^\sharp$ (by the very definition of $S^\sharp$). Therefore $S^\sharp$ is an immersed surface.

Second step: $S^\sharp$ is an injectively immersed spacelike concave surface. Let $y \in S^\sharp$. By definition of $S^\sharp$, there exists a point $x \in S$ such that $y = (P_{S,x})^\sharp$. We will prove that the totally geodesic plane $x^\sharp$ meets $S^\sharp$ at y, and that $S^\sharp$ is contained in the past of $x^\sharp$ in a neighbourhood of y.

Since the point x belongs to the totally geodesic plane $P_{S,x}$, the point $y = (P_{S,x})^\sharp$ belongs to the totally geodesic plane $x^\sharp$. In other words, the totally geodesic plane $x^\sharp$ meets the surface $S^\sharp$ at y. Now let us consider a sequence $(y_k)_{k\in\mathbb{N}}$ of points of $S^\sharp$ converging to y when $k \to \infty$. By Remark 13.2, for every $k \geq 0$, the totally geodesic plane $y_k^\flat$ is tangent to the surface S at some point x_k. Since S is strictly convex, this implies that S is in the future of the plane $y_k^\flat$ in some neighbourhood U_k of the point x_k (and the size of the neighbourhood U_k does not depend of k provided that the x_k's stay in a compact subset of S). Moreover, if k is large, the totally geodesic plane $y_k^\flat$ is close to the totally geodesic plane $y^\flat = ((P_{S,x})^\sharp)^\flat = P_{S,x}$. Since S is *strictly* convex, this implies that x_k is close to x when k is large. It follows that, for k large enough, the point x is in the neighbourhood U_k. In particular, for k large enough, the point x is in the future of the totally geodesic plane $y_k^\flat$. It follows that, for k large enough, the point $y_k = (y_k^\flat)^\sharp$ is in the past of the totally geodesic plane $x^\sharp$.

So, we have proved that the totally geodesic plane $x^\sharp$ meets the immersed surface $S^\sharp$ at y, and that there exists a neighbourhood U of the point y in $S^\sharp$ such that $S^\sharp$ is contained in the past of $x^\sharp$ in U. It follows that $x^\sharp$ is the unique totally geodesic plane which is tangent to $S^\sharp$ at y, and that $S^\sharp$ is "locally contained" in the past of this totally geodesic plane. Since y is an arbitrary point in $S^\sharp$, this shows that $S^\sharp$ is injectively immersed, spacelike and concave (see §4.3).

Third step: $S^\sharp$ is strictly concave. Since we already know that $S^\sharp$ is concave, we only need to prove that the second fundamental form of $S^\sharp$ is non-degenerate at each point of $S^\sharp$. According to Lemma 13.4, it is equivalent to prove that the map $P_{S^\sharp} : y \mapsto P_{S^\sharp,y}$ is an immersion of $S^\sharp$ in $\mathcal{P}_{\mathrm{AdS}_3}$.

We have proved during the second step that, for every point $y \in S^\sharp$, if x is the unique point of the surface S such that $y = (P_{S,x})^\sharp$, then $x^\sharp$ is the unique totally geodesic plane tangent to $S^\sharp$ at y, that is $P_{S^\sharp,y} = x^\sharp$. Moreover, we have proved in the first and second step above that the map $x \mapsto (P_{S,x})^\sharp$ is an injective immersion of the surface S in AdS_3, and that the range of this immersion is the surface $S^\sharp$. It follows that the map $(P_{S,x})^\sharp \mapsto x$ is an immersion of $S^\sharp$ in AdS_3. Since the map $x \mapsto x^\sharp$ is obviously a local diffeomorphism from AdS_3 to $\mathcal{P}_{\mathrm{AdS}_3}$, this shows that the map

$$\begin{array}{rccc} P_{S^\sharp} : & S^\sharp & \to & \mathcal{P}_{\mathrm{AdS}_3} \\ & y = (P_{S,x})^\sharp & \mapsto & x^\sharp = P_{S^\sharp,y} \end{array}$$

is an immersion. Therefore, the surface $S^\sharp$ is stricly convex. □

During the above proof of Proposition 13.3, we have seen that, for every point $y \in S^\sharp$, if x is the unique point of the surface S such that $y = (P_{S,x})^\sharp$, then $P_{S^\sharp,y} = x^\sharp$, and therefore $(P_{S^\sharp,y})^\flat = (x^\sharp)^\flat = x$. This shows that :

Corollary 13.5. *If S is a strictly convex (resp. strictly concave) spacelike surface, then*

$$(S^\sharp)^\flat = S$$

and the map $\left|\begin{array}{ccc} S^\sharp & \to & (S^\sharp)^\flat \\ y & \mapsto & (P_{S,y})^\flat \end{array}\right.$ *is the inverse of the map* $\left|\begin{array}{ccc} S & \to & S^\sharp \\ x & \mapsto & (P_{S,x})^\sharp \end{array}\right.$.

Of course, similar arguments show that:

Corollary 13.6. *If S is a strictly convex (resp. strictly concave) spacelike surface, then*

$$(S^\flat)^\sharp = S$$

and the map $\left|\begin{array}{ccc} S^\flat & \to & (S^\flat)^\sharp \\ y & \mapsto & (P_{S,y})^\sharp \end{array}\right.$ *is the inverse of the map* $\left|\begin{array}{ccc} S & \to & S^\flat \\ x & \mapsto & (P_{S,x})^\flat \end{array}\right.$.

Proposition 13.7. *Let S be a strictly convex or strictly concave spacelike surface. Let x_0 be a point in S, and $y_0 = (P_{S,x_0})^\sharp$ (resp. $y_0 = (P_{S,x_0})^\flat$) be the corresponding point in $S^\sharp$ (resp. $S^\flat$). If λ, μ are the principal curvatures of S at x_0, then the principal curvatures of $S^\sharp$ (resp. $S^\flat$) at y_0 are $-\lambda^{-1}, -\mu^{-1}$.*

Proof. For every $x \in S$, let $n_{S,x} \in T_x \operatorname{AdS}_3$ be the future-directed unit normal vector of S at x. Since AdS_3 is a quadric in $\mathbb{R}^4$, we can see $n_{S,x}$ as an element of $T_x\mathbb{R}^4$. The key point of the proof is the following observation :

Claim. For every $x \in S$, the canonical isomorphism between $T_x\mathbb{R}^4$ and $\mathbb{R}^4$ maps the vector $n_{S,x} \in T_x\mathbb{R}^4$ to the point $(P_{S,x})^\sharp = -(P_{S,x})^\flat \in \mathbb{R}^4$.

Let $y \in \mathbb{R}^4$ be the image of the vector $n_{S,x}$ under the canonical isomorphism between $T_x\mathbb{R}^4$ and $\mathbb{R}^4$. For every subspace V_x of $T_x\mathbb{R}^4$, denote by $\overline{V_x} \subset \mathbb{R}^4$ the image of V_x under this isomorphism. For every $A \subset \mathbb{R}^4$, denote by $A^\perp$ the $Q_{2,2}$ orthogonal of A. Since $P_{S,x}$ is a totally geodesic plane of AdS_3, there exists a hyperplane H of $\mathbb{R}^4$ such that $P_{S,x}$ is a connected component of $H \cap \operatorname{AdS}_3$. Clearly, H is the vector subspace of $\mathbb{R}^4$ spanned by $P_{S,x}$, and $H = \mathbb{R}.x \oplus \overline{T_x P_{S,x}}$. The vector $n_{s,x}$ belongs to $T_x \operatorname{AdS}_3$. Hence the point y belongs to $\overline{T_x \operatorname{AdS}_3} = x^\perp$. Moreover the vector $n_{s,x}$ is orthogonal to $T_xS = T_xP_{S,x}$. Hence the point y belongs to $(\overline{T_x P_{S,x}})^\perp$. So we have proved that y belongs to $(\mathbb{R}.x \oplus \overline{T_x P_{S,x}})^\perp = H^\perp = (P_{S,x})^\perp$. Moreover the vector $n_{S,x}$ has norm -1, and the Lorentzian metric on AdS_3 is induced by the quadratic form $Q_{2,2}$. Hence the point y belongs to $\operatorname{AdS}_3 = \{z \in \mathbb{R}^4 \mid Q_{2,2}(z) = -1\}$. So we have proved y belongs to $(P_{S,x})^\perp \cap \operatorname{AdS}_3 = \{(P_{S,x})^\flat, (P_{S,x})^\sharp\} = \{(P_{S,x})^\flat, (P_{S,x})^\sharp\}$. Finally, the fact that $n_{S,x}$ is future-oriented means that y is equal to $(P_{S,x})^\sharp$ rather than $(P_{S,x})^\flat$. This completes the proof of the claim.

Now, for every $x \in \operatorname{AdS}_3$, we identify $T_x\mathbb{R}^4$ with $\mathbb{R}^4$ (using the canonical isomorphism between these two spaces). The above claim shows that the Weingarten map of S at x_0 is identified with the derivative at x_0 of the map $x \mapsto (P_{S,x})^\sharp$ (which also the opposite of the derivative of the map $x \mapsto (P_{S,x})^\flat$, since $(P_{S,x})^\sharp$ and $(P_{S,x})^\flat$ are two antipodal points). Of course the same is true if we replace S by $S^\sharp$: the Weingarten map of $S^\sharp$ at y_0 is identified with the derivative at y_0 of the map $y \mapsto (P_{S^\sharp,y})^\sharp$ (which also the opposite of the derivative of the map $y \mapsto (P_{S^\sharp,y})^\flat$). Together with Corollary 13.5, this shows that the Weingarten map of S at x_0 is identified with the opposite of the inverse of the Weingarten map of $S^\sharp$ at y_0. The proposition follows. □

Corollary 13.8. *If S is a spacelike surface with constant K-curvature $\kappa \neq 0$, then $S^\sharp$ and $S^\flat$ are spacelike surfaces with constant K-curvature κ^{-1}.*

Proof. Every surface with non-zero constant K-curvature is either strictly convex or strictly concave. Therefore the corollary is an immediate consequence of Proposition 13.7. □

It is interesting to observe that the future dual $S^\sharp$ and the past dual $S^\flat$ of a spacelike surface can also be defined by pushing along orthogonal geodesics :

Proposition 13.9. *For every spacelike surface S in AdS_3, the future dual $S^\sharp$ (resp. the past dual $S^\flat$) is obtained by pushing S along orthogonal geodesic for a time $\pi/2$ (resp. a time $-\pi/2$).*

Proof. Let S be a spacelike surface in AdS_3. For every $x \in S$, we denote by $n_{S,x}$ the future-directed unit normal vector of S at x. For every $t \in \mathbb{R}$, let

$$S^t := \{\exp_x (t.n_{S,x}) \mid x \in S\}$$

the set of points at distance t from S along an orthogonal geodesic (S^t is not necessarily a surface when $|t|$ is large, but this is not relevant here). We shall prove that $S^{\pi/2} = S^\sharp$ (the same arguments apply to show that $S^{-\pi/2} = S^\flat$).

Consider a point y in $S^\sharp$. By definition of $S^\sharp$, there exists a point $x \in S$ such that $y = P^\sharp_{S,x}$. Since the surface S and the totally geodesic plane $P_{S,x}$ are tangent at x, the vector $n_{S,x}$ is orthogonal to $P_{S,x}$. Using Remark 13.1, it follows that the point $y = P^\sharp_{S,x}$ coincides with the point $\exp_x\left(\frac{\pi}{2}.n_{S,x}\right)$. In particular, $y \in S^{\pi/2}$. Conversely, consider a point $y \in S^{\pi/2}$. There exists a point $x \in S$ such that $y = \exp_x\left(\frac{\pi}{2}.n_{S,x}\right)$. The same argument as above shows that $(P_{S,x})^\sharp = \exp_x(\frac{\pi}{2}.n_{S,x}) = y$. In particular, $y \in S^\sharp$. □

13.1.3. *Duality for convex and concave surfaces in MGHC AdS_3-spacetimes.* Now we consider a MGHC spacetime M of constant curvature -1. According to Theorem 7.1, there exists a representation $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ (where Γ is the fundamental group of the Cauchy surfaces of M) and an open set $E(\Lambda^+_\rho) \subset \mathrm{AdS}_3$ such that M is isometric to the quotient $M(\rho) := \rho(\Gamma_g)\backslash E(\Lambda^+_\rho)$.

Let Σ be a strictly convex Cauchy surface in M. Consider the lift S of Σ in $E(\Lambda^+_\rho) \subset \mathrm{AdS}_3$. Then S is a strictly convex spacelike surface in AdS_3. So we can consider the future dual $S^\sharp$ of S as defined in the previous paragraph. By Proposition 13.3, $S^\sharp$ is a strictly concave spacelike surface in AdS_3. Since the surface S is $\rho(\Gamma)$-invariant, so is the surface $S^\sharp$ (the duality is clearly a $\mathrm{Isom}(Q_{2,2})$-equivariant operation). Moreover, since S is strictly convex, Proposition 11.10 shows that $\phi^t_S : S \to \mathrm{AdS}_3$ is an immersion for every $t \in [0, \pi/2)$, and step 1 of the proof of Proposition 13.3 shows that $\phi^{\pi/2}_S : S \to \mathrm{AdS}_3$ is an immersion. Hence, Lemma 11.15 shows that the surface $S^\sharp = S^{\pi/2}$ is contained in $E(\Lambda^+_\rho)$, and projects to a Cauchy surface $\Sigma^\sharp = \Sigma^{\pi/2}$ in $M \simeq \Gamma\backslash E(\Lambda^+_\rho)$. This is a compact strictly concave spacelike surface in M, i.e. equivalently, a strictly concave Cauchy surface in M. We say that $\Sigma^\sharp$ is the *future dual* of the Cauchy surface Σ.

Similarly, one can define the *past dual* $\Sigma^\flat$ of a strictly concave Cauchy surface Σ in M.

Remark 13.10. If Σ is a strictly convex Cauchy surface in M, and S is a lift of Σ in $E(\Lambda^+_\rho)$, then the past dual $S^\flat$ of S is well-defined. Yet, $S^\flat$ is not contained in $E(\Lambda^+_\rho)$ (actually it is contained in $E(\Lambda^-_\rho)$). In particular, $S^\flat$ cannot be projected in M. This is the reason why the past dual of Σ does not exist. Similarly, the future dual of a strictly concave Cauchy surface in M does not exist.

By Corollary 13.5, for every strictly convex Cauchy surface Σ in M, one has $(\Sigma^\sharp)^\flat = \Sigma$. Similarly, for every strictly concave Cauchy surface Σ in M, one has $(\Sigma^\flat)^\sharp = \Sigma$. By Corollary 13.8, if Σ is a strictly convex (resp. concave) Cauchy surface with constant K-curvature κ, then $\Sigma^\sharp$ (resp. $\Sigma^\flat$) has constant K-curvature κ^{-1}.

13.2. **Proof of Theorem 2.1 in the AdS_3-case.**

Let (M, g) be a non-elementary MGHC spacetime with constant curvature $\Lambda = -1$. We denote by M^- and M^+ the past tight region and the future tight region of M. In order to prove Theorem 2.1, we have to construct a K-slicing on M^-, and a K-slicing on M^+. The starting point of our construction will be the Cauchy surfaces Σ^- and Σ^+ provided by Proposition 11.14. Recall that Σ^- is a convex Cauchy surface and Σ^+ is a convex Cauchy surface, both with constant K-curvature -1.

13.2.1. *Construction of the K-slicings on $J^-(\Sigma_-)$ and $J^+(\Sigma_+)$.* The Cauchy surface Σ^- is convex and has constant K-curvature -1. Using Proposition 13.3, we get a convex Cauchy surface Σ^{--} in the past of Σ^- and a real number $\epsilon > 0$, such that (Σ^{--}, Σ^-) is a pair of κ-barriers for every $\kappa \in (-1-\epsilon, -1)$. Together with Theorem 3.2, this shows the existence, for every $\kappa \in (-1-\epsilon, -1)$, of a convex Cauchy surface Σ^-_κ with constant K-curvature κ. We set $\Sigma^-_1 := \Sigma^-$. Then $(\Sigma^-_\kappa)_{\kappa\in(-1-\epsilon,-1]}$ is a local K-slicing. Now, we consider a local K-slicing $(\Sigma_\kappa)_{\kappa\in I}$ which extends $(\Sigma_\kappa)_{\kappa\in(-1-\epsilon,-1]}$, and is maximal for this property. The same arguments as in §12.3 show that $(-\infty, -1] \subset I$ and that the support of $(\Sigma_\kappa)_{\kappa\in I}$ contains the past $J^-(\Sigma^-)$ of the Cauchy surface Σ^-. So, we have proved the following:

Proposition 13.11. *There exists a local K-slicing $(\Sigma^-_\kappa)_{\kappa\in(-\infty,-1]}$ whose support is the past $J^-(\Sigma^-)$ of the Cauchy surface Σ^-.* □

Of course, similar arguments show that:

Proposition 13.12. *There exists a local K-slicing $(\Sigma^+_\kappa)_{\kappa\in(-\infty,-1]}$ whose support is the future $J^+(\Sigma^+)$ of the Cauchy surface Σ^+.* □

13.2.2. *Construction of the K-slicings on $M^- \setminus J^-(\Sigma^-)$ and $M^+ \setminus J^+(\Sigma^+)$.* Proposition 13.11 provides us with a local K-slicing $(\Sigma^-_\kappa)_{\kappa\in(-\infty,-1]}$ whose support is the past of the Cauchy surface Σ^-. To complete the proof of Theorem 2.1 in the AdS_3-case, we need to extend this local K-slicing, and get a K-slicing $(\Sigma^-_\kappa)_{\kappa\in(-\infty,0)}$ whose support is the whole past tight region M^-. For this purpose, we use the duality between convex and concave Cauchy surfaces : for every $\kappa \in (-1, 0)$, we set

$$\Sigma^-_\kappa := \left(\Sigma^+_{1/\kappa}\right)^\flat.$$

(observe that this definition is not chasing its own tail since the surface $\Sigma^+_{1/\kappa}$ has already been defined for $\kappa \in (0, 1)$; see Proposition 13.12).

Claim 13.13. *The families of Cauchy surfaces $(\Sigma^-_\kappa)_{\kappa\in(-\infty,0)}$ is a local K-slicing whose support is included in the past tight region M^-.*

Proof. For $\kappa \leq -1$, we already know that the Cauchy surface Σ^-_κ is a convex and has constant K-curvature κ. Moreover, for $\kappa \in (-1, 0)$, we know that the Cauchy surface $\Sigma^+_{1/\kappa}$ is strictly concave and has constant K-curvature $\frac{1}{\kappa}$. Together with Proposition 13.7, this implies that, for $\kappa \in (-1, 0)$, the Cauchy

surface $\Sigma_\kappa^- := (\Sigma_{1/\kappa}^+)^\flat$ is a convex and has constant K-curvature κ. This shows that $(\Sigma_\kappa^-)_{\kappa\in(-\infty,0)}$ is a local K-slicing. Since every convex Cauchy surface in M is contained in M^-, the support of this K-slicing must be contained in M^-. □

Claim 13.14. *For every point $x \in M^-$, there exists $\kappa \in (-\infty, 0)$ such that x is in the past of the Cauchy surface Σ_κ^-.*

Proof. We denote by $\check\tau : M \to (0,\pi)$ the reverse cosmological time of M. Recall that the past tight region M^- of M can be characterized as follows:

$$M^- = \{x \in M \mid \check\tau(x) > \pi/2\}. \tag{5}$$

The support of the local K-slicing $(\Sigma_{1/\kappa}^+)_{\kappa\in[-1,0)} = (\Sigma_\kappa^+)_{\kappa\in(-\infty,-1]}$ is a neighbourhood of the future end of M (recall that this means that it contains the future of a Cauchy surface). Since $\check\tau$ is regular, this implies that

$$\lim_{\kappa\to 0} \sup_{x\in\Sigma_{1/\kappa}^+} \check\tau(x) = 0. \tag{6}$$

Now, recall that, for every $\kappa \in [-1,0)$, the surface $\Sigma_\kappa^- = (\Sigma_{1/\kappa}^+)^\flat$ is obtained by pushing the surface $\Sigma_{1/\kappa}^+$ under the time $(-\pi/2)$ along orthogonal geodesics. In particular, every point of Σ_κ^- is the past end of a geodesic segment orthogonal to $\Sigma_{1/\kappa}^+$ of length $-\pi/2$. This implies that, for every $\kappa \in [-1,0)$, the length of a timelike curve joining the surface Σ_κ^- to the surface $\Sigma_{1/\kappa}^+$ is at most $\pi/2$ (recall that, in a globally hyperbolic spacetime, the supremum of the lengths of the timelike curves joining a surface Σ to a point p is realized by a timelike geodesic orthogonal to Σ). Together with (6), this implies that

$$\lim_{\kappa\to 0} \sup_{x\in\Sigma_\kappa^-} \check\tau(x) \le \frac{\pi}{2} \tag{7}$$

(actually, this inequality turns out to be an equality since, for evey κ, the surface Σ_κ^- is convex, thus contained in $M^- = \{x \in M \mid \check\tau(x) > \pi/2\}$). Inequality (7) and equality (5) show that for every point $x \in M^-$, there exists $\kappa \in (-1,0)$ such that x is in the past of the Cauchy surface Σ_κ^-. □

Claim 13.14 together with Lemma 12.4 show that the support of the local K-slicing $(\Sigma_\kappa^-)_{\kappa\in(-\infty,0)}$ is the whole past tight region M^-. Of course, similar arguments show that the support of the K-slicing $(\Sigma_\kappa^+)_{\kappa\in(-\infty,0)}$ is the whole future tight region M^+. This completes the proof of Theorem 2.1 in the locally anti-de Sitter case.

14. K-SLICINGS OF HYPERBOLIC ENDS

In this section, we explain how Theorem 2.2 can be seen as a corollary of Theorem 2.1 (more precisely, of the existence of K-slicings in non-elementary MGHC dS_3-spacetimes). We will not go into too many details for several reasons:

- Theorem 2.2 is not a new result; it was proved by F. Labourie some fifteen years ago (using techniques rather different from ours);
- the main tool which allows to translate a result concerning non-elementary MGHC dS_3-spacetimes into a result concerning hyperbolic ends is the so-called $\mathrm{dS}_3 \leftrightarrow \mathbb{H}^3$ duality. This duality is completely similar to the $\mathrm{AdS}_3 \leftrightarrow \mathrm{AdS}_3$ duality described in the previous section.

– this duality was already observed and used by Mess in [47, § 6] to prove that locally de Sitter MGHC spacetimes admit K-slicings using Labourie's result. Here we adopt the reverse point of view: we explain how to deduce Labourie's result from the existence of K-slicings on locally de Sitter MGHC spacetimes.

14.1. $\mathrm{dS}_3 \leftrightarrow \mathbb{H}^3$ **duality.**

14.1.1. *Duality between points and totally geodesic planes.* Consider the quadratic form $Q_{1,3} = -x_1^2 + x_2^2 + x_3^2 + x_4^2$ on $\mathbb{R}^4$. Recall that dS_3 is the hyperboloid $\{Q_{1,3} = 1\}$ endowed with the Lorentzian metric induced by $Q_{1,3}$. Also recall the each of the two sheets of the hyperboloid $\{Q_{1,3} = -1\}$ is a copy of the 3-dimensional hyperbolic space; we denote them by H^3_- and H^3_+. For every set $A \subset \mathbb{R}^4$, we denote by $A^\perp$ the $Q_{2,2}$-orthogonal of A in $\mathbb{R}^4$. We choose an orientation on $\mathbb{R}^4$.

Let x be a point in $\mathrm{dS}_3 \subset \mathbb{R}^4$. Then $x^\perp$ is an hyperplane in $\mathbb{R}^4$, and the restriction of $Q_{1,3}$ to $x^\perp$ has signature $(+,+,+)$. It follows that $x^\perp \cap H^+_3$ is a totally geodesic plane in H^+_3. We denote this totally geodesic plane by x^*. The orientation of $\mathbb{R}^4$ and the choice of x (rather than $-x$) defines an orientation on $x^\perp$, and subsequently on x^*. Conversely, let P be an oriented totally geodesic plane in H^3_+. Then $P^\perp$ is a oriented line in $\mathbb{R}^4$, and the restriction of $Q_{1,3}$ to $P^\perp$ is negative definite. It follows that $P^\perp$ intersects dS_3 at two (antipodal) points, and the orientation on $P^\perp$ allows to distinguish choose one of these two points that we denote by P^*. Clearly, for every point $x \in \mathrm{dS}_3$ and every totally geodesic plane $P \subset H^3_+$, one has $(x^*)^* = x$ and $(P^*)^* = P$. This defines a duality between the points of dS_3 and the totally geodesic planes in $H^3_+ \simeq \mathbb{H}^3$.

Now, let x be a point in H^3_+. One can check that $x^\perp \cap \mathrm{dS}_3$ is a spacelike totally geodesic space in dS_3; we denote this totally geodesic plane by x^*. Conversely, let P be a spacelike totally geodesic plane in dS_3. The intersection $P^\perp \cap H^3_+$ is reduced to a point, that we denote by P^*. Clearly, for every point $x \in H^3_+$ and every spacelike totally geodesic plane $P \subset \mathrm{dS}_3$, one has $(x^*)^* = x$ and $(P^*)^* = P$. This defines a duality between the points of $H^3_+ \simeq \mathbb{H}^3$ and the spacelike totally geodesic planes in dS_3.

14.1.2. *Duality between convex surfaces in* dS_3 *and convex surfaces in* H^3_+. Let S be a strictly convex spacelike surface in dS_3. Using exactly the same construction as in §13.1.2 (but now using the duality between points and planes described in §14.1.1 instead of those described in §13.1.1), one can to define a strictly convex $S^* \subset H^+_3$ such that the support planes of S (resp. S^*) are the duals of the points of S^* (resp. S). Conversely, given a strictly convex surface in H^3_+, one can define a strictly convex spacelike surface in dS_3 such that the support planes of S (resp. S^* are the duals of the points of S^* (resp. S). Then, for every strictly convex (resp. strictly convex spacelike) surface S in H^3_+ (resp. dS_3), one has $(S^*)* = S$. Moreover, the same arguments as in the proof of Proposition 13.7 allow to prove:

Proposition 14.1. *Let S be a strictly convex (resp. strictly convex spacelike) surface in H^3_+ (resp. dS_3). If S has constant K-curvature κ, then S^* has constant curvature $-\kappa^{-1}$.*

14.1.3. *Duality between MGHC* dS_3*-spacetimes and hyperbolic ends.*

Definition 14.2. A (geometrically finite) *hyperbolic end* is a hyperbolic 3-manifold (M, g) such that:

– M is homeomorphic to $\Sigma \times (0, +\infty)$ where Σ is a closed surface,

– if $(\overline{M}, \bar{g})$ is the metric completion of M, and $\bar{\Sigma} = \overline{M} \setminus M$, then $(\bar{\Sigma}, \bar{g})$ is a hyperbolic surface homeomorphic to Σ,
– $(\bar{\Sigma}, \bar{g})$ is concave, i.e. there is no geodesic in M connecting two elements of $\bar{\Sigma}$.
– $(\bar{\Sigma}, \bar{g})$ is a pleated surface.

Let (M, g) be a hyperbolic end. It admits a conformal boundary homeomorphic to Σ. This conformal boundary is naturally endowed with a Möbius structure. As explained in §6.2, every compact surface with negative Euler caracteristic endowed with a Möbius structure defines a non-elementary MGHC dS_3-spacetime. We shall denote by (M^*, g^*) the non-elementary MGHC dS_3-spacetime associated to $(\bar{\Sigma}, \bar{g})$. We say that (M^*, g^*) is the dual of the hyperbolic end (M, g).

Conversely, for every non-elementary MGHC dS_3-spacetime (M, g), one can define a geometrically finite hyperbolic end (M^*, g^*). We say that (M^*, g^*) is the dual of (M, g).

14.1.4. *Duality between convex surfaces in MGHC* dS_3*-spacetimes and convex surfaces in hyperbolic ends.* Consider a non-elementary MGHC dS_3-spacetime (M, g) and the hyperbolic (M^*, g^*) which is dual to (M, g).

Let Σ be a strictly convex Cauchy surface in M. Then Σ lifts to a strictly convex immersed spacelike surface $S \subset \mathrm{dS}_3$. The dual surface S^* is a strictly convex immersed surface in H^3_+. It can be checked that S^* projects to a compact (strictly convex) surface Σ^* in M^*.

Conversely, if Σ is a strictly convex surface in M^*, one can lift Σ to a strictly convex surface S in H^3_+, consider the dual S^* of S , and project S^* to a strictly convex Cauchy surface Σ^* in $((M^*)^*, (g^*)^*) = (M, g)$.

For every strictly convex Cauchy (resp. compact) surface Σ in (M, g) (resp. in (M^*, g^*)), one has $(\Sigma^*)^* = \Sigma$. Moreover, if Σ has constant K-curvature a, then Σ^* has constant K-curvature $-1/a$.

14.2. **Sketch of proof of Theorem 2.2.**

Consider a hyperbolic end (M, g). Consider the future complete non-elementary MGHC dS_3-spacetimes (M^*, g^*) dual to the hyperbolic end (M, g). According to Theorem 2.1, (M^*, g^*) admits a K-time $\kappa^* : M^* \to (-\infty, -1)$. For every $a \in (-\infty, 0)$, the level set $\Sigma^*_a := (\kappa^*)^{-1}(a)$ is a strictly convex Cauchy surface in M^* with constant K-curvature equal to a. Then $\Sigma_a := (\Sigma^*_a)^*$ is a strictly convex compact surface in M with constant K-curvature $-1/a$. Then $\{\Sigma_a\}_{a \in]-\infty, -1[}$ is a family of compact surfaces with constant K-curvature in M. Using the fact that $\{\Sigma^*_a\}_{a \in]-\infty, -1[}$ is a trivial foliation of M^*, it is quite easy to prove that $\{\Sigma_a\}_{a \in]-\infty, -1[}$ is a trivial foliation of M. Theorem 2.2 follows.

15. Surfaces with prescribed K-curvature

In this section, we prove Theorem 2.4 and Corollary 2.5. This theorem will follow from :

– the existence K-times on MGHC spacetimes with constant curvature (Theorem 2.1) which will provide us with a pair of barriers,
– a generalisation of Theorem 3.2 (Theorem 15.2 below), which asserts that surfaces with prescribed K-curvature exist as soon as barriers exist.

Definition 15.1. Let (M, g) be a 3-dimensional globally hyperbolic spacetime with compact Cauchy surfaces. Let $f : M \to (-\infty, 0)$ be a smooth function. A *pair of f-barriers* is a pair of disjoint strictly convex Cauchy surfaces Σ^-, Σ^+ in M, such that:

a. Σ^- is in the past of Σ^+,
b. $\kappa^{\Sigma^-}(x) \leq f(x)$ for every $x \in \Sigma^-$,
c. $\kappa^{\Sigma^+}(x) \geq f(x)$ for every $x \in \Sigma^+$.

Theorem 15.2 (Gerhardt, [30])**.** *Let (M, g) be a 3-dimensional spatially compact globally hyperbolic spacetime, and $f : M \to]-\infty, 0[$ be a smooth function. Assume that M admits a pair of f-barriers (Σ^-, Σ^+). Then M admits a strictly convex Cauchy surface Σ such that $\kappa^{\Sigma}(x) = f(x)$ for all $x \in \Sigma$.*

Remark 15.3. We recall that the sign conventions of Gerhardt are different from ours (see Remark 3.3). This is the reason why the function f is required to be strictly positive in [30], whereas it is required to be strictly negative in the statement above.

Proof of Theorem 2.4. Consider a 3-dimensional non-elementary MGHC spacetime (M, g) with constant curvature Λ. Recall that in the $\Lambda \geq 0$ case we assume that (M, g) is future complete (see Remark 5.3). Let $f : M_0 \to \mathbb{R}$ be a smooth function such that the range of f is contained in a compact interval $[a, b] \subset]-\infty\,, \min(0, -\Lambda)[$. Define M_0 as follow : $M_0 = M$ is $\Lambda \geq 0$ (flat and locally de Sitter case), and M_0 is the past of the convex core of M if $\Lambda < 0$ (anti de Sitter case). By Theorem 2.1, there exists a K-time $\kappa : M_0 \to]-\infty, \min(-\Lambda, 0)[$. Consider the level sets of κ:

$$\Sigma^- := \kappa^{-1}(a) \quad \text{and} \quad \Sigma^+ := \kappa^{-1}(b).$$

By definition of a K-time, Σ^-, Σ^+ are two strictly convex Cauchy surfaces with constant K-curvarture respectively equal to a and b, and Σ^- is in the past of Σ^+. In particular, (Σ^-, Σ^+) is a pair of f-barriers. Therefore Theorem 15.2 applies in our situation and provides us with a strictly convex Cauchy surface Σ such that $\kappa^{\Sigma}(x) = f(x)$ for every x in Σ. □

Proof of Corollary 2.5. Write M as a topological product $\Sigma_0 \times \mathbb{R}$, consider the function $f : M \to \mathbb{R}$ defined by $f(x, t) = f_0(x)$ where $x \in \Sigma_0$ and $t \in \mathbb{R}$, and apply Theorem 2.4 to the function f. □

16. The Minkowski problem

The purpose of this section is to prove Theorem 2.6. As for the proof of Theorem 2.4, we will use Theorem 2.1 to get a pair of barriers, and then, a theorem of Gerhardt which asserts that surfaces with prescibed K-curvature exist as soon as barriers exist. As a preliminary step, we will need to translate Theorem 2.6 into a statement concerning a function defined on the unit tangent bundle of a non-elementary MGHC flat spacetime.

We begin by stating Gerhardt's result:

Theorem 16.1 (Gerhardt, see [31]; see also Remarks 3.3 and 15.3)**.** *Let (M, g) be a spatially compact globally hyperbolic spacetime. Let Ω be an open subset of M bounded by two disjoint strictly convex Cauchy surfaces Σ^- and Σ^+, where Σ^- is assumed to be in the past of Σ^+. Denote by $T^{-1}\Omega$ the bundle of future-oriented unit timelike tangent vectors over Ω:*

$$T^{-1}\Omega = \{(x, \nu) \in TM \mid x \in \Omega\,,\ \nu \text{ is future-oriented and } g(\nu, \nu) = -1\}.$$

Consider a smooth function $\Phi : T^{-1}\Omega \to \mathbb{R}$ with the two following properties:

(i) *there is a negative constant c_1 such that, for all $(x,\nu) \in T^{-1}\Omega$,*

$$\Phi(x,\nu) \leq c_1 < 0.$$

(ii) *there are some constants $c_2, c_3 \in \mathbb{R}$ such that, for all $(x,\nu) \in T^{-1}\Omega$,*

$$|||d_x\Phi(x,\nu)||| \leq c_2\left(1+\|\nu\|^2\right) \quad \textit{and} \quad |||d_\nu\Phi(x,\nu)||| \leq c_3\left(1+\|\nu\|\right)$$

where $d_x\Phi$ and $d_\nu\Phi$ are the derivatives of Φ with respect to x and ν, where $|||.|||$ is the operator norm associated to an arbitrary auxiliary Riemannian metric on M, and $\|\nu\|$ is the norm of ν for this Riemannian metric[14].

Assume that (Σ^-,Σ^+) is a pair of Φ-barriers, that is:

- *for every $x \in \Sigma^-$, one has $\kappa^{\Sigma^-}(x) \leq \Phi(x,\nu^{\Sigma^-}(x))$,*
- *for every $x \in \Sigma^+$, one has $\kappa^{\Sigma^+}(x) \geq \Phi(x,\nu^{\Sigma^+}(x))$.*

Then there exists a strictly convex Cauchy surface Σ in M such that, for every $x \in \Sigma$, one has

$$\kappa^{\Sigma}(x) = \Phi\left(x,\nu^{\Sigma}(x)\right).$$

In order to show that Theorem 16.1 applies in our situation, we shall need the following elementary lemma:

Lemma 16.2. *Let Γ is a co-compact Fuchsian group in SO$(1,2)$, and $f : \mathbb{H}^2 \to \mathbb{R}$ be a Γ-invariant function. See $\mathbb{H}^2$ as the upper sheet of the hyperboloid $x_1^2+x_2^2-x_3^2 = -1$ in $\mathbb{R}^3$, and consider an arbitrary euclidian norm $\|.\|$ on $\mathbb{R}^3$. Then there exists a constant c such that, for every $\mathrm{n} \in \mathbb{H}^2$,*

$$|||df(\nu)||| \leq c$$

where $|||.|||$ is the operator norm[15] *associated to the euclidian norm $\|.\|$.*

Proof. Let $\|.\|_{Lor} := \sqrt{x_1^2+x_2^2-x_3^2}$ be the Lorentzian pseudo-norm on $\mathbb{R}^3$ which is preserved by SO$(1,2)$, and $\langle .,. \rangle_{Lor}$ be the associated Lorentzian pseudo-scalar product. We will prove the lemma for the Euclidean norm $\|.\|_{Euc} := \sqrt{x_1^2+x_2^2+x_3^2}$; the general case of an arbitrary Euclidean norm will follow, up to a change of the constant c.

[14]More precisely, we choose a Riemannian metric η on M, we observe that for every $(x,\nu) \in TM$ there is a canonical identification between $T_{(x,\nu)}(T_xM) \simeq T_xM$, and we set:

$$\begin{aligned}
\|v\| &:= \sqrt{\eta(v,v)} \\
|||d_x\Phi(x,\nu)||| &:= \sup_{v\in T_xM} \frac{|d_x\Phi(x,\nu).v|}{\|v\|} \\
|||d_\nu\Phi(x,\nu)||| &:= \sup_{v\in T_{(x,\nu)}(T_x^{-1}\Omega)\subset T_{(x,\nu)}(T_xM)\simeq T_xM} \frac{|d_\nu\Phi(x,\nu).v|}{\|v\|}
\end{aligned}$$

[15]More precisely, $|||df(\nu)||| = \sup_{v\in T_\nu\mathbb{H}^2\subset T_\nu\mathbb{R}^3\simeq\mathbb{R}^3} \frac{|df(\nu).v|}{\|v\|}$.

We denote by ∇f the Lorentzian gradient of f (i.e. the gradient with respect to the hyperbolic norm on $\mathbb{H}^2$ induced by $\|.\|_{Lor}$). First observe that, for every $\nu \in \mathbb{H}^2$, and every vector $v \in T_\nu\mathbb{H}^2 \subset \mathbb{R}^3$, we have:

$$\|v\|_{Euc} \geq \|v\|_{Lor}$$

Now, observe that, for every $\nu \in \mathbb{H}^2$, the restriction of $\langle .,. \rangle_{Lor}$ to the plane $T_\nu\mathbb{H}^2$ is positive definite. Hence, for every $\nu \in \mathbb{H}^2$ and every $v \in T_\nu\mathbb{H}^2$, the Cauchy Schwarz inequality for the restriction of $\langle .,. \rangle_{Lor}$ to $T_\nu\mathbb{H}^2$ yields:

$$|\langle (\nabla f)(\nu), v\rangle_{Lor}| \leq \|(\nabla f)(\nu)\|_{Lor}\ \|v\|_{Lor}.$$

As a consequence, for every $\nu \in \mathbb{H}^2$ and every $v \in T_\nu\mathbb{H}^2$, we have:

$$\begin{aligned} |df(\nu).v| &= |\langle (\nabla f)(\nu), v\rangle_{Lor}| \\ &\leq \|(\nabla f)(\nu)\|_{Lor}\ \|v\|_{Lor} \\ &\leq \|(\nabla f)(\nu)\|_{Lor}\ \|v\|_{Euc} \end{aligned}$$

But since $\nabla f(\nu)$ is Γ-equivariant, where Γ is co-compact, its hyperbolic norm $\|\nabla f(\nu)\|_{Lor}$ is uniformly bounded from above. The lemma follows. □

Now we are in a position to prove Theorem 2.6. We consider a co-compact Fuchsian subgroup Γ in $\mathrm{SO}(1,2)$, a subgroup $\mathbf{\Gamma}$ of $\mathrm{SO}(1,2) \ltimes \mathbb{R}^3$ which projects bijectively on Γ, and a Γ-invariant function $f : \mathbb{H}^2 \to \mathbb{R}$.

Theorem 5.2 states that there exists a future-complete regular domain $E = E(\mathbf{\Gamma})$ in Min_3 such that the action of $\mathbf{\Gamma}$ on E is free and properly discontinuous, and such that $M := \mathbf{\Gamma}\backslash E$ is a non-elementary future-complete MGHC flat spacetime. We consider the bundle

$$T^{-1}\,\mathrm{Min}_3 = \{(x,\nu) \in T\,\mathrm{Min}_3 \mid \nu \text{ future-oriented, } \langle \nu,\nu\rangle = -1\},$$

and we canonically identify $T^{-1}\,\mathrm{Min}_3$ with $\mathrm{Min}_3 \times \mathbb{H}^2$. Then we consider the function $F : T^{-1}\,\mathrm{Min}_3 \simeq \mathrm{Min}_3 \times \mathbb{H}^2 \to (-\infty, 0)$ defined by

$$F(x,\nu) := f(\nu)$$

The group $\mathbf{\Gamma}$ acts on $T^{-1}\,\mathrm{Min}_3$ by

$$\mathrm{g}.(x,\nu) = (\mathbf{g}.(x), d\mathbf{g}.x(\nu)) = (\mathbf{g}.(x), \mathrm{g}(\nu))$$

(where $\mathbf{g} \in \mathbf{\Gamma} \subset \mathrm{SO}(1,2) \ltimes \mathbb{R}^3$ and $g \in \Gamma \subset \mathrm{SO}(1,2)$ is its linear part) and

$$\mathbf{\Gamma}\backslash(T^{-1}\,\mathrm{Min}_3) = T^{-1}M = \{(x,\nu) \in TM \mid \nu \text{ future-oriented, } \langle \nu,\nu\rangle = -1\}.$$

Since the function f is Γ-invariant, the function F is $\mathbf{\Gamma}$-invariant. Hence F induces a function $\Phi : T^{-1}M \to (-\infty, 0)$.

Consider a Cauchy surface Σ in $M = \mathbf{\Gamma}\backslash E$. Then Σ lifts to a $\mathbf{\Gamma}$-invariant spacelike surface S in $E \subset \mathrm{Min}_3$. Let ν^S, ν^Σ, κ^S, κ^Σ be respectively the Gauss map of S, the Gauss map of Σ, the K-curvature of S and the K-curvature of Σ. We have the following equivalences

$$\begin{array}{rrcll} & f(\nu) & = & \kappa^S \circ (\nu^S)^{-1}(\nu) & \text{for every } \nu \in \text{range}(\nu^S) \subset \mathbb{H}^2 \\ \iff & F(x,\nu^S(x)) & = & \kappa^S(x) & \text{for every } x \in S \\ \iff & \Phi(x,\nu^\Sigma(x)) & = & \kappa^\Sigma(x) & \text{for every } x \in \Sigma \end{array}$$

Therefore, in order to prove Theorem 2.6, it is enough to find a strictly convex Cauchy surface Σ in M such that

$$\Phi\left(x, \nu^\Sigma(x)\right) = \kappa^\Sigma(x) \quad \text{for every } x \in \Sigma \tag{8}$$

We want to use Theorem 16.1 to get such a Cauchy surface Σ. So, we have to prove that the function Φ satisfies the conditions (i) and (ii) in the statement of Theorem 16.1, and we have to find a pair of Φ-barriers.

The function $f : \mathbb{H}^2 \to (-\infty, 0)$ is continuous and Γ-invariant. The group Γ is co-compact. Hence, the range of f is a compact interval $[a, b] \subset (0, +\infty)$. The range of the function $\Phi : T^{-1}M \to (-\infty, 0)$ is the same compact interval $[a, b]$. In particular, condition (i) of Theorem 16.1 is satisfied.

By Theorem 2.1, the spacetime M admits a K-time $\kappa : M \to (-\infty, 0)$. We consider the Cauchy surfaces $\Sigma^- := \kappa^{-1}(a)$ and $\Sigma^+ := \kappa^{-1}(b)$. By definition of a K-time, the Cauchy surface Σ^- is strictly convex and has constant K-curvature a, the surface Σ^+ is strictly convex Cauchy and has constant K-curvature b, and Σ^- is in the past of Σ^+. In particular, (Σ^-, Σ^+) is a pair of Φ-barriers in the sense defined in the statement of Theorem 16.1.

We denote by Ω the open subset of M bounded by the Cauchy surfaces Σ^- and Σ^+. Observe that Ω is relatively compact in M. We denote by O the lift of Ω in M.

Since the function F is independant of x, we have $d_x F(x, \nu) = 0$ for every $(x, \nu) \in T^{-1}\,\mathrm{Min}_3$. Hence we have $d_x \Phi(x, \nu) = 0$ for every $(x, \nu) \in T^{-1}\,\mathrm{Min}_3$. In particular, the first inequality in condition (ii) of Theorem 16.1 is satisfied.

Let η be a Riemannian metric on M. One can lift η to a $\mathbf{\Gamma}$-invariant Riemannian metric h on $E \subset \mathrm{Min}_3$. For every $(x, \nu) \in T^{-1}M$, if $(\tilde{x}, \tilde{\nu}) \in T^{-1}\,\mathrm{Min}_3 \simeq \mathrm{Min}_3 \times \mathbb{H}^2$ is a lift of (x, ν), then

$$d_\nu \Phi(x, \nu) = d_{\tilde{\nu}} F(\tilde{x}, \tilde{\nu}) = df(\tilde{\nu}) \tag{9}$$

According to Lemma 16.2, for every $\widetilde{x} \in O$, there exists a constant $c_{\widetilde{x}}$, such that, for every $\tilde{\nu} \in \mathbb{H}^2$,

$$|||df(\tilde{\nu})|||_{\tilde{x}} \leq c_{\bar{x}}$$

where $|||.|||_{\tilde{x}}$ is the operator norm associated to the euclidian norm $h_{\tilde{x}}$ on $\mathbb{R}^3 \simeq T_{\tilde{x}}\,\mathrm{Min}_3$. Now observe that $\sup_{\tilde{\nu} \in \mathbb{H}^2} |||df(\tilde{\nu})|||_{\bar{x}}$ depends in a bounded way of $\tilde{x}$ as far as $\tilde{x}$ stays in O; indeed the Riemannian metric h and the derivative df are $\mathbf{\Gamma}$-equivariant and $\mathbf{\Gamma} \backslash O = \Omega$ is relatively compact. Hence, there exists a constant c such that, for every $\tilde{x} \in O$ and every $\tilde{\nu} \in \mathbb{H}^2$,

$$|||df(\tilde{\nu})|||_{\tilde{x}} \leq c.$$

Using equality (9), this implies that, for every $(x, \nu) \in T^{-1}\Omega$,

$$|||d_\nu \Phi(x, \nu)||| \leq c$$

where $|||d_\nu \Phi(x, \nu)|||$ is the operator norm of $d_\nu \Phi(x, \nu)$ associated to the Riemannian metric η. In particular, the second inequality in condition (ii) of Theorem 16.1 is satisfied.

So all the hypotheses of Theorem 16.1 are satisfied. Hence this theorem provides us with a strictly convex Cauchy surface Σ in M satisfying (8). In order to achieve he proof of Theorem 2.6 the only remaining point is to prove the uniqueness of $\mathbf{\Gamma}$-solution for a given uniform lattice $\mathbf{\Gamma}$ of $\mathrm{SO}(1, 2) \ltimes \mathbb{R}^3$.

Let S_1, S_2 be two such $\mathbf{\Gamma}$-invariant solutions. For any t, let S_i^t be the surface obtained by pushing S_i along the normal geodesics during the time t. Let t_1 be the minimal time such that $S_1^{t_1}$ lies in the future of S_2. Since S_1 and S_2 projects in the quotient as compact Cauchy surfaces, t_1 is well-defined and finite. Exchanging S_1 with S_2 if necessary, one can assume $t_1 \geq 0$. Then S_2 and $S_1^{t_1}$ are tangent at a common point x. This point is at lorentzian distance t_1 from a point y in S_1 along a timelike geodesic orthogonal to S_1 at y. This geodesic is also orthogonal to S^{t_1}, and thus to S_2, at x. Hence $\nu^{S_1}(y) = \nu^{S_2}(x)$. Since

they are both solutions of the same Minkowski problem, $\kappa^{S_1}(y) = \kappa^{S_2}(x)$. But the K-curvature strictly increases by pushing along normal geodesics (Remark 11.12): at one hand, the K-curvature of $S_1^{t_1}$ is bounded from above by κ^{S_2} (since it lies in the future of S_2); on the other hand, if $t_1 > 0$, it should be strictly bigger than $\kappa^{S_1}(y)$. It follows that t_1 is 0: S_1 lies in the future of S_2. In particular, $t_2 \geq 0$. Permuting the role of S_2 and S_1, we prove similarly that S_2 lies in the future of S_1. Hence $S_1 = S_2$. Theorem 2.6 follows.

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