



**HAL**  
open science

## Pruning a Lévy continuum random tree

Romain Abraham, Jean-François Delmas, Guillaume Voisin

► **To cite this version:**

Romain Abraham, Jean-François Delmas, Guillaume Voisin. Pruning a Lévy continuum random tree. 2008. hal-00270803v1

**HAL Id: hal-00270803**

**<https://hal.science/hal-00270803v1>**

Preprint submitted on 7 Apr 2008 (v1), last revised 12 Jul 2010 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# PRUNING A LÉVY CONTINUUM RANDOM TREE

ROMAIN ABRAHAM, JEAN-FRANÇOIS DELMAS, AND GUILLAUME VOISIN

ABSTRACT. Given a general critical or sub-critical branching mechanism, we define a pruning procedure of the associated Lévy continuum random tree. This pruning procedure is defined by adding some marks on the tree, using Lévy snake techniques. We then prove that the resulting sub-tree after pruning is still a Lévy continuum random tree. This last result is proved using the exploration process that codes the CRT, a special Markov property and martingale problems for exploration processes. We finally give the joint law under the excursion measure of the lengths of the excursions of the initial exploration process and the pruned one.

## 1. INTRODUCTION

Continuous state branching processes (CSBP) were first introduced by Jirina [17] and it is known since Lamperti [18] that these processes are the scaling limits of Galton-Watson processes. They hence model the evolution of a large population on a long time interval. The law of such a process is characterized by the so-called branching mechanism function  $\psi$ . We will be interested mainly by critical or sub-critical CSBP. In those cases, the branching mechanism  $\psi$  is given by

$$(1) \quad \psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell) \left( e^{-\lambda\ell} - 1 + \lambda\ell \right), \quad \lambda \geq 0,$$

with  $\alpha \geq 0$ ,  $\beta \geq 0$  and the Lévy measure  $\pi$  is a positive  $\sigma$ -finite measure on  $(0, +\infty)$  such that  $\int_{(0,+\infty)} (\ell \wedge \ell^2) \pi(d\ell) < \infty$ .

The aim of this paper is, given two sub-critical or critical branching mechanisms

$$\begin{aligned} \psi(\lambda) &= \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell) \left( e^{-\lambda\ell} - 1 + \lambda\ell \right), \\ \psi_0(\lambda) &= \alpha_0\lambda + \beta_0\lambda^2 + \int_{(0,+\infty)} \pi_0(d\ell) \left( e^{-\lambda\ell} - 1 + \lambda\ell \right), \end{aligned}$$

to construct a pair of processes  $((Y_t, Y_t^0), t \geq 0)$  such that

- The process  $Y$  (resp.  $Y^0$ ) is a CSBP with branching mechanism  $\psi$  (resp.  $\psi_0$ ),
- $Y_0^0 = Y_0$ ,
- For every  $t \geq 0$ ,  $Y_t^0 \leq Y_t$ .

In other words,  $Y^0$  represents the evolution of a sub-population of the total population  $Y$ . Therefore, it rather clear that this construction is only possible under some natural assumptions:

- The diffusion coefficient of  $Y$  and  $Y^0$  is the same,  $\beta = \beta_0$ .

---

*Date:* April 7, 2008.

*2000 Mathematics Subject Classification.* 60J25, 60G57, 60J80.

*Key words and phrases.* continuum random tree, Lévy snake, special Markov property.

- There exists a positive  $\sigma$ -finite measure  $\pi_1$  on  $(0, +\infty)$  such that  $\int_{(0,+\infty)} \ell \pi_1(d\ell) < +\infty$  and

$$(2) \quad \pi = \pi_0 + \pi_1.$$

Thus, the jumps of the total population  $Y$  are those of  $Y^0$  plus some additional ones.

- The drift coefficient

$$(3) \quad \alpha_1 = \alpha_0 - \alpha - \int_{(0,+\infty)} \ell \pi_1(d\ell)$$

is non-negative. Thus the total population  $Y$  decreases slower than the population  $Y^0$ .

A first approach for this construction is to consider the population  $Y^0$  as an initial Eve-population (recall the initial condition at  $t = 0$ ) and the total population  $Y$  is obtained by adding an immigration process. This is the purpose of [3] where  $Y$  is constructed as a CSBP with immigration rate proportional to the size of the population. Notice that [3] deals also with super-critical CSBP.

The alternative construction we present here is to use the continuum random trees that describe the genealogy of the CSBPs. Continuum random trees (CRT) were first introduced by Aldous [7, 6, 8], and then extended by Le Gall and Le Jan [20] and developed later by Duquesne and Le Gall [15]. Here we start with the CRT associated with the total population  $Y$  and we introduce a pruning procedure of this tree in order to recover a CRT associated with the sub-population  $Y^0$ . As we consider general critical or sub-critical branching mechanism, this work extends previous work from Abraham and Serlet [5] on Brownian CRT ( $\pi = 0$ ) and Abraham and Delmas [4] on CRT without Brownian part ( $\beta = 0$ ). See also Bertoin [10] for an approach using Galton-Watson trees and  $\pi_1 = 0$ .

This work has also others motivations. This result is a first step to construct a general fragmentation process associated with a general CRT. It also helps to understand the construction given in [2] on the Williams' decomposition for CRT with proportional immigration.

The next subsections give a brief presentation of the mathematical objects and state the main results. The last one describes the organization of the paper.

**1.1. Exploration process.** The coding of a tree by its height process is now well-known. For instance, the height process of Aldous' CRT [8] is a normalized Brownian excursion. In [20], Le Gall and Le Jan associated with a Lévy process with no negative jumps that does not drift to infinity,  $X = (X_t, t \geq 0)$ , a CSBP and a Lévy CRT which keeps track of the genealogy of the CSBP. Let  $\psi$ , given by formula (1), denote the Laplace exponent of  $X$ :  $\mathbb{E}[e^{-\lambda X_t}] = e^{t\psi(\lambda)}$  for  $\lambda \geq 0$ . Following [15], we shall also assume that  $X$  is of infinite variation a.s. which implies that  $\beta > 0$  or  $\int_{(0,1)} \ell \pi(d\ell) = \infty$ . Notice those hypothesis are fulfilled in the stable case:  $\psi(\lambda) = \lambda^\alpha$ ,  $\alpha \in (1, 2)$  and the quadratic case  $\psi(\lambda) = \lambda^2$ .

Informally, for the height process  $H = (H_t, t \geq 0)$  associated with  $X$ ,  $H_t$  gives the distance (which can be understood as the number of generations) between the individual labeled  $t$  and the root 0 of the CRT. An individual labeled  $t$  is an ancestor of  $s \geq t$  if  $H_t = \inf\{H_r, r \in [t, s]\}$ , and  $\inf\{H_r, r \in [u, v]\}$  is the "generation" of the most recent common ancestor of  $u$  and  $v$ . The height process is a key tool in this construction but it is not a Markov process. The so-called exploration process  $\rho = (\rho_t, t \geq 0)$  is a càd-làg Markov process taking values in  $\mathcal{M}_f(\mathbb{R}_+)$ , the set of measures with finite mass on  $\mathbb{R}_+$  endowed with the topology of weak convergence. The height process can easily be recovered from the exploration process as

$H_t = H(\rho_t)$ , where  $H(\mu)$  denotes the supremum of the closed support of the measure  $\mu$  (with the convention that  $H(0) = 0$ ).

To understand what the exploration process means, let us use the queuing system representation of [20] when  $\beta = 0$ . We consider a preemptive LIFO (Last In, First Out) queue with one server. A jump of  $X$  at time  $s$  corresponds to the arrival of a new customer requiring a service equal to  $\Delta_s := X_s - X_{s-}$ . The server interrupts his current job and starts immediately the service of this new customer (preemptive LIFO procedure). When this new service is finished, the server will resume the previous job. When  $\pi$  is infinite, all services will suffer interruptions. The customer (arrived at time)  $s$  will still be in the system at time  $t > s$  if and only if  $X_{s-} < \inf_{s \leq r \leq t} X_r$  and, in this case, the quantity  $\rho_t(H_s)$  represents the remaining service required by the customer  $s$  at time  $t$ . Observe that  $\rho_t([0, H_t])$  corresponds to the load of the server at time  $t$  and is equal to  $X_t - I_t$  where

$$I_t = \inf\{X_u, 0 \leq u \leq t\}.$$

Another process of interest will be the dual process  $(\eta_t, t \geq 0)$  which is also a measure-valued process. In the queuing system description, for a customer  $s$  still present in the system at time  $t$ , the quantity  $\eta_t(H_s)$  represents the amount of service of customer  $s$  already completed at time  $t$ , so that  $\rho_t(H_s) + \eta_t(H_s) = \Delta_s$  holds for any customer  $s$  still present in the system at time  $t$ .

Definition and properties of the height process, the exploration process and the dual process are recalled in Section 2.

**1.2. The pruned exploration process.** As described before, we start from a CRT constructed from a Lévy process  $X$  with Laplace exponent  $\psi$ . This CRT has a rather complicated structure with nodes of index three (if  $\beta > 0$ ) and nodes of infinite index (if  $\pi \neq 0$ ). Each infinite node corresponds to a jump of the Lévy process  $X$ . We want to prune this CRT and for that purpose, we will put some marks on the CRT and we will prune the CRT according to them. These marks are of two types: some lay on the nodes of infinite index of the tree and the others lay on the skeleton.

**1.2.1. Marks on the nodes ( $\pi_1 \neq 0$ ).** Recall that the nodes with infinite index are described by the jumps of the process  $X$  (i.e. by the Lévy measure  $\pi$ ) and (2). Therefore, we must distinguish the jumps of  $X$  that are due to measure  $\pi_0$  from those due to measure  $\pi_1$ . This is done by taking

$$X = X^{(0)} + X^{(1)}$$

where  $X^{(0)}$  and  $X^{(1)}$  are independent Lévy processes with respective Lévy measure  $\pi_0$  and  $\pi_1$  (see Section 2.1). We suppose that the jumps of  $X$  that are jumps of  $X^{(1)}$  are marked, which gives marked nodes on the CRT.

**1.2.2. Marks on the skeleton ( $\alpha_1 > 0$ ).** Those marks are only necessary when  $\alpha_1 > 0$ , see (3) for the definition of  $\alpha_1$ . Given an individual labeled by  $t$  (at a distance  $H_t$  from the root), we add some marks along its lineage according to a Poisson process with intensity  $\alpha_1$ . As the nodes of the CRT are of null Lebesgue measure on this lineage, these marks will lay on the skeleton of the tree. Of course, this procedure must be coherent so that, given two distinct individuals, the marks on their lineage must coincide between the root and their last common ancestor. Therefore, we use Lévy snakes techniques (see Duquesne-Le Gall, [15]) to define these marks. Let us remark that in [15], the height process  $H$  is supposed to be continuous for the construction of the Lévy snake. We will explain in the appendix how to remove this technical assumption.

1.2.3. *The Lévy Poisson snake and the pruned exploration process.* As we don't use the CRT framework but only the exploration processes that code the CRTs, all what has been described above must be translated in terms of CRT. Therefore, we define a measure-valued process  $\mathcal{S} := ((\rho_t, m_t^{\text{nod}}, m_t^{\text{ske}}), t \geq 0)$  called the Lévy Poisson snake, where the process  $\rho$  is the usual exploration process whereas the processes  $m^{\text{nod}}$  and  $m^{\text{ske}}$  keep track of the marks on the CRT respectively on the nodes and on the skeleton. We set  $A_t$  for the Lebesgue measure of the set of the individuals prior to  $t$  that contain no mark on their lineage (see (18) for a precise definition), we consider its right-continuous inverse  $C_t = \inf\{r > 0; A_r \geq t\}$  and we define the pruned exploration process  $\tilde{\rho}$  by

$$\tilde{\rho}_t = \rho_{C_t} \quad \text{for } t \geq 0.$$

**Theorem 1.1.** *The pruned exploration process  $\tilde{\rho}$  is distributed as the exploration process associated with a Lévy process with Laplace exponent  $\psi_0$ .*

The proof relies on a martingale problem for  $\tilde{\rho}$  and a special Markov property, Theorem 4.2, which covers the quadratic case (see Proposition 6 in [5] or Proposition 7 in [11]) and the case without quadratic term (see Theorem 3.12 in [4]).

Finally, we give the joint law of the length of the excursion of the exploration process and the length of the excursion of the pruned exploration process, see Proposition 6.1.

1.3. **Organization of the paper.** We first recall in the next Section the construction of the exploration process, how it codes a CRT and its main properties we shall use. We also define the marked Lévy snake that is used for pruning the tree. In Section 3, we define rigorously the pruned exploration process  $\tilde{\rho}$  and restate precisely Theorem 1.1. The rest of the paper is devoted to the proof of that theorem. In Section 4, we state and prove a special Markov property of the marked Lévy snake, that gives the law of the snake “above” the marks, conditionally on  $\tilde{\rho}$ . We use this special property in Section 5 to derive from the martingale problem satisfied by  $\rho$ , introduced in [1] when  $\beta = 0$ , a martingale problem for  $\tilde{\rho}$  which allows us to obtain the law of  $\tilde{\rho}$ . Finally, we compute in the last section, under the excursion measure, the joint law of the lengths of the excursions of  $\rho$  and  $\tilde{\rho}$ . The Appendix is devoted to some extension of the Lévy snake when the height process is not continuous.

## 2. LÉVY SNAKE: NOTATIONS AND PROPERTIES

We recall here the construction of the Lévy continuum random tree introduced in [20, 19] and developed later in [15]. We will emphasize on the height process and the exploration process which are the key tools to handle this tree. The results of this section are mainly extracted from [15], but for Section 2.4.

If  $E$  is a polish space, let  $\mathcal{B}(E)$  (resp.  $\mathcal{B}_+(E)$ ) be the set of real-valued measurable (resp. and non-negative) functions defined on  $E$  endowed with its Borel  $\sigma$ -field, and let  $\mathcal{M}(E)$  (resp.  $\mathcal{M}_f(E)$ ) be the set of  $\sigma$ -finite (resp. finite) measures on  $E$ , endowed with the topology of vague (resp. weak) convergence. For any measure  $\mu \in \mathcal{M}(E)$  and  $f \in \mathcal{B}_+(E)$ , we write

$$\langle \mu, f \rangle = \int f(x) \mu(dx).$$

2.1. **The underlying Lévy process.** We consider a  $\mathbb{R}$ -valued Lévy process  $X = (X_t, t \geq 0)$  with no negative jumps, starting from 0. Its law is characterized by its Laplace transform: for  $\lambda \geq 0$

$$\mathbb{E} \left[ e^{-\lambda X_t} \right] = e^{t\psi(\lambda)},$$

where its Laplace exponent  $\psi$  is given by

$$\psi(\lambda) = \alpha'\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell) \left( e^{-\lambda\ell} - 1 + \mathbf{1}_{\{\ell < 1\}}\lambda\ell \right),$$

where  $\beta \geq 0$  and the Lévy measure  $\pi$  is a non-negative  $\sigma$ -finite measure on  $(0, +\infty)$  such that  $\int_{(0,+\infty)} (1 \wedge \ell^2)\pi(d\ell) < \infty$ . In this paper, we assume that  $X$

- has first moments (i.e.  $\int_{(0,+\infty)} (\ell \wedge \ell^2)\pi(d\ell) < \infty$ ),
- is of infinite variation (i.e.  $\beta > 0$  or  $\int_{(0,1)} \ell\pi(d\ell) = +\infty$ ),
- does not drift to  $+\infty$ .

The Laplace exponent of  $X$  can then be written as (1), with drift  $\alpha \geq 0$  (as  $X$  does not drift to  $+\infty$ ), quadratic term  $\beta \geq 0$  and Lévy measure  $\pi$ , a non-negative  $\sigma$ -finite measure on  $(0, +\infty)$  such that  $\int_{(0,+\infty)} (\ell \wedge \ell^2)\pi(d\ell) < \infty$ , and  $\int_{(0,1)} \ell\pi(d\ell) = \infty$  or  $\beta > 0$ .

We shall write  $X$  as the sum of a Lévy process  $X^{(0)} = (X_t^{(0)}, t \geq 0)$  with no negative jumps and of an independent subordinator  $X^{(1)} = (X_t^{(1)}, t \geq 0)$ . Let  $\psi_0$  be the Laplace exponent of  $X^{(0)}$ , and  $\phi_1$  the Laplace exponent of  $X^{(1)}$  (notice we have a different sign convention for the Laplace exponent of the Lévy process with no negative jumps and the subordinator):

$$\mathbb{E} \left[ e^{-\lambda X_t^{(0)}} \right] = e^{t\psi_0(\lambda)} \quad \text{and} \quad \mathbb{E} \left[ e^{-\lambda X_t^{(1)}} \right] = e^{-t\phi_1(\lambda)},$$

with  $\psi = \psi_0 - \phi_1$  and

$$\begin{aligned} \psi_0(\lambda) &= \alpha_0\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi_0(d\ell) \left( e^{-\lambda\ell} - 1 + \lambda\ell \right), \\ \phi_1(\lambda) &= \alpha_1\lambda + \int_{(0,+\infty)} \pi_1(d\ell) \left( 1 - e^{-\lambda\ell} \right), \end{aligned}$$

where  $\alpha_0 \geq 0$ ,  $\alpha_1 \geq 0$ ,  $\pi_1$  is a non-negative  $\sigma$ -finite measure on  $(0, +\infty)$  with  $\int_{(0,+\infty)} \ell\pi_1(d\ell) < \infty$  and  $\pi_0$  is a non-negative  $\sigma$ -finite measure on  $(0, +\infty)$  with  $\int_{(0,+\infty)} (\ell \wedge \ell^2)\pi_0(d\ell) < \infty$ . Because of  $\psi = \psi_0 - \phi_1$ , we have (2) and (3). As  $X$  has a finite first moment, we get that the subordinator  $X^{(1)}$  has also a finite first moment. Notice the condition on the Lévy measure of the subordinator,  $\int_{(0,+\infty)} \ell\pi_1(d\ell) < \infty$ , is stronger than the usual condition of integrability of  $1 \wedge \ell$  because of the existence of this first moment.

We assume that  $\phi_1 \neq 0$ , so that

$$(4) \quad \alpha_0 > 0.$$

Since  $X^{(0)} + X^{(1)}$  is a Lévy process with Laplace exponent  $\psi_0 - \phi = \psi$ , we shall assume that  $X = X^{(0)} + X^{(1)}$ . For  $a \in \{0, 1\}$ , let  $\mathcal{J}^a = \{s \geq 0; X_s^{(a)} \neq X_{s-}^{(a)}\}$  be the set of jumping times of  $X^{(a)}$ . Since  $X^{(0)}$  and  $X^{(1)}$  are independent, we have that a.s.  $\mathcal{J}^0 \cap \mathcal{J}^1 = \emptyset$  and  $\mathcal{J} = \mathcal{J}^0 \cup \mathcal{J}^1$  is the set of jumping times of  $X$ . For  $s \in \mathcal{J}$ , we denote by

$$\Delta_s = X_s - X_{s-}$$

the jump of  $X$  at time  $s$  and  $\Delta_s = 0$  otherwise. The two processes  $\sum_{s \in \mathcal{J}^a} \delta_{(s, \Delta_s)}$ , for  $a \in \{0, 1\}$ , are independent Poisson point processes with intensity  $\pi_a$ .

As  $X$  and  $X^{(0)}$  are of infinite variation, we have, see Corollary VII.5 in [9],

$$(5) \quad \lim_{\lambda \rightarrow \infty} \frac{\lambda}{\psi(\lambda)} = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \frac{\lambda}{\psi_0(\lambda)} = 0.$$

It is easy to check, using  $\int_{(0,\infty)} \pi_1(d\ell)\ell < \infty$ , that

$$(6) \quad \lim_{\lambda \rightarrow \infty} \frac{\phi_1(\lambda)}{\lambda} = \alpha_1.$$

Let  $I = (I_t, t \geq 0)$  be the infimum process of  $X$ ,  $I_t = \inf_{0 \leq s \leq t} X_s$ , and let  $S = (S_t, t \geq 0)$  be the supremum process,  $S_t = \sup_{0 \leq s \leq t} X_s$ . We will also consider for every  $0 \leq s \leq t$  the infimum of  $X$  over  $[s, t]$ :

$$I_t^s = \inf_{s \leq r \leq t} X_r.$$

The point 0 is regular for the Markov process  $X - I$ , and  $-I$  is the local time of  $X - I$  at 0 (see [9], chap. VII). Let  $\mathbb{N}$  be the associated excursion measure of the process  $X - I$  away from 0, and let  $\sigma = \inf\{t > 0; X_t - I_t = 0\}$  be the length of the excursion of  $X - I$  under  $\mathbb{N}$ . We will assume that under  $\mathbb{N}$ ,  $X_0 = I_0 = 0$ .

Since  $X$  is of infinite variation, 0 is also regular for the Markov process  $S - X$ . The local time,  $L = (L_t, t \geq 0)$ , of  $S - X$  at 0 will be normalized so that

$$\mathbb{E}[e^{-\beta S_{L_t^{-1}}}] = e^{-t\psi(\beta)/\beta},$$

where  $L_t^{-1} = \inf\{s \geq 0; L_s \geq t\}$  (see also [9] Theorem VII.4 (ii)).

**2.2. The height process and the Lévy CRT.** For each  $t \geq 0$ , we consider the reversed process at time  $t$ ,  $\hat{X}^{(t)} = (\hat{X}_s^{(t)}, 0 \leq s \leq t)$  by:

$$\hat{X}_s^{(t)} = X_t - X_{(t-s)-} \quad \text{if } 0 \leq s < t,$$

and  $\hat{X}_t^{(t)} = X_t$ . The two processes  $(\hat{X}_s^{(t)}, 0 \leq s \leq t)$  and  $(X_s, 0 \leq s \leq t)$  have the same law. Let  $\hat{S}^{(t)}$  be the supremum process of  $\hat{X}^{(t)}$  and  $\hat{L}^{(t)}$  be the local time at 0 of  $\hat{S}^{(t)} - \hat{X}^{(t)}$  with the same normalization as  $L$ .

**Definition 2.1.** ([15], Definition 1.2.1, Lemma 1.2.1 and Lemma 1.2.4)

There exists a  $[0, \infty]$ -valued lower semi-continuous process  $H = (H_t, t \geq 0)$ , called the height process, such that  $H_0 = 0$  and for all  $t \geq 0$ , a.s.  $H_t = \hat{L}_t^{(t)}$ . And a.s. for all  $s < t$  s.t.  $X_{s-} \leq I_t^s$  and for  $s = t$  if  $\Delta_t > 0$  then  $H_t < \infty$  and for all  $t' > t \geq 0$ , the process  $H$  takes all the values between  $H_t$  and  $H_{t'}$  on the time interval  $[t, t']$ .

The height process  $(H_t, t \in [0, \sigma])$  under  $\mathbb{N}$  codes a continuous genealogical structure, the Lévy CRT, via the following procedure.

- (i) To each  $t \in [0, \sigma]$  corresponds a vertex at generation  $H_t$ .
- (ii) Vertex  $t$  is an ancestor of vertex  $t'$  if  $H_t = H_{t,t'}$ , where

$$(7) \quad H_{t,t'} = \inf\{H_u, u \in [t \wedge t', t \vee t']\}.$$

In general  $H_{t,t'}$  is the generation of the last common ancestor of  $t$  and  $t'$ .

- (iii) We put  $d(t, t') = H_t + H_{t'} - 2H_{t,t'}$  and identify  $t$  and  $t'$  ( $t \sim t'$ ) if  $d(t, t') = 0$ .

The Lévy CRT coded by  $H$  is then the quotient set  $[0, \sigma]/\sim$ , equipped with the distance  $d$  and the genealogical relation specified in (ii).

**2.3. The exploration process.** The height process is not Markov in general. But it is a very simple function of a measure-valued Markov process, the so-called exploration process.

The exploration process  $\rho = (\rho_t, t \geq 0)$  is a  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process defined as follows: for every  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,  $\langle \rho_t, f \rangle = \int_{[0,t]} d_s I_t^s f(H_s)$ , or equivalently

$$(8) \quad \rho_t(dr) = \beta \mathbf{1}_{[0, H_t]}(r) dr + \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} (I_t^s - X_{s-}) \delta_{H_s}(dr).$$

In particular, the total mass of  $\rho_t$  is  $\langle \rho_t, 1 \rangle = X_t - I_t$ .

For  $\mu \in \mathcal{M}(\mathbb{R}_+)$ , we set

$$(9) \quad H(\mu) = \sup \text{Supp } \mu,$$

where  $\text{Supp } \mu$  is the closed support of  $\mu$ , with the convention  $H(0) = 0$ . We have

**Proposition 2.2.** ([15], Lemma 1.2.2 and Formula (1.12))

Almost surely, for every  $t > 0$ ,

- $H(\rho_t) = H_t$ ,
- $\rho_t = 0$  if and only if  $H_t = 0$ ,
- if  $\rho_t \neq 0$ , then  $\text{Supp } \rho_t = [0, H_t]$ .
- $\rho_t = \rho_{t-} + \Delta_t \delta_{H_t}$ , where  $\Delta_t = 0$  if  $t \notin \mathcal{J}$ .

In the definition of the exploration process, as  $X$  starts from 0, we have  $\rho_0 = 0$  a.s. To state the Markov property of  $\rho$ , we must first define the process  $\rho$  started at any initial measure  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ .

For  $a \in [0, \langle \mu, 1 \rangle]$ , we define the erased measure  $k_a \mu$  by

$$(10) \quad k_a \mu([0, r]) = \mu([0, r]) \wedge (\langle \mu, 1 \rangle - a), \quad \text{for } r \geq 0.$$

If  $a > \langle \mu, 1 \rangle$ , we set  $k_a \mu = 0$ . In other words, the measure  $k_a \mu$  is the measure  $\mu$  erased by a mass  $a$  backward from  $H(\mu)$ .

For  $\nu, \mu \in \mathcal{M}_f(\mathbb{R}_+)$ , and  $\mu$  with compact support, we define the concatenation  $[\mu, \nu] \in \mathcal{M}_f(\mathbb{R}_+)$  of the two measures by:

$$\langle [\mu, \nu], f \rangle = \langle \mu, f \rangle + \langle \nu, f(H(\mu) + \cdot) \rangle, \quad f \in \mathcal{B}_+(\mathbb{R}_+).$$

Finally, we set for every  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$  and every  $t > 0$ ,

$$(11) \quad \rho_t^\mu = [k_{-I_t} \mu, \rho_t].$$

We say that  $(\rho_t^\mu, t \geq 0)$  is the process  $\rho$  started at  $\rho_0^\mu = \mu$ , and write  $\mathbb{P}_\mu$  for its law. Unless there is an ambiguity, we shall write  $\rho_t$  for  $\rho_t^\mu$ .

**Proposition 2.3.** ([15], Proposition 1.2.3)

The process  $(\rho_t, t \geq 0)$  is a càd-làg strong Markov process in  $\mathcal{M}_f(\mathbb{R}_+)$ .

*Remark 2.4.* From the construction of  $\rho$ , we get that a.s.  $\rho_t = 0$  if and only if  $-I_t \geq \langle \rho_0, 1 \rangle$  and  $X_t - I_t = 0$ . This implies that 0 is also a regular point for  $\rho$ . Notice that  $\mathbb{N}$  is also the excursion measure of the process  $\rho$  away from 0, and that  $\sigma$ , the length of the excursion, is  $\mathbb{N}$ -a.e. equal to  $\inf\{t > 0; \rho_t = 0\}$ .

Exponential formula for the Poisson point process of jumps of the inverse subordinator of  $-I$  gives (see also the beginning of Section 3.2.2. [15]) that for  $\lambda > 0$

$$(12) \quad \mathbb{N} \left[ 1 - e^{-\lambda \sigma} \right] = \psi^{-1}(\lambda).$$



**2.4. The Lévy Poisson snake.** We shall construct a Poisson snake in order to cut the Lévy CRT at marked nodes and at marks on the skeleton. The marked nodes will correspond to the jumps of  $X$  at times in  $\mathcal{J}^1$ . For marks on the skeleton, we will follow [5] and use an underlying Poisson process.

2.4.1. *Marks on the nodes.* For  $t \geq 0$ , we consider the measure on  $\mathbb{R}_+$ ,

$$m_t^{\text{nod}}(dr) = \sum_{\substack{0 < s \leq t, s \in \mathcal{J}^1 \\ X_{s-} < I_t^s}} (I_t^s - X_{s-}) \delta_{H_s}(dr).$$

The atoms of  $m_t^{\text{nod}}$  give the marked nodes of the exploration process at time  $t$ .

2.4.2. *Marks on the skeleton.* . Following Proposition 7.2, see also Proposition 4.1.1 in [15] when  $H$  is continuous, we consider a Lévy snake  $((\rho_t, W_t), t \geq 0)$  with underlying process a Poisson process with intensity  $\alpha_1$ . A.s., we have that  $W_t$  is càd-làg with jumps equal to one. Its derivative  $m_t^{\text{ske}}$  is an atomic measure on  $[0, H_t)$ ; it gives the marks on the skeleton of the exploration process at time  $t$ .

2.4.3. *The Lévy Poisson snake.* The mark process will be given by  $m = (m^{\text{nod}}, m^{\text{ske}})$ .

We call the process  $\mathcal{S} = ((\rho_t, m_t), t \geq 0)$  the Lévy marked snake started at  $\rho_0 = 0, m_0 = 0$ . To get the Markov property of the Lévy marked snake, we must define the process  $\mathcal{S}$  started at any initial value  $(\mu, \Pi^{\text{nod}}, \Pi^{\text{ske}}) \in \mathbb{S}$ , where  $\mathbb{S}$  is the set of  $(\mu, \Pi^{\text{nod}}, \Pi^{\text{ske}})$  such that  $\mu, \Pi^{\text{nod}} \in \mathcal{M}_f(\mathbb{R}_+)$ , with  $\Pi^{\text{nod}}$  absolutely continuous w.r.t. the singular part,  $\mu^{(s)}$ , of  $\mu$ , and  $\Pi^{\text{ske}} \in \mathcal{M}(\mathbb{R}_+)$ . Let  $\Pi = (\Pi^{\text{nod}}, \Pi^{\text{ske}})$ . For  $(\mu, \Pi) \in \mathbb{S}$ , we set  $H_t^\mu = H(k_{-I_t}\mu)$  and we define  $(m^{\text{nod}})_t^{(\mu, \Pi)} = [\frac{d\Pi^{\text{nod}}}{d\mu^{(s)}} k_{-I_t}\mu, m_t^{\text{nod}}]$  and  $(m^{\text{ske}})_t^{(\mu, \Pi)} = [\Pi^{\text{ske}} \mathbf{1}_{[0, H_{0,t})}, m_t^{\text{ske}}]$ . Notice the definition of  $(m^{\text{ske}})_t^{(\mu, \Pi)}$  is coherent with the construction of the Lévy snake, with  $W_0$  being the cumulative function of  $\Pi^{\text{ske}}$  over  $[0, H_0]$ .

We shall write  $m^{\text{nod}}$  for  $(m^{\text{nod}})_t^{(\mu, \Pi)}$  and similarly for  $m^{\text{ske}}$  and  $m$ . By construction and since  $\rho$  is an homogeneous Markov process, the Lévy marked snake  $\mathcal{S} = (\rho, m)$  is an homogeneous Markov process.

We now denote by  $\mathbb{P}_{\mu, \Pi}$  the law of the Lévy marked snake starting at time 0 from  $(\mu, \Pi)$ , and by  $\mathbb{P}_{\mu, \Pi}^*$  the law of the Lévy marked snake killed when  $\rho$  reaches 0. For convenience we shall write  $\mathbb{P}_\mu$  if  $\Pi = 0$  and  $\mathbb{P}$  if  $(\mu, \Pi) = 0$  and similarly for  $\mathbb{P}^*$ . Finally, we still denote by  $\mathbb{N}$  the distribution of  $\mathcal{S}$  when  $\rho$  is distributed under the excursion measure  $\mathbb{N}$ .

Let  $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$  be the filtration generated by  $\mathcal{S}$  completed the usual way. Using the strong Markov property of  $(X, X^{(1)})$  and Proposition 7.2 or Theorem 4.1.2 in [15] if  $H$  is continuous, we get the following result.

**Proposition 2.5.** *The marked snake  $\mathcal{S}$  is a càd-làg strong Markov process.*

By construction, we have the so called snake property: a.s.  $(\rho_t, m_t)(s) = (\rho_{t'}, m_{t'})(s)$  for all  $0 \leq s < H_{t, t'}$ .

**2.5. Poisson representation of the snake.** We decompose the path of  $\mathcal{S}$  under  $\mathbb{P}_{\mu, \Pi}^*$  according to excursions of the total mass of  $\rho$  above its minimum, see Section 4.2.3 in [15]. More precisely, let  $(\tilde{\alpha}_i, \tilde{\beta}_i), i \in \tilde{I}$  be the excursion intervals of  $X - I$  above 0 under  $\mathbb{P}_{\mu, \Pi}^*$ . For

every  $i \in \tilde{I}$ , we define  $h_i = H_{\tilde{\alpha}_i}$  and  $\mathcal{S}^i = (\rho^i, m^i)$  by the formulas: for  $t \geq 0$  and  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,

$$(13) \quad \langle \rho_t^i, f \rangle = \int_{(h_i, +\infty)} f(x - h_i) \rho_{(\tilde{\alpha}_i + t) \wedge \tilde{\beta}_i}(dx)$$

$$(14) \quad \langle (m_t^a)^i, f \rangle = \int_{(h_i, +\infty) \times [0, +\infty)} f(x - h_i) m_{(\tilde{\alpha}_i + t) \wedge \tilde{\beta}_i}^a(dx), \quad a \in \{\text{nod}, \text{ske}\},$$

with  $m^i = ((m^{\text{nod}})^i, (m^{\text{ske}})^i)$ . We set  $\sigma^i = \inf\{s > 0; \langle \rho_s^i, 1 \rangle = 0\}$ . It is easy to adapt Lemma 4.2.4. of [15] to get the following Lemma.

**Lemma 2.6.** *Let  $(\mu, \Pi) \in \mathbb{S}$ . The point measure  $\sum_{i \in \tilde{I}} \delta_{(h_i, \mathcal{S}^i)}$  is under  $\mathbb{P}_{\mu, \Pi}^*$  a Poisson point measure with intensity  $\mu(dr) \mathbb{N}[d\mathcal{S}]$ .*

**2.6. The dual process and representation formula.** We shall need the  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process  $\eta = (\eta_t, t \geq 0)$  defined by

$$\eta_t(dr) = \beta \mathbf{1}_{[0, H_t]}(r) dr + \sum_{\substack{0 < s \leq t \\ X_s - < I_t^s}} (X_s - I_t^s) \delta_{H_s}(dr).$$

The process  $\eta$  is the dual process of  $\rho$  under  $\mathbb{N}$  (see Corollary 3.1.6 in [15]).

The next Lemma on time reversibility can easily be deduced from Corollary 3.1.6 of [15] and the construction of  $m$ .

**Lemma 2.7.** *Under  $\mathbb{N}$ , the processes  $((\rho_s, \eta_s, \mathbf{1}_{\{m_s=0\}}), s \in [0, \sigma])$  and  $((\eta_{(\sigma-s)-}, \rho_{(\sigma-s)-}, \mathbf{1}_{\{m_{(\sigma-s)-}=0\}}), s \in [0, \sigma])$  have the same distribution.*

We present a Poisson representation of  $(\rho, \eta, m)$  under  $\mathbb{N}$ . Let  $\mathcal{N}_0(dx dl du)$ ,  $\mathcal{N}_1(dx dl du)$  and  $\mathcal{N}_2(dx)$  be independent Poisson point measures respectively on  $[0, +\infty)^3$ ,  $[0, +\infty)^3$  and  $[0, +\infty)$  with respective intensity

$$dx \ell \pi_0(dl) \mathbf{1}_{[0,1]}(u) du, \quad dx \ell \pi_1(dl) \mathbf{1}_{[0,1]}(u) du \quad \text{and} \quad \alpha_1 dx.$$

For every  $a > 0$ , let us denote by  $\mathbb{M}_a$  the law of the pair  $(\mu, \nu, m^{\text{nod}}, m^{\text{ske}})$  of measures on  $\mathbb{R}_+$  with finite mass defined by: for any  $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$(15) \quad \langle \mu, f \rangle = \int (\mathcal{N}_0(dx dl du) + \mathcal{N}_1(dx dl du)) \mathbf{1}_{[0,a]}(x) u \ell f(x) + \beta \int_0^a f(r) dr,$$

$$(16) \quad \langle \nu, f \rangle = \int (\mathcal{N}_0(dx dl du) + \mathcal{N}_1(dx dl du)) \mathbf{1}_{[0,a]}(x) (1-u) \ell f(x) + \beta \int_0^a f(r) dr,$$

$$(17) \quad \langle m^{\text{nod}}, f \rangle = \int \mathcal{N}_1(dx dl du) \mathbf{1}_{[0,a]}(x) u \ell f(x) \quad \text{and} \quad \langle m^{\text{ske}}, f \rangle = \int \mathcal{N}_2(dx) \mathbf{1}_{[0,a]}(x) f(x).$$

*Remark 2.8.* In particular  $\mu(dr) + \nu(dr)$  is defined as  $\mathbf{1}_{[0,a]}(r) d_r W_r$ , where  $W$  is a subordinator with Laplace exponent  $\psi' - \alpha$ .

We finally set  $\mathbb{M} = \int_0^{+\infty} da e^{-\alpha a} \mathbb{M}_a$ . Using the construction of the snake, it is easy to deduce from Proposition 3.1.3 in [15], the following Poisson representation.

**Proposition 2.9.** *For every non-negative measurable function  $F$  on  $\mathcal{M}_f(\mathbb{R}_+)^2$ ,*

$$\mathbb{N} \left[ \int_0^\sigma F(\rho_t, \eta_t, m_t) dt \right] = \int \mathbb{M}(d\mu d\nu dm) F(\mu, \nu, m),$$

where  $m = (m^{\text{nod}}, m^{\text{ske}})$  and  $\sigma = \inf\{s > 0; \rho_s = 0\}$  denotes the length of the excursion.

## 3. THE PRUNED EXPLORATION PROCESS

We define the following continuous additive functional of the process  $((\rho_t, m_t), t \geq 0)$ :

$$(18) \quad A_t = \int_0^t \mathbf{1}_{\{m_s=0\}} ds \quad \text{for } t \geq 0.$$

**Lemma 3.1.** *We have the following properties.*

- (i) For  $\lambda > 0$ ,  $\mathbb{N}[1 - e^{-\lambda A_\sigma}] = \psi_0^{-1}(\lambda)$ .
- (ii)  $\mathbb{N}$ -a.e.  $0$  and  $\sigma$  are points of increase for  $A$ . More precisely,  $\mathbb{N}$ -a.e. for all  $\varepsilon > 0$ , we have  $A_\varepsilon > 0$  and  $A_\sigma - A_{(\sigma-\varepsilon)\vee 0} > 0$ .
- (iii) If  $\lim_{\lambda \rightarrow \infty} \phi_1(\lambda) = +\infty$ , then  $\mathbb{N}$ -a.e. the set  $\{s; m_s \neq 0\}$  is dense in  $[0, \sigma]$ .

*Proof.* We first prove (i). Let  $\lambda > 0$ . Before computing  $v = \mathbb{N}[1 - \exp -\lambda A_\sigma]$ , notice that  $A_\sigma \leq \sigma$  implies, thanks to (12), that  $v \leq \mathbb{N}[1 - \exp -\lambda \sigma] = \psi^{-1}(\lambda) < +\infty$ . We have

$$v = \lambda \mathbb{N} \left[ \int_0^\sigma dA_t e^{-\lambda \int_t^\sigma dA_u} \right] = \lambda \mathbb{N} \left[ \int_0^\sigma dA_t \mathbb{E}_{\rho_t, m_t}^* [e^{-\lambda A_\sigma}] \right],$$

where we replaced  $e^{-\lambda \int_t^\sigma dA_u}$  in the last equality by  $\mathbb{E}_{\rho_t, m_t}^* [e^{-\lambda A_\sigma}]$ , its optional projection, and used that  $dA_t$ -a.e.  $m_t = 0$ . In order to compute this last expression, we use the decomposition of  $\mathcal{S}$  under  $\mathbb{P}_\mu^*$  according to excursions of the total mass of  $\rho$  above its minimum, see Lemma 2.6. Using the same notations as in this lemma, notice that under  $\mathbb{P}_\mu^*$ , we have  $A_\sigma = A_\infty = \sum_{i \in \tilde{I}} A_\infty^i$ , where for every  $T \geq 0$ ,

$$(19) \quad A_T^i = \int_0^T \mathbf{1}_{\{m_t^i=0\}} dt.$$

By Lemma 2.6, we get

$$\mathbb{E}_\mu^* [e^{-\lambda A_\sigma}] = e^{-\langle \mu, 1 \rangle \mathbb{N}[1 - \exp -\lambda A_\sigma]} = e^{-v \langle \mu, 1 \rangle}.$$

We have

(20)

$$\begin{aligned} v &= \lambda \mathbb{N} \left[ \int_0^\sigma dA_t e^{-v \langle \rho_t, 1 \rangle} \right] = \lambda \mathbb{N} \left[ \int_0^\sigma dt \mathbf{1}_{\{m_t=0\}} e^{-v \langle \rho_t, 1 \rangle} \right] \\ &= \lambda \int_0^{+\infty} da e^{-\alpha a} \mathbb{M}_a [\mathbf{1}_{\{m=0\}} e^{-v \langle \mu, 1 \rangle}] \\ &= \lambda \int_0^{+\infty} da e^{-\alpha a} \exp \left\{ -\alpha_1 a - \int_0^a dx \int_0^1 du \int_{(0, \infty)} \ell_1 \pi_1(d\ell_1) \right\} \\ &\quad \exp \left\{ -\beta v a - \int_0^a dx \int_0^1 du \int_{(0, \infty)} \ell_0 \pi_0(d\ell_0) (1 - e^{-v u \ell_0}) \right\} \\ (21) \quad &= \lambda \int_0^{+\infty} da \exp \left\{ -a \int_0^1 du \psi'_0(uv) \right\} \end{aligned}$$

(22)

$$= \lambda \frac{v}{\psi_0(v)},$$

where we used Proposition 2.9 for the third and fourth equalities, and for the last equality that  $\alpha_0 = \alpha + \alpha_1 + \int_{(0, \infty)} \ell_1 \pi_1(d\ell_1)$  as well as

$$(23) \quad \psi'_0(\lambda) = \alpha_0 + \int_{(0, \infty)} \pi_0(d\ell_0) \ell_0 (1 - e^{-\lambda \ell_0}).$$

Notice that if  $v = 0$ , then (21) implies  $v = \lambda/\psi'_0(0)$ , which is absurd since  $\psi'_0(0) = \alpha_0 > 0$  thanks to (4). Therefore we have  $v \in (0, \infty)$ , and we can divide (22) by  $v$  to get  $\psi_0(v) = \lambda$ . This proves (i).

Now, we prove (ii). If we let  $\lambda$  goes to infinity in (i) and use that  $\lim_{r \rightarrow \infty} \psi_0(r) = +\infty$ , then we get that  $\mathbb{N}[A_\sigma > 0] = +\infty$ . Notice that for  $(\mu, \Pi) \in \mathbb{S}$ , we have under  $\mathbb{P}_{\mu, \Pi}^*$ ,  $A_\infty \geq \sum_{i \in \tilde{I}} A_\infty^i$ , with  $A^i$  defined by (19). Thus Lemma 2.6 implies that if  $\mu \neq 0$ , then  $\mathbb{P}_{\mu, \Pi}^*$ -a.s.  $\tilde{I}$  is infinite and  $A_\infty > 0$ . Using the Markov property at time  $t$  of the snake under  $\mathbb{N}$ , we get that for any  $t > 0$ ,  $\mathbb{N}$ -a.e. on  $\{\sigma > t\}$ , we have  $A_\sigma - A_t > 0$ . This implies that  $\sigma$  is  $\mathbb{N}$ -a.e. a point of increase of  $A$ . By time reversibility, see Lemma 2.7, we also get that  $\mathbb{N}$ -a.e.  $0$  is a point of increase of  $A$ . This gives (ii).

If  $\alpha_1 > 0$  then the snake  $((\rho_s, W_s), s \geq 0)$  is non trivial. It is well known that, since the Lévy process  $X$  is of infinite variation, the set  $\{s; \exists t \in [0, H_s), W_s(t) \neq 0\}$  is  $\mathbb{N}$ -a.e. dense in  $[0, \sigma]$ . This implies that  $\{s; m_s \neq 0\}$  is  $\mathbb{N}$ -a.e. dense in  $[0, \sigma]$ .

If  $\alpha_1 = 0$  and  $\pi_1((0, \infty)) = \infty$ , then the set  $\mathcal{J}^1$  of jumping time of  $X$  is  $\mathbb{N}$ -a.e. dense in  $[0, \sigma]$ . This also implies that  $\{s; m_s \neq 0\}$  is  $\mathbb{N}$ -a.e. dense in  $[0, \sigma]$ .

If  $\alpha_1 = 0$  and  $\pi_1((0, \infty)) < \infty$ , then the set  $\mathcal{J}^1$  of jumping time of  $X$  is  $\mathbb{N}$ -a.e. finite. This implies that  $\{s; m_s \neq 0\} \cap [0, \sigma]$  is  $\mathbb{N}$ -a.e. a finite union of intervals, which, thanks to (i), is not dense in  $[0, \sigma]$ .

To get (iii), notice that  $\lim_{\lambda \rightarrow \infty} \phi_1(\lambda) = \infty$  if and only if  $\alpha_1 > 0$  or  $\pi_1((0, \infty)) = \infty$ .  $\square$

We set  $C_t = \inf\{r > 0; A_r > t\}$  the right continuous inverse of  $A$ , with the convention that  $\inf \emptyset = \infty$ . From excursion decomposition, see Lemma 2.6, (ii) of Lemma 3.1 implies the following Corollary.

**Corollary 3.2.** *For any initial measures  $(\mu, \Pi) \in \mathbb{S}$ ,  $\mathbb{P}_{\mu, \Pi}$ -a.s. the process  $(C_t, t \geq 0)$  is finite. If  $m_0 = 0$ , then  $\mathbb{P}_{\mu, \Pi}$ -a.s.  $C_0 = 0$ .*

We define the pruned exploration process  $\tilde{\rho} = (\tilde{\rho}_t = \rho_{C_t}, t \geq 0)$  and the pruned Lévy marked snake  $\tilde{\mathcal{S}} = (\tilde{\rho}, \tilde{m})$ , where  $\tilde{m} = (m_{C_t}, t \geq 0)$ . Notice  $C_t$  is a  $\mathcal{F}$ -stopping time for any  $t \geq 0$  and is finite a.s. from Corollary 3.2. Notice the process  $\tilde{\rho}$ , and thus the process  $\tilde{\mathcal{S}}$ , is càd-làg. We also set  $\tilde{H}_t = H_{C_t}$  and  $\tilde{\sigma} = \inf\{t > 0; \tilde{\rho}_t = 0\}$ .

Let  $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t, t \geq 0)$  be the filtration generated by the pruned Lévy Poisson snake  $\tilde{\mathcal{S}}$  completed the usual way. In particular  $\tilde{\mathcal{F}}_t \subset \mathcal{F}_{C_t}$ , where if  $\tau$  is an  $\mathcal{F}$ -stopping time, then  $\mathcal{F}_\tau$  is the  $\sigma$ -field associated with  $\tau$ .

We are now able to restate precisely Theorem 1.1

**Theorem 3.3.** *For every measure  $\mu$  with finite mass, the law of the pruned exploration process  $\tilde{\rho}$  under  $\mathbb{P}_{\mu, 0}$  is the law of the exploration process associated with a Lévy process with Laplace exponent  $\psi_0$  under  $\mathbb{P}_\mu$ .*

#### 4. A SPECIAL MARKOV PROPERTY

In order to define the excursions of the Lévy Poisson snake away from  $\{s \geq 0; m_s = 0\}$ , we define  $O$  as the interior of  $\{s \geq 0, m_s \neq 0\}$ . We shall see that the complementary of  $O$  has positive Lebesgue measure.

**Lemma 4.1.** *If the set  $\{s \geq 0, m_s \neq 0\}$  is non empty then  $O$  is non empty. If we have  $\lim_{\lambda \rightarrow \infty} \phi_1(\lambda) = \infty$ , then  $\mathbb{N}$ -a.e. the open set  $O$  is dense in  $[0, \sigma]$ .*

*Proof.* For any element  $s'$  in  $\{s \geq 0, m_s \neq 0\}$ , there exists  $u \leq H_{s'}$  such that  $m_{s'}([0, u]) \neq 0$  and  $\rho_{s'}([u, \infty)) > 0$ . Then we consider  $\tau_{s'} = \inf\{t > s', \rho_t([u, \infty)) = 0\}$ . By the right continuity of  $\rho$ , we have  $\tau_{s'} > s'$  and the snake property implies that  $\mathbb{N}$ -a.e.  $(s', \tau_{s'}) \subset O$ .

Use (iii) of Lemma 3.1 to get the last part.  $\square$

We write  $O = \bigcup_{i \in I} (\alpha_i, \beta_i)$  and say that  $(\alpha_i, \beta_i)_{i \in I}$  are the excursions intervals of the Lévy marked snake  $\mathcal{S} = (\rho, m)$  away from  $\{s \geq 0, m_s = 0\}$ . For every  $i \in I$ , let us define the measure-valued process  $\mathcal{S}^i = (\rho^i, m^i)$  by: for every  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,  $t \geq 0$ ,

$$(24) \quad \begin{aligned} \langle \rho_t^i, f \rangle &= \int_{[H_{\alpha_i}, +\infty)} f(x - H_{\alpha_i}) \rho_{(\alpha_i+t) \wedge \beta_i}(dx) \\ \langle (m^{(a)})_t^i, f \rangle &= \int_{(H_{\alpha_i}, +\infty)} f(x - H_{\alpha_i}) m_{(\alpha_i+t) \wedge \beta_i}^{(a)}(dx) \quad \text{with } a \in \{\text{nod, ske}\} \end{aligned}$$

and  $m_t^i = ((m^{(\text{nod})})_t^i, (m^{(\text{ske})})_t^i)$ . Notice that the mass located at  $H_{\alpha_i}$  is kept, if there is any, in the definition of  $\rho^i$  whereas it is removed in the definition of  $m^i$ . In particular, if  $\Delta_{\alpha_i} > 0$ , then  $\rho_0^i = \Delta_{\alpha_i} \delta_0$  and for every  $t < \beta_i - \alpha_i$ , the measure  $\rho_t^i$  charges 0. On the contrary, as  $m_0^i = 0$ , we have, for every  $t < \beta_i - \alpha_i$ ,  $m_t^i(\{0\}) = 0$ .

Let  $\tilde{\mathcal{F}}_\infty$  be the  $\sigma$ -field generated by  $\mathcal{S} = ((\rho_{C_t}, m_{C_t}), t \geq 0)$ . Recall that  $\mathbb{P}_{\mu, \Pi}^*(d\mathcal{S})$  denotes the law of the snake  $\mathcal{S}$  started at  $(\mu, \Pi) \in \mathbb{S}$  and stopped when  $\rho$  reaches 0. For  $\ell \in (0, +\infty)$ , we will write  $\mathbb{P}_\ell^*$  for  $\mathbb{P}_{\ell \delta_0, 0}^*$ .

If  $Q$  is a measure on  $\mathbb{S}$  and  $\varphi$  is a non-negative measurable function defined on the measurable space  $\mathbb{R}_+ \times \Omega \times \mathbb{S}$ , we denote by

$$Q[\varphi(u, \omega, \cdot)] = \int_{\mathbb{S}} \varphi(u, \omega, \mathcal{S}) Q(d\mathcal{S}).$$

In other words, the integration concerns only the third component of the function  $\varphi$ .

We can now state the Special Markov Property.

**Theorem 4.2** (Special Markov property). *Let  $\varphi$  be a non-negative measurable function defined on  $\mathbb{R}_+ \times \Omega \times \mathbb{S}$  such that  $t \mapsto \varphi(t, \omega, \mathcal{S})$  is progressively  $\tilde{\mathcal{F}}_\infty$ -measurable for any  $\mathcal{S} \in \mathbb{S}$ . Then, we have  $\mathbb{P}$ -a.s.*

$$(25) \quad \begin{aligned} \mathbb{E} \left[ \exp \left( - \sum_{i \in I} \varphi(A_{\alpha_i}, \omega, \mathcal{S}^i) \right) \middle| \tilde{\mathcal{F}}_\infty \right] \\ = \exp \left( - \int_0^\infty du \alpha_1 \mathbb{N} \left[ 1 - e^{-\varphi(u, \omega, \cdot)} \right] \right) \\ \exp \left( - \int_0^\infty du \int_{(0, \infty)} \pi_1(d\ell) \left( 1 - \mathbb{E}_\ell^* [e^{-\varphi(u, \omega, \cdot)}] \right) \right). \end{aligned}$$

Furthermore, the law of the excursion process  $\sum_{i \in I} \delta_{(A_{\alpha_i}, \rho_{\alpha_i}, \mathcal{S}^i)}(du d\mu d\mathcal{S})$ , given  $\tilde{\mathcal{F}}_\infty$ , is the law of a Poisson point measure with intensity  $\mathbf{1}_{\{u \geq 0\}} du \delta_{\bar{\rho}_u}(d\mu) \left( \alpha_1 \mathbb{N}(d\mathcal{S}) + \int_{(0, \infty)} \pi_1(d\ell) \mathbb{P}_\ell^*(d\mathcal{S}) \right)$ .

Informally speaking, this Theorem gives the distribution of the Lévy marked snake “above” the pruned CRT. The end of this section is now devoted to its proof.

Let us first remark that, if  $\lim_{\lambda \rightarrow +\infty} \phi_1(\lambda) < +\infty$ , we have  $\alpha_1 = 0$  and  $\pi_1$  is a finite measure. Hence, there is no marks on the skeleton and the number of marks on the nodes is finite on every bounded interval of time. The proof of Theorem 4.2 in that case is easy and left to the reader. For the rest of this Section, we assume that  $\lim_{\lambda \rightarrow +\infty} \phi_1(\lambda) = +\infty$ .

Fix  $t > 0$  and  $\eta > 0$ . For  $\mathcal{S} = (\mathcal{S}_s = (\rho_s, m_s), s \geq 0)$ , we set  $B = \{\sigma(\mathcal{S}) = -\infty\} \cup \{T_\eta(\mathcal{S}) = +\infty\} \cup \{L_\eta(\mathcal{S}) = -\infty\}$  where  $\sigma(\mathcal{S}) = \inf\{s > 0; \rho_s = 0\}$ ,  $T_\eta(\mathcal{S}) = \inf\{s \in$

$[0, \sigma(\mathcal{S})]; \langle \rho_s, 1 \rangle \geq \eta$  and  $H_s \geq \eta$  and  $L_\eta(\mathcal{S}) = \sup\{s \in [0, \sigma(\mathcal{S})]; \langle \eta_s, 1 \rangle \geq \eta\}$ , with the convention  $\inf \emptyset = +\infty$  and  $\sup \emptyset = 0$ .

We consider non-negative bounded functions  $\varphi$  satisfying the assumptions of Theorem 4.2 and these three conditions:

- (h<sub>1</sub>)  $\varphi(u, \omega, \mathcal{S}) = 0$  for any  $u \geq t$ .
- (h<sub>2</sub>)  $u \mapsto \varphi(u, \omega, \mathcal{S})$  is uniformly Lipschitz (with a constant that does not depend on  $\omega$  and  $\mathcal{S}$ ).
- (h<sub>3</sub>)  $\varphi(u, \omega, \mathcal{S}) = 0$  on  $B$ ; and if  $\mathcal{S} \in B^c$  then  $\varphi(u, \omega, \mathcal{S})$  depends on  $\mathcal{S}$  only through  $(\mathcal{S}_u, u \in [T_\eta, L_\eta])$ .

**Lemma 4.3.** *If  $\varphi$  satisfies (h<sub>1</sub>), (h<sub>2</sub>) and (h<sub>3</sub>), then the function  $w$  defined on  $(0, \infty) \times [0, \infty) \times \Omega$  by*

$$w(\ell, u, \omega) = \mathbb{E}_\ell^* [e^{-\varphi(u, \omega, \cdot)}]$$

*is uniformly continuous on  $(0, \infty) \times [0, \infty)$ .*

For convenience, we shall write  $w(\ell, u)$  instead of  $w(\ell, u, \omega)$ .

*Proof.* Let  $u > 0$  and  $\ell' > \ell$ . If we set  $\tau_\ell = \inf\{t \geq 0, \rho_t(\{0\}) = \ell\}$  we have, by the strong Markov property at time  $\tau_\ell$  and assumption (h<sub>3</sub>), that

$$\mathbb{E}_{\ell'}^* [e^{-\varphi(u, \omega, \cdot)}] = \mathbb{E}_{\ell'}^* [\mathbf{1}_{\{T_\eta > \tau_\ell\}} \mathbb{E}_\ell^* [e^{-\varphi(u, \omega, \cdot)}]] + \mathbb{E}_{\ell'}^* [e^{-\varphi(u, \omega, \cdot)} \mathbf{1}_{\{T_\eta \leq \tau_\ell\}}].$$

Therefore,

$$\begin{aligned} |w(\ell', u) - w(\ell, u)| &\leq \mathbb{E}_{\ell'}^* [\mathbf{1}_{\{T_\eta \leq \tau_\ell\}} \mathbb{E}_\ell^* [e^{-\varphi(u, \omega, \cdot)}]] + \mathbb{E}_{\ell'}^* [e^{-\varphi(u, \omega, \cdot)} \mathbf{1}_{\{T_\eta \leq \tau_\ell\}}] \\ &\leq 2\mathbb{P}_{\ell'}^*(T_\eta \leq \tau_\ell) \\ &= 2\mathbb{P}_{\ell' - \ell}^*(T_\eta < +\infty). \end{aligned}$$

Using Lemma 2.6, for  $\ell' - \ell < \eta$ , we get

$$|w(\ell', u) - w(\ell, u)| \leq 2 \left(1 - e^{-(\ell' - \ell)\mathbb{N}[T_\eta < \infty]}\right).$$

Since  $\mathbb{N}[T_\eta < \infty] < \infty$ , we then deduce there exists a finite constant  $c_\eta$  s.t. for all function  $\varphi$  satisfying (h<sub>1</sub>)–(h<sub>3</sub>),

$$(26) \quad |w(\ell', u) - w(\ell, u)| \leq c_\eta |\ell' - \ell|.$$

The absolute continuity with respect to  $u$  is a direct consequence of assumption (h<sub>2</sub>).  $\square$

The proof now goes along the next subsections.

**4.1. Stopping times.** Let  $R(dt, du)$  be a Poisson point measure on  $\mathbb{R}_+^2$  independent of  $\mathcal{F}_\infty$  with intensity the Lebesgue measure. For every  $\varepsilon > 0$ , the process  $R_t^\varepsilon := R([0, t] \times [0, 1/\varepsilon])$  is a Poisson process with intensity  $1/\varepsilon$ . We denote by  $(e_k^\varepsilon, k \geq 1)$  the time intervals between the jumps of  $(R_t^\varepsilon, t \geq 0)$ . The random variables  $(e_k^\varepsilon, k \geq 1)$  are i.i.d. exponential random variables with mean  $\varepsilon$ , and are independent of  $\mathcal{F}_\infty$ . They define a mesh of  $\mathbb{R}_+$  which is finer and finer as  $\varepsilon$  decreases to 0.

For  $\varepsilon > 0$ , we consider  $T_0^\varepsilon = 0$ ,  $M_0^\varepsilon = 0$  and for  $k \geq 0$ ,

$$(27) \quad \begin{aligned} M_{k+1}^\varepsilon &= \inf\{i > M_k^\varepsilon; m_{T_k^\varepsilon + \sum_{j=M_k^\varepsilon+1}^i e_j^\varepsilon} \neq 0\}, \\ S_{k+1}^\varepsilon &= T_k^\varepsilon + \sum_{j=M_k^\varepsilon+1}^{M_{k+1}^\varepsilon} e_j^\varepsilon, \\ T_{k+1}^\varepsilon &= \inf\{s > S_{k+1}^\varepsilon; m_s = 0\}, \end{aligned}$$

with the convention  $\inf \emptyset = +\infty$ . Notice  $T_k^\varepsilon$  and  $S_k^\varepsilon$  are  $\mathcal{F}$ -stopping times.

Now we introduce a notation for the process defined above the marks on the intervals  $[S_k^\varepsilon, T_k^\varepsilon]$ . We set, for  $a \geq 0$ ,

$$(28) \quad \bar{H}_a = \sup\{t > 0, m_a([0, t]) = 0\}, \quad \rho_a^- = \rho_a \mathbf{1}_{[0, \bar{H}_a)}$$

and  $\rho_a^+$  defined by  $\rho_a = [\rho_a^-, \rho_a^+]$ , that is for any  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,

$$(29) \quad \langle \rho_a^+, f \rangle = \int_{[\bar{H}_a, \infty)} f(r - \bar{H}_a) \rho_a(dr).$$

For  $k \geq 1$  and  $\varepsilon > 0$  fixed, we define  $\mathcal{S}^{k, \varepsilon} = (\rho^{k, \varepsilon}, m^{k, \varepsilon})$  by : for  $s > 0$  and  $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$\begin{aligned} \rho_s^{k, \varepsilon} &= \rho_{(S_k^\varepsilon + s) \wedge T_k^\varepsilon}^+, \\ \langle (m^{(a)})_s^{k, \varepsilon}, f \rangle &= \int_{(\bar{H}_{S_k^\varepsilon}, +\infty)} f(r - \bar{H}_{S_k^\varepsilon}) m_{(S_k^\varepsilon + s) \wedge T_k^\varepsilon}^{(a)}(dr), \quad \text{with } a \in \{\text{nod}, \text{ske}\}, \end{aligned}$$

and  $m_s^{k, \varepsilon} = ((m^{(\text{nod})})_s^{k, \varepsilon}, (m^{(\text{ske})})_s^{k, \varepsilon})$ . Notice that  $\rho_s^{k, \varepsilon}(\{0\}) = \rho_{S_k^\varepsilon}^\varepsilon(\{\bar{H}_{S_k^\varepsilon}\})$ . For  $k \geq 1$ , we consider the  $\sigma$ -field  $\mathcal{F}^{(\varepsilon), k}$  generated by the family of processes  $(\mathcal{S}_{(T_\ell^\varepsilon + s) \wedge S_{\ell+1}^\varepsilon}^\varepsilon, s > 0)_{\ell \in \{0, \dots, k-1\}}$ .

**4.2. Approximation of the functional.** Let  $\mathcal{S}$  be a Lévy snake and  $g$  be a function defined on  $\mathbb{S}$ . We decompose the path of  $\rho$  according to the excursions of the total mass of  $\rho$  above its minimum as in (24), but for the excursions corresponding to an eventual initial mass. We set  $Y_t = \langle \rho_t, 1 \rangle$  and  $J_t = \inf_{0 \leq u \leq t} Y_u$ . Recall that  $(Y_t, t \geq 0)$  is distributed under  $\mathbb{P}_\mu^*$  as a Lévy process with Laplace exponent  $\psi$  started at  $\langle \mu, 1 \rangle$  and killed when it reaches 0. Let  $(\tilde{\alpha}_i, \tilde{\beta}_i)$ ,  $i \in \tilde{I}$ , be the intervals excursion of  $Y - J$  away from 0. For every  $i \in \tilde{I}$ , we define  $h_i = H_{\tilde{\alpha}_i} = H_{\tilde{\beta}_i}$ . We set  $\mathcal{I} = \{i \in \tilde{I}; h_i > 0\}$  and for  $i \in \mathcal{I}$  let  $\mathcal{S}^i = (\rho^i, m^i)$  be defined by (24). Let  $i_0 \notin \mathcal{I}$ . We set  $\mathcal{I}^* = \mathcal{I} \cup \{i_0\}$ ,  $\tilde{\alpha}_{i_0} = \inf\{\tilde{\alpha}_i; i \in \tilde{I}, h_i = 0\}$ ,  $\tilde{\beta}_{i_0} = \sup\{\tilde{\beta}_i; i \in \tilde{I}, h_i = 0\}$  and  $\mathcal{S}^{i_0} = (\rho^{i_0}, m^{i_0})$  with

$$\begin{aligned} \langle \rho_t^{i_0}, f \rangle &= \int_{[0, +\infty)} f(x) \rho_{(\tilde{\alpha}_{i_0} + t) \wedge \tilde{\beta}_{i_0}}(dx) \\ \langle (m^{(a)})_t^{i_0}, f \rangle &= \int_{(0, +\infty)} f(x) m_{(\tilde{\alpha}_{i_0} + t) \wedge \tilde{\beta}_{i_0}}^{(a)}(dx) \quad \text{with } a \in \{\text{nod}, \text{ske}\}, \end{aligned}$$

and  $m_t^{i_0} = ((m^{(\text{nod})})_t^{i_0}, (m^{(\text{ske})})_t^{i_0})$ . See figure 4.2 to get the picture of the different excursions.

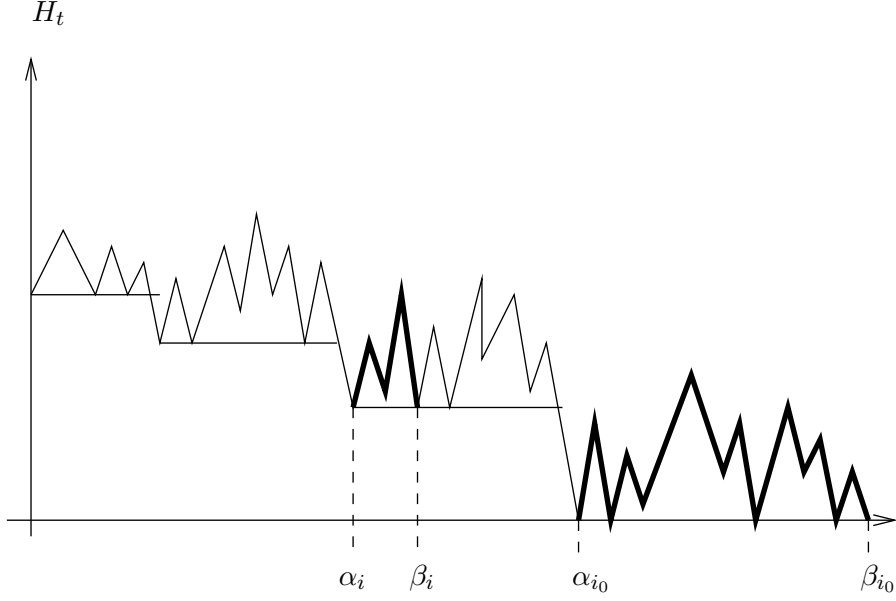


FIGURE 1. Definition of the different excursions

We define

$$(30) \quad g^*(\mathcal{S}) = \sum_{i \in \mathcal{I}^*} g(\mathcal{S}^i).$$

**Lemma 4.4.**  $\mathbb{P}$ -a.s., we have, for  $\varepsilon > 0$  small enough,

$$(31) \quad \sum_{i \in \mathcal{I}} \varphi(A_{\alpha_i}, \omega, \mathcal{S}^i) = \sum_{k=1}^{\infty} \varphi(A_{S_{k_j}^\varepsilon}, \omega, \mathcal{S}^{k_j, \varepsilon}) = \sum_{k=1}^{\infty} \varphi^*(A_{S_{k_j}^\varepsilon}, \omega, \mathcal{S}^{k_j, \varepsilon}),$$

where the sums have a finite number of non zero terms.

*Proof. First equality.* By assumptions  $(h_1)$  and  $(h_3)$ , as  $\mathbb{N}[T_\eta < +\infty] < +\infty$ , the set

$$\mathcal{J} = \{i \in \mathcal{I}, \varphi(A_{\alpha_i}, \omega, \mathcal{S}^i) \neq 0\}$$

is finite. Therefore, for  $\varepsilon$  small enough, for every  $j \in \mathcal{J}$ , the mesh defined by (27) intersects the interval  $(\alpha_j, \beta_j)$ : in other words, there exists an integer  $k_j$  such that  $S_{k_j}^\varepsilon \in (\alpha_j, \beta_j)$  (and that integer is unique).

Moreover, for every  $j \in \mathcal{J}$ , we can choose  $\varepsilon$  small enough so that  $S_{k_j}^\varepsilon < T_\eta(\rho^j)$ , which gives that, for  $\varepsilon$  small enough,

$$\varphi(A_{\alpha_j}, \omega, \mathcal{S}^j) = \varphi(A_{\alpha_j}, \omega, \mathcal{S}^{k_j, \varepsilon}).$$

Finally, as the mark at  $\alpha_j$  is still present at time  $S_{k_j}^\varepsilon$ , the additive functional  $A$  is constant on that time interval and

$$\varphi(A_{\alpha_j}, \omega, \mathcal{S}^j) = \varphi(A_{S_{k_j}^\varepsilon}, \omega, \mathcal{S}^{k_j, \varepsilon}).$$

*Second equality.* Let  $j \in \mathcal{J}$ . We consider the decomposition of  $\mathcal{S}^{k_j, \varepsilon}$  above its minimum described at the beginning of this Subsection. We must consider two cases :



First case : The mass at  $\alpha_j$  is on a node. Then, for  $\varepsilon > 0$  small enough, we have  $T_\eta > \tilde{\alpha}_{i_0}$  and

$$\varphi(A_{S_{k_j}^\varepsilon}, \omega, \mathcal{S}^{k_j, \varepsilon}) = \varphi(A_{S_{k_j}^\varepsilon}, \omega, \mathcal{S}^{i_0}) = \varphi^*(A_{S_{k_j}^\varepsilon}, \omega, \mathcal{S}^{k_j, \varepsilon})$$

as all the terms in the sum that defines  $\varphi^*$  are zero but for  $i = i_0$ .

Second case : The mass at  $\alpha_j$  is on the skeleton. In that case, we again can choose  $\varepsilon$  small enough so that the interval  $[T_\eta, L_\eta]$  is included in one excursion interval above the minimum of the exploration total mass process of  $S^{k_j, \varepsilon}$ . We then conclude as in the previous case.  $\square$

Notice that for  $k \in \mathbb{N}^*$

$$(32) \quad \mathcal{F}^{(\varepsilon), k} \subset \mathcal{F}_{S_k^\varepsilon}.$$

It is easy to check the following measurable result.

**Lemma 4.5.** *For any  $\varepsilon > 0$ ,  $k \in \mathbb{N}^*$ , the function  $\varphi(A_{S_k^\varepsilon}, \omega, \cdot)$  is  $\mathcal{F}^{(\varepsilon), k}$ -measurable.*

### 4.3. Computation of the conditional expectation of the approximation.

**Lemma 4.6.** *For every  $\tilde{\mathcal{F}}_\infty$ -measurable non-negative random variable  $Z$ , we have*

$$\mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^{\infty} \varphi^* \left( A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k, \varepsilon} \right) \right) \right] = \mathbb{E} \left[ Z \prod_{k=1}^{\infty} K_\varepsilon(A_{S_k^\varepsilon}, \omega) \right],$$

where  $\gamma = \psi^{-1}(1/\varepsilon)$  and

$$(33) \quad K_\varepsilon(r, \omega) = \frac{\psi(\gamma)}{\phi_1(\gamma)} \frac{\gamma - v(r)}{\psi(\gamma) - \psi(v(r))} \left( \alpha_1 + \int_0^1 du \int_{(0, \infty)} \ell_1 \pi_1(d\ell_1) w(u\ell_1, r) e^{-\gamma(1-u)\ell_1} \right).$$

where

$$(34) \quad w(\ell, r) = \mathbb{E}_\ell^* \left[ e^{-\varphi(r, \omega, \cdot)} \right] \quad \text{and} \quad v(r) = \mathbb{N} \left[ 1 - e^{-\varphi(r, \omega, \cdot)} \right].$$

*Proof.* Let  $p \geq 1$ . Recall that  $H_{t, t'}$  denotes the minimum of  $H$  between  $t$  and  $t'$ . We set

$$\begin{aligned} \xi_d^{p-1} &= \sup \left\{ t > T_{p-1}^\varepsilon; H_t = H_{T_{p-1}^\varepsilon, S_p^\varepsilon} \right\}, \\ \xi_g^p &= \inf \left\{ t > T_{p-1}^\varepsilon; H_t = \bar{H}_{S_p^\varepsilon} \quad \text{and} \quad H_{t, S_p^\varepsilon} = H_t \right\}. \end{aligned}$$

Roughly speaking,  $\xi_d^{p-1}$  is the time at which the height process reaches its minimum over  $[T_{p-1}^\varepsilon, S_p^\varepsilon]$  and  $\xi_g^p$  is the time at which appears the first mark of the snake at time  $S_p^\varepsilon$ , see figure 4.3 to help understanding.

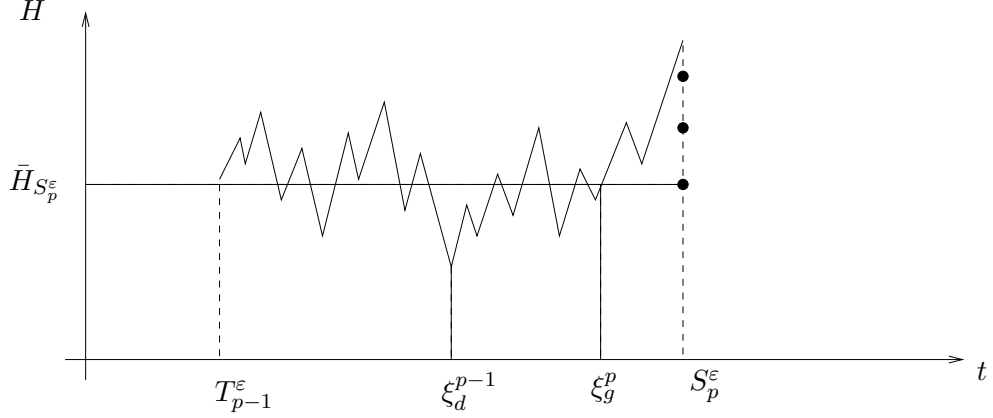


FIGURE 2. Position of various random times

We consider a bounded non-negative random variable  $Z$  of the form  $Z = Z_0 Z_1 Z_2 Z_3$ , where  $Z_0 \in \mathcal{F}^{(\varepsilon), p-1}$ ,  $Z_1 \in \sigma(\mathcal{S}_u, T_{p-1}^\varepsilon \leq u < \xi_d^{p-1})$ ,  $Z_2 \in \sigma(\mathcal{S}_u, \xi_d^{p-1} \leq u < \xi_g^p)$  and  $Z_3 \in \sigma(\mathcal{S}_{(T_k^\varepsilon+s) \wedge S_{k+1}^\varepsilon-}, s \geq 0, k \geq p)$  are bounded non-negative.

To compute  $\mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^p \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \right]$ , we first apply the strong Markov property at time  $T_p^\varepsilon$ . We obtain

$$\mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^p \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \right] = \mathbb{E} \left[ Z_0 Z_1 Z_2 \exp \left( - \sum_{k=1}^p \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \mathbb{E}_{\rho_{T_p^\varepsilon}, 0} [Z_3] \right].$$

Recall notation (28) and (29). Notice that  $\rho_{T_p^\varepsilon} = \rho_{S_p^\varepsilon}^-$ , and consequently  $\rho_{T_p^\varepsilon}$  is measurable with respect to  $\mathcal{F}_{S_p^\varepsilon}$ . So, when we use the strong Markov property at time  $S_p^\varepsilon$ , we get thanks to Lemma 4.5 and (32)

$$\begin{aligned} & \mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^p \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \right] \\ &= \mathbb{E} \left[ Z_0 Z_1 Z_2 \exp \left( - \sum_{k=1}^{p-1} \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \mathbb{E}_{\rho_{S_p^\varepsilon}^+}^* \left[ e^{-\varphi^*(A_{S_p^\varepsilon}, \omega, \cdot)} \right] \mathbb{E}_{\rho_{S_p^\varepsilon}^-} [Z_3] \right]. \end{aligned}$$

Using the strong Markov property at time  $T_{p-1}^\varepsilon$ , we get

$$(35) \quad \begin{aligned} & \mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^p \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \right] \\ &= \mathbb{E} \left[ Z_0 \exp \left( - \sum_{k=1}^{p-1} \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \mathbb{E}_{\rho_{T_{p-1}^\varepsilon}}^* \left[ Z_1 Z_2 F_{b_0}(\rho_{\tau'}^+, A_{\tau'}) G(\rho_{\tau'}^-) \right] \right]_{|b_0=A_{S_{p-1}^\varepsilon}}, \end{aligned}$$

with  $F_{b_0}(\mu, b) = \mathbb{E}_\mu^* [e^{-\varphi^*(b_0+b, \omega, \cdot)}]$ ,  $G(\mu) = \mathbb{E}_\mu^* [Z_3]$ , and  $\tau'$  is distributed as  $S_1^\varepsilon$ . In what follows, we shall write  $F$  for  $F_{b_0}$  in order to simplify the expressions when there is no confusion.

We then consider the Poisson decomposition of  $\mathcal{S}$  under  $\mathbb{P}_{\rho_{T_{p-1}^\varepsilon}}^*$  given in Lemma 2.6. Notice there exists a unique excursion  $i_1 \in \tilde{I}$  s.t.  $\tilde{\alpha}_{i_1} < \tau' < \tilde{\beta}_{i_1}$ .

By hypothesis on  $Z_1$ , we can write  $Z_1 = \mathcal{H}_1(\rho_{T_{p-1}^\varepsilon}, \sum_{i \in \tilde{I}; \tilde{\alpha}_i < \tilde{\alpha}_{i_1}} \delta_{h_i, \mathcal{S}^i})$  for a measurable function  $\mathcal{H}_1$ . We can also write  $Z_2 = \mathcal{H}_2(\rho_u, \xi_d^{p-1} \leq u < \xi_g^p)$  for a measurable function  $\mathcal{H}_2$  as  $m_u = 0$  for  $u \in [\xi_d^{p-1}, \xi_g^p)$ . Then, using compensation formula in excursion theory, see Corollary IV.11 and Theorem VII.1 in [9], we get

$$(36) \quad \mathbb{E}_{\mu'}^* [Z_1 Z_2 F(\rho_{\tau'}^+, A_{\tau'}) G(\rho_{\tau'}^-)] \\ = \mathbb{E} \left[ \int \mu'(dv) \mathbf{1}_{\{\tau' > \sum_{s < v} \sigma(\mathcal{S}^s)\}} \mathcal{H}_1(\mu', \sum_{s < v} \delta_{\mathcal{S}^s}) h_F^{\mu'}(r, \sum_{s < v} A_{\sigma(\mathcal{S}^s)}(\mathcal{S}^s)) \right],$$

where  $\sum_s \delta_{s, \mathcal{S}^s}$  is a Poisson point measure with intensity  $\mu'(ds) \mathbb{N}[d\mathcal{S}]$ ,  $\sigma(\mathcal{S}) = \inf\{r > 0, \mathcal{S}_r = 0\}$ ,  $A_t(\mathcal{S}^s) = \int_0^t dv' \mathbf{1}_{\{m_{v'}(\mathcal{S}^s) = 0\}}$  and

$$h_F^{\mu'}(s, b) = \mathbb{N} [F(\rho_{\tau'}^+, b + A_{\tau'}) G([k_s \mu', \rho_{\tau'}^-]) \mathcal{H}_2([k_s \mu', \rho_t], 0 \leq t < \xi_g^1) \mathbf{1}_{\{\tau' < \sigma\}}].$$

Let  $(R_k, k \geq 0)$  be the increasing sequence of the jumping times of a Poisson process of intensity  $1/\varepsilon$ , independent of  $\mathcal{S}$ . Then, by time-reversal, we have

$$h_F^{\mu'}(s, b) = \mathbb{N} \left[ \sum_{k=1}^{+\infty} \mathbf{1}_{\{m_{R_k} \neq 0\}} \mathbf{1}_{\{\forall k' > k, m_{R_{k'}} = 0\}} F(\eta_{R_k}^+, b + A_\sigma - A_{R_k}) \right. \\ \left. G([k_s \mu', \eta_{R_k}^-]) \mathcal{H}_2([k_s \mu', \eta_u], \tau_k < u \leq \sigma) \right],$$

where  $\tau_k = \inf\{t > R_k; m_t = 0\}$ . We then apply the strong Markov property at time  $R_k$  and the Poisson representation of the Poisson Lévy snake to get

$$h_F^{\mu'}(s, b) = \mathbb{N} \left[ \sum_{k=1}^{+\infty} \mathbf{1}_{\{m_{R_k} \neq 0\}} G([k_s \mu', \eta_{R_k}^-]) \right. \\ \left. \mathbb{E}_{\rho_{R_k}}^* \left[ \mathbf{1}_{\{\forall k' > 0, m_{R_{k'}} = 0\}} F(\eta', b + A_\sigma) \mathcal{H}_2([k_s \mu', \eta_u], \tau_0 < u \leq \sigma) \right]_{|\eta' = \eta_{R_k}^+} \right],$$

where  $\tau_0 = \inf\{t > 0; m_t = 0\}$ . Now, let us remark that, if  $m_0 \neq 0$ , then  $m_s \neq 0$  for  $s \in [0, \tau_0]$  and  $A_{\tau_0} = 0$ . Therefore,  $m_{R_1} = 0$  implies  $R_1 > \tau_0$ . The strong Markov property at time  $\tau_0$  gives, with  $\eta' = \eta_{R_k}^+$ ,

$$\mathbf{1}_{\{m_{R_k} \neq 0\}} \mathbb{E}_{\rho_{R_k}}^* \left[ \mathbf{1}_{\{\forall k' > 0, m_{R_{k'}} = 0\}} F(\eta', b + A_\sigma) \mathcal{H}_2([k_s \mu', \eta_u], \tau_0 < u \leq \sigma) \right] \\ = \mathbf{1}_{\{m_{R_k} \neq 0\}} \mathbb{P}_{\rho_{R_k}^+}^* (R_1 > \sigma) \mathbb{E}_{\rho_{R_k}^-}^* \left[ \mathbf{1}_{\{\forall k' > 0, m_{R_{k'}} = 0\}} F(\eta', b + A_\sigma) \mathcal{H}_2([k_s \mu', \eta_u], 0 < u \leq \sigma) \right].$$

We have, using the Poisson representation of Lemma 2.6 and (12), that  $\mathbb{P}_{\rho_{R_k}^+}^* (R_1 > \sigma) =$

$\mathbb{E}_{\rho_{R_k}^+}^* [e^{-\sigma/\varepsilon}] = e^{-\gamma \langle \rho_{R_k}^+, 1 \rangle}$ , as  $\gamma = \psi^{-1}(1/\varepsilon)$ . We obtain

$$h_F^{\mu'}(s, b) = \mathbb{N} \left[ \sum_{k=1}^{+\infty} \mathbf{1}_{\{m_{R_k} \neq 0\}} \tilde{G}(\rho_{R_k}^-, \eta_{R_k}^-, \rho_{R_k}^+, \eta_{R_k}^+) \right],$$

where

$$\begin{aligned} \tilde{G}(\mu, \nu, \rho', \eta') &= G([k_s \mu', \nu]) e^{-\gamma \langle \rho', 1 \rangle} \\ &\quad \mathbb{E}_\mu^* \left[ \mathbf{1}_{\{\forall k' > 0, m_{R_{k'}} = 0\}} F(\eta', b + A_\sigma) \mathcal{H}_2([k_s \mu', \rho_u], 0 < u \leq \sigma) \right]. \end{aligned}$$

As  $\sum_{k \geq 1} \delta_{R_k}$  is a Poisson point process with intensity  $1/\varepsilon$ , we deduce that

$$h_F^{\mu'}(s, b) = \frac{1}{\varepsilon} \mathbb{N} \left[ \int_0^\sigma dt \mathbf{1}_{\{m_t \neq 0\}} \tilde{G}(\rho_t^-, \eta_t^-, \rho_t^+, \eta_t^+) \right].$$

Using Proposition 2.9, we get

$$h_F^{\mu'}(s, b) = \frac{1}{\varepsilon} \int_0^\infty da e^{-\alpha a} \mathbb{M}_a \left[ \mathbf{1}_{\{m \neq 0\}} \tilde{G}(\mu^-, \nu^-, \mu^+, \nu^+) \right].$$

For  $r > 0$  and  $\mu$  a measure on  $\mathbb{R}_+$ , let us define the measures  $\mu_{\geq r}$  and  $\mu_{< r}$  by

$$\langle \mu_{\geq r}, f \rangle = \int f(x - r) \mathbf{1}_{\{x \geq r\}} \mu(dx) \quad \text{and} \quad \langle \mu_{< r}, f \rangle = \int f(x) \mathbf{1}_{\{x < r\}} \mu(dx).$$

We set  $q(du, dl_1) = \alpha_1 \delta_{(0,0)}(du dl_1) + du \ell_1 \pi_1(dl_1)$  and by convention  $\pi(\{0\}) = 0$ . Using Palm formula, we get

$$\begin{aligned} &\mathbb{M}_a \left[ \mathbf{1}_{\{m \neq 0\}} \tilde{G}(\mu^-, \nu^-, \mu^+, \nu^+) \right] \\ &= \int_0^a dr \int_{[0,1] \times [0,\infty)} q(du, dl_1) \\ &\quad \mathbb{M}_a \left[ \mathbf{1}_{\{m((0,r))=0\}} \tilde{G}(\mu_{< r}, \nu_{< r}, \mu_{\geq r} + u \ell_1 \delta_0, \nu_{\geq r} + (1-u) \ell_1 \delta_0) \right]. \end{aligned}$$

Using the independence of the Poisson point measures, we get

$$\begin{aligned} &\mathbb{M}_a \left[ \mathbf{1}_{\{m((0,r))=0\}} \tilde{G}(\mu_{< r}, \nu_{< r}, \mu_{\geq r} + u \ell_1 \delta_0, \nu_{\geq r} + (1-u) \ell_1 \delta_0) \right] \\ &= \int \mathbb{M}_r(d\mu, d\nu, dm) \int \mathbb{M}_{a-r}(d\rho', d\eta', dm') \mathbf{1}_{\{m=0\}} \tilde{G}(\mu, \nu, \rho' + u \ell_1 \delta_0, \eta' + (1-u) \ell_1 \delta_0). \end{aligned}$$

We deduce that

$$\begin{aligned} h_F^{\mu'}(s, b) &= \frac{1}{\varepsilon} \int_{[0,1] \times [0,\infty)} q(du, dl_1) \int \mathbb{M}(d\mu, d\nu, dm) \int \mathbb{M}(d\rho', d\eta', dm') \\ &\quad \mathbf{1}_{\{m=0\}} \tilde{G}(\mu, \nu, \rho' + u \ell_1 \delta_0, \eta' + (1-u) \ell_1 \delta_0). \end{aligned}$$

Plugging this result in (36), we get

$$(37) \quad \mathbb{E}_{\mu'} \left[ Z_1 Z_2 F(\rho_{\tau'}^+, A_{\tau'}) G(\rho_{\tau'}^-) \right] = \mathbb{E}_{\mu'} \left[ Z_1 Z_2 \frac{\Gamma_F(A_{\tau'})}{\Gamma_1} G(\rho_{\tau'}^-) \right],$$

where for  $f$  non-negative and defined on  $\mathcal{M}_f(\mathbb{R}_+) \times [0, \infty)$

$$\Gamma_f(b) = \int_{[0,1] \times [0,\infty)} q(du, dl_1) \int \mathbb{M}(d\rho', d\eta', dm') e^{-\gamma \langle \rho', 1 \rangle - \gamma u \ell_1} f(\eta' + (1-u) \ell_1 \delta_0, b).$$

We now use the particular form of  $F$  to compute  $\Gamma_F$ . Using (30) and Lemma 2.6, we get

$$F_{b_0}(\mu, b) = \mathbb{E}_\mu^* \left[ e^{-\varphi^*(b_0 + b, \omega, \cdot)} \right] = \mathbb{E}_{\mu(\{0\})}^* \left[ e^{-\varphi(b_0 + b, \omega, \cdot)} \right] e^{-\mu((0,\infty)) \mathbb{N} [1 - e^{-\varphi(b_0 + b, \omega, \cdot)}]}.$$

We  $w$  and  $v$  defined in (34), we get

$$\begin{aligned} \mathbb{M}_a \left[ e^{-\gamma\langle\mu,1\rangle - \gamma u \ell_1} F_{b_0}(\nu + (1-u)\ell_1\delta_0, b) \right] \\ = w((1-u)\ell_1, b_0 + b) e^{-\gamma u \ell_1} \mathbb{M}_a \left[ e^{-\gamma\langle\mu,1\rangle} e^{-v(b_0+b)\langle\nu,1\rangle} \right] \\ = w((1-u)\ell_1, b_0 + b) e^{-\gamma u \ell_1} e^{-a\left(\frac{\psi(\gamma) - \psi(v(b_0+b))}{\gamma - v(b_0+b)} - \alpha\right)}. \end{aligned}$$

We deduce that

$$\Gamma_{F_{b_0}}(b) = \frac{\gamma - v(b_0 + b)}{\psi(\gamma) - \psi(v(b_0 + b))} \left( \alpha_1 + \int_0^1 du \int_{(0,\infty)} \ell_1 \pi_1(d\ell_1) w(u\ell_1, b_0 + b) e^{-\gamma(1-u)\ell_1} \right),$$

and with  $F = 1$ ,  $\Gamma_1 = \frac{\gamma}{\psi(\gamma)} \frac{\phi_1(\gamma)}{\gamma} = \frac{\phi_1(\gamma)}{\psi(\gamma)}$ .

Use this and (37) in (35), arguments backward from (35) and definition (33) to get

$$\mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^p \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \right] = \mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^{p-1} \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) K_\varepsilon(A_{S_p^\varepsilon}, \omega) \right].$$

From monotone class Theorem, this equality holds also for any non-negative  $Z$  measurable w.r.t. the  $\sigma$ -field  $\bar{\mathcal{F}}_\infty = \sigma((\mathcal{S}_{C_t}, t \in [A_{T_k^\varepsilon}, A_{S_{k+1}^\varepsilon}]), k \geq 0)$ . Notice that  $K_\varepsilon(A_{S_p^\varepsilon}) = K_\varepsilon(A_{S_p^\varepsilon}, \omega)$  is measurable w.r.t.  $\bar{\mathcal{F}}_\infty$ . So, we may iterate the previous argument and let  $p$  goes to infinity to finally get that for any non-negative random variable  $Z \in \bar{\mathcal{F}}_\infty$ , we have

$$\mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^{\infty} \varphi^*(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) \right) \right] = \mathbb{E} \left[ Z \prod_{k=1}^{\infty} K_\varepsilon(A_{S_k^\varepsilon}, \omega) \right].$$

As  $\bar{\mathcal{F}}_\infty$  contains  $\tilde{\mathcal{F}}_\infty$ , the Lemma is proved.  $\square$

**4.4. Computation of the limit.** We first study the process  $t \mapsto N_{\varepsilon,t}$ . Recall notation of Section 4.1. Let  $A_s^\varepsilon$  be the Lebesgue measure of  $[0, s] \cap \left( \bigcup_{k \geq 0} [T_k^\varepsilon, S_{k+1}^\varepsilon] \right)$ . The process  $t \mapsto \sup\{i \in \mathbb{N}; \sum_{j=1}^i e_j^\varepsilon \leq A_t^\varepsilon\}$  is a Poisson process with intensity  $1/\varepsilon$  and the process  $s \mapsto N_{\varepsilon,t}$ , where

$$N_{\varepsilon,t} = \sup\{k \in \mathbb{N}; A_{S_k^\varepsilon}^\varepsilon \leq t\} = \sup\{k \in \mathbb{N}; \sum_{j=1}^{M_k^\varepsilon} e_j^\varepsilon \leq A_t^\varepsilon\},$$

is a marked Poisson process with intensity  $\mathbb{P}(m_\tau \neq 0)/\varepsilon$ , where  $\tau$  is an exponential random variable with mean  $\varepsilon$  independent of  $\mathcal{S}$ .

**Lemma 4.7.** *The process  $t \mapsto N_{\varepsilon,t}$  is a Poisson process with intensity  $\frac{\phi_1(\gamma)}{\varepsilon\psi_0(\gamma)}$ , where  $\gamma = \psi^{-1}(1/\varepsilon)$ .*

*Proof.* We have, by the similar computations as in the proof of Lemma 4.6,

$$\begin{aligned}\mathbb{P}(m_\tau = 0) &= \frac{1}{\varepsilon} \mathbb{E} \left[ \int_0^\infty dt e^{-t/\varepsilon} \mathbf{1}_{\{m_t=0\}} \right] \\ &= \frac{1}{\varepsilon} \int_0^\infty ds e^{-\gamma s} \mathbb{N} \left[ \int_0^\sigma dt e^{-t/\varepsilon} \mathbf{1}_{\{m_t=0\}} \right] \\ &= \frac{1}{\varepsilon \gamma} \mathbb{N} \left[ \int_0^\sigma dt e^{-t/\varepsilon} \mathbf{1}_{\{m_t=0\}} \right].\end{aligned}$$

By time reversibility and using optional projection and (12), we have

$$\begin{aligned}\mathbb{N} \left[ \int_0^\sigma dt e^{-t/\varepsilon} \mathbf{1}_{\{m_t=0\}} \right] &= \mathbb{N} \left[ \int_0^\sigma dt e^{-(\sigma-t)/\varepsilon} \mathbf{1}_{\{m_t=0\}} \right] \\ &= \mathbb{N} \left[ \int_0^\sigma dt e^{-\gamma \langle \rho_t, 1 \rangle} \mathbf{1}_{\{m_t=0\}} \right].\end{aligned}$$

The proof of Lemma 3.1, see (20) and (22), gives that  $\mathbb{P}(m_\tau = 0) = \frac{1}{\varepsilon \psi_0(\gamma)}$ . Since  $\varepsilon^{-1} = \psi(\gamma) = \psi_0(\gamma) - \phi_1(\gamma)$ , we get  $\frac{1}{\varepsilon} \mathbb{P}(m_\tau \neq 0) = \frac{\phi_1(\gamma)}{\varepsilon \psi_0(\gamma)}$ .  $\square$

We then get the following Corollary.

**Corollary 4.8.** *There exists a sub-sequence  $(\varepsilon_j, j \in \mathbb{N})$  decreasing to 0, s.t.  $\mathbb{P}$ -a.s. for any  $t_0 \geq 0$  and any bounded measurable function  $h$  defined on  $\mathbb{R}_+ \times \Omega$  such that  $u \mapsto h(u, \omega)$  is a.s. continuous and  $h(u, \omega) = 0$  for  $u \geq t_0$ , we have, with  $\gamma_j = \psi^{-1}(1/\varepsilon_j)$ ,*

$$\lim_{j \rightarrow \infty} \phi_1(\gamma_j)^{-1} \sum_{k=1}^{\infty} h(A_{S_k^{\varepsilon_j}}, \omega) = \int_0^\infty h(u, \omega) du.$$

*Proof.* Notice that as a direct consequence of (5) and (6), we get

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \psi_0(\gamma) = 1.$$

Recall that  $(A_{S_k^\varepsilon}, k \geq 1)$  are the jumping time of the Poisson process  $t \mapsto N_{\varepsilon, t}$  with parameter  $\phi_1(\gamma)/\varepsilon \psi_0(\gamma)$ . Standard results on Poisson process implies the vague convergence in distribution (see also Lemma XI.11.1 in [13]) of  $\phi_1(\gamma)^{-1} \sum_{k=1}^{\infty} \delta_{A_{S_k^\varepsilon}}(dr)$  towards the Lebesgue measure on  $\mathbb{R}_+$  as  $\varepsilon$  goes down to 0. Since the limit is deterministic, the convergence holds in probability and a.s. along a decreasing sub-sequence  $(\varepsilon_j, j \in \mathbb{N})$ . In particular, as  $h$  is continuous and  $h(u, \omega) = 0$  for  $u \geq t_0$ , we have that a.s.

$$\lim_{j \rightarrow \infty} \phi_1(\gamma_j)^{-1} \sum_{k=1}^{\infty} h(A_{S_k^{\varepsilon_j}}, \omega) = \int_0^\infty h(u, \omega) du.$$

Notice that a.s. for all  $u \geq 0$ ,  $\lim_{\varepsilon' \rightarrow 0} A_u^{\varepsilon'} = A_u$ , and the convergence is uniform over any bounded interval as  $A^{\varepsilon'}$  and  $A$  are non-decreasing. From the continuity of  $h$  and since  $h(u, \omega) = 0$  for  $u \geq t_0$ , we deduce that a.s.

$$\lim_{j \rightarrow \infty} \phi_1(\gamma_j)^{-1} \sum_{k=1}^{\infty} |h(A_{S_k^{\varepsilon_j}}, \omega) - h(A_{S_k^{\varepsilon_j}}, \omega)| = 0.$$

This ends the proof of the Corollary.  $\square$

We now study  $K_\varepsilon$  given by (33). We keep the same notation as in Lemma 4.6.

**Lemma 4.9.** *There exists a deterministic function  $R$  s.t.  $\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 0$  and  $\mathbb{P}$ -a.s for all  $\varepsilon > 0$ , we have:*

$$\sup_{r \geq 0} \left| \phi_1(\gamma) \log(K_\varepsilon(r, \omega)) - \alpha_1 v(r) - \int_{(0, \infty)} \pi_1(d\ell_1) (1 - w(\ell_1, r)) \right| \leq R(\varepsilon).$$

*Proof.* We have

$$\begin{aligned} K_\varepsilon(r, \omega) &= \frac{\psi(\gamma)}{\psi(\gamma) - \psi(v(r))} \frac{\gamma - v(r)}{\gamma} \frac{1}{\phi_1(\gamma)} \left( \alpha_1 \gamma + \gamma \int_0^1 du \int_{(0, \infty)} \ell_1 \pi_1(d\ell_1) w(u\ell_1, r) e^{-\gamma(1-u)\ell_1} \right) \\ &= \frac{\psi(\gamma)}{\psi(\gamma) - \psi(v(r))} \frac{\gamma - v(r)}{\gamma} \frac{1}{\phi_1(\gamma)} \left( \alpha_1 \gamma + \int_{(0, \infty)} \pi_1(d\ell_1) \int_0^{\gamma\ell_1} e^{-s} ds w(\ell_1 - \frac{s}{\gamma}, r) \right) \\ &= \frac{\psi(\gamma)}{\psi(\gamma) - \psi(v(r))} \frac{\gamma - v(r)}{\gamma} \frac{1}{\phi_1(\gamma)} \\ &\quad \left( \phi_1(\gamma) - \int_{(0, \infty)} \pi_1(d\ell_1) \int_0^{\gamma\ell_1} e^{-s} ds \left( 1 - w(\ell_1 - \frac{s}{\gamma}, r) \right) \right). \end{aligned}$$

In particular, we have  $\phi_1(\gamma) \log(K_\varepsilon(r)) = -A_1 + A_2 + A_3$ , where

$$\begin{aligned} A_1(r) &= \phi_1(\gamma) \log \left( 1 - \psi(v(r))/\psi(\gamma) \right), \\ A_2(r) &= \phi_1(\gamma) \log(1 - v(r)/\gamma), \\ A_3(r) &= \phi_1(\gamma) \log \left( 1 - \int_{(0, \infty)} \pi_1(d\ell_1) \int_0^{\gamma\ell_1} e^{-s} ds \left( 1 - w(\ell_1 - \frac{s}{\gamma}, r) \right) / \phi_1(\gamma) \right). \end{aligned}$$

Thanks to (h<sub>3</sub>), there exists a finite constant  $a > 0$  s.t.  $\mathbb{P}$ -a.s.  $v(r) < a$  for all  $r \geq 0$ . We deduce there exists  $\varepsilon_0 > 0$  and a finite constant  $c > 0$  s.t.  $\mathbb{P}$ -a.s for all  $\varepsilon \in (0, \varepsilon_0]$ ,

$$(38) \quad \sup_{r \geq 0} |A_1(r)| \leq c \frac{\phi_1(\gamma)}{\psi(\gamma)} \quad \text{and} \quad \sup_{r \geq 0} |A_2(r) - \alpha_1 v(r)| \leq \frac{c}{\gamma} + c \left| \frac{\phi_1(\gamma)}{\gamma} - \alpha_1 \right|.$$

We have

$$\begin{aligned} &\int_{(0, \infty)} \pi_1(d\ell_1) \int_0^{\gamma\ell_1} e^{-s} ds \left( 1 - w(\ell_1 - \frac{s}{\gamma}, r) \right) - \int_{(0, \infty)} \pi_1(d\ell_1) (1 - w(\ell_1, r)) \\ &= \int_{(0, \infty)} \pi_1(d\ell_1) e^{-\gamma\ell_1} (w(\ell_1, r) - 1) \\ &\quad + \int_{(0, \infty)} \pi_1(d\ell_1) \int_0^\infty e^{-s} ds \left( w(\ell_1, r) - w(\ell_1 - \frac{s}{\gamma}, r) \right) \mathbf{1}_{\{s \leq \gamma\ell_1\}}. \end{aligned}$$

It is then easy to get, using (h<sub>4</sub>) and (26), that  $\mathbb{P}$ -a.s

$$\phi_2(\gamma) = \sup_{r \geq 0} \left| \int_{(0, \infty)} \pi_1(d\ell_1) \int_0^{\gamma\ell_1} e^{-s} ds \left( 1 - w(\ell_1 - \frac{s}{\gamma}, r) \right) - \int_{(0, \infty)} \pi_1(d\ell_1) (1 - w(\ell_1, r)) \right|$$

converges to 0 as  $\gamma$  goes to infinity.

Recall that we assumed that  $\lim_{\gamma \rightarrow \infty} \phi_1(\gamma) = +\infty$ . Thus, there exist  $\varepsilon_0 > 0$  and a finite constant  $c > 0$  s.t.  $\mathbb{P}$ -a.s for all  $\varepsilon \in (0, \varepsilon_0]$ ,

$$(39) \quad \sup_{r \geq 0} \left| A_3(r) - \int_{(0, \infty)} \pi_1(d\ell_1) (1 - w(\ell_1, r)) \right| \leq \frac{c}{\phi_1(\gamma)} + \phi_2(\gamma).$$

Using (38) and (39), we get that there exists a deterministic function  $R$  s.t.  $\mathbb{P}$ -a.s

$$\sup_{r \geq 0} \left| \phi_1(\gamma) \log(K_\varepsilon(r)) - \alpha_1 v(r) - \int_{(0, \infty)} \pi_1(d\ell_1) (1 - w(\ell_1, r)) \right| \leq R(\varepsilon),$$

where  $\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 0$ , thanks to (5) and (6). □

The previous results allow us to compute the following limit. We keep the same notation as in Lemma 4.6.

**Lemma 4.10.** *Let  $\varphi$  satisfying condition  $(h_1)$ – $(h_3)$ . There exists a sub-sequence  $(\varepsilon_j, j \in \mathbb{N})$  decreasing to 0, s.t.  $\mathbb{P}$ -a.s.*

$$\lim_{j \rightarrow \infty} \prod_{k=1}^{\infty} K_{\varepsilon_j}(A_{S_k^{\varepsilon_j}}) = \exp - \int_0^{\infty} du \left( \alpha_1 v(u) + \int_{(0, \infty)} \pi_1(d\ell) (1 - w(\ell, u)) \right).$$

*Proof.* Notice that thanks to  $(h_1)$ , the functions  $v$  and  $w(\ell, \cdot)$  are continuous and that for  $r \geq t$ ,  $v(r) = 0$  and  $w(\ell, r) = 1$ . The result is then a direct consequence of Corollary 4.8 and Lemma 4.9. □

**4.5. Proof of Theorem 4.2.** Now we can prove the special Markov property in the case  $\lim_{\gamma \rightarrow \infty} \phi_1(\gamma) = +\infty$ .

Let  $Z \in \tilde{\mathcal{F}}_\infty$  non-negative such that  $\mathbb{E}[Z] < \infty$ . Let  $\varphi$  satisfying hypothesis of Theorem 4.2,  $(h_1)$ – $(h_3)$ . We have, using notation of the previous sections

$$\begin{aligned} \mathbb{E} \left[ Z \exp \left( - \sum_{i \in I} \varphi(A_{\alpha_i}, \omega, \mathcal{S}^i) \right) \right] &= \lim_{j \rightarrow \infty} \mathbb{E} \left[ Z \exp \left( - \sum_{k=1}^{\infty} \varphi^* \left( A_{S_k^{\varepsilon_j}}, \omega, \mathcal{S}^{k, \varepsilon_j} \right) \right) \right] \\ &= \lim_{j \rightarrow \infty} \mathbb{E} \left[ Z \prod_{k=1}^{\infty} K_{\varepsilon_j}(A_{S_k^{\varepsilon_j}}) \right] \\ &= \mathbb{E} \left[ Z e^{- \int_0^{\infty} du \left( \alpha_1 v(u) + \int_{(0, \infty)} \pi_1(d\ell) (1 - w(\ell, u)) \right)} \right], \end{aligned}$$

where we used Lemma 4.4 and dominated convergence for the first equality, Lemma 4.6 for the second equality, Lemma 4.10 and dominated convergence for the last equality. By monotone class Theorem and monotonicity, we can remove hypothesis  $(h_1)$ – $(h_3)$ . To end the proof of the first part, notice that  $\int_0^t du \left( \alpha_1 v(u) + \int_{(0, \infty)} \pi_1(d\ell) (1 - w(\ell, u)) \right)$  is  $\tilde{\mathcal{F}}_\infty$ -measurable and so this is  $\mathbb{P}$ -a.e. equal to the conditional expectation (i.e. the left hand side term of (25)).

The second part of Theorem 4.2 is just a consequence of the following Lemma.

**Lemma 4.11.** *Let  $\varphi$  be a bounded non-negative measurable function defined on the product space  $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+) \times \mathbb{S}$ .  $\mathbb{N}$ -a.e., we have*

$$\sum_{i \in I} \varphi(A_{\alpha_i}, \rho_{\alpha_i}, \mathcal{S}^i) = \sum_{i \in I} \tilde{\varphi}(A_{\alpha_i}, \omega, \mathcal{S}^i),$$



where  $\tilde{\varphi}(t, \omega, \mathcal{S}) = \varphi(t, \tilde{\rho}_t(\omega), \mathcal{S})$ .

*Proof.* First we assume that  $\varphi$  satisfies  $(h_3)$ . The same arguments as those used to prove Lemma 4.4 yields that  $\mathbb{P}$ -a.e. for  $\varepsilon > 0$  small enough, we have

$$\sum_{i \in I} \varphi(A_{\alpha_i}, \rho_{\alpha_i-}, \mathcal{S}^i) = \sum_{k \geq 1} \varphi(A_{S_k^\varepsilon}, \rho_{S_k^\varepsilon}^-, \mathcal{S}^{k,\varepsilon}).$$

Notice that by construction,  $\rho_{S_k^\varepsilon}^- = \rho_{T_k^\varepsilon}$  and that  $m_{T_k^\varepsilon} = 0$ . Using the strong Markov property at time  $T_k^\varepsilon$  and the second part of Corollary 3.2, we deduce that  $\mathbb{P}$ -a.e. for all  $k \in \mathbb{N}^*$ ,

$$(40) \quad C_{A_{T_k^\varepsilon}} = T_k^\varepsilon.$$

Therefore, as  $A_{S_k^\varepsilon} = A_{T_k^\varepsilon}$ , we have  $\mathbb{N}$ -a.e.

$$\tilde{\rho}_{A_{S_k^\varepsilon}} = \tilde{\rho}_{A_{T_k^\varepsilon}} = \rho_{T_k^\varepsilon} = \rho_{S_k^\varepsilon}^-.$$

Hence, we have that  $\mathbb{N}$ -a.e. for  $\varepsilon > 0$  small enough,

$$\sum_{i \in I} \varphi(A_{\alpha_i}, \rho_{\alpha_i-}, \mathcal{S}^i) = \sum_{k \geq 1} \tilde{\varphi}(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}),$$

with  $\tilde{\varphi}(t, \omega, \mathcal{S}) = \varphi(t, \tilde{\rho}_t(\omega), \mathcal{S})$ . Now, we complete the proof using Lemma 4.4. The condition  $(h_3)$  is removed using the monotone class Theorem.  $\square$

## 5. LAW OF THE PRUNED EXPLORATION PROCESS

Let  $\rho^{(0)}$  be the exploration process of a Lévy process with Laplace exponent  $\psi_0$ . The aim of this section is to prove the following Theorem.

**Theorem 5.1.** *For every finite measure  $\mu$ , the law of the pruned process  $\tilde{\rho}$  under  $\mathbb{P}_{\mu,0}$  is the law of the exploration process  $\rho^{(0)}$  associated with a Lévy process with Laplace exponent  $\psi_0$  under  $\mathbb{P}_\mu$ .*

**5.1. A martingale problem for  $\tilde{\rho}$ .** Let  $\tilde{\sigma} = \inf\{t > 0, \tilde{\rho}_t = 0\}$ . In this section, we shall compute the law of the total mass process  $(\langle \tilde{\rho}_{t \wedge \tilde{\sigma}}, 1 \rangle, t \geq 0)$  under  $\mathbb{P}_\mu = \mathbb{P}_{\mu,0}$ , using martingale problem characterization. We will first show how a martingale problem for  $\rho$  can be translated into a martingale problem for  $\tilde{\rho}$ , see also [1]. Unfortunately, we were not able to use standard techniques of random time change, as developed in Chapter 6 of [16] and used for Poisson snake in [5], mainly because  $t^{-1}(\mathbb{E}_\mu[f(\rho_t)\mathbf{1}_{\{m_t=0\}}] - f(\mu))$  may not have a limit as  $t$  goes down to 0, even for exponential functionals.

Let  $F, K \in \mathcal{B}(\mathcal{M}_f(\mathbb{R}_+))$  be bounded. We suppose that  $\mathbb{N} \left[ \int_0^\sigma |K(\rho_s)| ds \right] < \infty$ , that for any  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ ,  $\mathbb{E}_\mu^* \left[ \int_0^\sigma |K(\rho_s)| ds \right] < \infty$  and that  $M_t = F(\rho_{t \wedge \sigma}) - \int_0^{t \wedge \sigma} K(\rho_s) ds$ , for  $t \geq 0$ , defines an  $\mathcal{F}$ -martingale. In particular, notice that  $\mathbb{E}_\mu^* [\sup_{t \geq 0} |M_t|] < \infty$ . Thus, we can define for  $t \geq 0$ ,

$$N_t = \mathbb{E}_\mu^* [M_{C_t} | \tilde{\mathcal{F}}_t].$$

**Proposition 5.2.** *The process  $N = (N_t, t \geq 0)$  is an  $\tilde{\mathcal{F}}$ -martingale. And we have the representation formula for  $N_t$ :*

$$(41) \quad N_t = F(\tilde{\rho}_{t \wedge \tilde{\sigma}}) - \int_0^{t \wedge \tilde{\sigma}} du \tilde{K}(\tilde{\rho}_u),$$

with

$$(42) \quad \tilde{K}(\nu) = K(\nu) + \alpha_1 \mathbb{N} \left[ \int_0^\sigma K([\nu, \rho_s]) ds \right] + \int_{(0, \infty)} \pi_1(d\ell) \mathbb{E}_\ell^* \left[ \int_0^\sigma K([\nu, \rho_s]) ds \right].$$

*Proof.* Notice that  $N = (N_t, t \geq 0)$  is an  $\tilde{\mathcal{F}}$ -martingale. Indeed, we have for  $t, s \geq 0$ ,

$$\begin{aligned} \mathbb{E}_\mu[N_{t+s} | \tilde{\mathcal{F}}_t] &= \mathbb{E}_\mu[\mathbb{E}_\mu[M_{C_{t+s}} | \tilde{\mathcal{F}}_{t+s}] | \tilde{\mathcal{F}}_t] \\ &= \mathbb{E}_\mu[M_{C_{t+s}} | \tilde{\mathcal{F}}_t] \\ &= \mathbb{E}_\mu[\mathbb{E}_\mu[M_{C_{t+s}} | \mathcal{F}_{C_t}] | \tilde{\mathcal{F}}_t] \\ &= \mathbb{E}_\mu[M_{C_t} | \tilde{\mathcal{F}}_t], \end{aligned}$$

where we used the optional stopping time Theorem for the last equality. To compute  $\mathbb{E}_\mu[M_{C_t} | \tilde{\mathcal{F}}_t]$ , we write  $M_{C_t} = N'_t - M'_{C_t}$ , where for  $u \geq 0$ ,

$$M'_u = \int_0^{u \wedge \sigma} K(\rho_s) \mathbf{1}_{\{m_s \neq 0\}} ds.$$

Recall that  $C_0 = 0$   $\mathbb{P}_\mu$ -a.s. by Corollary 3.2. In particular, we get

$$\begin{aligned} N'_t = M_{C_t} + M'_{C_t} &= F(\rho_{C_t \wedge \sigma}) - \int_0^{C_t \wedge \sigma} K(\rho_s) \mathbf{1}_{\{m_s = 0\}} ds \\ &= F(\tilde{\rho}_{t \wedge \tilde{\sigma}}) - \int_0^{C_t \wedge \sigma} K(\rho_s) dA_s \\ &= F(\tilde{\rho}_{t \wedge \tilde{\sigma}}) - \int_0^{t \wedge \tilde{\sigma}} K(\tilde{\rho}_u) du, \end{aligned}$$

where we used the time change  $u = A_s$  for the last equality. In particular, as  $\tilde{\sigma}$  is an  $\tilde{\mathcal{F}}$ -stopping time, we get that the process  $(N'_t, t \geq 0)$  is  $\tilde{\mathcal{F}}$ -adapted. Since  $N_t = N'_t - \mathbb{E}_\mu[M'_{C_t} | \tilde{\mathcal{F}}_t]$ , we are left with the computation of  $\mathbb{E}_\mu[M'_{C_t} | \tilde{\mathcal{F}}_t]$ .

We keep the notations of Section 4. We consider  $(\rho^i, m^i)$ ,  $i \in I$  the excursions of the process  $(\rho, m)$  outside  $\{s, m_s = 0\}$  before  $\sigma$  and let  $(\alpha_i, \beta_i)$ ,  $i \in I$  be the corresponding interval excursions. In particular we can write

$$\int_0^{C_t \wedge \sigma} |K(\rho_s)| \mathbf{1}_{\{m_s \neq 0\}} ds = \sum_{i \in I} \Phi(A_{\alpha_i}, \rho_{\alpha_i -}, \rho^i),$$

with

$$\Phi(u, \mu, \rho) = \mathbf{1}_{\{u < t\}} \int_0^{\sigma(\rho)} |K([\mu, \rho_s])| ds,$$

where  $\sigma(\rho) = \inf\{v > 0; \rho_v = 0\}$ . We deduce from the second part of Theorem 4.2, that  $\mathbb{P}_\mu$ -a.s.

$$(43) \quad \mathbb{E}_\mu \left[ \int_0^{C_t \wedge \sigma} |K(\rho_s)| \mathbf{1}_{\{m_s \neq 0\}} ds | \tilde{\mathcal{F}}_\infty \right] = \int_0^{\tilde{\sigma}} \mathbf{1}_{\{u < t\}} \hat{K}(\tilde{\rho}_u) du,$$

with,  $\hat{K}$  defined for  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$  by

$$\hat{K}(\nu) = \alpha_1 \mathbb{N} \left[ \int_0^\sigma |K([\nu, \rho_s])| ds \right] + \int_{(0, \infty)} \pi_1(d\ell) \mathbb{E}_\ell^* \left[ \int_0^\sigma |K([\nu, \rho_s])| ds \right].$$

Since  $\mathbb{E}_\mu \left[ \int_0^{C_t \wedge \sigma} |K(\rho_s)| \mathbf{1}_{\{m_s \neq 0\}} ds \right] \leq \mathbb{E}_\mu \left[ \int_0^\sigma |K(\rho_s)| ds \right] < \infty$ , we deduce from (43) that  $\mathbb{P}_\mu$ -a.s.  $du$ -a.e.  $\mathbf{1}_{\{u < \tilde{\sigma}\}} \hat{K}(\tilde{\rho}_u)$  is finite.

We define  $\bar{K} \in \mathcal{B}(\mathcal{M}_f(\mathbb{R}_+))$  for  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$  by

$$(44) \quad \bar{K}(\nu) = \alpha_1 \mathbb{N} \left[ \int_0^\sigma K([\nu, \rho_s]) ds \right] + \int_{(0, \infty)} \pi_1(d\ell) \mathbb{E}_\ell^* \left[ \int_0^\sigma K([\nu, \rho_s]) ds \right]$$

if  $\hat{K}(\nu) < \infty$ , or by  $\bar{K}(\nu) = 0$  if  $\hat{K}(\nu) = +\infty$ . In particular, we have  $|\bar{K}(\nu)| \leq \hat{K}(\nu)$  and  $\mathbb{P}_\mu$ -a.s.  $\int_0^{\tilde{\sigma}} |\bar{K}(\tilde{\rho}_u)| du$  is finite. Using the special Markov property once again (see (43)), we get that  $\mathbb{P}_\mu$ -a.s.,

$$\mathbb{E}_\mu \left[ M'_{C_t} | \tilde{\mathcal{F}}_\infty \right] = \mathbb{E}_\mu \left[ \int_0^{C_t \wedge \sigma} K(\rho_s) \mathbf{1}_{\{m_s \neq 0\}} ds | \tilde{\mathcal{F}}_\infty \right] = \int_0^{t \wedge \tilde{\sigma}} \bar{K}(\tilde{\rho}_u) du.$$

Finally, as  $N_t = N'_t - \mathbb{E}_\mu \left[ M'_{C_t} | \tilde{\mathcal{F}}_\infty \right]$ , this gives (41).  $\square$

**Corollary 5.3.** *Let  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ . The law of the total mass process  $(\langle \tilde{\rho}_t, 1 \rangle, t \geq 0)$  under  $\mathbb{P}_{\mu,0}^*$  is the law of the total mass process of  $\rho^{(0)}$  under  $\mathbb{P}_\mu^*$ .*

*Proof.* Let  $X = (X_t, t \geq 0)$  be under  $\mathbb{P}_x^*$  a Lévy process with Laplace transform  $\psi$  started at  $x > 0$  and stopped when it reached 0. Under  $\mathbb{P}_\mu$ , the total mass process  $(\langle \rho_{t \wedge \sigma}, 1 \rangle, t \geq 0)$  is distributed as  $X$  under  $\mathbb{P}_{\langle \mu, 1 \rangle}^*$ . Let  $c > 0$ . From Lévy processes theory, we know that the process  $e^{-cX_t} - \psi(c) \int_0^t e^{-cX_s} ds$ , for  $t \geq 0$  is a martingale. We deduce from the stopping time Theorem that  $M = (M_t, t \geq 0)$  is an  $\mathcal{F}$ -martingale under  $\mathbb{P}_\mu$ , where  $M_t = F(\rho_{t \wedge \sigma}) - \int_0^{t \wedge \sigma} K(\rho_s) ds$ , with  $F, K \in \mathcal{B}(\mathcal{M}_f(\mathbb{R}_+))$  defined by  $F(\nu) = e^{-c\langle \nu, 1 \rangle}$  for  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$  and  $K = \psi(c)F$ . Notice  $K \geq 0$ . We have by dominated convergence and monotone convergence.

$$e^{-c\langle \mu, 1 \rangle} = \lim_{t \rightarrow \infty} \mathbb{E}_\mu[M_t] = \mathbb{E}_\mu[e^{-c\langle \rho_\sigma, 1 \rangle}] - \psi(c) \mathbb{E}_\mu \left[ \int_0^\sigma e^{-c\langle \rho_s, 1 \rangle} ds \right].$$

This implies that, for any  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ ,  $\mathbb{E}_\mu \left[ \int_0^\sigma |K(\rho_s)| ds \right]$  is finite. Using the Poisson representation, see Proposition 2.9, it is easy to get that

$$(45) \quad \mathbb{N} \left[ \int_0^\sigma dt e^{-c\langle \rho_t, 1 \rangle} \right] = \frac{c}{\psi(c)}.$$

In particular, it is also finite.

From Proposition 5.2, we get that  $N = (N_t, t \geq 0)$  is under  $\mathbb{P}_\mu$  an  $\tilde{\mathcal{F}}$ -martingale, where: for  $t \geq 0$ ,

$$N_t = e^{-c\langle \tilde{\rho}_{t \wedge \tilde{\sigma}}, 1 \rangle} - \int_0^{t \wedge \tilde{\sigma}} \tilde{K}(\tilde{\rho}_u) du$$

and  $\tilde{K}$  given by (42). We can compute  $\tilde{K}$ :

$$\begin{aligned}\tilde{K}(\nu) &= \psi(c) e^{-c\langle\nu,1\rangle} \left( 1 + \alpha_1 \mathbb{N} \left[ \int_0^\sigma e^{-c\langle\rho_s,1\rangle} ds \right] + \int_{(0,\infty)} \pi_1(d\ell) \mathbb{E}_\ell^* \left[ \int_0^\sigma e^{-c\langle\rho_s,1\rangle} ds \right] \right) \\ &= \psi(c) e^{-c\langle\nu,1\rangle} \left( 1 + \alpha_1 \frac{c}{\psi(c)} + \int_{(0,\infty)} \pi_1(d\ell) \int_0^\ell dr e^{-cr} \mathbb{N} \left[ \int_0^\sigma e^{-c\langle\rho_s,1\rangle} ds \right] \right) \\ &= e^{-c\langle\nu,1\rangle} \left( \psi(c) + \alpha_1 c + \int_{(0,\infty)} \pi_1(d\ell) (1 - e^{-c\ell}) \right) \\ &= \psi_0(c) e^{-c\langle\nu,1\rangle},\end{aligned}$$

where we used (45) and the excursion decomposition for the second equality, and  $\psi_0 = \psi + \phi_1$  for the last one.

Thus, the process  $(N_t, t \geq 0)$  with for  $t \geq 0$

$$N_t = e^{-c\langle\tilde{\rho}_{t\wedge\tilde{\sigma}},1\rangle} - \psi_0(c) \int_0^{t\wedge\tilde{\sigma}} e^{-c\langle\tilde{\rho}_u,1\rangle} du$$

is under  $\mathbb{P}_\mu$  an  $\tilde{\mathcal{F}}$ -martingale.

Notice that  $\tilde{\sigma} = \inf\{s \geq 0; \langle\tilde{\rho}_s,1\rangle = 0\}$ . Let  $X^{(0)} = (X_t^{(0)}, t \geq 0)$  be under  $\mathbb{P}_x^*$  a Lévy process with Laplace transform  $\psi_0$  started at  $x > 0$  and stopped when it reached 0. The two non-negative càd-làg processes  $(\langle\tilde{\rho}_{t\wedge\tilde{\sigma}},1\rangle, t \geq 0)$  and  $X^{(0)}$  solves the martingale problem: for any  $c \geq 0$ , the process defined for  $t \geq 0$  by

$$e^{-cY_{t\wedge\sigma'}} - \psi_0(c) \int_0^{t\wedge\sigma'} e^{-cY_s} ds,$$

where  $\sigma' = \inf\{s \geq 0; Y_s \leq 0\}$ , is a martingale. From Corollary 4.4.4 in [16], we deduce that those two processes have the same distribution. To finish the proof, notice that the total mass process of  $\rho^{(0)}$  under  $\mathbb{P}_\mu^*$  is distributed as  $X^{(0)}$  under  $\mathbb{P}_{\langle\mu,1\rangle}^*$ .  $\square$

**5.2. Identification of the law of  $\tilde{\rho}$ .** To begin with, let us mention some useful properties of the process  $\tilde{\rho}$ .

**Lemma 5.4.** *We have the following properties for the process  $\tilde{\rho}$ .*

- (i)  $\tilde{\rho}$  is a càd-làg Markov process.
- (ii) The sojourn time at 0 of  $\tilde{\rho}$  is 0.
- (iii) 0 is recurrent for  $\tilde{\rho}$ .

*Proof.* (i) This is a direct consequence of the strong Markov property of the process  $(\rho, m)$ .

(ii) We have for  $r > 0$ , with the change of variable  $t = A_s$ , a.s.

$$\int_0^r \mathbf{1}_{\{\tilde{\rho}_t=0\}} dt = \int_0^r \mathbf{1}_{\{\rho_{C_t}=0\}} dt = \int_0^{C_r} \mathbf{1}_{\{\rho_s=0\}} dA_s = \int_0^{C_r} \mathbf{1}_{\{\rho_s=0\}} ds = 0,$$

as the sojourn time of  $\rho$  at 0 is 0 a.s.

(iii) Since  $\tilde{\sigma} = A_\sigma$  and  $\sigma < +\infty$  a.s., we deduce that 0 is recurrent for  $\tilde{\rho}$  a.s.  $\square$

Since the processes  $\tilde{\rho}$  and  $\rho^{(0)}$  are both Markov processes, to show that they have the same law, it is enough to show that they have the same one-dimensional marginals. We first prove that result under the excursion measure.

**Proposition 5.5.** *For every  $\lambda > 0$  and every non-negative bounded measurable function  $f$ ,*

$$\mathbb{N} \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] = \mathbb{N} \left[ \int_0^{\sigma^{(0)}} e^{-\lambda t - \langle \rho_t^{(0)}, f \rangle} dt \right].$$

*Proof.* On one hand, we compute, using the definition of the pruned process  $\tilde{\rho}$ ,

$$\mathbb{N} \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] = \mathbb{N} \left[ \int_0^{A_\sigma} e^{-\lambda t - \langle \rho_{C_t}, f \rangle} dt \right].$$

We now make the change of variable  $t = A_u$  to get

$$\begin{aligned} \mathbb{N} \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] &= \mathbb{N} \left[ \int_0^\sigma e^{-\lambda A_u} e^{-\langle \rho_u, f \rangle} dA_u \right] \\ &= \mathbb{N} \left[ \int_0^\sigma e^{-\lambda A_u} e^{-\langle \rho_u, f \rangle} \mathbf{1}_{\{m_u=0\}} du \right]. \end{aligned}$$

By a time reversibility argument, see Lemma 2.7, we obtain

$$\begin{aligned} \mathbb{N} \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] &= \mathbb{N} \left[ \int_0^\sigma \mathbf{1}_{\{m_u=0\}} e^{-\langle \eta_u, f \rangle} e^{-\lambda(A_\sigma - A_u)} du \right] \\ &= \mathbb{N} \left[ \int_0^\sigma \mathbf{1}_{\{m_u=0\}} e^{-\langle \eta_u, f \rangle} \mathbb{E}_{\rho_u, 0}^* \left[ e^{-\lambda A_\sigma} \right] du \right] \\ &= \mathbb{N} \left[ \int_0^\sigma \mathbf{1}_{\{m_u=0\}} e^{-\langle \eta_u, f \rangle - \psi_0^{-1}(\lambda) \langle \rho_u, 1 \rangle} du \right], \end{aligned}$$

where we applied Lemma 3.1 (i) for the last equality. Now, using Proposition 2.9, we have

$$\mathbb{N} \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] = \int_0^\infty da e^{-\alpha a} \mathbb{M}_a \left[ \mathbf{1}_{\{m=0\}} e^{-\langle \nu, f \rangle - \psi_0^{-1}(\lambda) \langle \mu, 1 \rangle} \right].$$

Using usual properties of point Poisson measures, we have, with  $c = \alpha_1 + \int_{(0, \infty)} \ell \pi_1(d\ell)$ ,

$$\mathbb{M}_a \left[ \mathbf{1}_{\{m=0\}} F(\mu, \nu) \right] = e^{-ca} \mathbb{M}_a \left[ F(\mu^0, \nu^0) \right],$$

where with the notations of Proposition 2.9, for any  $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$\begin{aligned} \langle \mu^0, f \rangle &= \int \mathcal{N}_0(dx d\ell du) \mathbf{1}_{[0, a]}(x) u \ell f(x) + \beta \int_0^a f(r) dr, \\ \langle \nu^0, f \rangle &= \int \mathcal{N}_0(dx d\ell du) \mathbf{1}_{[0, a]}(x) (1-u) \ell f(x) + \beta \int_0^a f(r) dr. \end{aligned}$$

As  $\alpha_0 = \alpha + c$ , we have

$$\mathbb{N} \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] = \int_0^\infty da e^{-\alpha_0 a} \mathbb{M}_a \left[ e^{-\langle \nu^0, f \rangle - \psi_0^{-1}(\lambda) \langle \mu^0, 1 \rangle} \right].$$

Proposition 3.1.3 in [15] directly implies that the left-hand side of the previous equality is equal to  $\mathbb{N} \left[ \int_0^{\sigma^{(0)}} e^{-\langle \eta_t^{(0)}, f \rangle - \psi_0^{-1}(\lambda) \langle \rho_t^{(0)}, 1 \rangle} dt \right]$ . On the other hand, similar computations as above yields that this quantity is equal to  $\mathbb{N} \left[ \int_0^{\sigma^{(0)}} e^{-\lambda t - \langle \rho_t^{(0)}, f \rangle} dt \right]$ . This ends the proof.  $\square$

Now, we prove the same result under  $\mathbb{P}_{\mu, 0}^*$ , that is:

**Proposition 5.6.** *For every  $\lambda > 0$ ,  $f \in \mathcal{B}_+(\mathbb{R}_+)$  bounded and every finite measure  $\mu$ ,*

$$\mathbb{E}_{\mu,0}^* \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] = \mathbb{E}_{\mu}^* \left[ \int_0^{\sigma^{(0)}} e^{-\lambda t - \langle \rho_t^{(0)}, f \rangle} dt \right].$$

*Proof.* From the Poisson representation, see Lemma 2.6, and using notations of this Lemma and of (19) we have

$$\begin{aligned} \mathbb{E}_{\mu,0}^* \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] &= \mathbb{E}_{\mu,0}^* \left[ \int_0^{\sigma} e^{-\lambda A_u - \langle \rho_u, f \rangle} dA_u \right] \\ &= \mathbb{E}_{\mu,0}^* \left[ \sum_{i \in J} e^{-\lambda A_{\alpha_i} - \langle k_{-I_{\alpha_i}}, f \rangle} \int_0^{\sigma_i} e^{-\langle \rho_s^i, f - I_{\alpha_i} \rangle - \lambda A_s^i} dA_s^i \right], \end{aligned}$$

where the function  $f_r$  is defined by  $f_r(x) = f(H_r^{(\mu)} + x)$  and  $H_r^{(\mu)} = H(k_r \mu)$  is the maximal element of the closed support of  $k_r \mu$  (see (9)). We recall that  $-I$  is the local time at 0 of the reflected process  $X - I$ , and that  $\tau_r = \inf\{s; -I_s > r\}$  is the right continuous inverse of  $-I$ . From excursion formula, and using the time change  $-I_s = r$  (or equivalently  $\tau_r = s$ ), we get

$$\begin{aligned} \mathbb{E}_{\mu,0}^* \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f \rangle} dt \right] &= \mathbb{E}_{\mu,0}^* \left[ \int_0^{\tau^{(\mu,1)}} d(-I_s) e^{-\langle k_{-I_s} \mu, f \rangle - \lambda A_s} G(-I_s) \right] \\ (46) \qquad \qquad \qquad &= \mathbb{E}_{\mu,0}^* \left[ \int_0^{\langle \mu, 1 \rangle} dr e^{-\langle k_r \mu, f \rangle - \lambda A_{\tau_r}} G(r) \right], \end{aligned}$$

where the function  $G(r)$  is given by

$$G(r) = \mathbb{N} \left[ \int_0^{\sigma} e^{-\langle \rho_s, f_r \rangle - \lambda A_s} dA_s \right] = \mathbb{N} \left[ \int_0^{\tilde{\sigma}} e^{-\lambda t - \langle \tilde{\rho}_t, f_r \rangle} dt \right].$$

The same kind of computation gives

$$(47) \qquad \mathbb{E}_{\mu}^* \left[ \int_0^{\sigma^{(0)}} e^{-\lambda t - \langle \rho_t^{(0)}, f \rangle} dt \right] = \mathbb{E} \left[ \int_0^{\langle \mu, 1 \rangle} dr e^{-\langle k_r \mu, f \rangle - \lambda \tau_r^{(0)}} G^{(0)}(r) \right]$$

where the function  $G^{(0)}$  is defined by

$$G^{(0)}(r) = \mathbb{N} \left[ \int_0^{\sigma^{(0)}} e^{-\lambda s - \langle \rho_s^{(0)}, f_r \rangle} ds \right]$$

and  $\tau^{(0)}$  is the right-continuous inverse of the infimum process  $-I^{(0)}$  of the Lévy process with Laplace exponent  $\psi_0$ .

Proposition 5.5 says that the functions  $G$  and  $G^{(0)}$  are equal. Moreover, as the total mass processes have the same law (see Corollary 5.3), we know that the proposition is true for  $f$  constant. And, for  $f$  constant, the functions  $G$  and  $G^{(0)}$  are also constant. Therefore, we have for  $f$  constant equal to  $c \geq 0$ ,

$$\mathbb{E}_{\mu,0}^* \left[ \int_0^{\langle \mu, 1 \rangle} dr e^{-c(\langle \mu, 1 \rangle - r)} e^{-\lambda A_{\tau_r}} \right] = \mathbb{E} \left[ \int_0^{\langle \mu, 1 \rangle} dr e^{-c(\langle \mu, 1 \rangle - r)} e^{-\lambda \tau_r^{(0)}} \right].$$

As this is true for any  $c \geq 0$ , uniqueness of the Laplace transform gives the equality

$$\mathbb{E}_{\mu,0}^* \left[ e^{-\lambda A_{\tau_r}} \right] = \mathbb{E} \left[ e^{-\lambda \tau_r^{(0)}} \right] \quad dr - \text{a.e.}$$

In fact this equality holds for every  $r$  by right-continuity.

Finally as  $G = G^{(0)}$ , we have thanks to (46) and (47), that, for every bounded non-negative measurable function  $f$ ,

$$\int_0^{\langle \mu, 1 \rangle} dr e^{-\langle k_r \mu, f \rangle} \mathbb{E}_{\mu, 0}^* \left[ e^{-\lambda A_{\tau_r}} \right] G(r) = \int_0^{\langle \mu, 1 \rangle} dr e^{-\langle k_r \mu, f \rangle} \mathbb{E} \left[ e^{-\lambda \tau_r^{(0)}} \right] G^{(0)}(r)$$

which ends the proof.  $\square$

**Corollary 5.7.** *The process  $\tilde{\rho}$  under  $\mathbb{P}_{\mu, 0}^*$  is distributed as  $\rho^{(0)}$  under  $\mathbb{P}_\mu^*$ .*

*Proof.* Let  $f \in \mathcal{B}_+(\mathbb{R}_+)$  bounded. Proposition 5.6 can be re-written as

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E}_{\mu, 0}^* \left[ e^{-\langle \tilde{\rho}_t, f \rangle} \mathbf{1}_{\{t \leq \tilde{\sigma}\}} \right] dt = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_\mu^* \left[ e^{-\langle \rho_t^{(0)}, f \rangle} \mathbf{1}_{\{t \leq \sigma^{(0)}\}} \right] dt.$$

By uniqueness of the Laplace transform, we deduce that, for almost every  $t > 0$ ,

$$\mathbb{E}_{\mu, 0}^* \left[ e^{-\langle \tilde{\rho}_t, f \rangle} \mathbf{1}_{\{t \leq \tilde{\sigma}\}} \right] = \mathbb{E}_\mu^* \left[ e^{-\langle \rho_t^{(0)}, f \rangle} \mathbf{1}_{\{t \leq \sigma^{(0)}\}} \right].$$

In fact this equality holds for every  $t$  by right-continuity. As the Laplace functionals characterize the law of a random measure, we deduce that, for fixed  $t > 0$ , the law of  $\tilde{\rho}_t$  under  $\mathbb{P}_{\mu, 0}^*$  is the same as the law of  $\rho_t^{(0)}$  under  $\mathbb{P}_\mu^*$ .

The Markov property then gives the equality in law for the càd-làg processes  $\tilde{\rho}$  and  $\rho^{(0)}$ .  $\square$

*Proof of Theorem 5.1.* 0 is recurrent for the Markov càd-làg processes  $\tilde{\rho}$  and  $\rho^{(0)}$ . These two processes have no sojourn time at 0, and when killed on the first hitting time of 0, they have the same law, thanks to Lemma 5.7. From Theorem 4.2 of [12], Section 5, we deduce that  $\tilde{\rho}$  under  $\mathbb{P}_{\mu, 0}$  is distributed as  $\rho^{(0)}$  under  $\mathbb{P}_\mu$ .  $\square$

## 6. LAW OF THE EXCURSION LENGTHS

Recall  $\tilde{\sigma} = \int_0^\sigma \mathbf{1}_{\{m_s = 0\}} ds$  denotes the length of the excursion of the pruned exploration process. We can compute the joint law of  $(\tilde{\sigma}, \sigma)$ . This will determine uniquely the law of  $\tilde{\sigma}$  conditionally on  $\sigma = r$ .

**Proposition 6.1.** *For all non-negative  $\gamma, \kappa$ , the value  $v$  defined by  $v = \mathbb{N} \left[ 1 - e^{-\psi(\gamma)\sigma - \kappa\tilde{\sigma}} \right]$  is the unique non-negative solution of the equation*

$$\psi_0(v) = \kappa + \psi_0(\gamma).$$

*Proof.* Excursion theory implies that the special Markov property, Theorem 4.2, also holds under  $\mathbb{N}$ , with the integration of  $u$  over  $[0, \tilde{\sigma} = A_\sigma]$  instead of  $[0, \infty)$ . Taking  $\phi(\mathcal{S}) = \psi(\gamma)\sigma$ , we have

$$\begin{aligned} v = \mathbb{N} \left[ 1 - e^{-\kappa\tilde{\sigma} - \psi(\gamma)\sigma} \right] &= \mathbb{N} \left[ 1 - e^{-(\kappa + \psi(\gamma))\tilde{\sigma} - \psi(\gamma) \int_0^\sigma \mathbf{1}_{\{m_s \neq 0\}} ds} \right] \\ &= \mathbb{N} \left[ 1 - e^{-(\kappa + \psi(\gamma))\tilde{\sigma} - \tilde{\sigma} \left( \alpha_1 \mathbb{N}[1 - e^{-\psi(\gamma)\sigma}] + \int_{(0, +\infty)} \pi_1(d\ell) (1 - \mathbb{E}_\ell^* [e^{-\psi(\gamma)\sigma}] \right)} \right]. \end{aligned}$$

Notice that  $\sigma$  under  $\mathbb{P}_\ell^*$  is distributed as  $\tau_\ell$ , the first time for which the infimum of  $X$ , started at 0, reaches  $-\ell$ . Since  $\tau_\ell$  is distributed as a subordinator with Laplace exponent  $\psi^{-1}$  at time  $\ell$ , we have

$$\mathbb{E}_\ell^* [e^{-\psi(\gamma)\sigma}] = \mathbb{E} \left[ e^{-\psi(\gamma)\tau_\ell} \right] = e^{-\ell\gamma}.$$

Thanks to (12), we get  $\mathbb{N}[1 - e^{-\psi(\gamma)\sigma}] = \gamma$ . We deduce that

$$\begin{aligned} v &= \mathbb{N} \left[ 1 - e^{-(\kappa + \psi(\gamma))\tilde{\sigma} - \tilde{\sigma} \left( \alpha_1 \gamma + \int_{(0, +\infty)} \pi_1(d\ell)(1 - e^{-\gamma\ell}) \right)} \right] \\ &= \mathbb{N} \left[ 1 - e^{-(\kappa + \psi_0(\gamma))\tilde{\sigma}} \right] \\ &= \psi_0^{-1}(\kappa + \psi_0(\gamma)). \end{aligned}$$

Since  $\psi_0$  is increasing and continuous, we get the result.  $\square$

## 7. APPENDIX

We shall present in this appendix, how one can extend the construction of the Lévy snake from [15] to the case of non continuous height process.

Let  $D$  be a distance on  $\mathcal{M}_f(\mathbb{R}_+)$  which defines the topology of weak convergence. Let us recall that  $(\mathcal{M}_f(\mathbb{R}_+), D)$  is a Polish space, see [14], section 3.1.

Let  $E$  be a Polish space, whose topology is defined by a metric  $\delta$ , and  $\partial$  be a cemetery point added to  $E$ . Let  $\mathcal{W}_x$  be the space of all  $E$ -valued weighted killed paths started at  $x \in E$ . An element  $\bar{w} = (\mu, w)$  of  $\mathcal{W}_x$  is a mass measure  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$  and a càd-làg mapping  $w : [0, \langle \mu, 1 \rangle) \rightarrow E$  s.t.  $w(0) = x$ . By convention the point  $x$  is also considered as a weighted killed path with mass measure  $\mu = 0$ . We set  $\mathcal{W} = \bigcup_{x \in E} \mathcal{W}_x$  and equip  $\mathcal{W}$  with the distance

$$d((\mu, w), (\mu', w')) = \delta(w(0), w'(0)) + D(\mu, \mu') + \int_0^{\langle \mu, 1 \rangle \wedge \langle \mu', 1 \rangle} dt (d_t(w_{\leq t}, w'_{\leq t})),$$

where  $d_t$  is the Skorohod metric on the space  $\mathbb{D}([0, t], E)$  and  $w_{\leq t}$  denote the restriction of  $w$  to the interval  $[0, t]$ . It is then elementary to check that the space  $(\mathcal{W}, d)$  is a Polish space. We shall write  $\mu_{\bar{w}}$  instead of  $\mu$  when  $\bar{w} = (\mu, w)$ .

Recall (9). We consider a family of probability measures  $\bar{\Pi}_{x, \mu}$ , for  $x \in E$  and the mass measure  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$  on  $\mathcal{W}_x$ , s.t.

- a)  $\mu_{\bar{w}} = \mu$ ,  $\bar{\Pi}_{x, \mu}(d\bar{w})$ -a.s.;
- b)  $w(0) = x$ ,  $\bar{\Pi}_{x, \mu}(d\bar{w})$ -a.s.;
- c)  $w$  has no fixed discontinuity: for all  $s \in [0, \langle \mu, 1 \rangle)$ ,  $\bar{\Pi}_{x, \mu}(w(s-) = w(s)) = 1$ ;
- d) If  $H(\mu) < \infty$ , then  $w(\langle \mu, 1 \rangle -)$  exists  $\bar{\Pi}_{x, \mu}(d\bar{w})$ -a.s.;
- e) If  $H(\mu) < \infty$  and  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$ , then under  $\bar{\Pi}_{x, [\mu, \nu]}$ ,  $(w(r), r \in [0, \langle \mu, 1 \rangle))$  is distributed as  $(w(r), r \in [0, \langle \mu, 1 \rangle))$  under  $\bar{\Pi}_{x, \mu}$  and, conditionally on  $(w(r), r \in [0, \langle \mu, 1 \rangle))$ ,  $(w(r + \langle \mu, 1 \rangle), r \in [0, \langle \nu, 1 \rangle))$  is distributed as  $(w(r), r \in [0, \langle \nu, 1 \rangle))$  under  $\bar{\Pi}_{w(\langle \mu, 1 \rangle -), \nu}$ .

The last property corresponds to the Markov property conditionally on the mass measure. We shall assume that the mapping  $(x, \mu) \mapsto \bar{\Pi}_{x, \mu}$  is measurable.

Let  $\rho$  be an exploration process starting at  $\mu$ . We set  $Y_t = \langle \rho_t, 1 \rangle$ . Recall that  $(Y_t, t \geq 0)$  is distributed as a Lévy process with Laplace exponent  $\psi$  started at  $\langle \mu, 1 \rangle$ . For  $0 \leq s < t$ , we set  $J_{s,t} = \inf_{s \leq u \leq t} Y_u$  and  $\rho_{s,t} = k_{(Y_s - J_{s,t})} \rho_s = k_{(Y_t - J_{s,t})} \rho_t$ , the last equality being a consequence of the construction of the exploration process. We also define  $\bar{\rho}_t^{(s)}$  as the unique measure  $\nu$  s.t.  $[\rho_{s,t}, \nu] = \rho_t$ .

Conditionally on  $\rho$ , we define a probability transition semi-group  $R_{s,t}^\rho$  on  $\mathcal{W}_x$  as follows: for  $0 \leq s < t$  s.t.  $J_{s,t} < Y_s$  or  $w(\langle \rho_s, 1 \rangle -)$  exists and  $\mu_{\bar{w}} = \rho_s$ , under  $R_{s,t}^\rho(\bar{w}, d\bar{w}')$  we have

- i)  $\mu_{\bar{w}'} = \rho_t$ ,
- ii)  $(w'(r), r \in [0, \langle \rho_{s,t}, 1 \rangle)) = (w(r), r \in [0, \langle \rho_{s,t}, 1 \rangle))$ ,
- iii)  $(w'(r), r \in [\langle \rho_{s,t}, 1 \rangle, \langle \rho_t, 1 \rangle))$  is distributed according to  $\bar{\Pi}_{w(\langle \rho_{s,t}, 1 \rangle -), \bar{\rho}_t^{(s)}}$ .



In (iii), by convention, if  $\rho_{s,t} = 0$ , then  $w(\langle \rho_{s,t}, 1 \rangle -) = x$ . Notice that for fixed  $s < t$ , a.s.  $J_{s,t} < Y_s$  so that, with the previous convention  $w(\langle \rho_{s,t}, 1 \rangle -)$  is a.s. well defined. Notice that if  $(\rho_s, w)$  is distributed as  $\bar{\Pi}_{x, \rho_s}$ , then  $(\rho_t, w')$  is distributed as  $\bar{\Pi}_{x, \rho_t}$  thanks to condition e) on  $\bar{\Pi}$ . Thus we can use the Kolmogorov extension theorem to get that there exists a unique probability measure  $\mathbb{P}_{(\mu, w)}$  on  $(\mathcal{W}_x)^{\mathbb{R}^+}$  s.t. for  $0 = s_0 < s_1 < \dots < s_n$ ,

$$\begin{aligned} & \mathbb{P}_{(\mu, w)}(W'_{s_0} \in A_0, \rho_{s_0} \in B_0, \dots, W'_{s_n} \in A_n, \rho_{s_n} \in B_n) \\ &= \mathbb{E}_{\mu} \left[ \mathbf{1}_{\{\rho_{s_0} \in B_0, \dots, \rho_{s_n} \in B_n\}} \mathbf{1}_{\{w \in A_0\}} \int_{A_1 \times \dots \times A_n} R_{s_0, s_1}^{\rho}(w, dw_{s_1}) \cdots R_{s_{n-1}, s_n}^{\rho}(w_{s_{n-1}}, dw_{s_n}) \right]. \end{aligned}$$

We set  $\bar{W}_s = (\rho_s, W'_s)$ . Notice that  $W'_s(r) = W'_t(r)$  for  $r \in [0, \langle \rho_{s,t}, 1 \rangle)$  and thus that

$$d(\bar{W}_s, \bar{W}_t) \leq D(\rho_s, \rho_t) + |Y_s \wedge Y_t - J_{s,t}|.$$

Since  $\rho$  and  $Y$  are  $\mathbb{P}_{\mu}$ -a.s. càd-làg, this implies that the mapping  $s \mapsto \bar{W}_s$  is  $\mathbb{P}_{(\mu, w)}$ -a.s. càd-làg on  $[0, \infty) \cap \mathbb{Q}$ . Hence there is a unique càd-làg extension to the positive real line, we shall still denote by  $\mathbb{P}_{(\mu, w)}$ . The process  $(\bar{W}_s, s \geq 0)$  is under  $\mathbb{P}_{(\mu, w)}$  a time-homogeneous Markov process living in  $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W})$ . We call this distribution the distribution of the weighted Lévy snake associated with  $\bar{\Pi}$ .

We denote by  $(\mathcal{F}_s, s \geq 0)$  the canonical filtration on  $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W})$ . One can readily adapt the proofs of Propositions 4.1.1 and 4.1.2 of [15] to get the following result.

**Theorem 7.1.** *The process  $(\bar{W}_s, s \geq 0)$  is a càd-làg Markov process and is strong Markov with respect to the filtration  $(\mathcal{F}_{s+}, s \geq 0)$ .*

Let us remark that, when the family of probability measures  $\bar{\pi}_{x, \mu}$  is just the law of a homogeneous Markov process  $\xi$  starting at  $x$  and stopped at time  $\langle \mu, 1 \rangle$ , the previous construction gives a snake with spatial motion  $\xi$  and lifetime process  $X - I$ .

However, we need some dependency between the spatial motion and the exploration process  $\rho$  in order to recover the usual Lévy snake from the weighted Lévy snake. Informally, we keep the spatial motion from moving when time  $t$  is “on a mass” of  $\rho_s$ . This idea can be compared to a subordination and has already been used in the snake framework by Bertoin, Le Gall and Le Jan in [11] in order to construct a kind of Lévy snake from the usual Brownian snake.

Let  $\Pi_x$  be the distribution of  $\xi$  a càd-làg Markov process taking values in  $E$  with no fixed discontinuities and starting at  $x$ , such that the mapping  $x \mapsto \Pi_x$  is measurable. Recall (10) and set  $\hat{\mu}_r = k_{\langle \mu, 1 \rangle - r} \mu$  for  $r \in [0, \langle \mu, 1 \rangle)$ . We define  $\bar{\Pi}_{x, \mu}$  as the distribution of  $(\mu, w)$  with  $w = (\xi_{H(\hat{\mu}_r)}, r \in [0, \langle \mu, 1 \rangle))$  under  $\Pi_x$ . Notice that  $\xi_{r'} = w(\langle \mu, \mathbf{1}_{[0, r']} \rangle)$  for  $r' \in [0, H(\mu))$ . We have that  $\bar{\Pi}$  satisfies condition a)-e).

Let  $((\rho_s, W'_s), s \geq 0)$  be the corresponding weighted Lévy snake. For  $s \geq 0, r \geq 0$ , we set  $W_s(r) = W'_s(\langle \rho_s, \mathbf{1}_{[0, r]} \rangle)$ . The process  $((\rho_s, W_s), s \geq 0)$  is the Lévy snake defined in Section 4 of [15] with underlying motion  $\xi$ , when  $H$  is continuous. Notice that the construction here allows to remove the continuity condition of  $H$ . As a consequence of Theorem 7.1, we get that the (general) Lévy snake is strong Markov.

**Proposition 7.2.** *The process  $((\rho_s, W_s), s \geq 0)$  is a càd-làg Markov process and is strong Markov with respect to the filtration  $(\mathcal{F}_{s+}, s \geq 0)$ .*

## REFERENCES

- [1] Romain ABRAHAM and Jean-François DELMAS. Feller property and infinitesimal generator of the exploration process. *J. Theor. Probab.*, 20(3):355–370, 2007.

- [2] Romain ABRAHAM and Jean-François DELMAS. Williams' decomposition of the Lévy continuous random tree and simultaneous extinction probability for populations with neutral mutations. *Preprint CERMICS*, 2007.
- [3] Romain ABRAHAM and Jean-François DELMAS. Changing the branching mechanism of a continuous state branching process using immigration. *To appear in Ann. Inst. Henri Poincaré*, 2008.
- [4] Romain ABRAHAM and Jean-François DELMAS. Fragmentation associated to Lévy processes using snake. *Probab. Th. Rel. Fields*, 141:113–154, 2008.
- [5] Romain ABRAHAM and Laurent SERLET. Poisson snake and fragmentation. *Elect. J. of Probab.*, 7, 2002.
- [6] David ALDOUS. The continuum random tree II: an overview. In *Proc. Durham Symp. Stochastic Analysis*, pages 23–70. Cambridge univ. press edition, 1990.
- [7] David ALDOUS. The continuum random tree I. *Ann. Probab.*, 19(1):1–28, 1991.
- [8] David ALDOUS. The continuum random tree III. *Ann. Probab.*, 21(1):248–289, 1993.
- [9] Jean BERTOIN. *Lévy processes*. Cambridge University Press, Cambridge, 1996.
- [10] Jean BERTOIN. The structure of the allelic partition of the total population for Galton-Watson processes with neutral mutations. *arXiv:0711.3852*, 2008.
- [11] Jean BERTOIN, Jean-François LE GALL, and Yves LE JAN. Spatial branching processes and subordination. *Canad. J. of Math.*, 49(1):24–54, 1997.
- [12] R.M. BLUMENTHAL. *Excursions of Markov processes*. Birkhäuser, Boston, 1992.
- [13] D. J. DALEY and D. VERE-JONES. *An introduction to the theory of point processes. Vol. II. Probability and its Applications (New York)*. Springer-Verlag, New York, second edition, 2008. General theory and structures.
- [14] Donald A. DAWSON. Measure-valued Markov processes. In *École d'été de probabilité de Saint-Flour 1991*, volume 1541 of *Lect. Notes Math.*, pages 1–260. Springer Verlag, Berlin, 1993.
- [15] Thomas DUQUESNE and Jean-François LE GALL. *Random trees, Lévy processes and spatial branching processes*, volume 281. Astérisque, 2002.
- [16] Stewart N. ETHIER and Thomas G. KURTZ. *Markov processes*. Wiley, 1986.
- [17] M. JIRINA. Stochastic branching processes with continuous state space. *Czech. Math. J.*, 83(8):292–312, 1958.
- [18] John LAMPERTI. The limit of a sequence of branching processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 7:271–288, 1967.
- [19] Jean-François LE GALL and Yves LE JAN. Branching processes in Lévy processes: Laplace functionals of snake and superprocesses. *Ann. Probab.*, 26:1407–1432, 1998.
- [20] Jean-François LE GALL and Yves LE JAN. Branching processes in Lévy processes: The exploration process. *Ann. Probab.*, 26:213–252, 1998.

ROMAIN ABRAHAM, MAPMO, CNRS UMR 6628, FÉDÉRATION DENIS POISSON FR 2964, UNIVERSITÉ D'ORLÉANS, B.P. 6759, 45067 ORLÉANS CEDEX 2 FRANCE.

*E-mail address:* `romain.abraham@univ-orleans.fr`

JEAN-FRANÇOIS DELMAS, CERMICS, UNIVERSITÉ PARIS-EST, 6-8 AV. BLAISE PASCAL, CHAMPS-SUR-MARNE, 77455 MARNE LA VALLÉE, FRANCE.

*E-mail address:* `delmas@cermics.enpc.fr`

GUILLAUME VOISIN, MAPMO CNRS UMR 6628, FÉDÉRATION DENIS POISSON FR 2964, UNIVERSITÉ D'ORLÉANS, B.P. 6759, 45067 ORLÉANS CEDEX 2 FRANCE.

*E-mail address:* `guillaume.voisin@univ-orleans.fr`