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Abstract

We establish new convergence results, in strong topologies, for solutions of the parabolic-parabolic Keller–Segel system in the plane, to the corresponding solutions of the parabolic-elliptic model, as a physical parameter goes to zero. Our main tools are suitable space-time estimates, implying the global existence of slowly decaying (in general, nonintegrable) solutions for these models, under a natural smallness assumption.

Key words and phrases: chemotaxis system, local and global in time solutions, convergence of solutions

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1 Introduction

We consider two related nonlinear parabolic systems which are frequently used as models for a description of chemotactic phenomena, including the aggregation of microorganisms caused by a chemoattractant, i.e. a chemical whose concentration gradient governs the oriented movement of those microorganisms. The parabolic character of the systems comes from the diffusion described by the Laplacians. A version of the system (PE) below is also

used in astrophysics as a model of the evolution of a cloud of self-gravitating particles in the mean field approximation.

The first one is the classical parabolic-elliptic Keller–Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \varphi), \\ \Delta \varphi + u = 0, & x \in \mathbb{R}^2, \ t > 0, \\ u(0) = u_0. \end{cases} \tag{PE}$$

Here, $u = u(x,t)$, $\varphi = \varphi(x,t)$ are either functions or suitable (tempered) distributions. When $u \geq 0$, $\varphi \geq 0$, they may be interpreted as concentrations (densities) of microorganisms and chemicals, respectively.

The second one is the parabolic-parabolic system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \varphi), \\ \tau \varphi_t = \Delta \varphi + u, & x \in \mathbb{R}^2, \ t > 0, \\ u(0) = u_0, \quad \varphi(0) = 0, \end{cases} \tag{PP}$$

where $\tau > 0$ is a fixed parameter. Each of the models can be considered as a single nonlinear parabolic equation for u with a *nonlocal* (either in x or in (x,t)) nonlinearity since the term $\nabla \varphi$ can be expressed as a linear integral operator acting on u. In the latter model, the variations of the concentration φ are governed by the linear nonhomogeneous heat equation, and therefore are slower than in the former system, where the response of φ to the variations of u is instantaneous, and described by the integral operator $(-\Delta)^{-1}$ whose kernel has a singularity. Thus, one may expect that the evolution described by (PE) might be faster than that for (PP), especially for large values of τ when the diffusion of φ is rather slow compared to that of u. Moreover, the nonlinear effects for (PE) should manifest themselves faster than for (PP).

The theory of the system (PE) is relatively well developed, in particular when this is studied in a bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, with the homogeneous Neumann conditions for u and φ at the boundary of the domain. One of the most intriguing properties of (PE) considered for *positive* and *integrable* solutions u in $d = 2$ case is the existence of a threshold value 8π of mass $M \equiv \int u(x,t)\,\mathrm{d}x$, see the pioneering work [17] and [1, 3]. Namely, if $u_0 \geq 0$ is such that $\int u_0(x)\,\mathrm{d}x > 8\pi$, then any regular, positive solution u of (PE) cannot be global in time. We refer the reader for a fine description of the asymptotic behaviour of integrable solutions of (PE) in the subcritical case $M < 8\pi$ to [11] and to [10] for the limit case $M = 8\pi$. See also [8, 9] in the radially symmetric case. The higher dimensional versions of (PE) have been also extensively studied, cf., e.g., [2, 6, 5], and [1] for blow up phenomena.

The doubly parabolic system (PP) has been a bit less studied. For instance, it is known that if for the initial data u_0 one has $M < 8\pi$, then positive solutions are global in time, see [12] in the case of a bounded planar domain, and also [3, Theorem 5]. However, it is not known whether $M \leq 8\pi$ is, in general, a *necessary* condition for the existence of global in time solutions. That is, it is not known whether the blow up occurs for solutions, except for a specific example in [16] of a particular blowing up solution for

a system close to (PP). Even, it is an open question what is the exact range of M guaranteeing the existence of integrable *self-similar solutions*. For the system (PE) it is proved that $M \in [0, 8\pi)$, and the self-similar solutions (unique for a given $M \in [0, 8\pi)$) describe the generic asymptotic behaviour of global in time, positive and integrable solutions of (PE). Concerning (PP), it is known that $M < M(\tau)$ with $M(\tau)$ linear in τ, is a necessary condition for the existence of self-similar solutions, cf. [4]. For (PP) with small M such special solutions are also important in the study of space-time decay of general solutions, see [24]. The analysis if any $M > 8\pi$ may correspond to a self-similar solution is under way, see [7]. For a different point of view about self-similar solutions for higher dimensional models of (PP), see also [18]. Usual proofs of a blow up for (PE) involve calculations of moments of a solution and then symmetrization, cf. [11, 1]. These methods seem do not work for (PP), hence another approach is needed to show a blow up for that system. For a numerical insight on blow-up issues we refer, e.g., to [14].

A nice result in [25] shows that the solutions of the systems (PP) and (PE) enjoy a kind of stability property as $\tau \searrow 0$: solutions of (PP) converge in a suitable sense to those of (PE). It had been an old question raised by J. J. L. Velázquez and D. Wrzosek, recently solved in [25]. However, this result obtained for suitably small solutions in quite a big functional space of pseudomeasures, gives no indication on the behaviour of possible ("large") blowing up solutions.

The solvability of the systems (PE), (PP) has been studied in various classes of functions and distributions, like Lebesgue, Morrey, Besov, etc., with an immediate motivation to include the *a priori* strongest possible critical singularities of either solutions or initial data which appear to be point measures in the two-dimensional case and the multiples of $|x|^{-2}$ function in the higher dimensional case. In particular, "vast" functional spaces suitable for analysis of the two-dimensional systems include measure and pseudomeasure spaces, cf. [2, 3, 6, 25].

We show in this paper a result on the existence of (in general, nonintegrable) solutions in a class $\mathcal{X}$ of functions with natural space-time decay properties, see Theorem 2.1 and 3.1. Here, the space E of admissible initial conditions also contains Dirac measures. The corresponding solutions may be positive and "large" in the sense of their nonintegrability. Nevertheless, they are defined globally in time. Unlike the paper [25], we work in (x, t) space, when [25] has dealt with the Fourier variables ξ, cf. the formulation (36) below. Such results are obtained by an extension and refinement of techniques used in [5] for (PE) in higher dimensions, but neither for (PP), nor in the two-dimensional case of (PE) which often requires a specific treatement. Moreover, the function spaces that we employ here allow us to deal with data that can be more singular than those considered in [5]. The spaces $\mathcal{X}$ and E defined in the next section are, in a sense, critical for that analysis, and have been already considered, in slightly different forms, e.g. in the studies of the Navier–Stokes system in [13, 22].

Our main results are contained in Section 4, where we address the problem of the convergence as $\tau \searrow 0$ of solutions u^τ of the system (PP) to the

corresponding solutions u of (PE), in the space $\mathcal{X}$, arising from small data in E. Mathematically, our stability result is not included in, and does not imply, that of [25]. However, it seems to us that the use of the natural (x,t) variables provides a more immediate physical interpretation. Furthermore, our method looks more flexible, and can be used to prove the stability of the system with respect to stronger topologies. For example, we establish also the convergence in the $L^\infty_t(L^1_x)$-norm for data belonging to $E \cap L^1$, and in the $L^\infty_{t,x}$-norm for data in $E \cap L^\infty$. Motivated by [19], we will also address this issue in the more general setting of *shift invariant spaces of local measures*. The main difficulty for obtaining the convergence $u^\tau \to u$ in strong norms is that $\nabla\varphi^\tau$ enjoys some kind of *instability* as $\tau \to 0$, in particular in weighted spaces.

Moreover, we give a nonexistence (blow up) result for solutions of (PP) in $\mathbb{R}^d$, $d \geq 1$, with the positive Fourier transform of $\widehat{u_0}$ in the spirit of [23], see Theorem 5.1. These are complex valued solutions with no straightforward physical/biological interpretation. However, such a result tells us that there is no hope to prove the global existence of solutions to (PP) and similar models for arbitrarily large data relying only on size estimates.

2 The parabolic-elliptic system

In order to study the systems (PE) and (PP) we introduce the Banach space $\mathcal{X}$ of functions $u = u(x,t)$ and the Banach space E of tempered distributions $u_0 \in \mathcal{S}'(\mathbb{R}^2)$ by defining the norms

$$\|u\|_{\mathcal{X}} = \operatorname{ess\,sup}_{t>0, x\in\mathbb{R}^2} (t + |x|^2)\, |u(x,t)|, \tag{1}$$

and

$$\|u_0\|_E = \|e^{t\Delta}u_0\|_{\mathcal{X}}. \tag{2}$$

Here, $e^{t\Delta}$ denotes the heat semigroup defined by the Gaussian kernel g_t, $g_t(x) = \frac{1}{4\pi t} e^{-\frac{1}{4t}|x|^2}$. For example, the Dirac mass $u_0 = \delta$ in $\mathbb{R}^2$ is an element of E. Notice that, by the definition, E is continuously embedded into the weak Hardy space $\mathcal{H}^1_w$, which is the space consisting of all tempered distributions f such that $\sup_{t>0} |e^{t\Delta}f|$ belongs to the Lorentz space $L^{1,\infty}(\mathbb{R}^2)$. See [21].

Let us define the bilinear form B_0 by

$$B_0(u,v)(t) \equiv \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \nabla (-\Delta)^{-1} v)(s)\, \mathrm{d}s. \tag{3}$$

Here, $(-\Delta)^{-1}$ is the convolution operator on functions defined on $\mathbb{R}^2$ with the kernel $K(x) = -\frac{1}{2\pi}\log|x|$. With this notation, the equivalent integral (mild) formulation to (PE), called also the Duhamel formula, reads

$$u(t) = e^{t\Delta}u_0 - B_0(u,u). \tag{4}$$

We begin by establishing the following simple result

Theorem 2.1 *There exist two absolute constants $\epsilon, \beta > 0$ with the following property. Let $u_0 \in E$ be such that $\|u_0\|_E < \epsilon$. Then there exists a unique (mild) solution $u \in \mathcal{X}$ of (PE) such that $\|u\|_{\mathcal{X}} \leq \epsilon\beta$.*

The proof of Theorem 2.1 will follow from a series of lemmata.

First, we have the following estimate of the leading term in $\nabla\varphi$

Lemma 2.1 *Let $u \in \mathcal{X}$ and φ be such that $\Delta\varphi + u = 0$. Then*

$$\nabla\varphi(x,t) = \frac{c_0\, x}{|x|^2} \int_{|y|\leq|x|/2} u(y,t)\,\mathrm{d}y + \mathcal{R}(x,t) \tag{5}$$

with $c_0 = -\frac{1}{2\pi}$ and the remainder $\mathcal{R}$ satisfying

$$|\mathcal{R}(x,t)| \leq C\|u\|_{\mathcal{X}} \left(t^{\frac{1}{2}} + |x|\right)^{-1}.$$

Proof. Indeed, let us represent the partial derivatives of φ, for $j = 1, 2$, as

$$\partial_j\varphi = \frac{c_0 x_j}{|x|^2} * u \equiv I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{|y|\leq|x|/2} \frac{c_0(x_j - y_j)}{|x-y|^2}\, u(y,t)\,\mathrm{d}y.$$

The terms I_2 and I_3 are obtained by taking the integration domains $\{|x-y| \leq |x|/2\}$ and $\{|x-y| \geq |x|/2,\, |y| \geq |x|/2\}$, respectively, in the convolution integrals defining $\partial_j\varphi$. It is straightforward to prove that I_2 and I_3 can be bounded by $C\|u\|_{\mathcal{X}} \left(t^{\frac{1}{2}} + |x|\right)^{-1}$. On the other hand, we can rewrite I_1 as

$$\frac{c_0 x_j}{|x|^2} \int_{|y|\leq|x|/2} u(y,t)\,\mathrm{d}y + R_1(x,t).$$

An application of the Taylor formula shows that the above bound holds also for R_1. ∎

We immediately deduce from (5) the following useful estimate

Lemma 2.2 *Let $u \in \mathcal{X}$ and $\Delta\varphi + u = 0$. Then*

$$\|\nabla\varphi(t)\|_{L^\infty} \leq C\|u\|_{\mathcal{X}}\, t^{-\frac{1}{2}}.$$

The last lemma that we need is the following

Lemma 2.3 *Let $u, v \in \mathcal{X}$. Then, for some constant C_0 independent of u, v*

$$\|B_0(u,v)\|_{\mathcal{X}} \leq C_0\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}.$$

Proof. We can assume, without any restriction, that $\|u\|_{\mathcal{X}} = \|v\|_{\mathcal{X}} = 1$. Lemma 2.2 implies

$$|u\nabla(-\Delta)^{-1}v|(x,t) \le C|x|^{-\frac{3}{2}}t^{-\frac{3}{4}}, \tag{6a}$$

and also

$$|u\nabla(-\Delta)^{-1}v|(x,t) \le C|x|^{-2}t^{-\frac{1}{2}}. \tag{6b}$$

We denote the gradient of the heat semigroup kernel g_t by

$$G(\cdot,t) \equiv \nabla_x \left(\frac{1}{4\pi t}e^{-\frac{1}{4t}|\cdot|^2}\right). \tag{7}$$

Then we may represent B_0 as

$$\begin{aligned} B_0(u,v)(x,t) &= \int_0^t\int G(x-y,t-s)(u\nabla(-\Delta)^{-1}v)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\equiv J_1 + J_2, \end{aligned}$$

where $J_1 = \int_0^t\int_{|y|\le|x|/2}\dots$ and $J_2 = \int_0^t\int_{|y|\ge|x|/2}\dots$. Using the estimate

$$|G(x-y,t-s)| \le C|x-y|^{-\frac{5}{2}}(t-s)^{-\frac{1}{4}}$$

and inequality (6a), we get the bound

$$|J_1(x,t)| \le C|x|^{-2}. \tag{8}$$

Another possible estimate is

$$|J_1(x,t)| \le C|x|^{-\frac{3}{2}}t^{-\frac{1}{4}}, \tag{9}$$

which is obtained using the bound

$$|G(x-y,t-s)| \le C|x-y|^{-2}(t-s)^{-\frac{1}{2}}.$$

On the other hand, from the property

$$\|G(\cdot,t-s)\|_{L^1} = c(t-s)^{-\frac{1}{2}} \tag{10}$$

and inequality (6b), we obtain

$$|J_2(x,t)| \le C|x|^{-2}. \tag{11}$$

As before, we have also the bound

$$|J_2(x,t)| \le C|x|^{-\frac{3}{2}}t^{-\frac{1}{4}}. \tag{12}$$

This second estimate is deduced from (6a).

Then, using (8), (11), we obtain the space decay estimate

$$|B_0(u,v)|(x,t) \le C|x|^{-2} \tag{13}$$

and from (9), (12) — a provisory (not optimal) time decay estimate

$$\|B_0(u,v)(t)\|_{L^{\frac{4}{3},\infty}} \le Ct^{-\frac{1}{4}}.$$

But we may represent B_0 as

$$B_0(u,v)(t) = e^{\frac{t}{2}\Delta} B(u,v)(t/2) + \int_{t/2}^{t} G(t-s) * (u\nabla(-\Delta)^{-1}v)(s)\,\mathrm{d}s.$$

Thus, applying the weak Young-type inequality for convolutions in Lorentz spaces $L^{\frac{4}{3},\infty} * L^{4,1} \subset L^{\infty}$, see [19], and the equality obtained from the scaling laws in Lorentz spaces,

$$\left\| \frac{1}{4\pi t} e^{-\frac{1}{4t}|\cdot|^2} \right\|_{L^{4,1}} = ct^{-\frac{3}{4}}, \tag{14}$$

we finally get

$$\begin{aligned} \|B_0(u,v)(t)\|_{L^\infty} &\le Ct^{-1} + C\int_{t/2}^{t} (t-s)^{-\frac{1}{2}} \|u\nabla(-\Delta)^{-1}v(s)\|_{L^\infty}\,\mathrm{d}s \\ &\le Ct^{-1}. \end{aligned} \tag{15}$$

Combining inequalities (13) and (15) we get $B(u,v) \in \mathcal{X}$, together with its continuity with respect to u and v. □

Proof. Note that (using the duality $\mathcal{S} - \mathcal{S}'$) we have $e^{t\Delta}u_0 \to u_0$ in $\mathcal{S}'$ as $t \to 0$. The conclusion of Theorem 2.1 follows in a standard way (cf., e.g., [19, 2, 6]) from the contraction fixed point theorem. □

3 The parabolic-parabolic system

Let $\tau > 0$ be a fixed parameter. We consider the system (PP) whose equivalent integral formulation reads

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\left[u(s)\frac{1}{\tau}\nabla \int_0^s e^{\frac{1}{\tau}(s-\sigma)\Delta} u(\sigma)\,\mathrm{d}\sigma \right] \mathrm{d}s. \tag{16}$$

We introduce for all $\tau \ge 0$ the bilinear form B_τ (recall that G is defined by the expression (7))

$$B_\tau(u,v)(x,t) \equiv \int_0^t\!\!\int G(x-y,t-s)\big(u\,W_\tau(v)\big)(y,s)\,\mathrm{d}y\,\mathrm{d}s, \tag{17}$$

where $W_\tau(v)$ is the linear operator acting on v

$$W_\tau(v)(x,t) = \int_0^t \frac{1}{\tau}\left[G\Big(\frac{t-\sigma}{\tau}\Big) * v(\sigma) \right](x,\sigma)\,\mathrm{d}\sigma \quad \text{for } \tau > 0, \tag{18a}$$

with a natural convention

$$W_0(v)(x,t) = \big(\nabla(-\Delta)^{-1}v\big)\,(x,t). \tag{18b}$$

In this way, the system (PP) is also rewritten in a compact form (cf. (4)) as

$$u = e^{t\Delta}u_0 - B_\tau(u,u). \tag{19}$$

We are going to solve (19) in the space $\mathcal{X}$ exactly as was in the parabolic-elliptic case. An additional estimate, however, is needed:

Lemma 3.1 *Let $u \in \mathcal{X}$ and $\tau > 0$. Then there exists a constant $C^* > 0$, independent of u and τ, such that*

$$\|W_\tau(u)(t)\|_{L^\infty} \le C^* t^{-\frac{1}{2}} \|u\|_{\mathcal{X}}. \tag{20}$$

Proof. As usual we can and do assume $\|u\|_{\mathcal{X}} = 1$. Then, for all $1 < p \le \infty$ we have

$$|u(x,\sigma)| \le |x|^{-\frac{2}{p}} \sigma^{-1+\frac{1}{p}}.$$

This implies, for $1 < p \le \infty$,

$$\|u(\sigma)\|_{L^{p,\infty}} \le \sigma^{-1+\frac{1}{p}}.$$

Now, we represent

$$W_\tau(u) = I_1 + I_2,$$

where $I_1 = \int_0^{t/2} \dots$ and $I_2 = \int_{t/2}^t \dots$. Evidently, we obtain the bound

$$\begin{aligned} \|I_1(t)\|_{L^\infty} &\le C \int_0^{t/2} \left\| \frac{1}{\tau} G\Big(\frac{t-\sigma}{\tau}\Big) \right\|_{L^{2,1}} \|u(\sigma)\|_{L^{2,\infty}} \, \mathrm{d}\sigma \\ &\le C t^{-\frac{1}{2}}. \end{aligned}$$

For the integral I_2, let us begin with a rough bound

$$\begin{aligned} \|I_2(t)\|_{L^\infty} &\le C \int_{t/2}^t \left\| \frac{1}{\tau} G\Big(\frac{t-\sigma}{\tau}\Big) \right\|_{L^1} \|u(\sigma)\|_{L^\infty} \, \mathrm{d}\sigma \\ &\le C \tau^{-\frac{1}{2}} t^{-\frac{1}{2}}. \end{aligned} \tag{21}$$

This bound gives the required estimate, excepted when τ belongs to a neighbourhood of the origin. Thus, in the sequel, it is enough to consider the case $0 < \tau < \frac{1}{2}$. Now, we further decompose

$$I_2 \equiv I_{2,1} + I_{2,2},$$

where $I_{2,1} = \int_{t/2}^{t-\tau t} \dots$ and $I_{2,2} = \int_{t-\tau t}^t \dots$. Next, we are going to improve (21) writing

$$\begin{aligned} \|I_{2,1}(t)\|_{L^\infty} &\le C \int_{t/2}^{t-\tau t} \left\| \frac{1}{\tau} G\Big(\frac{t-\sigma}{\tau}\Big) \right\|_{L^{3,1}} \|u(\sigma)\|_{L^{\frac{3}{2},\infty}} \, \mathrm{d}\sigma \\ &\le C \tau^{\frac{1}{6}} \int_{t/2}^{t-\tau t} (t-\sigma)^{-\frac{7}{6}} \sigma^{-\frac{1}{3}} \, \mathrm{d}\sigma \\ &\le C t^{-\frac{1}{2}}. \end{aligned} \tag{22}$$

The last estimate is

$$\begin{aligned} \|I_{2,2}(t)\|_{L^\infty} &\le C \int_{t-\tau t}^t \left\| \frac{1}{\tau} G\Big(\frac{t-\sigma}{\tau}\Big) \right\|_{L^1} \|u(\sigma)\|_{L^\infty} \, \mathrm{d}\sigma \\ &\le C \tau^{-\frac{1}{2}} \int_{t-\tau t}^t (t-\sigma)^{-\frac{1}{2}} \sigma^{-1} \, \mathrm{d}\sigma \\ &\le C t^{-\frac{1}{2}}. \end{aligned} \tag{23}$$

The conclusion of Lemma 3.1 follows from (21), (22), (23). □

Lemma 3.1 allows us to see that, if u, $v \in \mathcal{X}$ with $\|u\|_{\mathcal{X}} = \|v\|_{\mathcal{X}} = 1$, then

$$|u\, W_\tau(v)|(x,t) \le C|x|^{-\frac{3}{2}} t^{-\frac{3}{4}}, \tag{24a}$$

and

$$|u\, W_\tau(v)|(x,t) \le C|x|^{-2} t^{-\frac{1}{2}}, \tag{24b}$$

for some constant $C > 0$ independent of $\tau > 0$. These are the analogous estimates as those we obtained in the parabolic-elliptic case (see inequalities (6a) and (6b)). Then, using exactly the same arguments as in the previous section, we arrive at the following existence result

Theorem 3.1 *There exist two absolute constants $\epsilon^*, \beta^* > 0$ with the following property. Let $u_0 \in E$ be such that $\|u_0\|_E < \epsilon^*$. Then there exists a unique (mild) solution $u \in \mathcal{X}$ of (PP) such that $\|u\|_{\mathcal{X}} \le \epsilon^* \beta^*$.*

Remark. The case of nonzero initial data $\varphi(0)$ can be studied in a quite similar way.

Remark. A closer look at the proofs of estimates for $\nabla\varphi$ in (PE) and (PP) reveals that the behaviour of $\nabla\varphi$ is a bit different in these two cases. Namely, if $0 \not\equiv u$ and $u(x,t) \sim (t+|x|^2)^{-1}$ (in the sense that $c_1(t+|x|^2)^{-1} \le u(x,t) \le c_2(t+|x|^2)^{-1}$ for some c_1, $c_2 > 0$), then it follows from (5) that $\nabla\varphi(x,t) \sim |x|^{-1}\log(t+|x|^2)$ for (PE), while $\nabla\varphi$ is more regular: $|\nabla\varphi(x,t)| \le c\left(t^{\frac{1}{2}} + |x|\right)^{-1}$ in (PP) case. In other words, letting $\mathcal{Y}$ be the space of functions $f = f(x,t)$ such that $f^2 \in \mathcal{X}$, we have $\varphi^\tau \in \mathcal{Y}$ for $\tau > 0$, but (φ^τ) does not converge in $\mathcal{Y}$ as $\tau \to 0$. However, such an instability does not prevent from the convergence of the densities $u^\tau \to u$ for vanishing τ.

4 Study of the $\tau \searrow 0$ limit

We now study the convergence as $\tau \searrow 0$ of solutions u^τ of the system (PP) to the corresponding solution u of (PE). A result in this direction has been obtained recently by A. Raczyński in [25], who established the convergence $u^\tau \to u$ in the norm $\mathcal{Y}_\alpha$, for $\alpha \in (1,2)$, defined as

$$\|u\|_{\mathcal{Y}_\alpha} = \operatorname*{ess\,sup}_{t>0,\, x\in\mathbb{R}^2} \left(1 + t^{\frac{1}{2}}|\xi|\right)^\alpha |\widehat{u}(\xi,t)|. \tag{25}$$

We will obtain in subsection 4.2 a similar result using the $\mathcal{X}$-norm.

4.1 Regularity properties of solutions of (PE)

In this subsection we prepare some preliminary material. The first Proposition consists of a regularity result with respect to the space variable for solutions of (PE). The second Proposition describes their regularity properties with respect to the time variable.

Proposition 4.1 *For all $r \in (1,2)$ there exists a constant ϵ_r, with $0 < \epsilon_r \le \epsilon$ (the absolute constant of Theorem 2.1) such that, if $\|u_0\|_E < \epsilon_r$, then the solution of (PE) constructed in Theorem 2.1 satisfies*

$$\|\nabla u(t)\|_{L^{r,\infty}} \le Ct^{-\frac{3}{2}+\frac{1}{r}}, \tag{26}$$

for some constant $C = C(u_0, r)$ independent of t.

Proof. We use a standard argument involving the subspace $\mathcal{X}_r \subset \mathcal{X}$ defined by

$$\mathcal{X}_r = \Big\{ u \in \mathcal{X},\ \exists\, C\colon \|\nabla u(t)\|_{L^{r,\infty}} \le Ct^{-\frac{3}{2}+\frac{1}{r}} \Big\},$$

and equipped with its natural norm. Recalling that the kernel $\frac{c_0 x}{|x|^2}$ of the operator $\nabla(-\Delta)^{-1}$ belongs to $L^{2,\infty}$, first we deduce from the Young inequality

$$\big\|\nabla^2(-\Delta)^{-1}v(t)\big\|_{L^{\alpha,\infty}} \le Ct^{-\frac{3}{2}+\frac{1}{r}}\|v\|_{\mathcal{X}_r}, \qquad \tfrac{1}{\alpha} = \tfrac{1}{r} - \tfrac{1}{2}. \tag{27}$$

Next, from the Hölder inequality (noticing that $\|u(t)\|_{L^{2,\infty}} \le Ct^{-\frac{1}{2}}\|u\|_{\mathcal{X}}$),

$$\|u\nabla^2(-\Delta)^{-1}v(t)\|_{L^{r,\infty}} \le Ct^{-2+\frac{1}{r}}\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}_r}. \tag{28}$$

The generalization of the classical inequalities to Lorentz spaces can be found, e.g., in [19].

We claim that the bilinear operator B_0 introduced in (3) is boundedly defined: $B_0\colon \mathcal{X}_r \times \mathcal{X}_r \to \mathcal{X}_r$. Indeed, for $\|u\|_{\mathcal{X}_r} = \|v\|_{\mathcal{X}_r} = 1$, we combine the estimates

$$\|\nabla G(t-s)\|_1 \le C(t-s)^{-1}$$

and (10) with the inequality (a consequence of $u \in \mathcal{X}$)

$$\|u(s)\|_{L^{r,\infty}} \le Cs^{-1+\frac{1}{r}}, \tag{29}$$

the estimate $\|\nabla(-\Delta)^{-1}v(s)\|_\infty \le Cs^{-\frac{1}{2}}$ obtained from Lemma 2.2, and (28). Then we arrive at

$$\begin{aligned}
&\|\nabla B_0(u,v)(t)\|_{L^{r,\infty}} \\
&\quad\le C\int_0^{t/2}(t-s)^{-1}\|u(s)\|_{L^{r,\infty}}\|\nabla(-\Delta)^{-1}v(s)\|_\infty\,\mathrm{d}s \\
&\qquad + C\int_{t/2}^{t}(t-s)^{-\frac{1}{2}}\Big(\|\nabla u(s)\|_{L^{r,\infty}}\|\nabla(-\Delta)^{-1}v(s)\|_\infty + s^{-2+\frac{1}{r}}\Big)\,\mathrm{d}s \\
&\quad\le Ct^{-\frac{3}{2}+\frac{1}{r}}.
\end{aligned}$$

Moreover, for $u_0 \in E$, we have $|e^{\frac{t}{2}\Delta}u_0(x)| \le C(t+|x|^2)^{-1}$. Hence, $\|e^{\frac{t}{2}\Delta}u_0\|_{L^{\gamma,\infty}} \le Ct^{-1+\frac{1}{\gamma}}$ for $1 < \gamma < \infty$. We now choose $\beta, \gamma \in (1,\infty)$ such that $1 + \frac{1}{r} = \frac{1}{\beta} + \frac{1}{\gamma}$. Then, the semigroup property of the heat kernel g_t, and the fact that $\nabla g_{t/2} \in L^{\beta,1}$, imply

$$\|\nabla e^{t\Delta}u_0\|_{L^{r,\infty}} \le Ct^{-\frac{3}{2}+\frac{1}{r}}.$$

Now the usual the application of the contraction mapping theorem, in a closed ball of small radius in the space $\mathcal{X}_r$, allows us to conclude. ❑

The following proposition is the first crucial tool for our stability result. It provides the Hölder regularity, with respect to the time variable, of solutions of (PE) in Lorentz spaces.

Proposition 4.2 *Let $1 < r < 2$ and $u_0 \in E$, such that $\|u_0\|_E < \epsilon_r$. Then the solution u of (PE) constructed in Proposition 4.1 satisfies for all $0 < t' < t$*

$$\|u(t) - u(t')\|_{L^{r,\infty}} \leq C(t-t')^{\frac{1}{2}}(t')^{-\frac{3}{2}+\frac{1}{r}}, \tag{30}$$

for some $C = C(u_0, r)$ independent of t and t'.

Proof. It is enough to show that both $e^{t\Delta}u_0 - e^{t'\Delta}u_0$ and $B_0(u,u)(t) - B_0(u,u)(t')$ satisfy the required bound in the $L^{r,\infty}$-norm.

From the identity

$$\begin{aligned} e^{t\Delta}u_0(x) - e^{t'\Delta}u_0(x) &= \int \Big(e^{t'\Delta}u_0(x-y) - e^{t'\Delta}u_0(x)\Big) g_{t-t'}(y)\,\mathrm{d}y \\ &= -\int\!\!\int_0^1 \nabla e^{t'\Delta}u_0(x-\theta y)\cdot y g_{t-t'}(y)\,\mathrm{d}y\,\mathrm{d}\theta, \end{aligned}$$

we get

$$\begin{aligned} \|e^{t\Delta}u_0 - e^{t'\Delta}u_0\|_{L^{r,\infty}} &\leq C\|\nabla e^{t'\Delta}u_0\|_{L^{r,\infty}}\|y g_{t-t'}\|_1 \\ &\leq C(t-t')^{\frac{1}{2}}(t')^{-\frac{3}{2}+\frac{1}{r}}. \end{aligned}$$

Now, we can write

$$B_0(u,u)(t) - B_0(u,u)(t') = A_1 + A_2,$$

with

$$A_1 \equiv \int_0^{t'} \big(G(t-s) - G(t'-s)\big) * \big(u\nabla(-\Delta)^{-1}u\big)(s)\,\mathrm{d}s$$

and

$$A_2 \equiv \int_{t'}^{t} G(t-s) * \big(u\nabla(-\Delta)^{-1}u\big)(s)\,\mathrm{d}s.$$

Recall that from $u \in \mathcal{X}$ we deduce (29). Combining this with the estimate of Lemma 2.2 we get

$$\big\|u\nabla(-\Delta)^{-1}u(s)\big\|_{L^{r,\infty}} \leq Cs^{-\frac{3}{2}+\frac{1}{r}}. \tag{31}$$

This immediately yields

$$\|A_2\|_{L^{r,\infty}} \leq \int_{t'}^{t} (t-s)^{-\frac{1}{2}} s^{-\frac{3}{2}+\frac{1}{r}}\,\mathrm{d}s \leq C(t-t')^{\frac{1}{2}}\big(t'\big)^{-\frac{3}{2}+\frac{1}{r}}.$$

The estimate of A_1 is slightly more involved. We start with the identity

$$G(t-s) - G(t'-s) = \Big(e^{(t-t')\Delta} - \mathrm{Id}\Big)\, G(t'-s).$$

The action of the convolution operator with the function on the right hand side is studied via the following variant of a result established in [20].

Lemma 4.1 *Denote the Calderón operator by* $\Lambda = (-\Delta)^{\frac{1}{2}}$. *Then, for some constant* C *depending only on* $r \in (1, \infty)$

$$\left\| \left(e^{t\Delta} - \mathrm{Id}\right) f \right\|_{L^{r,\infty}} \leq C t^{\frac{1}{2}} \|\Lambda f\|_{L^{r,\infty}}.$$

Proof. Writing $f = \Lambda^{-1}\Lambda f$, we see that

$$\left(e^{t\Delta} - \mathrm{Id}\right) f = \Phi_t * (\Lambda f),$$

where,

$$\widehat{\Phi}_t(\xi) = t^{\frac{1}{2}} \widehat{\Phi}(t^{\frac{1}{2}}\xi),$$

and

$$\widehat{\Phi}(\xi) = (e^{-|\xi|^2} - 1)|\xi|^{-1}.$$

It only remains to show that $\Phi \in L^1(\mathbb{R}^2)$, which is immediate. Indeed, it is well known, and easy to check, that $\widehat{\Psi}(\xi) = |\xi| e^{-|\xi|^2}$ defines a function $\Psi \in L^1(\mathbb{R}^2)$ (for example, with the method described in [15], one obtains $|\Psi(x)| \leq C(1 + |x|)^{-3}$ and $|\nabla\Psi(x)| \leq C(1 + |x|)^{-4}$). We conclude applying the Bochner inequality to the identity

$$\Phi(x) = -2 \int_1^\infty \Psi(\eta\, x)\, \mathrm{d}\eta.$$

□

Using this Lemma we deduce

$$\begin{aligned} \|A_1\|_{L^{r,\infty}} &\leq C(t - t')^{\frac{1}{2}} \int_0^{t'} \|\Lambda\nabla g_{t'-s} * \left(u\nabla(-\Delta)^{-1}u\right)(s)\|_{L^{r,\infty}}\, \mathrm{d}s \\ &= C(t - t')^{\frac{1}{2}}(A_{1,1} + A_{1,2}), \end{aligned}$$

where $A_{1,1}$ and $A_{1,2}$ are obtained splitting the integral at $s = t'/2$.

But, as the function Ψ introduced in the proof of Lemma 4.1 satisfies $\nabla\Psi \in L^1(\mathbb{R}^2)$, we see by a simple rescaling that

$$\|\Lambda\nabla g_{t'-s}\|_1 \leq C(t' - s)^{-1}.$$

Combining this estimate with inequality (31), we get

$$A_{1,1} \leq C(t')^{-\frac{3}{2}+\frac{1}{r}}, \qquad 1 < r < 2.$$

For treating $A_{1,2}$, we combine the estimate

$$\|\Lambda g_{t'-s}\|_1 \leq C(t' - s)^{-\frac{1}{2}}$$

with the inequality (for $1 < r < 2$)

$$\|(\nabla u)\left(\nabla(-\Delta)^{-1}u\right)(s)\|_{L^{r,\infty}} + \|u\left(\nabla^2(-\Delta)^{-1}u\right)(s)\|_{L^{r,\infty}} \leq C s^{-2+\frac{1}{r}},$$

obtained by applying (26), Lemma 2.2, and (28) with $u = v$. We get as before

$$A_{1,2} \leq C(t')^{-\frac{3}{2}+\frac{1}{r}}, \qquad 1 < r < 2,$$

and this concludes the proof of Proposition 4.2. □

4.2 The vanishing τ limit

After Proposition 4.2, the second crucial step for the study of the limit as $\tau \searrow 0$ consists in the asymptotic analysis of the linear operators W_τ, $\tau \geq 0$, introduced in (18a)–(18b). This is the purpose of the following lemma.

Lemma 4.2 *Let $\varepsilon = \varepsilon(\tau)$ be an arbitrary function, strictly increasing and continuous on $[0,1]$, such that $\varepsilon(0) = 0$. Let also $1 < r < 2$ and u be a function satisfying, for $0 < t' < t$,*

$$\begin{aligned} &\|u(t)\|_{L^{r,\infty}} \leq C t^{-1+\frac{1}{r}}, \\ &\|u(t) - u(t')\|_{L^{r,\infty}} \leq C(t-t')^{\frac{1}{2}} (t')^{-\frac{3}{2}+\frac{1}{r}}, \end{aligned} \tag{32}$$

with a constant C independent of t, t'. Then, for all $t > 0$, $\tau \in [0,1]$, and for another constant C, independent of t and τ,

$$\varepsilon(\tau) \sup_{t>0} t^{\frac{1}{2}} \left\| \left(W_\tau(u) - W_0(u)\right)(t) \right\|_\infty \leq C \tau^{\frac{1}{r}-\frac{1}{2}}. \tag{33}$$

In particular, if $u_0 \in E$ is small enough (for example, $\|u_0\|_E \leq \epsilon_{3/2}$), then the corresponding solution u of (PE) constructed in Proposition 4.1 satisfies

$$\lim_{\tau \to 0} \sup_{t>0} t^{\frac{1}{2}} \left\| \left(W_\tau(u) - W_0(u)\right)(t) \right\|_\infty = 0. \tag{34}$$

Proof. Without any restriction we can assume that $0 < \varepsilon(\tau) < \frac{1}{2}$ for positive τ. Define $\tilde{\varepsilon}(\tau)$ such that $\varepsilon = \tilde{\varepsilon}^{\frac{1}{r}-\frac{1}{2}}$. Borrowing from [25] the idea of splitting the time integral using intervals depending on τ, we write

$$W_\tau(u) - W_0(u) \equiv J_1 + J_2 + J_3,$$

where

$$J_1(t) = \int_0^{t(1-\tilde{\varepsilon}(\tau))} \left[\frac{1}{\tau} G\Big(\frac{t-s}{\tau}\Big) * u(s) \right] \mathrm{d}s,$$

next

$$J_2(t) = \int_{t(1-\tilde{\varepsilon}(\tau))}^{t} \frac{1}{\tau} G\Big(\frac{t-s}{\tau}\Big) * u(t)\, \mathrm{d}s \;-\; W_0(u)(t),$$

and

$$J_3(t) = \int_{t(1-\tilde{\varepsilon}(\tau))}^{t} \frac{1}{\tau} G\Big(\frac{t-s}{\tau}\Big) * \big[u(s) - u(t)\big]\, \mathrm{d}s.$$

From the first relation of (32) and the Young inequality in Lorentz space (using that, by (7), $G(\cdot,t) \in L^{r',1}(\mathbb{R}^2)$, where r' is the conjugate exponent), we get

$$\begin{aligned} \|J_1(t)\|_\infty &\leq C \tau^{\frac{1}{r}-\frac{1}{2}} \int_0^{t(1-\tilde{\varepsilon}(\tau))} (t-s)^{-\frac{1}{2}-\frac{1}{r}} s^{-1+\frac{1}{r}}\, \mathrm{d}s \\ &\leq C \left(\frac{\tau}{\tilde{\varepsilon}(\tau)} \right)^{\frac{1}{r}-\frac{1}{2}} t^{-\frac{1}{2}}. \end{aligned}$$

Notice that this estimate of J_1 is exactly what we need for (33).

As for J_2, we see from a simple computation via the Fourier transform that

$$J_2(t) = -g_{t\tilde{\varepsilon}(\tau)/\tau} * W_0(u)(t).$$

If $\frac{1}{\alpha} = \frac{1}{r} - \frac{1}{2}$, then we deduce from the usual weak-convolution estimates that $W_0(u) = \nabla(-\Delta)^{-1}u$ is bounded in the $L^{\alpha,\infty}$-norm, by $Ct^{-1+\frac{1}{r}}$. Applying once more the Young inequality (using now $g_{t\tilde{\varepsilon}(\tau)/\tau} \in L^{\alpha',1}$), we get, as before,

$$\|J_2(t)\|_\infty \leq C\left(\frac{\tau}{\tilde{\varepsilon}(\tau)}\right)^{\frac{1}{r}-\frac{1}{2}} t^{-\frac{1}{2}}.$$

Applying the second of inequalities (32), we obtain immediately

$$\|J_3(t)\|_\infty \leq C\tau^{\frac{1}{r}-\frac{1}{2}} t^{-\frac{1}{2}},$$

which is even better than what we need. This proves the inequality (33). Choosing, for example, $r = \frac{3}{2}$ and $\varepsilon(\tau) = \frac{1}{2}\tau^{1/12}$ proves the last claim (34) of Lemma 4.2. □

We are now in the position of establishing our first main result

Theorem 4.1 *There exists an absolute constant $\epsilon' > 0$ (a priori smaller than the constants $\epsilon, \epsilon^* > 0$ in Theorems 2.1 and 3.1), such that if $u_0 \in E$, $\|u_0\|_E < \epsilon'$, then denoting by $u \in \mathcal{X}$ the solution of (PE) and $u^\tau \in \mathcal{X}$ the solution of (PP) constructed in the previous theorems, we have as $\tau \searrow 0$*

$$u^\tau \to u \qquad \text{in } \mathcal{X}.$$

Proof. The proof follows easily from Lemma 4.2. Indeed, from the integral equations (4) and (19), the bilinearity of B_τ and B_0, and the smallness of the solutions u^τ and u, we have (similarly as in [25], where two terms in the bilinear expansion can be absorbed by the left hand side)

$$\|u^\tau - u\|_{\mathcal{X}} \leq C\|B_\tau(u,u) - B_0(u,u)\|_{\mathcal{X}}.$$

But, by the definition of W_τ and W_0 (see (18a)-(18b)),

$$B_\tau(u,u)(t) - B_0(u,u)(t) = \int_0^t G(t-s) * \big(u(W_\tau(u) - W_0(u))\big)(s)\, ds.$$

Argueing as in the proof of Lemma 2.3, we obtain

$$\|B_\tau(u,u) - B_0(u,u)\|_{\mathcal{X}} \leq C\|u\|_{\mathcal{X}}\Big(\sup_{t>0} t^{\frac{1}{2}}\|W_\tau(u) - W_0(u)(t)\|_\infty\Big).$$

If $\epsilon' > 0$ is small enough, then Lemma 4.2 can be applied to the solution u of (PE), implying that the right hand side of the above inequality has a vanishing limit for small τ. This finally gives

$$\|u^\tau - u\|_{\mathcal{X}} \to 0 \qquad \text{as} \quad \tau \searrow 0.$$

□

Remark 4.1 Notice that, the smaller the norm $\|u_0\|_E$, the faster the convergence $u^\tau \to u$ as $\tau \to 0$. This is due to the fact that for very small data it is possible to apply Lemma 4.2 with r close to 1 (despite the constants in our estimates blow up as $r \searrow 1$). More precisely, our arguments show that for any $0 < \delta < \frac{1}{2}$, one can find a constant $C > 0$ and $\epsilon(\delta) > 0$ such that, for $\|u_0\|_E \leq \epsilon(\delta)$, one has $\|u^\tau - u\|_{\mathcal{X}} \leq C\tau^{\frac{1}{2}-\delta}$ for all $0 \leq \tau \leq 1$.

4.3 The parabolic-elliptic limit in stronger topologies

If $u_0 \geq 0$ is small in the E-norm, and belongs to a smaller space, for example, $u_0 \in E \cap L^1$, then the solutions u^τ and u of (PP) and (PE) will remain in L^1, uniformly in time, during their evolution. Hence, it is natural to ask whether the convergence $u^\tau \to u$ holds also in the natural norm of $L^\infty((0,\infty); L^1)$. Our next theorem provides a positive answer. As the proof of this fact does not really depend on a particular topology under consideration, it seems appropriate to consider a more abstract setting.

We denote by $\mathcal{L}$ any *shift invariant Banach space of local measures*, see [19, Ch. 4] for their definition and main properties. These are Banach spaces of distributions, continuously embedded in $\mathcal{D}'(\mathbb{R}^2)$. Moreover, they are known to satisfy the following properties (for some constant $C > 0$ depending only on $\mathcal{L}$),

1. For all $f \in \mathcal{L}$, $g \in L^1(\mathbb{R}^2)$, the convolution product $f * g$ is well defined in $\mathcal{L}$ and $\|f * g\|_{\mathcal{L}} \leq C\|f\|_{\mathcal{L}}\|g\|_1$.
2. For all $f \in \mathcal{L}$, $h \in L^\infty(\mathbb{R}^2)$, the pointwise product fh is well defined in $\mathcal{L}$ and $\|fh\|_{\mathcal{L}} \leq C\|f\|_{\mathcal{L}}\|h\|_\infty$.
3. Each bounded sequence $\{f_k\} \subset \mathcal{L}$ has a subsequence convergent in $\mathcal{L}$, in the distributional sense.

Obvious examples of spaces satisfying these properties (and which are indeed shift invariant space of local measures) are the L^p-spaces, $1 < p \leq \infty$, the Lorentz spaces $L^{p,q}$, $1 < p < \infty$, $1 < q \leq \infty$ and the space of bounded Borel measures $M(\mathbb{R}^2) = \mathcal{C}_0(\mathbb{R}^2)^*$. In the latter case, such duality relations ensures Property 3. Other interesting examples include the Morrey–Campanato spaces $M^{p,q}$, $(1 < p \leq q < \infty)$ and suitable multiplier spaces, see [19, Ch. 17].

On the other hand, the space of pseudomeasures, i.e., the space of tempered distributions f such that $\widehat{f} \in L^\infty$) does not fulfill the second requirement. Therefore, the stability result in the pseudomeasure topology will not be encompassed by our next Theorem, but requires a specific (and more involved) treatment, see [25].

Notice that, because of the conservation of the total mass for positive solutions of (PE) and (PP), the L^1-norm remains constant during the evolution. This observation will allow us to handle the case of data $u_0 \in E \cap L^1$, despite Property 3 breaks down for L^1.

Theorem 4.2 *Let $u_0 \in E \cap \mathcal{L}$, where $\mathcal{L}$ is either a shift invariant Banach space of local measures, or $\mathcal{L} = L^1$. In the latter case we require either $u_0 \geq 0$ or, for signed u_0, that $\|u_0\|_1$ is sufficiently small. Then there exists a positive constant $\tilde{\epsilon}$, depending only on $\mathcal{L}$, such that, if*

$$\|u_0\|_E < \tilde{\epsilon},$$

then the systems (PE) and (PP) possess unique solutions u and u^τ, respectively, such that for an absolute constant $\tilde{\beta} > 0$, $\|u\|_{\mathcal{X}} \leq \tilde{\beta}\tilde{\epsilon}$ and $\|u^\tau\|_{\mathcal{X}} \leq \tilde{\beta}\tilde{\epsilon}$. In addition,

$$\sup_{t>0} \|u(t)\|_{\mathcal{L}} < \infty \qquad \textit{and} \qquad \sup_{t>0} \|u^\tau(t)\|_{\mathcal{L}} < \infty.$$

Moreover, the conclusion $\|u^\tau(t)-u(t)\|_{\mathcal{X}} \to 0$ *of Theorem 4.1 is strengthened*

$$\sup_{t>0} \|u^\tau(t) - u(t)\|_{\mathcal{L}} \to 0 \qquad as \quad \tau \searrow 0.$$

Proof. Obviously we have, for some constant $C_0 > 0$ independent of t

$$\|e^{t\Delta}u_0\|_{\mathcal{L}} \le C_0.$$

Moreover, for each $\tau \ge 0$ (we include in this way the analysis of (PE)), we have the estimate

$$\|B_\tau(u,v)(t)\|_{\mathcal{L}} \le \tilde{C}\Big(\sup_{t>0} \|u(t)\|_{\mathcal{L}}\Big)\|v\|_{\mathcal{X}}, \tag{35}$$

where $\tilde{C} > 0$ depends only on $\mathcal{L}$. This follows from (17) written as

$$B_\tau(u,v)(t) = \int_0^t G(t-s) * \big(uW_\tau(v)\big)(s)\,\mathrm{d}s$$

for each $\tau \ge 0$, with the convention (18b). Therefore, using the usual estimate (10), we have

$$\|B_\tau(u,v)(t)\|_{\mathcal{L}} \le C\Big(\sup_{t>0} \|u(t)\|_{\mathcal{L}}\Big)\Big(\sup_{t>0} t^{\frac{1}{2}}\|W_\tau(v)(t)\|_\infty\Big).$$

The last factor is bounded by $\|v\|_{\mathcal{X}}$ owing to Lemma 2.2 in the case $\tau = 0$, and to Lemma 2.3 for $\tau > 0$. This yields (35).

Now, we can consider, for $\tau \ge 0$, the sequence of approximating solutions

$$u_k^\tau = e^{t\Delta}u_0 - B_\tau(u_{k-1}^\tau, u_{k-1}^\tau), \qquad k = 1, 2, \dots\ .$$

When $\tilde{\epsilon} < \min\{\epsilon, \epsilon^*\}$, we know by the proofs of Theorems 2.1 and 3.1 that the sequence u_k^τ converges in $\mathcal{X}$ to the solution u^τ of (PE) or (PP). Here, of course, $u = u^0$ for the solutions of (PE).

On the other hand, applying recursively (35), we get $u_k^\tau(t) \in \mathcal{L}$ for all k and

$$\sup_{t>0} \|u_k^\tau(y)\|_{\mathcal{L}} \le C_0 + \tilde{\beta}\tilde{C}\tilde{\epsilon}\Big(\sup_{t>0} \|u_{k-1}^\tau(y)\|_{\mathcal{L}}\Big),$$

with $\tilde{\beta} = \max\{\beta, \beta^*\}$ (the constants obtained in Theorems 2.1 and 3.1). Iterating this inequality we arrive at

$$\sup_{t>0} \|u_k^\tau(t)\|_{\mathcal{L}} \le C' < \infty,$$

provided $\tilde{\beta}\tilde{C}\tilde{\epsilon} < 1$. If $\mathcal{L}$ is a shift invariant Banach space of local measures, from Property 3 we get for all $\tau \ge 0$

$$\sup_{t>0} \|u^\tau(t)\|_{\mathcal{L}} \le C' < \infty,$$

where $C' > 0$ is independent on τ. Of course, the last claim remains valid in the case $\mathcal{L} = L^1$ and $u_0 \ge 0$ (notice that the smallness of $\|u_0\|_E$ prevents blow up results that could occur, otherwise, when the second moment of u_0 are

finite and $\int u_0 > 8\pi$. See the introduction and the references therein quoted). If we remove the assumption $u_0 \geq 0$, we can obtain the same conclusion provided $\|u_0\|_1$ is sufficiently small. Indeed, we see from inequality (35) that the fixed point argument applies in the space $L^\infty_t(L^1) \cap \mathcal{X}$.

We now discuss the stability, including also the case $\mathcal{L} = L^1$. From the bilinearity of B_τ, the mixed estimate (35) and the smallness of the solutions u and u^τ (allowing two terms of the bilinear expansions to be absorbed by the left hand side), we obtain

$$\|u^\tau(t) - u(t)\|_{\mathcal{L}} \leq C \sup_{t>0} \|B_\tau(u,u) - B_0(u,u)(t)\|_{\mathcal{L}}.$$

Argueing as at the end of the proof of Theorem 4.1, we arrive at

$$\|B_\tau(u,u) - B_0(u,u)(t)\|_{\mathcal{L}} \leq C\Big(\sup_{t>0} \|u(t)\|_{\mathcal{L}}\Big)\Big(\sup_{t>0} t^{\frac{1}{2}} \|(W_\tau(u) - W_0(u))(t)\|_\infty\Big),$$

and the conclusion follows from Lemma 4.2. □

Remark. As an application of this general result, let us observe that taking $\mathcal{L} = L^\infty$, we obtain for $u_0 \in E \cap L^\infty$, with u_0 small in the E-norm,

$$u^\tau \to u$$

as $\tau \searrow 0$, uniformly in $(x,t) \in \mathbb{R}^2 \times [0,\infty)$.

5 Blow up of complex valued solutions of the parabolic-parabolic system

Consider the system (PP) in the space $\mathbb{R}^d$ with any $d \geq 1$. Passing to the Fourier transforms, we may write the Duhamel formula (16) in the form

$$\begin{aligned}\widehat{u}(\xi,t) &= e^{-t|\xi|^2}\widehat{u}_0(\xi) \\ &\quad + \int_0^t \int_0^s \int_{\mathbb{R}^d} \frac{\xi\cdot\eta}{\tau} e^{-(t-s)|\xi|^2} e^{-\frac{1}{\tau}(s-\sigma)|\eta|^2} \widehat{u}(\xi-\eta,s)\widehat{u}(\eta,\sigma)\,\mathrm{d}\eta\,\mathrm{d}\sigma\,\mathrm{d}s.\end{aligned} \tag{36}$$

Our goal is to construct a class of complex valued initial data, such that the corresponding solutions blow up in finite time, in any classical norm. For $a \in \mathbb{R}$, we denote by $\dot{B}^{a,\infty}_\infty$ the homogeneous Besov space, which can also be identified with the Hölder–Zygmund space $\dot{\mathcal{C}}^a$. As it is well known (see [19, 23]), most of the classical functional spaces (including all homogeneous Triebel–Lizorkin, and thus Lebesgue or Sobolev spaces) are continuously embedded in $\dot{B}^{a,\infty}_\infty$ for some real a.

Theorem 5.1 *There exists $w_0 \in \mathcal{S}(\mathbb{R}^d)$ such that $\widehat{w}_0(\xi) \geq 0$, $\|\widehat{w}_0\|_{L^1} = 1$. If A is sufficiently large, then any (tempered) distributional solution of* (36) *with $u_0 = Aw_0$ (and thus any solution of (PP)) blows up in a finite time. More precisely, for some time $t^* < \infty$ and each $a \in \mathbb{R}$, $\|u(t^*)\|_{\dot{B}^{a,\infty}_\infty} = \infty$ holds.*

Our approach is closely related to that in [5, Theorem 3.1] which followed the argument in [23] for the "cheap" Navier–Stokes equations. We produce some estimates from below of the Fourier transform of any solution with $u(0) = u_0$ that can be obtained via the iteration procedure for (36) with

$$u_{k+1} = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\frac{1}{\tau}\left[u_k(s)\nabla\int_0^s e^{\frac{1}{\tau}(s-\sigma)\Delta}u_k(\sigma)\,\mathrm{d}\sigma\right]\mathrm{d}s,$$

for $k = 1, 2, \dots$. This recurrence relation, in general, does not preserve the positivity of the Fourier transform of u_0. This leads us to restrict our attention to data of the form $\widehat{u}_0 = A\widehat{w}_0$ with $w_0 \in \mathcal{S}(\mathbb{R}^d)$, such that $\operatorname{supp}\widehat{w}_0 \subset \left\{\xi \in \mathbb{R}^d\colon 2^{-1} \le \xi_1 \le |\xi| \le 1\right\}$. Define, for $k = 0, 1, \dots$ the set

$$E_k = \left\{\xi \in \mathbb{R}^d\colon 2^{k-1} \le \xi_1 \le |\xi| \le 2^k\right\}.$$

Then we see that for $w_k = w_0^{2^k}$, we have $\widehat{w}_k = (2\pi)^{-d}\widehat{w}_{k-1} * \widehat{w}_{k-1}$, and therefore, $\operatorname{supp}\widehat{w}_k \subset E_k$. This implies that if, in addition, $\widehat{w}_0(\xi) \ge 0$, then the positivity of the Fourier transform will be preserved by the sequence u_k, and so by u. Next Lemma tells us more:

Lemma 5.1 *For all $k = 0, 1, 2, \dots$, we have*

$$\widehat{u}(\xi, t) \ge \beta_k e^{-2^k t}\mathbf{1}_{t_k \le t < t^*}(t)\widehat{w}_k(\xi), \tag{37}$$

where $\{\beta_k\}$ and $\{t_k\}$ are two sequences defined below in (38) *and* (39).

Proof. For $k = 0$, the conclusion immediately follows from

$$\widehat{u}(\xi, t) \ge Ae^{-t|\xi|^2}\widehat{w}_0(\xi),$$

provided we choose $\beta_0 = A$ and $t_0 = 0$. Let $k \ge 1$. Assume that the inequality of the lemma holds for $k-1$. Then, for all $t_k \le t < t^*$,

$$\begin{aligned}\widehat{u}(\xi, t) &\ge \int_{t_{k-1}}^t \int_{t_{k-1}}^s \int_{\mathbb{R}^d} \frac{\xi_1\eta_1}{\tau}e^{-(t-s)|\xi|^2}e^{-\frac{1}{\tau}(s-\sigma)|\eta|^2}\beta_{k-1}^2 e^{-2^{k-1}s}e^{-2^{k-1}\sigma}\times \\ &\qquad\qquad \times \frac{\widehat{w}_{k-1}(\xi-\eta)\widehat{w}_{k-1}(\eta)}{(2\pi)^d}\,\mathrm{d}\eta\,\mathrm{d}\sigma\,\mathrm{d}s \\ &\ge \int_{t_{k-1}}^t \int_{\mathbb{R}^d} (s - t_{k-1})\frac{\xi_1 2^{k-2}}{\tau}e^{-(t-s)|\xi|^2}e^{-\frac{1}{\tau}(t^*-t_{k-1})2^{2k-2}}\beta_{k-1}^2 e^{-2^k s}\times \\ &\qquad\qquad \times \frac{\widehat{w}_{k-1}(\xi-\eta)\widehat{w}_{k-1}(\eta)}{(2\pi)^d}\,\mathrm{d}\eta\,\mathrm{d}s\end{aligned}$$

Thus, we can bound $\widehat{u}(\xi,t)$ from below as follows

$$\begin{aligned}
\widehat{u}(\xi,t) &\geq \int_{t_{k-1}}^{t} (s-t_{k-1})\frac{2^{2k-3}}{\tau} e^{-(t-s)2^{2k}} e^{-\frac{1}{\tau}(t^*-t_{k-1})2^{2k-2}} \beta_{k-1}^2 e^{-2^k s}\widehat{w}_k(\xi)\,\mathrm{d}s \\
&\geq \left(\int_{t_{k-1}}^{t} (s-t_{k-1})e^{-(t-s)2^{2k}}\,\mathrm{d}s\right)\frac{2^{2k-3}}{\tau} e^{-\frac{1}{\tau}(t^*-t_{k-1})2^{2k-2}} \beta_{k-1}^2 e^{-2^k t}\widehat{w}_k(\xi) \\
&= \left(\frac{t-t_{k-1}}{2^{2k}} - \frac{1-e^{-(t-t_{k-1})2^{2k}}}{2^{4k}}\right)\frac{2^{2k-3}}{\tau} e^{-\frac{1}{\tau}(t^*-t_{k-1})2^{2k-2}} \beta_{k-1}^2 e^{-2^k t}\widehat{w}_k(\xi) \\
&\geq \left((t_k-t_{k-1}) - \frac{1-e^{-(t^*-t_{k-1})2^{2k}}}{2^{2k}}\right)\frac{2^{-3}}{\tau} e^{-\frac{1}{\tau}(t^*-t_{k-1})2^{2k-2}} \beta_{k-1}^2 e^{-2^k t}\widehat{w}_k(\xi).
\end{aligned}$$

This suggests us to set, for some $\delta > 0$, $t^* - t_{k-1} = \delta\tau 2^{-2k+2}$. With this choice, putting $t_0 = 0$, we have

$$t^* = \delta\tau, \qquad t_k = \delta\tau(1-2^{-2k}). \tag{38}$$

Then, $t_k - t_{k-1} = 3\delta\tau 2^{-2k}$. We get, for $t_k \leq t < t^*$,

$$\widehat{u}(\xi,t) \geq (3\delta\tau - 1 + e^{-4\delta\tau})2^{-2k-3}\tau^{-1}e^{-\delta}\beta_{k-1}^2 e^{-2^k\, t}\widehat{w}_k(\xi).$$

This inequality is interesting only if the right hand side is positive. Therefore we will assume that $3\delta\tau \geq 1$. We choose $\{\beta_k\}$ in such a way that

$$\beta_0 = A, \qquad \beta_k = (3\delta\tau - 1 + e^{-4\delta\tau})2^{-2k-3}\tau^{-1}e^{-\delta}\beta_{k-1}^2, \qquad k = 1, 2, \dots\ .$$

This choice leads to inequality (37).

In order to compute β_k, we introduce $M_{\delta,\tau}$, such that

$$2^{M_{\delta,\tau}} = (3\delta\tau - 1 + e^{-4\delta\tau})e^{-\delta}\, 2^{-3}\tau^{-1}.$$

Notice that we have $\beta_k = 2^{M_{\delta,\tau}-2k}\beta_{k-1}^2$. We claim that, for some $a, b, c \in \mathbb{R}$,

$$\beta_k = A^{2^k} 2^{a+bk+c2^k}.$$

Indeed, from an easy calculation we find $b = 2$, $a = 4 - M_{\delta,\tau}$ and finally $c = M_{\delta,\tau} - 4$, which is needed to ensure $\beta_0 = A$.

Hence, we have

$$\beta_k = \left(A\, 2^{M_{\delta,\tau}-4}\right)^{2^k} 2^{4-M_{\delta,\tau}+2k}, \qquad k = 0, 1, \dots\ . \tag{39}$$

□

We conclude that when

$$A \geq 2^{4-M_{\delta,\tau}},$$

we have by (39) $\beta_k \to \infty$ and, in particular, $\|u(t_k)\|_{L^1} = \|\widehat{u}(t_k)\|_\infty \to \infty$ for $k \to \infty$. The above size condition on A can be rewritten in an equivalent form as

$$(3\delta\tau - 1 + e^{-4\delta\tau})A \geq 2^7 e^{\delta}\,\tau.$$

A further analysis of the lower bounds obtained for the Fourier transform of a candidate solution permits us to conclude, as was in [5], that $\|u(t^*)\|_{\dot{B}^{a,\infty}_\infty} = \infty$ for each $a \in \mathbb{R}$, so that all Besov (and also L^p or Triebel–Lizorkin) norms of u blow up not later than t^*. Notice that, for a blow-up at $t^* = 1$, we need $A \geq Ce^{1/\tau}\tau$, cf. (38).

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