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Limit theorems for subcritical Branching Process in Random Environment depending on the initial number of particles

Vincent Bansaye *

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Abstract

Asymptotic behaviors for subcritical Branching Processes in Random Environment (BPRE) starting with several particles depend on whether the BPRE is strongly subcritical (SS), intermediate subcritical (IS) or weakly subcritical (WS) (see [12]). Descendants of particles for BPRE are not independent. In the (SS+IS) case, the asymptotic probability of survival is proportional to the initial number of particles. And conditionally on the survival of the population, only one initial particle survives a.s. These two properties do not hold in the (WS) case and different asymptotics are established, which require to prove new results on random walk with negative drift. We provide an interpretation of these results by characterizing the sequence of environments selected when we condition by the survival of particles. This also raises the problem of the dependence of the Yaglom quasistationary distributions on the initial number of particles and the asymptotic behavior of the Q-process associated with a subcritical BPRE.

Key words. Branching process in random environment (BPRE). Yaglom distribution. Q-process. Random walk with negative drift.

A.M.S. Classification. 60J80, 60J85, 60K37.

1 Introduction

We consider a Branching Process in Random Environment (BPRE) $(Z_n)_{n \in \mathbb{N}}$ specified by a sequence of iid generating functions $(f_n)_{n \in \mathbb{N}}$ distributed as f [2, 4, 5, 12]. More precisely, conditionally on the environment $(f_n)_{n \in \mathbb{N}}$, particles at generation n reproduce independently of each other and their offspring have generating function f_n . Then Z_n is

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the number of particles at generation n and Z_{n+1} is the sum of Z_n independent random variables with generating function f_n . That is, for every $n \in \mathbb{N}$,

$$\mathbb{E}(s^{Z_{n+1}} | Z_0, \dots, Z_n; f_0, \dots, f_n) = f_n(s)^{Z_n} \quad (0 \leq s \leq 1).$$

Thus, denoting by $F_n := f_0 \circ \dots \circ f_{n-1}$, we have for every $k \in \mathbb{N}$,

$$\mathbb{E}_k(s^{Z_{n+1}} | f_0, \dots, f_n) = \mathbb{E}(s^{Z_{n+1}} | Z_0 = k, f_0, \dots, f_n) = F_n(s)^k \quad (0 \leq s \leq 1).$$

When the environments are deterministic (i.e. f is a deterministic generating function), this process is the Galton Watson process (GW) with reproduction law Z , where f is the generating function of Z .

In this paper, we consider the subcritical case :

$$\mathbb{E}(\log(f'(1))) < 0.$$

This is the case where extinction occurs a.s., that is

$$\mathbb{P}(\exists n \in \mathbb{N} : Z_n = 0) = 1.$$

For a subcritical GW process, if $\mathbb{E}(Z_1 \log^+(Z_1)) < \infty$, there exists $c > 0$ such that $\mathbb{P}(Z_n > 0) \sim cf'(1)^n$ when n tends to infinity (see [6]). In random environments, this asymptotic depends on whether the BPRE is strongly subcritical (SS), intermediate subcritical (IS) or weakly subcritical (WS) (see [12] or the Preliminaries Section for details). A subcritical GW process is strongly subcritical (SS).

We study the role of the initial number of particles in the limit theorems. For a GW process, particles are independent. As a consequence, limit theorems starting with several particles can be directly derived from the case with one single initial particle.

In random environment, particles do not reproduce independently; more precisely independence holds only conditionally on the environments. This explains why asymptotics for (WS) BPRE starting with several particles are different from the analogous results for a GW process.

When the BPRE is (SS) or (IS), conditioning on the survival of the population at generation n , only one initial particle survives in generation n when $n \rightarrow \infty$, just as for a GW process. But this does not hold in the (WS) case (see forthcoming Proposition 2). Thus, (WS) BPRE conditioned to survive have a supercritical behavior, as previously observed in [2].

We give an interpretation of these results in terms of environments (see Section 3.3 for details). Conditioning on non-extinction induces a selection of environments with high reproduction law. In the (SS+IS) case, we prove that the survival probability of the branching process in the environments selected is still zero. This is obvious if environments are a.s. subcritical, i.e. $f'(1) < 1$ a.s. But in the (WS) case, conditioning by the survival of the population select only supercritical environments. That is, the sequence of environments selected has a.s. a positive survival probability (Theorem 3). Finally we make the initial number of particles tend to infinity and the sequence of environments becomes subcritical again.

We determine how the asymptotic survival probability depends on the initial number of particles. In that view, we define

$$\alpha_k := \lim_{n \rightarrow \infty} \mathbb{P}_k(Z_n > 0) / \mathbb{P}_1(Z_n > 0).$$

For a GW process, $\alpha_k = k$. That is, the asymptotic survival probability is proportional to the initial number of particles. This equality still holds in the (SS+IS) case for BPRE, but not in the (WS) case where a different asymptotic as $k \rightarrow \infty$ is established (see forthcoming Theorem 1). For the proof, we need an asymptotic result on random walks with negative drift (Section 5), which gives the product of the means of the successive environments.

In the supercritical case, see [13] for asymptotics of the extinction probability when the number of initial particles tends to infinity.

In Section 3.3, we are interested in the characterization of the Yaglom quasistationary distribution, that is the limit as $n \rightarrow \infty$ of the number of particles at generation n , conditioned to be nonzero, starting with k particles.

Finally, in Section 3.4, we focus on the Q-process associated to the subcritical BPRE, which is defined for all $l_1, l_2, \dots, l_n \in \mathbb{N}$, by

$$\mathbb{P}_k(Y_1 = l_1, \dots, Y_n = l_n) = \lim_{p \rightarrow \infty} \mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n | Z_{n+p} > 0).$$

See [6] for details on the Q-process associated to GW.

2 Preliminaries

We recall limit theorems for subcritical BPRE. Note that $s \in \mathbb{R}^+ \mapsto \mathbb{E}(f'(1)^s)$ is a convex function and define γ and α in $[0, 1]$ such that

$$\gamma := \inf_{\theta \in [0, 1]} \{\mathbb{E}(f'(1)^\theta)\} = \mathbb{E}(f'(1)^\alpha). \quad (1)$$

From now on, we assume $\mathbb{E}(f'(1)|\log(f'(1))|) < \infty$. Note that $0 < \gamma < 1$, $\gamma \leq \mathbb{E}(f'(1))$, and

$$\gamma = \mathbb{E}(f'(1)) \Leftrightarrow \mathbb{E}(f'(1)\log(f'(1))) \leq 0.$$

There are three subcases (see [12]).

- ★ The strongly subcritical case (SS), where $\mathbb{E}(f'(1)\log(f'(1))) < 0$. In this case, assuming further

$$\mathbb{E}(Z_1 \log^+(Z_1)) < \infty,$$

then there exist $c, \alpha_k > 0$ such that, as $n \rightarrow \infty$:

$$\mathbb{P}_k(Z_n > 0) \sim c\alpha_k \mathbb{E}(f'(1))^n, \quad \alpha_1 = 1. \quad (2)$$

- ★ The intermediate subcritical case (IS), where $\mathbb{E}(f'(1)\log(f'(1))) = 0$. In this case, assuming further

$$\mathbb{E}(f'(1)\log^2(f'(1))) < \infty, \quad \mathbb{E}([1 + \log^-(f'(1))]f''(1)) < \infty,$$

then there exist $c, \alpha_k > 0$ such that as $n \rightarrow \infty$:

$$\mathbb{P}_k(Z_n > 0) \sim c\alpha_k n^{-1/2} \mathbb{E}(f'(1))^n, \quad \alpha_1 = 1. \quad (3)$$

★ The weakly subcritical case (WS), where $0 < \mathbb{E}(f'(1)\log(f'(1))) < \infty$. In this case, assuming further

$$\mathbb{E}(f''(1)/f'(1)^{1-\alpha}) < \infty, \quad \mathbb{E}(f''(1)/f'(1)^{2-\alpha}) < \infty,$$

then there exist $c, \alpha_k > 0$ such that as $n \rightarrow \infty$:

$$\mathbb{P}_k(Z_n > 0) \sim c\alpha_k n^{-3/2}\gamma^n, \quad \alpha_1 = 1. \quad (4)$$

In the rest of the paper, for each case, we take the integrability assumptions above for granted. See [22] for asymptotics with weaker hypothesis in the (IS) case.

It is also known that the process Z_n starting from k particles and conditioned to be non zero converges to a finite positive random variable Υ_k , called the Yaglom quasistationary distribution (see [12]) :

$$\mathbb{E}_k(s^{Z_n} | Z_n > 0) \xrightarrow{n \rightarrow \infty} \mathbb{E}(s^{\Upsilon_k}).$$

See Section 3.3 for discussions about $(\Upsilon_k)_{k \in \mathbb{N}}$.

Actually, in [12], the result and the proof of these convergences are given for $k = 1$. They can be generalized to $k \geq 1$ with the following modifications. Set

$$S_i := \log(k) + \log(f'_0(1)) + \dots + \log(f'_{n-1}(1)), \quad g_0(s) := \frac{1}{1 - f_0(s)^k} - \frac{1}{k f'_0(1)(1 - s)},$$

and recall that

$$f_{k,l} := \begin{cases} f_k \circ f_{k+1} \circ \dots \circ f_{l-1}, & k < l \\ f_{k-1} \circ f_{k-2} \circ \dots \circ f_l, & k > l \\ \text{id}, & k = l. \end{cases}$$

Then $1 - \mathbb{E}_k(s^{Z_n} | Z_n > 0) = \mathbb{E}(1 - f_{0,n}^k(s)) / \mathbb{P}_k(Z_n > 0)$. Lemma 2.1 of [12] still holds replacing $f_{0,n}$ by $f_{0,n}^k$ and $\mathbb{P}(Z_n > 0)$ by $\mathbb{P}_k(Z_n > 0)$. Lemma 2.2 also still holds and Lemma 2.3. becomes

Lemma 1. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of probability generating functions. Then, for all $k \geq 1$, $i \geq 0$ and $0 \leq s \leq 1$,*

$$\exp(-S_{i+1})(1 - f_{i+1,0}^k(s)) \leq \exp(-S_i)(1 - f_{i,0}^k(s)) \leq 1 - s^k.$$

In particular,

$$\lim_{n \rightarrow \infty} \exp(-S_n)(1 - f_{n,0}(s)) \text{ exists,} \quad (0 \leq s \leq 1).$$

Finally, we consider the case where reproduction laws are a.s. linear fractional, since in that case survival probabilities can be computed explicitly. Thus, there are two random variables $A \in [0, \infty[$ and $B \in [0, 1[$ with $A + B \leq 1$ such that

$$f(s) = 1 - \frac{A}{1 - B} + \frac{As}{1 - Bs} \quad a.s. \quad (0 \leq s \leq 1). \quad (5)$$

In this case, setting for every $i \in \mathbb{N}$,

$$P_i := f'_{n-i}(1) \dots f'_{n-1}(1), \quad (P_0 = 1),$$

we have (see [3], [14] or [18])

$$\mathbb{P}(Z_n > 0 | Z_0 = 1, f_0, \dots, f_{n-1}) = 1 - F_n(0) = \left(1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{2f'_{n-i-1}(1)} P_i\right)^{-1} P_n. \quad (6)$$

As conditionally on (f_0, \dots, f_{n-1}) , $(Z_n^{(i)}, i \geq 1)$ is an iid sequence, we get

$$\mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(k)} > 0 | f_0, \dots, f_{n-1}) = \left(1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{2f'_{n-i-1}(1)} P_i\right)^{-k} P_n^k. \quad (7)$$

For a general BPRE, we use now that for every probability generating function f_i , we can find \tilde{f}_i linear fractional probability generating function such that for every $s \in [0, 1]$, $\tilde{f}_i(s) \geq f_i(s)$, $\tilde{f}'_i(1) = f'_i(1)$, $\tilde{f}''_i(1) = 2f''_i(1)$ (see [14] or [18]). Then, $\tilde{F}_n(0) \geq F_n(0)$ *a.s.* ensures that

$$\mathbb{P}(Z_n > 0 | f_0, \dots, f_{n-1}) \geq \mathbb{P}(\tilde{Z}_n > 0 | f_0, \dots, f_{n-1}) \quad a.s. \quad (8)$$

More generally, for every $k \geq 1$,

$$\begin{aligned} & \mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \dots, Z_n^{(k)} > 0 | f_0, \dots, f_{n-1}) \\ &= (1 - F_n(0))^k \\ &\geq (1 - \tilde{F}_n(0))^k \\ &= \mathbb{P}(\tilde{Z}_n^{(1)} > 0, \tilde{Z}_n^{(2)} > 0, \dots, \tilde{Z}_n^{(k)} > 0 | f_0, \dots, f_{n-1}) \quad a.s. \end{aligned} \quad (9)$$

3 Subcriticality starting from several particles

We give here the asymptotic of survival probabilities starting with k particles. Then we determine how many initial particles survive conditionally on non extinction of particles and we characterize the sequence of environments which are selected by this conditioning. Finally we consider the Yaglom quasistationary distributions of $(Z_n)_{n \in \mathbb{N}}$ and the associated Q-process. In the (SS) case, results are those expected, i.e. they are analogous to those of a GW process. In the (IS) case, results are different for the Yaglom quasistationary distribution and the Q-process. In the (WS) case, all results are different.

We label by $i \in \mathbb{N}$ each particle of the initial population and denote by $Z_n^{(i)}$ the number of descendants of particle i at generation n .

Thus $(Z_n^{(i)})_{n \in \mathbb{N}}$ are identically distributed BPRE ($i \in \mathbb{N}$), with common distribution $(Z_n)_{n \in \mathbb{N}}$ starting with one particle. Conditionally on the environments, these processes are independent. In other words, for all $n, k, l_i \in \mathbb{N}$,

$$\mathbb{P}(Z_n^{(i)} = l_i, 1 \leq i \leq k | f_0, \dots, f_{n-1}) = \prod_{i=1}^k \mathbb{P}(Z_n^{(1)} = l_i | f_0, \dots, f_{n-1}).$$

We denote by \mathbb{P}_k the probability associated with k initial particles. Then, under \mathbb{P}_k , $(Z_n)_{n \in \mathbb{N}}$ is *a.s.* equal to

$$\left(\sum_{i=1}^k Z_n^{(i)}\right)_{n \in \mathbb{N}}.$$

3.1 Survival probabilities starting with several particles

Note that $x \mapsto \mathbb{E}(f'(1)^x \log(f'(1)))$ increases with x .

Proposition 1. *For every $k \in \mathbb{N}^*$,*

(i) *If $\mathbb{E}(f'(1)^k \log(f'(1))) < 0$, then there exists $c_k > 0$ such that*

$$\mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \dots, Z_n^{(k)} > 0) \stackrel{n \rightarrow \infty}{\sim} c_k \mathbb{E}(f'(1)^k)^n$$

and $\mathbb{E}(f'(1)^k) < \mathbb{E}(f'(1)^{k-1}) < \dots < \mathbb{E}(f'(1))$.

(ii) *If $\mathbb{E}(f'(1)^k \log(f'(1))) = 0$, then there exists $c_k > 0$ such that*

$$\mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \dots, Z_n^{(k)} > 0) \stackrel{n \rightarrow \infty}{\sim} c_k n^{-1/2} \mathbb{E}(f'(1)^k)^n.$$

(iii) *If $\mathbb{E}(f'(1)^k \log(f'(1))) > 0$, then there exists $c_k > 0$ such that*

$$\mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \dots, Z_n^{(k)} > 0) \stackrel{n \rightarrow \infty}{\sim} c_k n^{-3/2} \tilde{\gamma}^n,$$

with $\tilde{\gamma} = \inf_{s \in \mathbb{R}^+} \{\mathbb{E}(f'(1)^s)\} \in]0, 1[$ and $c = c_1 \geq c_2 \geq \dots \geq c_k$.

Moreover, in the (IS+WS) case, $\tilde{\gamma} = \gamma$. In the (SS) case, $\tilde{\gamma} < \gamma = \mathbb{E}(f'(1))$.

The proof is given in Section 4.1 and uses the case where the probability generating function f is a.s. linear fractional. Indeed in this case the survival probability in a given environment can then be computed explicitly since linear fractional generating functions are stable by composition (see Preliminaries Section).

In the (SS+IS) case, the asymptotic probability of survival of particles is proportional to the number of initial particles, as stated below. This is not surprising and well known for subcritical GW process. But this does not hold in the (WS) case. Recall that α_k is defined as $\lim_{n \rightarrow \infty} \mathbb{P}_k(Z_n > 0) / \mathbb{P}_1(Z_n > 0)$.

Theorem 1. *In the (SS+IS) case, for every $k \in \mathbb{N}$, $\alpha_k = k$.*

In the (WS) case, $\alpha_k \rightarrow \infty$ as $k \rightarrow \infty$ and there exists $M_+ > 0$ such that

$$\alpha_k \leq M_+ \log(k) k^\alpha, \quad (k \geq 2),$$

where $\alpha \in]0, 1[$ is given by (1).

Assuming further $\mathbb{E}(f'(1)^{1/2} \log(f'(1))) > 0$ (i.e. $\alpha < 1/2$) and $f''(1)/f'(1)$ is bounded, there exists $M_- > 0$ such that

$$\alpha_k \geq M_- \log(k) k^\alpha, \quad (k \in \mathbb{N}).$$

One can naturally conjecture that the last result still holds for $1/2 \leq \alpha < 1$. The proof also uses the linear fractional case where, conditionally given the environments, the survival probability is related to a random walk whose jumps are the log of means of the reproduction law of the environments. That's why we first need to prove a result about random walk with negative drift conditioned to be larger than $-x < 0$ (see Appendix). One way to generalize the last result of the theorem above to the case $\mathbb{E}(f'(1)^{1/2} \log(f'(1))) > 0$ (i.e. $\alpha < 1/2$) would be to improve Lemma 6.

3.2 Survival of initial particles conditionally on non-extinction

We wonder now how many initial particles survive when we condition by the survival of the whole population of particles. We have the following elementary consequence of Proposition 1.

Proposition 2. *In the (SS+IS) case, for every $k \geq 1$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}_k(\exists i \neq j, 1 \leq i, j \leq k, Z_n^{(i)} > 0, Z_n^{(j)} > 0 | Z_n > 0) = 0.$$

In the (WS) case, for every $k \geq 1$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_k(\forall i, 1 \leq i \leq k, Z_n^{(i)} > 0 | Z_n > 0) > 0.$$

Thus, for (SS+IS) BPRE, conditionally on the survival of the population, only one initial particle survives, as for GW. But for (WS) BPRE, several initial particles survive with positive probability. Forthcoming Theorem 3 gives an interpretation of this property in terms of selection of favorable environments by conditioning on non-extinction. See Section 6.3 in [7] for an application of this result to a branching model for cell division with parasite infection, where we need to determine if several parasites survive in contaminated cells. In the same vein, see [21] for results on the reduced process associated with subcritical BPRE in the linear fractional case. In the (WS) case, the number of particles of the reduced process is not a.s. equal to 1 in the first generations.

What happens when the number of initial particles tends to infinity in the (WS) case ? As stated below, conditionally on non-extinction, the number of initial particles which survive is finite a.s. but not bounded, when the initial number of particles tend to infinity. More precisely, denote by N_n the number of particles in generation 0 whose descendance is alive in generation n . That is, starting with k initial particles :

$$N_n := \#\{1 \leq i \leq k : Z_n^{(i)} > 0\}.$$

Theorem 2. *In the (WS) case, assuming $\mathbb{E}(f'(1)^{1/2} \log(f'(1))) > 0$ (i.e. $\alpha < 1/2$) and $f''(1)/f'(1)$ is bounded, there exist $A_l \downarrow_{l \rightarrow \infty} 0$ such that for all $k \geq l \geq 0$,*

$$\limsup_{n \rightarrow \infty} \mathbb{P}_k(N_n \geq l | Z_n > 0) \leq A_l.$$

Moreover, for every $l \in \mathbb{N}^*$,

$$\liminf_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P}_k(N_n = l | Z_n > 0) > 0.$$

Thus, in the conditions of the theorem,

$$\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_k(N_n \geq l | Z_n > 0) \leq A_l, \quad \text{with } A_l \downarrow_{l \rightarrow \infty} 0.$$

3.3 Selection of environments conditionally on non-extinction

We characterize here the sequence of environments which are selected by conditioning on the survival of particles.

We denote by \mathcal{F} the set of generating functions and for every $\mathbf{g}_n = (g_0, \dots, g_{n-1}) \in \mathcal{F}^n$, we denote by $Z_{\mathbf{g}_n}$ the value at generation n of the time inhomogeneous branching process whose reproduction law at generation $l \leq n-1$ has generating function g_l . Thus, for every $k \geq 1$,

$$\mathbb{E}_k(s^{Z_{\mathbf{g}_n}}) = g_0 \circ g_1 \circ \dots \circ g_{n-1}(s)^k \quad (0 \leq s \leq 1). \quad (10)$$

And we denote by $p(\mathbf{g}_n)$ the survival probability of a particle in environment \mathbf{g}_n . That is,

$$p(\mathbf{g}_n) := \mathbb{P}_1(Z_{\mathbf{g}_n} > 0). \quad (11)$$

Denote by \mathbf{f}_n the sequence of environments until time n , i.e.

$$\mathbf{f}_n := (f_0, f_1, \dots, f_{n-1}).$$

In the subcritical case, $p(\mathbf{f}_n) = 0$ a.s. since $(Z_n)_{n \in \mathbb{N}}$ becomes extinct a.s. Roughly speaking, the sequences of environments have a.s. zero survival probability. In the (SS+IS) case, conditioning on the survival of particles does not change this fact, but it does in the (WS) case, as we can guess using Proposition 2. Actually, we prove that in the (WS) case, the sequence of environments which are selected by conditioning by $Z_n > 0$ have a.s. a positive survival probability. Thus, they are 'supercritical'. In [2], authors had already remarked this supercritical behavior of the BPRE $(Z_n)_{n \in \mathbb{N}}$ in the (WS) case by giving an analogy of the Kesten-Stigum theorem, i.e. the convergence of Z_n/m^n .

Theorem 3. *In the (SS+IS) case, for all $k \in \mathbb{N}^*$, $\epsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) = 0.$$

In the (WS) case, for every $k \in \mathbb{N}^$,*

$$\liminf_{n \rightarrow \infty} \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) \xrightarrow{\epsilon \rightarrow 0^+} 1.$$

This supercritical behavior in the (WS) case disappears as k tends to infinity. That is, the survival probability of selected sequences of environments tends to 0 as the number of particles grows to infinity.

Proposition 3. *In the (WS) case, for every $\epsilon > 0$,*

$$\limsup_{n \rightarrow \infty} \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) \xrightarrow{k \rightarrow \infty} 0.$$

In other words, conditionally on the survival of Z_n , the more initial particles there are, the less environments need to be favorable to allow the survival of particles, and the less likely it is for a given particle to survive. That's why, letting the number of initial particles tend to infinity does not make the number of survival initial particles tend to infinity, as stated in Theorem 2.

3.4 Yaglom quasistationary distributions

We focus now on the Yaglom quasistationary distribution of $(Z_n)_{n \in \mathbb{N}}$ (see Preliminaries for existence and references). For the GW process, this distribution does not depend on the initial number of particles and is characterized by a functional equation. This result still holds for (SS) BPRE. Indeed, starting with several particles, conditionally on the survival of one given particle, the others become extinct (see Proposition 2).

Theorem 4. *In the (SS+IS) case, for every $k \geq 1$, the BPRE Z_n starting from k and conditioned to be nonzero converges in distribution as $n \rightarrow \infty$ to a r.v. Υ which does not depend on k . Moreover, the generating function G of Υ verifies*

$$\mathbb{E}(G(f(s))) = \mathbb{E}(f'(1))G(s) + 1 - \mathbb{E}(f'(1)), \quad G(0) = 0.$$

In the (SS) case, G is the unique generating function which satisfies the functional equation above and $G'(1) < \infty$.

In the (WS) case, for every $k \geq 1$, the BPRE Z_n starting from k and conditioned to be nonzero converges in distribution as $n \rightarrow \infty$ to a r.v. Υ_k , whose generating function G_k verifies

$$\mathbb{E}(G_k(f(s))) = \gamma G_k(s) + 1 - \gamma, \quad G_k(0) = 0.$$

In the (WS) case, an open question is to determine if the quasistationary distribution Υ_k depends on the initial number k of particles. We know that for every $k \geq 1$, G_k verifies the same functional equation given above but we do not know if the solution is unique.

Moreover, we can prove the equality of the quasistationary distributions starting with two different numbers of particles in the following case. If $Z_1 \in \{0, 1, N\}$ for some $N \in \mathbb{N}^*$, then $\Upsilon_1 \stackrel{d}{=} \Upsilon_N$. (see Section 4.6 for the proof). Other observations also lead us to believe that quasistationary distributions Υ_k might not depend on k .

3.5 Q-process associated with a BPRE

The Q-process $(Y_n)_{n \in \mathbb{N}}$ starting from k particles associated to the BPRE $(Z_n)_{n \in \mathbb{N}}$ is defined for all $l_1, l_2, \dots, l_n \in \mathbb{N}$, by

$$\mathbb{P}_k(Y_1 = l_1, \dots, Y_n = l_n) = \lim_{p \rightarrow \infty} \mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n | Z_{n+p} > 0).$$

This is the BPRE $(Z_n)_{n \in \mathbb{N}}$ conditioned to survive in the distant future. See [6] for details in the case of GW processes. In the (SS) case, the Q-process converges in distribution to the size biased Yaglom distribution, as for GW process. Finer results have been obtained in [1]. In the (IS+WS) case, the Q-process is transient. That is, the population needs to grow largely in the first generations so that it can survive.

Proposition 4. *★ In the (SS) case, for every $k \in \mathbb{N}^*$, for all $l_1, l_2, \dots, l_n \in \mathbb{N}$,*

$$\mathbb{P}_k(Y_1 = l_1, \dots, Y_n = l_n) = \mathbb{E}(f'(1))^{-n} \frac{l_n}{k} \mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n).$$

Moreover $(Y_n)_{n \in \mathbb{N}}$ converges in distribution to the size biased Yaglom distribution.

$$\forall l \geq 0, \quad \mathbb{P}_k(Y_n = l) \xrightarrow{n \rightarrow \infty} \frac{l \mathbb{P}(\Upsilon = l)}{\mathbb{E}(\Upsilon)}.$$

★ In the (IS) case, for every $k \in \mathbb{N}^*$, for all $l_1, l_2, \dots, l_n \in \mathbb{N}$,

$$\mathbb{P}_k(Y_1 = l_1, \dots, Y_n = l_n) = \mathbb{E}(f'(1))^{-n} \frac{l_n}{k} \mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n).$$

Moreover $Y_n \rightarrow \infty$ in probability as $n \rightarrow \infty$.

★ In the (WS) case, for every $k \in \mathbb{N}^*$, for all $l_1, l_2, \dots, l_n \in \mathbb{N}$,

$$\mathbb{P}_k(Y_1 = l_1, \dots, Y_n = l_n) = \gamma^{-n} \frac{\alpha_{l_n}}{\alpha_k} \mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n).$$

Moreover Y_n tends to infinity a.s.

We focus now on the environments of the Q-process. We endow \mathcal{F} with distance d given by the infinity norm

$$d(f, g) = \|f - g\|_\infty$$

and we denote by $\mathcal{B}(\mathcal{F})$ the Borel σ -field.

We introduce the probability ν_k on $(\mathcal{F}^{\mathbb{N}}, \mathcal{B}(\mathcal{F})^{\otimes \mathbb{N}})$ which gives the distribution of the environments when the BPRE $(Z_n)_{n \in \mathbb{N}}$ starting from k particles is conditioned to survive. Using Kolomogorov Theorem, it can be specified by its projection on $(\mathcal{F}^p, \mathcal{B}(\mathcal{F})^{\otimes p})$ for every $p \in \mathbb{N}$, denoted by $\nu_k|_{\mathcal{F}^p}$,

$$\begin{aligned} \nu_k|_{\mathcal{F}^p}(\mathbf{d}_{\mathbf{g}_p}) &:= \lim_{n \rightarrow \infty} \mathbb{P}_k(\mathbf{f}_p \in \mathbf{d}_{\mathbf{g}_p} | Z_{n+p} > 0) \\ &= \gamma^{-p} \mathbb{P}(\mathbf{f}_p \in \mathbf{d}_{\mathbf{g}_p}) \sum_{l=1}^{\infty} \mathbb{P}_k(Z_{\mathbf{g}_p} = l) \frac{\alpha_l}{\alpha_k}, \end{aligned} \quad (12)$$

with $\mathbf{f}_p = (f_0, \dots, f_{p-1})$ and $\gamma = \mathbb{E}(f'(1))$ in the (SS+IS) case. The limit is the weak limit of probabilities on $(\mathcal{F}^p, \mathcal{B}(\mathcal{F})^{\otimes p})$ (see [8] for definition and Section 4.5 for the proof), which we endow with the distance d_p given by

$$d_p((g_0, \dots, g_{p-1}), (h_0, \dots, h_{p-1})) = \sup\{\|g_i - h_i\|_\infty : 0 \leq i \leq p-1\}. \quad (13)$$

For every $\mathbf{g} \in \mathcal{F}^{\mathbb{N}}$, we denote by $\mathbf{g}|_n$ the first n coordinates of $\mathbf{g} \in \mathcal{F}^{\mathbb{N}}$ and we introduce the survival probability in environment $\mathbf{g} \in \mathcal{F}^{\mathbb{N}}$:

$$p(\mathbf{g}) = \lim_{n \rightarrow \infty} \downarrow \mathbb{P}(Z_{\mathbf{g}|_n} > 0).$$

One can naturally conjecture an analogous of Theorem 3 and Proposition 3. That is, for every $k \in \mathbb{N}^*$,

In the (SS+IS) case, $\nu_k(\{\mathbf{g} \in \mathcal{F}^{\mathbb{N}} : p(\mathbf{g}) = 0\}) = 1$.

In the (WS) case, $\nu_k(\{\mathbf{g} \in \mathcal{F}^{\mathbb{N}} : p(\mathbf{g}) > 0\}) = 1$ and $\nu_k(p(\mathbf{f}) \in dx) \xrightarrow{n \rightarrow \infty} \delta_0(dx)$.

A perspective is to characterize the tree of particles when we condition by the survival of particles, i.e. the tree of particles of the Q-process. Informally, for GW process, this gives a spine with finite iid subtrees (see [11, 19]). This fact still holds in the (SS+IS) case but we will observe a 'multispine tree' in the (WS) case.

4 Proofs

First we give the main Notations and results for the proofs. We use the particular case when generating functions are a.s. linear fractional. In that case, the survival probability for a given environment is a functional of the random walk which sums the *log* of the successive means of environments (see (6)). Using results the random walk with negative drift proved in Appendix (Section 5), this enables us to control the survival probability conditionally on the environments, in the linear fractional case and then for general BPFE (Lemma 2 below). Using that conditionally on the sequence of environments, particles are independent, we get survival probabilities starting with several particles and then integrate with respect to environments.

Set for every $n \in \mathbb{N}$,

$$X_n := \log(f'_n(1)), \quad S_n = \sum_{i=0}^{n-1} X_i \quad (S_0 = 0),$$

and

$$L_n = \min\{S_i : 1 \leq i \leq n\}.$$

Recall that \mathcal{F} is the set of generating functions and

$$\mathbf{f}_n = (f_0, f_1, \dots, f_{n-1}).$$

For every $\mathbf{g}_n = (g_0, \dots, g_{n-1}) \in \mathcal{F}^n$, we denote by $Z_{\mathbf{g}_n}$ the value at generation p of the time inhomogeneous branching process whose reproduction law at generation l has generating function g_l . And we denote by $p(\mathbf{g}_n)$ the survival probability of a particle in environment \mathbf{g}_n . That is,

$$p(\mathbf{g}_n) := \mathbb{P}_1(Z_{\mathbf{g}_n} > 0) = \mathbb{P}_1(Z_n > 0 \mid \mathbf{f}_n = \mathbf{g}_n). \quad (14)$$

Roughly speaking, we prove now that

$$p(\mathbf{f}_n) \approx e^{L_n} \quad \text{a.s.}$$

Lemma 2. *For every $n \in \mathbb{N}$, we have*

$$p(\mathbf{f}_n) \leq e^{L_n} \quad \text{a.s.}$$

Moreover if $\mathbb{E}(f'(1)^{1/2} \log(f'(1))) > 0$ (i.e. $0 < \alpha < 1/2$) and $f''(1)/f'(1)$ is bounded, then there exists $\mu \geq 1$ such that for all $n \in \mathbb{N}$ and $x \in]0, 1]$,

$$\mathbb{P}(p(\mathbf{f}_n) \geq x) \geq \mathbb{P}(L_n \geq \log(\mu x))/4.$$

Proof. For the upper bound, note that for every $\mathbf{g}_n \in \mathcal{F}^n$, we have,

$$p(\mathbf{g}_n) = \mathbb{P}_1(Z_{\mathbf{g}_n} > 0) \leq \mathbb{E}_1(Z_{\mathbf{g}_n}) = \prod_{i=0}^{n-1} g'_i(1).$$

Thus $p(\mathbf{f}_n) \leq e^{S_n}$ a.s. As is usual, adding that $p(\mathbf{f}_n)$ decreases a.s. ensures that

$$p(\mathbf{f}_n) \leq e^{L_n} \quad \text{a.s.}$$

For the lower bound, recall that

$$P_i := f'_{n-i}(1) \dots f'_{n-1}(1), \quad (P_0 = 1),$$

and use (9) and (6) to get

$$p(\mathbf{f}_n) = \mathbb{P}(Z_n > 0 \mid \mathbf{f}_n) \geq \frac{\tilde{P}_n}{1 + \sum_{i=0}^{n-1} \frac{\tilde{f}''_{n-i-1}(1)}{2\tilde{f}'_{n-i-1}(1)} \tilde{P}_i} = \frac{P_n}{1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{f'_{n-i-1}(1)} P_i} \quad \text{a.s.}$$

We assume now that $C = (1 + \text{ess sup}(\frac{f''(1)}{f'(1)}))^{-1} > 0$. Denote by

$$S'_i := \log(f'_{n-i}(1)) + \dots + \log(f'_{n-1}(1)) \quad i \geq 1, \quad S'_0 = 0,$$

so that $P_i = \exp(S'_i)$. We then have

$$p(\mathbf{f}_n) \geq C \frac{e^{S'_n}}{2 \sum_{i=0}^{n-1} e^{S'_i}} \geq \frac{C}{2} \frac{e^{S'_n - \max\{S'_j: 0 \leq j \leq n\}}}{\sum_{i=0}^n e^{S'_i - \max\{S'_j: 0 \leq j \leq n\}}} \quad \text{a.s.}$$

Thus,

$$p(\mathbf{f}_n) \geq \frac{C}{2} \frac{e^{L_n}}{\sum_{i=0}^n e^{L_n - S_i}}. \quad (15)$$

As $\alpha < 1/2$, forthcoming Corollary 1 in Appendix (Section 5) ensures that there exists $\beta > 0$ such that for all $n \in \mathbb{N}$ and $x > 0$,

$$\begin{aligned} \mathbb{P}(p(\mathbf{f}_n) \geq x) &\geq \mathbb{P}(L_n \geq \log(2\beta x/C)) \mathbb{P}\left(\sum_{i=0}^n e^{L_n - S_i} \leq \beta \mid L_n \geq \log(2\beta x/C)\right) \\ &\geq \mathbb{P}(L_n \geq \log(\mu x))/4, \end{aligned}$$

writing $\mu = \min(1, 2\beta/C)$. □

Moreover, using independence of particles conditionally on environments, we have

$$\begin{aligned} \mathbb{P}_k(Z_n > 0) &= \int_{\mathcal{F}^n} \mathbb{P}_k(Z_{\mathbf{g}_n} > 0) \mathbb{P}(\mathbf{f}_n \in d\mathbf{g}_n) \\ &= \int_{\mathcal{F}^n} (1 - (1 - \mathbb{P}_1(Z_{\mathbf{g}_n} > 0))^k) \mathbb{P}(\mathbf{f}_n \in d\mathbf{g}_n) \\ &= \int_{\mathcal{F}^n} \mathbb{P}(p(\mathbf{f}_n) \in dx) (1 - (1 - x)^k), \end{aligned} \quad (16)$$

and

$$\begin{aligned} \alpha_k &= \lim_{n \rightarrow \infty} \mathbb{P}_k(Z_n > 0) / \mathbb{P}_1(Z_n > 0) \\ &= \lim_{n \rightarrow \infty} \int_0^1 (1 - (1 - x)^k) \frac{\mathbb{P}(p(\mathbf{f}_n) \in dx)}{\mathbb{P}_1(Z_n > 0)}. \end{aligned} \quad (17)$$

4.1 Proofs of Section 3.1

We split the proof of Proposition 1 into three parts.

Proof of Proposition 1 (i). We follow the proof of Theorem 1.2 (a) in [14] and introduce the probability $\tilde{\mathbb{P}}$ such that under $\tilde{\mathbb{P}}$, the environments still are iid and their law is given by

$$\tilde{\mathbb{P}}(f \in dg) = \mathbb{E}(f'(1)^k)^{-1} g'(1)^k \mathbb{P}(f \in dg).$$

Then, writing $P_n = f'_0(1) \dots f'_{n-1}(1)$ ($P_0 = 1$), we have

$$\mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(k)} > 0) = \mathbb{E}((1 - F_n(0))^k) = \mathbb{E}(f'(1)^k)^n \tilde{\mathbb{E}}((1 - F_n(0))/P_n)^k).$$

As $\mathbb{E}(f'(1)^k \log(f'(1))) < 0$, then $\tilde{\mathbb{E}}(\log(f'(1))) < 0$ and Theorem 5 in [4] ensures that

$$C = \lim_{n \rightarrow \infty} \frac{1 - F_n(0)}{P_n}$$

exists $\tilde{\mathbb{P}}$ a.s. and belongs to $]0, 1]$. Thus, as $n \rightarrow \infty$,

$$\mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(k)} > 0) \sim \mathbb{E}(f'(1)^k)^n \tilde{\mathbb{E}}(C^k).$$

Add that $s \mapsto \mathbb{E}(f'(1)^s)$ decreases for $s \in [0, \alpha]$ and $k < \alpha$ to complete the proof. \square

Proof of Proposition 1 (iii). We follow the proof of Theorem 1.2 (c) in [14].

STEP 1. First we consider the linear fractional case (see (5)). In that case, by (7),

$$\mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(k)} > 0 \mid f_0, \dots, f_{n-1}) = \left(1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{2f'_{n-i-1}(1)} P_i\right)^{-k} P_n^k.$$

Define $\tilde{\gamma}$ by

$$\tilde{\gamma} = \inf_{s \in \mathbb{R}^+} \{\mathbb{E}(f'(1)^s)\} = \mathbb{E}(f'(1)^{\tilde{\alpha}}),$$

where $0 < \tilde{\alpha} < k$ since $\mathbb{E}(f'(1)^k \log(f'(1))) > 0$. Let $\mathbb{P}_{\tilde{\alpha}}$ be the probability given by

$$\mathbb{P}_{\tilde{\alpha}}(f \in dg) = \tilde{\gamma}^{-1} g'(1)^{\tilde{\alpha}} \mathbb{P}(f \in dg).$$

Then

$$\mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(k)} > 0) = \tilde{\gamma}^n \mathbb{E}_{\tilde{\alpha}} \left[\left(1 + \sum_{i=0}^{n-1} \frac{f''_i(1)}{2f'_i(1)} P_i\right)^{-k} P_n^{k-\tilde{\alpha}} \right].$$

As $\mathbb{E}_{\tilde{\alpha}}(\log(f'(1))) = 0$, we apply Theorem 2.1 in [14] with

$$\phi(x) = x^{k-\tilde{\alpha}}, \quad \psi(x) = (1+x)^{-k}, \quad 0 < k - \tilde{\alpha} < k,$$

so there exists $c_k > 0$ such that, as $n \rightarrow \infty$,

$$\mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(k)} > 0) \sim c_k \tilde{\gamma}^n n^{-3/2}.$$

STEP 2. For the general case, we can use STEP 1. Indeed, by (9), there exists a BPRE $(\tilde{Z}_n)_{n \in \mathbb{N}}$ such that \tilde{f} is a.s. linear fractional, $\tilde{f}'(1) = f'(1)$ and

$$\mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \dots, Z_n^{(k)} > 0) \geq \mathbb{P}(\tilde{Z}_n^{(1)} > 0, \tilde{Z}_n^{(2)} > 0, \dots, \tilde{Z}_n^{(k)} > 0).$$

By STEP 1, this leads to the existence of $c_k(1) > 0$ such that

$$\mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \dots, Z_n^{(k)} > 0) \geq c_k(1) \gamma^n n^{-3/2}. \quad (18)$$

Note that by inclusion-exclusion principle, we have

$$\mathbb{P}_k(Z_n > 0) = \sum_{i=1}^k (-1)^{i+1} \binom{k}{i} \mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(i)} > 0). \quad (19)$$

Moreover, (4) ensure the convergence of

$$\gamma^{-n} n^{3/2} \mathbb{P}_1(Z_n > 0)$$

to $c\alpha_k$. By induction, it gives the convergence of

$$\gamma^{-n} n^{3/2} \mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0, \dots, Z_n^{(k)} > 0).$$

to a constant c_k , which is positive by (18).

To complete the proof note that $\gamma = \tilde{\gamma}$ iff $\mathbb{E}(f'(1)^s)]'(1) \geq 0$, i.e. in the (IS+WS) case. \square

Proof of Proposition 1 (ii). The proof is close to the previous one. First, we consider the linear fractional case and Introducing again the probability $\tilde{\mathbb{P}}$ defined by

$$\tilde{\mathbb{P}}(f \in dg) = \mathbb{E}(f'(1)^k)^{-1} g'(1)^k \mathbb{P}(f \in dg).$$

Using again (7), we get then

$$\mathbb{P}(Z_n^{(1)} > 0, \dots, Z_n^{(k)} > 0) = \mathbb{E}(f'(1)^k)^n \tilde{\mathbb{E}} \left[\left(1 + \sum_{i=0}^{n-1} \frac{f''_{n-i-1}(1)}{2f'_{i-1}(1)} P_i \right)^{-k} \right].$$

As $\tilde{\mathbb{E}}(\log(f'(1))) = 0$, we can use again Theorem 2.1 in [14] and conclude in the linear fractional case.

The general case can be proved following STEP 2 in the previous proof. \square

Proof of Theorem 1. Computation of α_k in the (SS+IS) case. In the (SS+IS) case, Proposition 2 and (19) ensure that for every $k \in \mathbb{N}$,

$$\mathbb{P}_k(Z_n > 0) \sim k\mathbb{P}(Z_n > 0), \quad (n \rightarrow \infty).$$

Then,

$$\alpha_k = k.$$

This gives the first result.

Limit of α_k in the (WS) case. Note that $\mathbb{P}_1(Z_{p+n} > 0) = \sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k)\mathbb{P}_k(Z_n > 0)$. Then,

$$\frac{\mathbb{P}_1(Z_{p+n} > 0)}{\mathbb{P}_1(Z_n > 0)} = \sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k) \frac{\mathbb{P}_k(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)}. \quad (20)$$

First, $\mathbb{P}(\cup_{i=1}^k \{Z_n^{(i)} > 0\}) \leq \sum_{i=1}^k \mathbb{P}(Z_n^{(i)} > 0)$, which gives

$$\mathbb{P}_k(Z_n > 0)/\mathbb{P}_1(Z_n > 0) \leq k.$$

Moreover $\sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k)k = \mathbb{E}(Z_p) < \infty$ and

$$\mathbb{P}_k(Z_n > 0)/\mathbb{P}_1(Z_n > 0) \xrightarrow{n \rightarrow \infty} \alpha_k,$$

then, by bounded convergence, we get

$$\sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k) \frac{\mathbb{P}_k(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)} \xrightarrow{n \rightarrow \infty} \sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k)\alpha_k.$$

Then, using again (4), letting $n \rightarrow \infty$ in (20) yields

$$\gamma^p = \sum_{k=1}^{\infty} \mathbb{P}_1(Z_p = k)\alpha_k.$$

Assuming that $(\alpha_k)_{k \in \mathbb{N}}$ is bounded by A leads to

$$\gamma^p \leq A\mathbb{P}_1(Z_p > 0).$$

Letting $p \rightarrow \infty$ leads to a contradiction with (4). Adding that α_k increases ensures that $\alpha_k \rightarrow \infty$ as $k \rightarrow \infty$.

Upper bound of α_k in the (WS) case. Recall (17),

$$\alpha_k = \lim_{n \rightarrow \infty} \int_0^1 (1 - (1 - x)^k) \frac{\mathbb{P}(p(\mathbf{f}_n) \in dx)}{\mathbb{P}_1(Z_n > 0)}.$$

Using the first inequality of Lemma 2 and $x \mapsto 1 - (1 - x)^k$ grows with x on $[0, 1]$, we have

$$\alpha_k \leq \limsup_{n \rightarrow \infty} \int_0^1 (1 - (1 - x)^k) \frac{\mathbb{P}(\exp(L_n) \in dx)}{\mathbb{P}_1(Z_n > 0)}.$$

By (26), we can use Fatou's Lemma and (27) gives

$$\alpha_k \leq \int_0^1 (1 - (1-x)^k) \nu_+(dx) \limsup_{n \rightarrow \infty} \frac{n^{-3/2} \gamma^n}{\mathbb{P}_1(Z_n > 0)}.$$

Thus, by (4) and definition of ν_+ ,

$$\alpha_k \leq c^{-1} c_+ [1 + \int_0^1 (1 - (1-x)^k) \log(1/x) x^{-\alpha-1} dx].$$

Finally, using $1 - (1-x)^k \leq kx$ and integration by parts,

$$\begin{aligned} & \int_0^1 (1 - (1-x)^k) \log(1/x) x^{-\alpha-1} dx \\ = & \int_0^{1/k} (1 - (1-x)^k) \log(1/x) x^{-\alpha-1} dx + \int_{1/k}^1 (1 - (1-x)^k) \log(1/x) x^{-\alpha-1} dx \\ \leq & k \int_0^{1/k} \log(1/x) x^{-\alpha} dx + \log(k) \int_{1/k}^1 x^{-\alpha-1} dx \\ \leq & k(1-\alpha)^{-1} [\log(k) k^{\alpha-1} + \int_0^{1/k} x^{-\alpha} dx] + \alpha^{-1} \log(k) (k^\alpha - 1) \\ \leq & M(1 + \log(k)) k^\alpha, \end{aligned}$$

for some $M > 0$. This ensures that $\alpha_k \leq c^{-1} c_+ [1 + M(1 + \log(k)) k^\alpha]$ and ends the proof.

Lower bound of α_k in the (WS) case assuming further $\mathbb{E}(f^{1/2}(1) \log(f'(1))) > 0$ (i.e. $\alpha < 1/2$) and $f''(1)/f'(1)$ is bounded.

By (4), Lemma 2 and (27), for every $x > 0$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\mathbb{P}(p(\mathbf{f}_n) \geq x)}{\mathbb{P}_1(Z_n > 0)} &= \liminf_{n \rightarrow \infty} \frac{\gamma^n n^{-3/2}}{\mathbb{P}_1(Z_n > 0)} \frac{\mathbb{P}(p(\mathbf{f}_n) \geq x)}{\gamma^n n^{-3/2}} \\ &\geq c^{-1} \frac{\mathbb{P}(L_n \geq \log(\mu x))}{\gamma^n n^{-3/2}} \\ &\geq (4c)^{-1} \nu_-([\mu x, 1]). \end{aligned}$$

Using (17), Fatou's Lemma ensures that,

$$\begin{aligned} \alpha_k &\geq (4c)^{-1} \int_0^{\mu^{-1}} (1 - (1-x)^k) \nu_-(d(\mu x)) \\ &\geq (4c)^{-1} c_- \mu^\alpha \int_0^{1/k} (1 - (1-x)^k) [\log(1/x) - \log(\mu)] x^{-\alpha-1} dx. \end{aligned}$$

For all $k \geq \mu^2$ and $x \in]0, 1/k]$, $\log(1/x) \geq 2 \log(\mu)$. So for every $k \geq \mu^2$,

$$\begin{aligned}
\alpha_k &\geq 2^{-1} \int_0^{1/k} (1 - (1-x)^k) \log(1/x) x^{-\alpha-1} dx \\
&\geq 2^{-1} k \inf_{x \in]0, 1/k]} \left\{ \frac{1 - (1-x)^k}{kx} \right\} \int_0^{1/k} \log(1/x) x^{-\alpha} dx \\
&\geq 2^{-1} k (1 - (1 - 1/k)^k) \int_0^{1/k} \log(1/x) x^{-\alpha} dx \\
&\geq 2^{-1} k (1 - (1 - 1/k)^k) \log(k) \int_0^{1/k} x^{-\alpha} dx \\
&\geq 2^{-1} \log(k) k^\alpha / (1 - \alpha).
\end{aligned}$$

This completes the proof. \square

4.2 Proofs of Section 3.2

Proof of Proposition 2. The first part (i.e. the (SS+IS) case) follows from

$$\mathbb{P}_k(\exists i \neq j, 1 \leq i, j \leq k, Z_n^{(i)} > 0, Z_n^{(j)} > 0 \mid Z_n > 0) \leq \binom{k}{2} \frac{\mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0)}{\mathbb{P}_k(Z_n > 0)},$$

asymptotics given by Proposition 1 (i-ii-iii) and equations (2) and (3).

The second part (i.e. the (WS) case) is directly derived from Proposition 1 (iii) and (4). \square

Proof of Theorem 2. Denote by $N(\mathbf{g}_n)$ the number of initial particles which survive until generation n where successive reproduction laws are given by \mathbf{g}_n (i.e. conditionally on $\mathbf{f}_n = \mathbf{g}_n$). Then, for all $1 \leq l \leq k$,

$$\begin{aligned}
\mathbb{P}_k(N_n = l) &= \int_{\mathcal{F}^n} \mathbb{P}(\mathbf{f}_n \in d\mathbf{g}_n) \mathbb{P}_k(N(\mathbf{g}_n) = l) \\
&= \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in dx) \binom{k}{l} x^l (1-x)^{k-l}.
\end{aligned}$$

Note that $x \in [0, 1] \mapsto x^l (1-x)^{k-l}$ is positive, increases on $[0, l/k]$ and decreases on $[l/k, 1]$.

First, we prove the upper bound. Recalling Lemma 2,

$$p(\mathbf{f}_n) \leq \exp(L_n),$$

we get

$$\begin{aligned}
& \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in dx) x^l (1-x)^{k-l} \\
&= \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in dx, \exp(L_n) \leq l/k) x^l (1-x)^{k-l} + \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in dx, \exp(L_n) > l/k) x^l (1-x)^{k-l} \\
&\leq \int_0^1 \mathbb{P}(\exp(L_n) \in dx) x^l (1-x)^{k-l} + \mathbb{P}(\exp(L_n) \in]l/k, 1]) (l/k)^l (1-l/k)^{k-l}.
\end{aligned}$$

By (25),

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{P}(\exp(L_n) \in]l/k, 1])}{\gamma^n n^{-3/2}} \leq u(\log(k/l))(k/l)^\alpha.$$

Second, using again the variations of $x \in [0, 1] \mapsto x^l (1-x)^{k-l}$ and (26), we get

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_0^1 \frac{\mathbb{P}(\exp(L_n) \in dx)}{n^{-3/2} \gamma^n} x^l (1-x)^{k-l} \\
&\leq \int_0^{l/k} \nu_+(dx) x^l (1-x)^{k-l} + \nu_+([l/k, 1]) (l/k)^l (1-l/k)^{k-l} \\
&\leq c_+ \int_0^1 \log(1/x) x^{-\alpha-1} x^l (1-x)^{k-l} dx + c_+ (1 + \int_{l/k}^1 \log(1/x) x^{-\alpha-1} dx) (l/k)^l (1-l/k)^{k-l} \\
&\leq c_+ \int_0^1 \log(1/x) x^{-\alpha-1} x^l (1-x)^{k-l} dx + c_+ (1 + \log(k/l) \frac{(k/l)^\alpha - 1}{\alpha}) (l/k)^l (1-l/k)^{k-l}.
\end{aligned}$$

Putting the three last inequalities together and using $u(\log(k/l)) \leq C(1 + \log(k/l))$ for some $C > 0$ ensures that there exists $D > 0$ such that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \int_0^1 \frac{\mathbb{P}(p(\mathbf{f}_n) \in dx)}{n^{-3/2} \gamma^n} x^l (1-x)^{k-l} \\
&\leq c_+ \int_0^1 \log(1/x) x^{-\alpha-1} x^l (1-x)^{k-l} dx + D(1 + \log(k/l)(k/l)^\alpha) (l/k)^l (1-l/k)^{k-l}.
\end{aligned}$$

Moreover, denoting by B is the Beta function, we have

$$\begin{aligned}
& \int_0^1 \log(x) x^{-\alpha-1} x^l (1-x)^{k-l} dx \\
&= \int_0^{1/k} \log(1/x) x^{l-\alpha-1} (1-x)^{k-l} dx + \int_{1/k}^1 \log(1/x) x^{l-\alpha-1} (1-x)^{k-l} dx \\
&\leq \int_0^{1/k} \log(1/x) x^{l-\alpha-1} dx + \log(k) \int_{1/k}^1 x^{l-\alpha-1} (1-x)^{k-l} dx \\
&\leq (l-\alpha)^{-1} [\log(k) k^{\alpha-l} + (l-\alpha)^{-1} k^{\alpha-l}] + \log(k) B(l-\alpha, k-l+1),
\end{aligned}$$

by integration by parts. By Stirling formula, there exists $C > 0$, and then $C', C'' > 0$ such that for all $1 \leq l \leq k$,

$$\begin{aligned}
\binom{k}{l} k^{-\alpha} B(l-\alpha, k-l+1) &\leq C \frac{k^{k-\alpha+1/2}}{l^{l+1/2} (k-l)^{k-l+1/2}} \frac{(l-\alpha)^{l-\alpha-1/2} (k-l+1)^{k-l+1/2}}{(k-\alpha+1)^{k-\alpha+1/2}} \\
&\leq C' \frac{(l-\alpha)^{l-\alpha-1/2} (k-l+1)^{k-l+1/2}}{l^{l+1/2} (k-l)^{k-l+1/2}} \\
&\leq C'' \frac{1}{l^{1+\alpha}}, \tag{21}
\end{aligned}$$

where the last inequality comes from the fact that $(1/x + 1/2)\log(1+x)$ is bounded for $x \in [0, 1]$, so that $(k-l+1/2)\log(1+1/(k-l))$ is bounded for $1 \leq l < k$.

Then, combining the three last inequalities gives

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}_k(N_n = l)}{k^\alpha \log(k) n^{-3/2} \gamma^n} &= \limsup_{n \rightarrow \infty} \frac{\binom{k}{l}}{k^\alpha \log(k)} \int_0^1 \frac{\mathbb{P}(\exp(L_n) \in dx)}{n^{-3/2} \gamma^n} x^l (1-x)^{k-l} \\ &\leq (l-\alpha)^{-1} \left[\binom{k}{l} k^{-l} + (l-\alpha)^{-1} k^{-l} / \log(k) + C'' \frac{1}{l^{1+\alpha}} \right] \\ &\quad + D \binom{k}{l} (\log(k) k^{-\alpha} + l^{-\alpha}) (l/k)^l (1-l/k)^{k-l}. \end{aligned}$$

Again Stirling formula ensures that there exists $C''' > 0$ such that

$$\binom{k}{l} k^{-l} \leq C''' \frac{k^{k-l}}{(k-l)^{k-l} e^l l!} = C''' \frac{e^{-(k-l)\log(1-l/k)}}{e^l l!}.$$

As for every $x \in [0, 1[$, $-\log(1-x) \leq x/(1-x)$, then $-(k-l)\log(1-l/k) \leq l$. As a consequence

$$\binom{k}{l} k^{-l} \leq C''' \frac{1}{l!}. \quad (22)$$

Then, there exists $D' > 0$ such that

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{P}_k(N_n \geq l)}{k^\alpha \log(k) n^{-3/2} \gamma^n} \leq D' \left[\frac{1}{l^{1+\alpha}} + \frac{1}{l!} + \binom{k}{l} l^{-\alpha} (l/k)^l (1-l/k)^{k-l} \right].$$

Then,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}_k(N_n \geq l)}{k^\alpha \log(k) n^{-3/2} \gamma^n} &= \limsup_{n \rightarrow \infty} \sum_{l'=l}^k \frac{\mathbb{P}_k(N_n = l')}{k^\alpha \log(k) n^{-3/2} \gamma^n} \\ &= \sum_{l'=l}^k \limsup_{n \rightarrow \infty} \frac{\mathbb{P}_k(N_n = l')}{k^\alpha \log(k) n^{-3/2} \gamma^n} \\ &\leq D \sum_{l'=l}^k \left[\frac{1}{l'^{1+\alpha}} + \frac{1}{l'!} + \binom{k}{l'} l'^{-\alpha} (l'/k)^{l'} (1-l'/k)^{k-l'} \right] \\ &\leq D \left[\sum_{l'=l}^k \left[\frac{1}{l'^{1+\alpha}} + \frac{1}{l'!} \right] + l^{-\alpha} \right] \end{aligned}$$

Recalling that $\mathbb{P}_k(Z_n > 0) \sim c\alpha_k n^{-3/2} \gamma^n$, ($n \rightarrow \infty$) and $\alpha_k \geq M_- \log(k) k^\alpha$, ($k \in \mathbb{N}$) (see Theorem 1), we have

$$\limsup_{n \rightarrow \infty} \mathbb{P}_k(N_n \geq l \mid Z_n > 0) = \limsup_{n \rightarrow \infty} \frac{\mathbb{P}_k(N_n \geq l)}{c\alpha_k n^{-3/2} \gamma^n} \leq (cM_-)^{-1} D \left[\sum_{l'=l}^k \left[\frac{1}{l'^{1+\alpha}} + \frac{1}{l'!} \right] + l^{-\alpha} \right].$$

This gives the first inequality of the proposition with $A_l = (cM_-)^{-1} D \left[\sum_{l'=l}^\infty \left[\frac{1}{l'^{1+\alpha}} + \frac{1}{l'!} \right] + l^{-\alpha} \right]$.

We can prove similarly the lower bound. By Lemma 2, for every $x > 0$,

$$\mathbb{P}(p(\mathbf{f}_n) \geq x) \geq \mathbb{P}(L_n \geq \log(x\mu))/4.$$

Then, using also (2), for all $0 \leq l < k$ and $N > 0$,

$$\begin{aligned} \mathbb{P}(p(\mathbf{f}_n) \in [l/k, Nl/k]) &= \mathbb{P}(p(\mathbf{f}_n) \geq l/k) - \mathbb{P}(p(\mathbf{f}_n) \geq Nl/k) \\ &\geq \mathbb{P}(L_n \geq \log(\mu l/k))/4 - \mathbb{P}(\exp(L_n) \geq Nl/k). \end{aligned}$$

By (25), we get

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{P}(p(\mathbf{f}_n) \in [l/k, Nl/k])}{n^{-3/2}\gamma^n} \geq (k/l)^\alpha [\mu^{-\alpha} u(\log(k) - \log(\mu l))/4 - N^{-\alpha} u(\log(k) - \log(Nl))].$$

Then, as u is linearly growing, we can fix $N \geq 1$ so that there exists $C > 0$ such that

$$\liminf_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{\mathbb{P}(p(\mathbf{f}_n) \in [l/k, Nl/k])}{k^\alpha \log(k) n^{-3/2}\gamma^n} \geq l^{-\alpha} C. \quad (23)$$

Using that

$$\mathbb{P}_k(N_n = l) = \int_0^1 \mathbb{P}(p(\mathbf{f}_n) \in dx) \binom{k}{l} x^l (1-x)^{k-l},$$

and $x \rightarrow x^l (1-x)^{k-l}$ decreases on $[l/k, 1]$, we have, for every $k \geq Nl$,

$$\mathbb{P}_k(N_n = l) \geq \mathbb{P}(p(\mathbf{f}_n) \in [l/k, Nl/k]) \binom{k}{l} (Nl/k)^l (1 - Nl/k)^{k-l}.$$

Then (23) and $\lim_{k \rightarrow \infty} \binom{k}{l} (Nl/k)^l (1 - Nl/k)^{k-l} > 0$ ensures that

$$\liminf_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{\mathbb{P}_k(N_n = l)}{k^\alpha \log(k) n^{-3/2}\gamma^n} > 0.$$

Use $\mathbb{P}_k(Z_n > 0) \sim c\alpha_k n^{-3/2}\gamma^n$ and the upperbound of α_k given in Theorem 1 to conclude. \square

4.3 Proofs of Section 3.3

Proof of Theorem 3. In the (WS+IS) case, recall that (see Section 3.1),

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0) = \mathbb{E}(p(\mathbf{f}_n)^2).$$

Thus, for every $\epsilon > 0$,

$$\mathbb{P}_k(Z_n^{(1)} > 0, Z_n^{(2)} > 0 \mid Z_n > 0) \geq \epsilon^2 \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0).$$

By Proposition 2, we get

$$\mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) \xrightarrow{n \rightarrow \infty} 0.$$

In the (WS) case, by (16), for every $\epsilon \in]0, 1[$:

$$\mathbb{P}_k(Z_n > 0) \geq \int_0^\epsilon \mathbb{P}(p(\mathbf{f}_n) \in dx)(1 - (1 - x)^k).$$

Moreover

$$\begin{aligned} & \left| \int_0^\epsilon \mathbb{P}(p(\mathbf{f}_n) \in dx)(1 - (1 - x)^k) - \int_0^\epsilon \mathbb{P}(p(\mathbf{f}_n) \in dx)kx \right| \\ & \leq k \sup_{x \in [0, \epsilon[} \left\{ \left| \frac{1 - (1 - x)^k}{kx} - 1 \right| \right\} \int_0^\epsilon \mathbb{P}(p(\mathbf{f}_n) \in dx)x \\ & \leq k \sup_{x \in [0, \epsilon[} \left\{ \left| \frac{1 - (1 - x)^k}{kx} - 1 \right| \right\} \mathbb{P}_1(Z_n > 0). \end{aligned}$$

Putting these two inequalities together yields

$$\mathbb{P}_k(Z_n > 0) \geq k \int_0^\epsilon \mathbb{P}(p(\mathbf{f}_n) \in dx)x - k \sup_{x \in [0, \epsilon[} \left\{ \left| \frac{1 - (1 - x)^k}{kx} - 1 \right| \right\} \mathbb{P}_1(Z_n > 0).$$

Then

$$\begin{aligned} \mathbb{P}_1(p(\mathbf{f}_n) \in [0, \epsilon[), Z_n > 0) &= \int_0^\epsilon \mathbb{P}(p(\mathbf{f}_n) \in dx)x \\ &\leq \mathbb{P}_k(Z_n > 0)/k + \sup_{x \in [0, \epsilon[} \left\{ \left| \frac{1 - (1 - x)^k}{kx} - 1 \right| \right\} \mathbb{P}_1(Z_n > 0). \end{aligned}$$

Dividing by $\mathbb{P}_1(Z_n > 0)$ and letting $n \rightarrow \infty$ ensure that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}_1(p(\mathbf{f}_n) \in [0, \epsilon[) \mid Z_n > 0) &\leq \limsup_{n \rightarrow \infty} \frac{\mathbb{P}_k(Z_n > 0)}{k\mathbb{P}_1(Z_n > 0)} + \sup_{x \in [0, \epsilon[} \left\{ \left| \frac{1 - (1 - x)^k}{kx} - 1 \right| \right\} \\ &\leq \frac{\alpha_k}{k} + \sup_{x \in [0, \epsilon[} \left\{ \left| \frac{1 - (1 - x)^k}{kx} - 1 \right| \right\}. \end{aligned}$$

Finally recall Theorem 1 and use

$$\alpha_k/k \xrightarrow{k \rightarrow \infty} 0, \quad \forall k \in \mathbb{N}^*, \quad \sup_{x \in [0, \epsilon[} \left\{ \left| \frac{1 - (1 - x)^k}{kx} - 1 \right| \right\} \xrightarrow{\epsilon \rightarrow 0} 0,$$

to get $\lim_{\epsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} \mathbb{P}_k(p(\mathbf{f}_n) \leq \epsilon \mid Z_n > 0) = 0$. \square

Proof of Proposition 3. Recall that for every $\mathbf{g}_n \in \mathcal{F}^n$, $\mathbb{P}_k(Z_{\mathbf{g}_n} > 0) = 1 - (1 - p(\mathbf{g}_n))^k$. Thus,

$$\begin{aligned} \mathbb{P}_k(p(\mathbf{f}_n) \in dx \mid Z_n > 0) &= \frac{\mathbb{P}(p(\mathbf{f}_n) \in dx)(1 - (1 - x)^k)}{\mathbb{P}_k(Z_n > 0)} \\ &= \mathbb{P}_1(p(\mathbf{f}_n) \in dx \mid Z_n > 0) \frac{\mathbb{P}_1(Z_n > 0)}{\mathbb{P}_k(Z_n > 0)} \frac{(1 - (1 - x)^k)}{x}. \end{aligned}$$

Then, for every $\epsilon > 0$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}_k(p(\mathbf{f}_n) \geq \epsilon \mid Z_n > 0) &= \frac{1}{\alpha_k} \limsup_{n \rightarrow \infty} \int_{\epsilon}^1 \mathbb{P}_1(p(\mathbf{f}_n) \in dx \mid Z_n > 0) \frac{(1 - (1 - x)^k)}{x} \\ &\leq \frac{1}{\epsilon \alpha_k}, \end{aligned}$$

and the left hand part tends to zero as k tends to infinity by Theorem 1. This ends up the proof. \square

4.4 Proofs of section 3.4

To prove Theorem 4, we first prove that the probability generating function G_k of the quasistationary distributions Υ_k verify the same functional equation. And we prove that in the (SS+IS) case, the quasistationary distributions do not depend on k . Then we prove a lemma which ensures the uniqueness of the solution of this functional equation in the (SS) case.

Lemma 3. *In the subcritical case, the generating function G_k of Υ_k verifies*

$$\mathbb{E}(G_k(f(s))) = \gamma G_k(s) + 1 - \gamma, \quad G_k(0) = 0.$$

In the (IS+SS) case, for every $k \geq 1$, $\Upsilon_k = \Upsilon_1$. In the (SS) case, $G'_1(1) < \infty$.

Proof. Let f_0 be distributed as f and independent of $(Z_n)_{n \in \mathbb{N}}$. For every $n \in \mathbb{N}$,

$$\begin{aligned} 1 - \mathbb{E}_k(s^{Z_{n+1}} \mid Z_{n+1} > 0) &= \frac{\mathbb{E}_k(1 - s^{Z_{n+1}})}{\mathbb{P}_k(Z_{n+1} > 0)} \\ &= \frac{1}{\mathbb{P}_k(Z_{n+1} > 0)} \sum_{i=1}^{\infty} \mathbb{P}_k(Z_n = i) \mathbb{E}_k(1 - s^{Z_{n+1}} \mid Z_n = i) \\ &= \frac{\mathbb{P}_k(Z_n > 0)}{\mathbb{P}_k(Z_{n+1} > 0)} \sum_{i=1}^{\infty} \mathbb{P}_k(Z_n = i \mid Z_n > 0) \mathbb{E}(1 - f_0(s)^i) \\ &= \frac{\mathbb{P}_k(Z_n > 0)}{\mathbb{P}_k(Z_{n+1} > 0)} \mathbb{E}_k(1 - f_0^{Z_n}(s) \mid Z_n > 0). \end{aligned}$$

Then letting n tend to infinity and using asymptotics given in Preliminaries gives

$$1 - G_k(s) = \gamma^{-1} \mathbb{E}(1 - G_k(f_0(s))),$$

where $\gamma = \mathbb{E}(f'(1))$ in the (SS+IS) case. This gives the equation of the lemma.

In the (SS) case, the fact that $G'_k(1) < \infty$ is proved in [12] for $k = 1$. The proof can be generalized to $k \geq 1$. And we can then use the uniqueness of the solution of the functional equation given below to prove that for every $k \geq 1$, $G_k = G_1$.

But in the (SS+IS) case, we can also directly prove uniqueness of all quasistationary distributions. Indeed, for every $i \geq 1$,

$$\mathbb{P}_2(Z_n = i) = \mathbb{P}(Z_n^{(1)} = i, Z_n^{(2)} = 0) + \mathbb{P}(Z_n^{(1)} = 0, Z_n^{(2)} = i) + \mathbb{P}_2(Z_n = i, Z_n^{(1)} > 0, Z_n^{(2)} > 0).$$

Moreover $|\mathbb{P}(Z_n^{(1)} = i, Z_n^{(2)} = 0) - P_1(Z_n = i)| \leq \mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0)$, then

$$|\mathbb{P}_2(Z_n = i) - 2\mathbb{P}_1(Z_n = i)| \leq 3\mathbb{P}(Z_n^{(1)} > 0, Z_n^{(2)} > 0).$$

Thus, using Proposition 2,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}_2(Z_n = i)}{\mathbb{P}_2(Z_n > 0)} = \lim_{n \rightarrow \infty} \frac{2\mathbb{P}_1(Z_n = i)}{\mathbb{P}_2(Z_n > 0)}.$$

As $\alpha_2 = \lim_{n \rightarrow \infty} \mathbb{P}_2(Z_n > 0)/\mathbb{P}_1(Z_n > 0) = 2$, we have

$$\begin{aligned} \mathbb{P}(\Upsilon_2 = i) &= \lim_{n \rightarrow \infty} \mathbb{P}_2(Z_n = i \mid Z_n > 0) \\ &= \lim_{n \rightarrow \infty} \frac{2\mathbb{P}_1(Z_n = i \mid Z_n > 0)\mathbb{P}_1(Z_n > 0)}{\mathbb{P}_2(Z_n > 0)} \\ &= \mathbb{P}(\Upsilon_1 = i). \end{aligned}$$

Then $\Upsilon_1 \stackrel{d}{=} \Upsilon_2$ and the same argument ensures that for every $k \geq 1$, $\Upsilon_k = \Upsilon_1$. \square

To prove the uniqueness of the functional equation in the (SS) case, we need the following result.

Lemma 4. *If $H : [0, 1] \rightarrow \mathbb{R}$ is a power series continuous on $[0, 1]$, $H(1) = 0$ and*

$$H(s) = \frac{\mathbb{E}(H(f(s))f'(s))}{\mathbb{E}(f'(1))}, \quad (0 \leq s \leq 1), \quad (24)$$

then $H \equiv 0$.

Proof. FIRST CASE : There exists $s_0 \in [0, 1[$ such that $\mathbb{E}(f'(s_0)) = \mathbb{E}(f'(1))$.

The monotonicity of f' implies

$$f'(s_0) = f'(1) \quad \text{a.s.},$$

and f' is a.s. constant on $[s_0, 1]$. As it is a power series, f' is a.s. constant.

Thus

$$f(s) = f'(1)s + (1 - f'(1)) \quad (0 \leq s \leq 1), \quad f'(1) \leq 1 \quad \text{a.s.}$$

Moreover, let $|H(\alpha)| = \sup\{|H(s)|, s \in [0, 1]\}$ with $\alpha \in [0, 1[$, and note that

$$\mathbb{E}(f'(1)(H(\alpha) - H(f(\alpha)))) = 0.$$

Thus $H(f(\alpha)) = H(\alpha)$ a.s. and by induction, recalling that $F_n = f_0 \circ f_1 \cdots \circ f_{n-1}$, we have

$$H(F_n(\alpha)) = H(\alpha) \quad \text{a.s.}$$

The orbit of $(F_n(\alpha))_{n \in \mathbb{N}}$ has a point of accumulation at 1, since $\alpha < 1$ and Z_n is subcritical. As H is a power series, then H is constant and equals to zero since $H(1) = 0$.

SECOND CASE : For every $s_0 \in [0, 1[$, $\mathbb{E}(f'(s_0)) < \mathbb{E}(f'(1))$.
If $H \neq 0$ then there exists $\alpha \in [0, 1[$ such that

$$\sup\{|H(s)| : s \in [0, \alpha]\} > 0$$

Let $\alpha_n \in [0, 1[$ such that $\alpha_n \xrightarrow{n \rightarrow \infty} 1$. Then, for every $n \in \mathbb{N}$, there exists $\beta_n \in [0, \alpha_n]$ such that :

$$\begin{aligned} \sup\{|H(s)| : s \in [0, \alpha_n]\} &= |H(\beta_n)| \\ &\leq \frac{\mathbb{E}(f'(\beta_n))}{\mathbb{E}(f'(1))} \sup\{|H(s)|, 0 \leq s \leq 1\} \\ &< \sup\{|H(s)|, 0 \leq s \leq 1\}, \end{aligned}$$

since $\sup\{|H(s)|, 0 \leq s \leq 1\} > 0$ and $\mathbb{E}(f'(\beta_n)) < \mathbb{E}(f'(1))$. As $I \cap J = \emptyset$, $\sup I < \sup(I \cup J) \Rightarrow \sup I < \sup J$, we get

$$\sup\{|H(s)| : s \in [0, \alpha_n]\} < \sup\{|H(s)| : s \in]\alpha_n, 1]\}.$$

And $H(s) \xrightarrow{s \rightarrow 1} 0$ leads to a contradiction letting $n \rightarrow \infty$. So $H = 0$. \square

We can now easily prove the uniqueness in the (SS) case in Theorem 4.

Lemma 5. *There exists at most one probability generating function G satisfying*

$$\mathbb{E}(G(f(s))) = \mathbb{E}(f'(1))G(s) + 1 - \mathbb{E}(f'(1)) \quad (0 \leq s \leq 1), \quad G(0) = 0, \quad G'(1) < \infty.$$

Proof. Assume that G_1 and G_2 are two probability generating functions which verify the equation above. By differentiation, G'_1 and G'_2 satisfy

$$\mathbb{E}(G'(f(s))f'(s)) = \mathbb{E}(f'(1))G'(s).$$

Then $H := G'_2(1)G'_1 - G'_1(1)G'_2$ verifies the conditions of Lemma 4. As a consequence,

$$G'_2(1)G'_1 = G'_1(1)G'_2.$$

And $G_1(0) = G_2(0) = 0$, $G_2(1) = G_1(1) = 1$ ensure that $G_1 = G_2$, which give the uniqueness. \square

Finally, we prove that if $Z_1 \in \{0, 1, N\}$ for some $N \geq 1$, then $\Upsilon_1 \stackrel{d}{=} \Upsilon_N$.

Proof. For every $s \in [0, 1]$, we have

$$\begin{aligned} \mathbb{E}_1(s^{Z_{n+1}} | Z_{n+1} > 0) &= \frac{\mathbb{E}_1(\mathbb{1}(Z_1 = 1, Z_{n+1} > 0)s^{Z_{n+1}} + \mathbb{1}(Z_1 = N, Z_{n+1} > 0)s^{Z_{n+1}})}{\mathbb{P}_1(Z_{n+1} > 0)} \\ &= \frac{\mathbb{P}_1(Z_1 = 1)\mathbb{P}_1(Z_n > 0)}{\mathbb{P}_1(Z_{n+1} > 0)} \mathbb{E}_1(s^{Z_n} | Z_n > 0) \\ &+ \frac{\mathbb{P}_1(Z_1 = N)\mathbb{P}_N(Z_n > 0)}{\mathbb{P}_1(Z_{n+1} > 0)} \mathbb{E}_N(s^{Z_n} | Z_n > 0). \end{aligned}$$

For every $k \in \mathbb{N}$, letting $n \rightarrow \infty$ using (1) yields

$$\mathbb{E}(s^{\Upsilon_1}) = \frac{\mathbb{P}_1(Z_1 = 1)}{\gamma} \mathbb{E}(s^{\Upsilon_1}) + \frac{\mathbb{P}_1(Z_1 = N)\alpha_N}{\gamma} \mathbb{E}(s^{\Upsilon_N}),$$

which proves $\Upsilon_1 \stackrel{d}{=} \Upsilon_N$. □

4.5 Proof of Section 3.5

Proof of Proposition 4. First, we have

$$\mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n | Z_{n+p} > 0) = \mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n) \frac{\mathbb{P}_{l_n}(Z_p > 0)}{\mathbb{P}_k(Z_{n+p} > 0)}.$$

Then, using (2), (3), (4), we get

$$\lim_{p \rightarrow \infty} \mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n | Z_{n+p} > 0) = \gamma^{-n} \frac{\alpha_{l_n}}{\alpha_k} \mathbb{P}_k(Z_1 = l_1, \dots, Z_n = l_n).$$

and recall $\alpha_l = l$ in the (SS+IS) case to get the distribution of $(Y_n)_{n \in \mathbb{N}}$.

To get the limit distribution of $(Y_n)_{n \in \mathbb{N}}$, note that, for every $l \in \mathbb{N}^*$,

$$\mathbb{P}_k(Y_n = l) = \gamma^{-n} \frac{\alpha_l}{\alpha_k} \mathbb{P}_k(Z_n = l) = \gamma^{-n} \mathbb{P}_k(Z_n > 0) \frac{\alpha_l}{\alpha_k} \mathbb{P}_k(Z_n = l | Z_n > 0).$$

Use respectively (2) and (3) to get the limit in distribution in the (SS) case and the (IS).

Finally, in the (WS) case, by (4), there exists $C > 0$ such that

$$\begin{aligned} \mathbb{P}_k(Y_n \leq l) &\leq C n^{-3/2} \frac{\alpha_l}{\alpha_k} \mathbb{P}_k(Z_n \leq l | Z_n > 0) \\ &\leq C n^{-3/2} \frac{\alpha_l}{\alpha_k}. \end{aligned}$$

Then Borel-Cantelli Lemma ensures that Y_n tends a.s. to infinity as $n \rightarrow \infty$. □

Proof of (12). To prove the convergence and the equality, note that

$$\begin{aligned} \mathbb{P}_k(\mathbf{f}_p \in \mathbf{d}_{\mathbf{g}_p} | Z_{n+p} > 0) &= \frac{\mathbb{P}(\mathbf{f}_p \in \mathbf{d}_{\mathbf{g}_p}) \mathbb{E}_k(\mathbb{P}_{Z_{\mathbf{g}_p}}(Z_n > 0))}{\mathbb{P}_k(Z_{n+p} > 0)} \\ &= \frac{\mathbb{P}_1(Z_n > 0)}{\mathbb{P}_k(Z_{n+p} > 0)} \sum_{l=1}^{\infty} \mathbb{P}_k(Z_{\mathbf{g}_p} = l) \frac{\mathbb{P}_l(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)}. \end{aligned}$$

Asymptotics given in Introduction ensure that

$$\frac{\mathbb{P}_1(Z_n > 0)}{\mathbb{P}_k(Z_{n+p} > 0)} \xrightarrow{n \rightarrow \infty} \frac{1}{\gamma^p \alpha_k},$$

and using the bounded convergence Theorem with

$$\frac{\mathbb{P}_l(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)} \xrightarrow{n \rightarrow \infty} \alpha_l, \quad \frac{\mathbb{P}_l(Z_n > 0)}{\mathbb{P}_1(Z_n > 0)} \leq l, \quad \mathbb{E}(Z_{\mathbf{g}_p}) < \infty.$$

ensures that

$$\lim_{n \rightarrow \infty} \mathbb{P}_k(\mathbf{f}_p \in \mathbf{d}_{\mathbf{g}_p} | Z_{n+p} > 0) = \gamma^{-p} \mathbb{P}(\mathbf{f}_p \in \mathbf{d}_{\mathbf{g}_p}) \sum_{l=1}^{\infty} \mathbb{P}_k(Z_{\mathbf{g}_p} = l) \frac{\alpha_l}{\alpha_k}.$$

This completes the proof. \square

5 Appendix : Random walk with negative drift

We study here the random walk $(S_n)_{n \in \mathbb{N}}$ with negative drift. Indeed, in the linear fractional case, the survival probability is a functional of the random walk obtained by summing the successive means of environments (see (6)). In the general case, the random walk appears in the lowerbound of the survival probability (see (15)). More precisely, we need to control the successive values of the random walk with negative drift conditioned to stay above $-x < 0$.

More specifically, let $(X_i)_{i \in \mathbb{N}}$ iid random variables distributed as X with

$$\mathbb{E}(X) < 0.$$

We assume that for every $z \in [0, 1]$, $\mathbb{E}(\exp(zX)) < \infty$ and $\mathbb{E}(X \exp(\alpha X)) = 0$ for some $0 < \alpha < 1$. Set $\gamma := \mathbb{E}(\exp(\alpha X))$,

$$S_n := \sum_{i=0}^{n-1} X_i, \quad (S_0 = 0),$$

and for all $n \in \mathbb{N}$, $k \in \mathbb{N}$,

$$L_n = \min\{S_i, 0 \leq i \leq n\}.$$

Its asymptotic is given in Lemma 4.1 in [12] or Lemma 7 in [16]. There exists a linearly increasing positive function u such that, as $n \rightarrow \infty$

$$\mathbb{P}(L_n \geq -x) \sim e^{\alpha x} u(x) n^{-3/2} \gamma^n, \quad (25)$$

for $x \geq 0$ if the distribution X is non-lattice, and for $x \in \lambda\mathbb{Z}$ if the distribution of X is supported by a centered lattice $\lambda\mathbb{Z}$.

Moreover for each $\theta > \alpha$, there exists $c_\theta > 0$ such that

$$\mathbb{P}(L_n \geq -x) \leq c_\theta e^{\theta x} n^{-3/2} \gamma^n, \quad (x \geq 0, n \in \mathbb{N}). \quad (26)$$

Finally, using (25) and the fact that u grows linearly, there exist $c_-, c_+ > 0$ such that the two following positive measures on $[0, 1]$,

$$\nu_-(dx) = c_- \log(1/x) x^{-\alpha-1} dx, \quad \nu_+(dx) = c_+ (\delta_1(dx) + \log(1/x) x^{-\alpha-1} dx),$$

verify for every $x \in]0, 1]$

$$\nu_-([x, 1]) \leq \lim_{n \rightarrow \infty} \frac{\mathbb{P}(e^{L_n} \geq x)}{n^{-3/2}\gamma^n} \leq \nu_+([x, 1]). \quad (27)$$

We need to control the successive values of the random walk conditioned to stay above $-x$ ($x \geq 0$). Under integrability conditions, it is known that the process $(S_{[nt]}/n^{1/2} | L_n \geq 0)$ converges weakly to Brownian meander as $n \rightarrow \infty$ (see [17]). Moreover Durrett [10] has proved that if there exists $q > 2$ such that $P\{X_1 > x\} \sim x^{-q}L(x)$ as $x \rightarrow \infty$, where L is slowly varying, then $(S_{[nt]}/n | L_n \geq 0)$ converges weakly to a non degenerate limit with a single jump.

We prove here that the random walk conditioned to stay above $-x$ ($x \geq 0$) spends very few time close to its minimum, by giving an upperbound of the number of visits of a level of the random walk reflected on its minimum. To be more specific, define

$$N_n(k) = \text{card}\{i \in \mathbb{N}, i \leq n, k \leq S_i - L_n < k + 1\}.$$

Lemma 6. *For every $\theta > \alpha$, there exists $d > 0$ such that*

$$\limsup_{n \rightarrow \infty} \mathbb{P}(N_n(k) \geq l | L_n \geq -x) \leq de^{\theta k} / \sqrt{l}, \quad (k, l \in \mathbb{N}, x \geq 0).$$

Moreover for all $\theta > \alpha$ and $x \geq 0$, there exists $C > 0$ such that

$$\mathbb{P}(N_n(k) \geq l | L_n \geq -x) \leq Ce^{\theta k} / \sqrt{l}, \quad (k, n, l \in \mathbb{N}). \quad (28)$$

Moreover, we will use the following consequence of the preceding lemma.

Corollary 1. *If $\alpha < 1/2$, there exists $\beta > 0$ such that for all $x \geq 0$ and $n \in \mathbb{N}$,*

$$\mathbb{P}\left(\sum_{i=0}^n \exp(L_n - S_i) \leq \beta | L_n \geq -x\right) \geq 1/4.$$

For the sake of simplicity, we assume that $X \in \mathbb{Z}$ a.s. for the proof of Lemma 6. Thus

$$\forall k, n \in \mathbb{N}^2, \quad N_n(k) = \text{card}\{i \in \mathbb{N}, i \leq n, S_i - L_n = k\},$$

and we denote by $(T_j : 1 \leq j \leq N_n(k))$ the successive times before n when $(S_i - L_n)_{i \in \mathbb{N}}$ visits k . That is

$$T_1 = \inf\{0 \leq i \leq n : S_i - L_n = k\}, \quad T_{j+1} = \inf\{T_j < i \leq n : S_i - L_n = k\}.$$

First, cutting the path of the random walk between two of these passage times enables us to prove the following result.

Lemma 7. *If $X \in \mathbb{Z}$ a.s., then for all n, k, l, i and $0 \leq h \leq n$, we have*

$$\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h) \leq (k+1)\mathbb{P}(L_{n-h} \geq -k)\mathbb{P}(L_h \geq -i),$$

and

$$\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_1 + n - T_l = h) \leq (k+1)\mathbb{P}(L_{n-h} \geq -k)\mathbb{P}(L_h \geq -i).$$

Proof. We introduce M_n the first reaching time of the minimum L_n before time n and $R_n(l)$ the last passage time of l before time n

$$M_n = \inf\{j \in [1, n] : S_j = L_n\}, \quad R_n(l) := \sup\{j \in [1, n] : S_j = l\}.$$

First, we consider the case where $M_n \in [0, T_l] \cup [T_{N_n(k)}, n]$ and split the path of the random walk between times T_l and $T_{N_n(k)}$. For all $j \leq 0$, $k \geq 0$ and $0 \leq n_1 < n_2 \leq n$, introduce then

$$\begin{aligned} A(j, n_1, n_2) &= \{L_n = j, N_n(k) \geq 2l, T_l = n_1, T_{N_n(k)} = n_2, M_n \in [0, n_1] \cup [n_2, n]\}, \\ B(j, n_1, n_2) &= \{\forall m \in [1, n_1] : S_m \geq j, S_{n_1} = S_{n_2} = j + k, \\ &\quad \forall m \in [n_2 + 1, n] : S_m \geq j, S_m \neq j + k, \exists a \in [0, n_1] \cup [n_2, n], S_a = j\}, \\ C(j, n_1, n_2) &= \{\forall m \in [n_1, n_2] : S_m \geq j, S_{n_1} = S_{n_2} = j + k\}. \end{aligned}$$

Note that conditionally on $D(n_1, n_2) := \{S_{n_1} = S_{n_2} = j + k\}$, $B(j, n_1, n_2)$ and $C(j, n_1, n_2)$ are independent,

$$\mathbb{P}(C(j, n_1, n_2) \mid S_{n_1} = j + k) \leq \mathbb{P}(L_{n_2 - n_1} \geq -k),$$

and

$$A(j, n_1, n_2) \subset B(j, n_1, n_2) \cap C(j, n_1, n_2).$$

Then, noting also that

$$\mathbb{P}(C(j, n_1, n_2) \mid D(n_1, n_2)) = \mathbb{P}(C(j, n_1, n_2) \mid S_{n_1} = j + k) \mathbb{P}(S_{n_1} = j + k) / \mathbb{P}(D(n_1, n_2)),$$

we have

$$\begin{aligned} \mathbb{P}(A(j, n_1, n_2)) &\leq \mathbb{P}(D(n_1, n_2)) \mathbb{P}(B(j, n_1, n_2) \mid D(n_1, n_2)) \mathbb{P}(C(j, n_1, n_2) \mid D(n_1, n_2)) \\ &= \mathbb{P}(S_{n_1} = j + k) \mathbb{P}(B(j, n_1, n_2) \mid D(n_1, n_2)) \mathbb{P}(C(j, n_1, n_2) \mid S_{n_1} = j + k) \\ &\leq \mathbb{P}(L_{n_2 - n_1} \geq -k) \mathbb{P}(S_{n_1} = j + k) \mathbb{P}(B(j, n_1, n_2) \mid D(n_1, n_2)). \end{aligned} \quad (29)$$

Moreover,

$$\begin{aligned} \{L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M_n \in [0, T_l] \cup [T_{N_k(n)}, n]\} \\ = \bigsqcup_{\substack{j \geq -i, \\ 1 \leq n_1 < n_2 \leq n, n_1 + n - n_2 = h}} A(j, n_1, n_2). \end{aligned}$$

Then, using the last two relations,

$$\begin{aligned} &\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M_n \in [0, T_l] \cup [T_{N_k(n)}, n]) \\ &= \sum_{\substack{j \geq -i, \\ 1 \leq n_1 < n_2 \leq n, n_2 - n_1 = n - h}} \mathbb{P}(A(j, n_1, n_2)) \\ &\leq \mathbb{P}(L_{n-h} \geq -k) \sum_{\substack{j \geq -i, \\ 1 \leq n_1 < n_2 \leq n, n_1 + n - n_2 = h}} \mathbb{P}(S_{n_1} = j + k) \mathbb{P}(B(j, n_1, n_2) \mid D(n_1, n_2)). \end{aligned}$$

Concatenating the path of the random walk before time n_1 and after time n_2 gives

$$\begin{aligned}
& \mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M_n \in [0, T_l] \cup [T_{N_k(n)}, n]) \\
\leq & \mathbb{P}(L_{n-h} \geq -k) \sum_{\substack{j \geq -i, \\ 1 \leq n_1 < n_2 \leq n, n_1 + n - n_2 = h}} \mathbb{P}(L_{n_1+n-n_2} = j, R_{n_1+n-n_2}(j+k) = n_1) \\
\leq & \mathbb{P}(L_{n-h} \geq -k) \sum_{j \geq -i} \mathbb{P}(L_h = j) \\
= & \mathbb{P}(L_{n-h} \geq -k) \mathbb{P}(L_h \geq -i). \tag{30}
\end{aligned}$$

Second, we consider the case where $M_n \in [T_l, T_{N_n(k)}]$ and split the path of the random walk between times T_1 and T_l ; For all $j, j' \leq 0, k \geq 0$ and $0 \leq n_1 < n_2 \leq n$, introduce then

$$\begin{aligned}
A'(j, n_1, n_2) &= \{L_n = -j, N_n(k) \geq 2l, T_l = n_1, T_{N_n(k)} = n_2, M_n \in [n_1, n_2]\}, \\
B'(j, j', n_1, n_2) &= \{\forall m \in [1, n_1] : S_m \geq j', S_{n_1} = S_{n_2} = j+k, \\
& \quad \forall m \in [n_2, n] : S_m \geq j', S_m \neq j+k, \exists a \in [0, n_1] \cup [n_2, n] : S_a = j'\}, \\
C'(j, n_1, n_2) &= \{\forall m \in [n_1, n_2] : S_m \geq j, S_{n_1} = S_{n_2} = k+j, \exists a \in [n_1, n_2] : S_a = j\}.
\end{aligned}$$

Note that conditionally on $D(n_1, n_2) = \{S_{n_1} = S_{n_2} = j+k\}$, $B'(j, j', n_1, n_2)$ and $C'(j, n_1, n_2)$ are independent,

$$A'(j, n_1, n_2) \subset \bigsqcup_{j'=j}^{j+k} B'(j, j', n_1, n_2) \cap C'(j, n_1, n_2).$$

and we get the analogue of (29),

$$\mathbb{P}(A'(j, n_1, n_2)) \leq \sum_{j'=j}^{j+k} \mathbb{P}(L_{n_2-n_1} \geq -k) \mathbb{P}(S_{n_1} = j+k) \mathbb{P}(B'(j, j', n_1, n_2) \mid D(n_1, n_2)).$$

Moreover

$$\begin{aligned}
& \{L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M \in [T_l, T_{N_k(n)}]\} \\
= & \bigsqcup_{\substack{j \geq -i, \\ 1 \leq n_1 < n_2 \leq n, n_1 + n - n_2 = h}} A'(j, n_1, n_2).
\end{aligned}$$

Then, following the proof of (30), we get

$$\begin{aligned}
& \mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h, M_n \in [T_l, T_{N_k(n)}]) \\
\leq & \mathbb{P}(L_{n-h} \geq -k) \sum_{j' \geq -i} \sum_{\substack{j \in [j'-k, j'] \\ 1 \leq n_1 < n_2 \leq n, \\ n_1 + n - n_2 = h}} \mathbb{P}(S_{n_1} = j+k) \mathbb{P}(B'(j, j', n_1, n_2) \mid D(n_1, n_2)) \\
\leq & \mathbb{P}(L_{n-h} \geq -k) \sum_{j' \geq -i} k \max_{j \in [j'-k, j']} \sum_{\substack{1 \leq n_1 < n_2 \leq n, \\ n_1 + n - n_2 = h}} \mathbb{P}(S_{n_1} = j+k) \mathbb{P}(B'(j, j', n_1, n_2) \mid D(n_1, n_2)) \\
\leq & \mathbb{P}(L_{n-h} \geq -k) \sum_{j' \geq -i} k \mathbb{P}(L_h = j') \\
\leq & k \mathbb{P}(L_{n-h} \geq -k) \mathbb{P}(L_h \geq -i). \tag{31}
\end{aligned}$$

Combining the inequalities (30) and (31), we get

$$\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h) \leq (k+1)\mathbb{P}(L_{n-h} \geq -k)\mathbb{P}(L_h \geq -i),$$

which proves the first inequality of the lemma. The second one can be proved similarly concatenating the random walk between $[0, T_1]$ and $[T_{N_n(k)}, n]$. \square

Proof of Lemma 6. Let $h \in \mathbb{N}$ such that $h \geq n/2$. The first inequality of Lemma 7 below ensures that

$$\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h) \leq (k+1)\mathbb{P}(L_h \geq -i)\mathbb{P}(L_{n-h} \geq -k).$$

Using (26),

$$\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h) \leq c_\theta(k+1)\mathbb{P}(L_h \geq -i)e^{\theta k}(n-h)^{-3/2}\gamma^{n-h}.$$

Moreover, using (25), for every $i \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n_0/2 \leq n/2 \leq h$,

$$\mathbb{P}(L_h \geq -i) \leq 2e^{i\alpha}u(i)h^{-3/2}\gamma^{-h} \leq 2.2^{3/2}e^{i\alpha}u(i)n^{-3/2}\gamma^h. \quad (32)$$

Then, writing $c'_\theta = 2.2^{3/2}.c_\theta$,

$$\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h) \leq c'_\theta e^{\alpha i}u(i)(k+1)e^{\theta k}\gamma^n n^{-3/2}(n-h)^{-3/2}. \quad (33)$$

Similarly, for every h such that $n_0/2 \leq n/2 \leq h$, the second inequality of Lemma 7 below ensures that

$$\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_1 + n - T_l = h) \leq c'_\theta e^{\alpha i}u(i)(k+1)e^{\theta k}\gamma^n n^{-3/2}(n-h)^{-3/2}. \quad (34)$$

Noting that a.s.

$$\{N_n(k) \geq 2l\} = \bigcup_{h=n/2}^{n-l} \{N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h\} \bigcup_{h=n/2}^{n-l} \{N_n(k) \geq 2l, T_1 + n - T_l = h\},$$

we can combine the last two inequalities (33) and (34), which give for every $n \geq n_0$,

$$\begin{aligned} \mathbb{P}(L_n \geq -i, N_n(k) \geq 2l) &\leq \sum_{n/2 \leq h \leq n-l} \mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_l + n - T_{N_n(k)} = h) \\ &\quad + \sum_{n/2 \leq h \leq n-l} \mathbb{P}(L_n \geq -i, N_n(k) \geq 2l, T_1 + n - T_l = h) \\ &\leq 2c'_\theta e^{\alpha i}u(i)\gamma^n n^{-3/2}(k+1)e^{\theta k} \sum_{n/2 \leq h \leq n-l} (n-h)^{-3/2} \\ &\leq 2c'_\theta e^{\alpha i}u(i)\gamma^n n^{-3/2}(k+1)e^{\theta k} \sum_{h \geq l} h^{-3/2} \\ &\leq 2.2c'_\theta e^{\alpha i}u(i)\gamma^n n^{-3/2}(k+1)e^{\theta k}/\sqrt{l}, \quad (n \geq n_0). \end{aligned}$$

Then, using again (25),

$$\limsup_{n \rightarrow \infty} \mathbb{P}(L_n \geq -i, N_n(k) \geq 2l) / \mathbb{P}(L_n \geq -i) \leq 4c'_\theta c_0^{-1} (k+1) e^{\theta k} / \sqrt{l}.$$

Using that $(k+1)e^{\theta k} = o(e^{\theta' k})$ if $\theta' > \theta$, this completes the proof of the first inequality of the lemma for $X \in \mathbb{Z}$. The general case can be proved similarly.

Note that, for every $\theta > \alpha$, when $h \geq n/2$, we can replace (32) by

$$\mathbb{P}(L_h \geq -i) \leq 2^{3/2} \cdot c_\theta e^{\theta i} n^{-3/2} \gamma^h, \quad (i, h, n \in \mathbb{N}).$$

Following the proof above ensures that there exists $c''_\theta > 0$ such for all $i, n, l \in \mathbb{N}$,

$$\mathbb{P}(L_n \geq -i, N_n(k) \geq 2l) \leq c''_\theta e^{\theta i} \gamma^n n^{-3/2} e^{\theta k} / \sqrt{l}.$$

Thus, by (25), for every $x \geq 0$, there exists $C_x > 0$ such that

$$\mathbb{P}(N_n(k) \geq l \mid L_n \geq -x) \leq 2c''_\theta C_x (k+1) e^{\theta k} / \sqrt{l}, \quad (k, n, l \in \mathbb{N}),$$

which gives the second inequality of the lemma. \square

Proof of Corollary 1. Let $\alpha < 1/2$ and $d > 0$ given by Theorem 1. Fix $\alpha < \theta < \mu/2 < 1/2$. Choose also $k_0 \in \mathbb{N}$ such that

$$d \sum_{k \geq k_0} e^{(\theta - \mu/2)k} < 1/2.$$

By (28), for every $x \geq 0$, there exists $D > 0$ such that for every $n \in \mathbb{N}$,

$$\mathbb{P}(N_n(k) \geq e^{\mu k} \mid L_n \geq -x) \leq D e^{(\theta - \mu/2)k}$$

which is summable with respect to k . Thus, by Fatou's lemma,

$$\limsup_{n \rightarrow \infty} \sum_{k \geq k_0} \mathbb{P}(N_n(k) \geq e^{\mu k} \mid L_n \geq -x) \leq \sum_{k \geq k_0} \limsup_{n \rightarrow \infty} \mathbb{P}(N_n(k) \geq e^{\mu k} \mid L_n \geq -x).$$

By Lemma 6, this gives, for every $x > 0$,

$$\limsup_{n \rightarrow \infty} \sum_{k \geq k_0} \mathbb{P}(N_n(k) \geq e^{\mu k} \mid L_n \geq -x) \leq d \sum_{k \geq k_0} e^{(\theta - \mu/2)k}.$$

Then,

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq k_0} \{N_n(k) \geq e^{\mu k}\} \mid L_n \geq -x\right) < 1/2.$$

By Lemma 6 again, fix $N \in \mathbb{N}$ such that

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{0 \leq k < k_0} \{N_n(k) \geq N\} \mid L_n \geq -x\right) \leq 1/4.$$

Then

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\bigcup_{0 \leq k < k_0} \{N_n(k) \geq N\} \bigcup_{k \geq k_0} \{N_k(k) \geq e^{\mu k}\} \mid L_n \geq -x \right) < 3/4.$$

Noting that

$$\sum_{i=0}^n \exp(L_n - S_i) \leq \sum_{k=0}^{\infty} N_n(k) e^{-k},$$

this ensures that for every $x \geq 0$,

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left(\sum_{i=0}^n \exp(L_n - S_i) \leq \beta \mid L_n \geq -x \right) > 1/4,$$

with $\beta := \sum_{0 \leq k < k_0} N e^{-k+1} + \sum_{k \geq k_0} e^{\mu k} e^{-k+1}$. This gives the result. \square

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References

- [1] V. I. Afanasyev, J. Geiger, G. Kersting, V.A. Vatutin (2005). Functional limit theorems for strongly subcritical branching processes in random environment. *Stochastic Process. Appl.* 115, no. 10, 1658–1676.
- [2] V. I. Afanasyev, J. Geiger, G. Kersting, V. A. Vatutin. Branching processes in random environment. A phase transition in the subcritical regime. Available via www.mathematik.uni-bielefeld.de/~trutnau/kersting.pdf.
- [3] A. Agresti (1975). On the extinction times of varying and random environment branching processes. *J. Appl. Prob.* 12, 39-46.
- [4] K. B. Athreya, S. Karlin (1971). On branching processes with random environments, I : Extinction probability. *Ann. Math. Stat.* 42. 1499-1520.
- [5] K. B. Athreya, S. Karlin (1971). On branching processes with random environments, II : limit theorems. *Ann. Math. Stat.* 42. 1843-1858.
- [6] K. B. Athreya, P. E. Ney (2004). *Branching processes*. Dover Publications, Inc., Mineola, NY.
- [7] V. Bansaye (2007). Proliferating parasites in dividing cells : Kimmel’s branching model revisited. To appear in *Ann. Appl. Prob.*.
- [8] P. Billingsley (1999). *Convergence of probability measures*. Second edition. Wiley Series in Probability.

- [9] F.M. Dekking (1988). On the survival probability of a branching process in a finite state iid environment. *Stochastic Processes and their Applications* 27, 151-157.
- [10] R. Durrett (1980). Conditioned Limit Theorem for Random Walks with Negative Drift. *Z. Wahrsch. Verw. Gebiete* 52, no. 3, 277-287.
- [11] J. Geiger (1999). Elementary new proofs of classical limit theorems for Galton-Watson processes. *J. Appl. Prob.* 36, 301-309.
- [12] J. Geiger, G. Kersting, V A. Vatutin (2003). Limit Theorems for subcritical branching processes in random environment. *Ann. I. H. Poincaré* 39, no. 4. 593-620.
- [13] D.R. Grey, L. Zhunwei (1993). The Asymptotic Behaviour of Extinction Probability in the Smith Wilkinson Branching Processes. *Adv. Appl. Prob.* 25, No 2, 263-289.
- [14] Y. Guivarc'h, Q. Liu (2001). Asymptotic properties of branching processes in random environment. *C.R. Acad. Sci. Paris*, t.332, Série I. 339-344.
- [15] J. Guyon (2007). Limit theorems for bifurcating Markov chains. Application to the detection of cellular aging. *Ann. Appl. Probab.* 17, No 5/6, 1538-1569.
- [16] K. Hirano (1998). Determination of the limiting coefficient for exponentials functional of random walks with positive drift. *J. Math. Sci. Univ. Tokyo* 5, 299-332.
- [17] D. L. Iglehart (1974). Iglehart, Donald L. Conditioned limit theorems for random walks. *Stochastic processes and related topics* (Proc. Summer Res. Inst. Statist. Inference for Stochastic Processes, Indiana Univ., Bloomington, Ind., 1974, Vol. 1; dedicated to Jerzy Neyman), 167-194.
- [18] M.V. Kozlov (1976). On the asymptotic behaviour of the probability of non-extinction for critical branching processes in a random environment. *Theory Probab. Appl.* 21 (4), 742-751.
- [19] R. Lyons, R. Pemantle, Y. Peres (1995). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* 23 (3), 1125-1138.
- [20] W.L. Smith, W. Wilkinson (1969). On branching processes in random environments. *Ann. Math. Stat.* 40, No 3, 814-827.
- [21] V. Vatutin (1999). Reduced subcritical Galton-Watson processes in a random environment. *Adv. in Appl. Probab.* 31 , no. 1, 88-111.
- [22] V. Vatutin (2004). A limit theorem for an intermediate subcritical branching process in a random environment. *Theory Probab. Appl.* 48 , no. 3, 481-492.