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Fast rotating condensates in an asymmetric harmonic trap

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We investigate the effect of the anisotropy of a harmonic trap on the behaviour of a fast rotating Bose-Einstein condensate. Fast rotation is reached when the rotational velocity is close to the smallest trapping frequency, thereby deconfining the condensate in the corresponding direction. A striking new feature is the non-existence of visible vortices for the ground state. The condensate can be described with the lowest Landau level set of states, but using distorted complex coordinates. We find that the coarse grained atomic density behaves like an inverted parabola with large radius in the deconfined direction, and like a fixed Gaussian in the other direction. It has no visible vortices, but invisible vortices which are needed to recover the mixed Thomas-Fermi Gaussian profile. There is a regime of small anisotropy and intermediate rotational velocity where the behaviour is similar to the isotropic case: a hexagonal Abrikosov lattice of vortices, with an inverted parabola profile.

PACS numbers:

Vortices appear in many quantum systems such as superconductors and superfluid liquid helium. Rotating atomic gaseous Bose-Einstein condensates constitute a novel many body system where vortices have been observed [1] and various aspects of macroscopic quantum physics can be studied. In a harmonically trapped condensate rotating at a frequency close to the trap frequency, interesting features have emerged, presenting a strong analogy with quantum Hall physics. In the mean field regime, the vortices form a triangular Abrikosov lattice [2] and the coarse grained density approaches an inverted parabola [5, 6, 7]. At very fast rotation, when the number of vortices becomes close to the number of atoms, the states are strongly correlated and the vortex lattice is expected to melt [3]. In the mean field regime, Ho [4] observed that the low lying states in a symmetric 2D trap are analogous to those in the lowest Landau level (LLL) for a charged particle in a uniform magnetic field. This analogy allows a simplified description of the gas by the location of the vortices: the wave function describing the condensate is a Gaussian multiplied by an analytic function of the complex variable $z = x + iy$. The zeroes of the analytic function are the location of the vortices. It is the distortion of the vortex lattice on the edges of the condensate which allows to create the inverted parabola profile [5, 6, 7, 8] for the coarse grained atomic density in the LLL.

The experimental achievement of rotating BEC involves anisotropic traps. An anisotropy of the trap can drastically change the picture in the fast rotation regime. In this case, in the rotating frame, the condensate becomes very elongated in one direction and forms

a novel quantum fluid in a narrow channel. The investigation of the vortex pattern has been performed for an infinite strip which corresponds to the situation where the rotational frequency has reached the smaller trapping frequency [10, 11], and for an elongated condensate [12, 13, 14]. As pointed out by Fetter [14], the description of the condensate can still be made in the framework of the lowest Landau level, defined by an anisotropic Gaussian, multiplied by an analytic function of $x + i\beta y$, where β is related to the anisotropy of the trap and the rotational frequency. We are going to show that the coarse grained density profile and the location of the vortices are very different from the isotropic case, and in particular, if the rotation is fast enough, there are only invisible vortices: the behaviour is an inverted parabola with large expansion in the deconfined direction, and a fixed Gaussian in the other direction. Contrary to the isotropic case, the wave function in the LLL can display an inverted parabola profile in one direction without visible vortices.

We consider a 2D gas of N atoms rotating at frequency Ω around the z axis. The gas is confined in a harmonic potential, with frequencies $\omega_x = \sqrt{1 - \nu^2}$, $\omega_y = \sqrt{1 + \nu^2}$ along the x, y axis respectively. The state of the gas is described by a macroscopic wave function ψ normalized to unity, which minimizes the Gross-Pitaevskii energy functional. In the following, we choose ω , $\hbar\omega$, and $\sqrt{\hbar/(m\omega)}$, as units of frequency, energy and length, respectively. The dimensionless coefficient $G = Na_s$ characterizes the strength of atomic interactions (here a_s is the atom scat-

tering length). The energy in the rotating frame is

$$E[\psi] = \int \left(\psi^* \left[H_{\Omega}^{(1)} \psi \right] + \frac{G}{2} |\psi|^4 \right) dx dy \quad (1)$$

where $H_{\Omega}^{(1)}$ is defined by

$$H_{\Omega}^{(1)} = -\frac{1}{2} \nabla^2 + \frac{1-\nu^2}{2} x^2 + \frac{1+\nu^2}{2} y^2 - \Omega L_z \quad (2)$$

and $L_z = i(y\partial_x - x\partial_y)$ is the angular momentum. We are going to study the fast rotation regime where Ω^2 approaches the critical velocity $\Omega_c^2 := 1 - \nu^2$ from below. Thus, we define the small parameter ε by $\varepsilon^2 = 1 - \nu^2 - \Omega^2$. The spectrum of the Hamiltonian (2) has a Landau level structure. The lowest Landau level is defined as (see [14])

$$f(x + i\beta y) e^{-\frac{\gamma}{8\beta}(x^2 + (\beta y)^2)} - i\frac{\nu^2}{2\Omega} xy, \quad f \text{ is analytic} \quad (3)$$

where γ and β are some constants related to Ω and ν given in the appendix. For such functions, $\langle H_{\Omega}^{(1)} \psi, \psi \rangle$ can be simplified (see the appendix and [14]), and in the small ε limit (with $\varepsilon \ll \nu$), we are left with the study of

$$E_{LLL}(\psi) = \int \frac{1}{2} (\varepsilon^2 x^2 + \kappa^2 y^2) |\psi|^2 + \frac{G}{2} |\psi|^4 dx dy \quad (4)$$

where $\kappa^2 \sim (\nu^2 + \varepsilon^2/2)(2 - \nu^2)/(1 - \nu^2)$. This energy only depends on the modulus of ψ . Hence, it is possible to forget the phase of ψ , and use the definition of the LLL (γ is defined in the appendix):

$$\psi(x, y) = f(x + i\beta y) e^{-\frac{\gamma}{8\beta}(x^2 + \beta^2 y^2)}, \quad f \text{ is analytic.} \quad (5)$$

The minimization of (4) without the analytic constraint provides the Thomas-Fermi profile for the coarse grain density:

$$|\psi|^2 = \rho_{TF} := \frac{2}{\pi R_x R_y} \left(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} \right), \quad (6)$$

where $R_x = \left(\frac{4G\kappa}{\pi\varepsilon^3}\right)^{1/4}$, $R_y = \left(\frac{4G\varepsilon}{\pi\kappa^3}\right)^{1/4}$. Note that in the isotropic case $\nu = 0$ (that is $\kappa = \varepsilon$), one recovers the standard circular shape, with $R_x = R_y = [4G/(\pi\varepsilon^2)]^{1/4}$. If ν is not zero, we always have that R_x is large since $\kappa \gg \varepsilon$. On the other hand, the behaviour of R_y depends on the respective values of ε and κ , that is ε and ν . Since κ is of order of ν , we find that R_y is large if $\nu \ll \varepsilon^{1/3}$ while R_y shrinks if $\nu \gg \varepsilon^{1/3}$. In the first case, which we will call the weakly anisotropic case, the profile (6) is reached in the fast rotation limit in the LLL using a vortex lattice, exactly as in the isotropic case. The only effect of the anisotropy is to change the coarse-grained profile, from an isotropic inverted parabola to an anisotropic one, as displayed in figure 1, but the lattice is still hexagonal. Figure 1 provides a typical vortex configuration, together with the corresponding density plot. It is obtained by

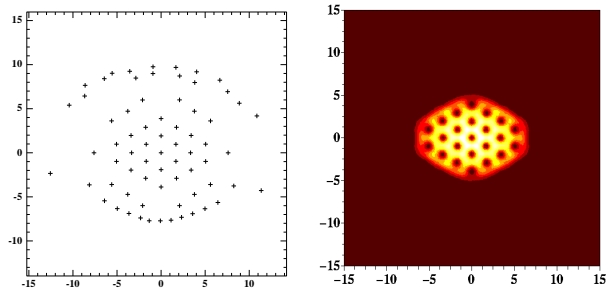


FIG. 1: An example of (a): a configuration of the zeroes (b): density plot. There are 58 vortices with 23 visible vortices. $\nu = 0.06$, $\Omega = 0.9977$, $\varepsilon^2 = 10^{-3}$, $G = 3$.

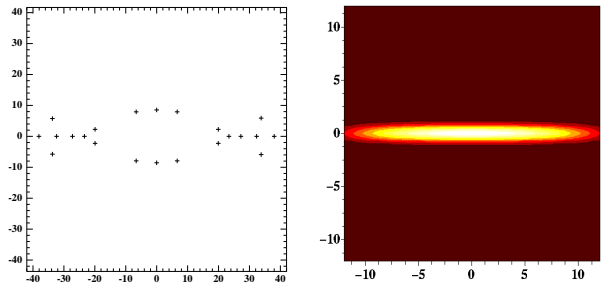


FIG. 2: An example of (a): a configuration of the zeroes (b): density plot. There are only invisible vortices (32 vortices). Here, $\nu = 0.48$, $\Omega = 0.8767$, $\varepsilon^2 = 10^{-3}$, $G = 3$. The extension in the y direction is given by (8)

minimizing the energy as a function of the location of vortices z_i with a conjugate gradient method.

However, when $\nu \gg \varepsilon^{1/3}$, that is for fast rotation, the behaviour is very different as illustrated in Figure 2: the shrinking of the condensate in the y direction is indeed not allowed in the LLL. This is due to the fact that the operator y^2 is bounded below by a positive constant in the LLL (the space (5)):

$$\forall \psi \in LLL, \quad \int_{\mathbb{R}^2} y^2 |\psi(x, y)|^2 dx dy \geq \frac{1}{\gamma\beta} \int_{\mathbb{R}^2} |\psi|^2. \quad (7)$$

The ground state of the operator y^2 is

$$u(x, y) = \left(\frac{\gamma\beta}{2\pi}\right)^{1/4} \exp\left(-\frac{\gamma\beta}{4} y^2 + i\frac{\gamma}{4} xy\right) \quad (8)$$

and every function in the LLL cannot be more localized in the y direction than (8). In this regime of parameters, the ground state of the energy is close to the Gaussian (8) in the y direction multiplied by an inverted parabola in the x direction. There is no vortex lattice. Only invisible vortices are needed to recover the inverted parabola profile in the x direction as we will explain below.

We recall that the orthogonal projection of $L^2(\mathbb{R}^2)$ onto the LLL is explicit [15]:

$$\Pi_{LLL}(\psi) = \frac{\gamma}{4\pi} \int e^{-\frac{\gamma}{8\beta}(|z|^2 - 2z\bar{z}' + |z'|^2)} \psi(x', y') dx' dy', \quad (9)$$

where $z = x + i\beta y$ and $z' = x' + i\beta y'$. If an LLL function ψ (i.e. ψ satisfies (5)) minimizes the energy (4), it is a solution of the projected Gross-Pitaevskii equation:

$$\Pi_{LLL} \left[\left(\frac{\varepsilon^2}{2} x^2 + \frac{\kappa^2}{2} y^2 + G|\psi|^2 - \mu \right) \psi \right] = 0, \quad (10)$$

where μ is the chemical potential.

The weakly anisotropic case. In the case where the anisotropy is small compared to $\Omega_c - \Omega$, namely $\nu \ll \varepsilon^{1/3}$, equation (10) can be approximated by $\Pi_{LLL} [G|\psi|^2 \psi] = \mu\psi$, which is the equation of the Abrikosov problem (see [2, 8]). A solution can be constructed using the Theta function (see [8] for the details):

$$\phi(x, y; \tau) = e^{\frac{\tau}{8\beta}(z^2 - |z|^2)} \Theta \left(\sqrt{\frac{\tau I \gamma}{4\pi\beta}} z, \tau \right), \quad (11)$$

where $z = x + i\beta y$ and $\tau = \tau_R + i\tau_I$ is the lattice parameter. The zeroes of the function ϕ lie on the lattice $\sqrt{\frac{4\pi\beta}{\tau I \gamma}} (\mathbb{Z} \oplus \mathbb{Z}\tau)$ and $|\phi|$ is periodic. The optimal lattice, that is the one minimizing $\mu(\tau) = \int |\phi|^4 / (\int |\phi|^2)^2$ is hexagonal, which corresponds to $\tau = e^{2i\pi/3}$ (the integrals are taken on one period).

As in the isotropic case [8], we can construct an approximate ground state by multiplying the solution (11) of the Abrikosov problem by a profile ρ varying at the same scale as ρ_{TF} defined in (6), that is, at the scale $\min(R_x, R_y) \sim (\varepsilon/\nu^3)^{1/4}$, which is large. Since this product is not in the LLL, we project it onto the LLL and define $v = \Pi_{LLL}(\rho(x, y)\phi(x, y; \tau))$. Estimating the energy of v yields

$$E_{LLL}(v) = \int_{\mathbb{R}^2} \left(\frac{\varepsilon^2}{2} x^2 + \frac{\kappa^2}{2} y^2 \right) \rho(x, y) + \frac{G\mu(\tau)}{2} \rho(x, y)^2 dx dy + O \left[\sqrt{\kappa\varepsilon} \left(\frac{\kappa^3}{\varepsilon} \right)^{1/2} \right].$$

This computation uses that ϕ and ρ do not vary on the same scale, hence the integrals can be decoupled. Then, minimizing with respect to ρ provides that ρ must be a Thomas-Fermi profile (6) with G changed into $\mu(\tau)G$: $\rho(x, y) = \frac{1}{\sqrt{\mu(\tau)}} \rho_{TF} \left(\frac{x}{\mu(\tau)^{1/4}}, \frac{y}{\mu(\tau)^{1/4}} \right)$. The condensate indeed expands in both directions, and a coarse-grained density profile is close to the inverted parabola.

The fast rotating case. In the case where $\varepsilon^{1/3} \ll \nu$, that is fast rotation, the limit of the projected Gross-Pitaevskii equation (10), is now:

$$\Pi_{LLL} \left[\left(\frac{\kappa^2}{2} y^2 + G|\psi|^2 \right) \psi \right] = \mu\psi. \quad (12)$$

The Θ function is no longer a solution to this equation, but a particular solution is given by the Gaussian (8). We are going to prove that in this regime, the ground state behaves like the Gaussian (8) in the y direction and an inverted parabola in the x direction, without visible vortices.

Indeed, in the LLL, we have the identity $\int (\partial_x |\psi|)^2 + (1/\beta^2)(\partial_y |\psi|)^2 = (\gamma/4\beta) \int |\psi|^2$ (see [9]). By adding and subtracting $\kappa^2/(2\gamma\beta) \int |\psi|^2$ to the energy, and using this identity, we find

$$E_{LLL}(\psi) = -\frac{\kappa^2}{2\gamma\beta} + \int \left(\frac{2\kappa^2}{\gamma^2\beta^2} (\partial_y |\psi|)^2 + \frac{\kappa^2}{2} y^2 |\psi|^2 \right) + \int \left(\frac{2\kappa^2}{\gamma^2} (\partial_x |\psi|)^2 + \frac{\varepsilon^2}{2} x^2 |\psi|^2 + \frac{G}{2} |\psi|^4 \right).$$

The ground state of $-(2/\gamma^2\beta^2)\partial_y^2 + (1/2)y^2$ is (8) and the ground energy is $1/(\gamma\beta)$. Projecting a general function of the LLL onto the space generated by (8) times a general function of x , and using that $\varepsilon^{1/3}/\nu$ is small, we find that $E_{LLL}(u) \geq (\kappa^2/(2\gamma\beta)) + E_{1D}(p(x))$ where

$$E_{1D}(p) = \int_{\mathbb{R}} \left(\frac{2\kappa^2}{\gamma^2} (p')^2 + \frac{1}{2} \varepsilon^2 x^2 p^2 + \frac{G}{4} \sqrt{\frac{\gamma\beta}{\pi}} p^4 \right) dx. \quad (13)$$

The minimizer of E_{1D} is of Thomas-Fermi type when $\varepsilon^{1/3}/\nu$ is small, namely

$$p(x) = \sqrt{\frac{3}{4R}} \left(1 - \frac{x^2}{R^2} \right)_+^{1/2}, \quad R = \left(\frac{3G}{4\varepsilon^2} \sqrt{\frac{\gamma\beta}{\pi}} \right)^{1/3} \quad (14)$$

As expected, $R \propto \varepsilon^{-2/3}$. This gives the energy estimate

$$\min E_{LLL} \approx \frac{\kappa^2}{2\gamma\beta} + \frac{3}{10} \left(\frac{3\varepsilon}{4} G \sqrt{\frac{\gamma\beta}{\pi}} \right)^{2/3}. \quad (15)$$

Let us point out that one can construct a test function in the LLL reproducing the behaviour of (8) in the y direction, and an inverted parabola in the x direction: we project a Dirac mass in the y direction times an inverted parabola in x , that is

$$v(x, y) = A \Pi_{LLL} [\delta_0(y) p(x)] = A \frac{\gamma}{4\pi} e^{-\frac{\gamma\beta}{8} y^2} \int_{\mathbb{R}} e^{-\frac{\gamma}{8\beta}((x-x')^2 - 2ix'\beta y)} p(x') dx', \quad (16)$$

where p is the function (14). The constant $A = (2\pi/\gamma\beta)^{1/4}$ is a normalization factor. The fact that p varies on a scale of order $\varepsilon^{-2/3}$ allows to expand (16) in powers of $\varepsilon^{2/3}$ [16], giving (up to a normalization constant), and with an error of order $\varepsilon^{4/3}$

$$v(x, y) = \left(\frac{\gamma\beta}{2\pi} \right)^{1/4} p(x) \exp \left(-\frac{\gamma\beta}{4} y^2 + i\frac{\gamma}{4} xy \right) + \varepsilon^{2/3} \left(\frac{\gamma\beta}{2\pi} \right)^{1/4} p'(x) i y \exp \left(-\frac{\gamma\beta}{4} y^2 + i\frac{\gamma}{4} xy \right) + O \left(\varepsilon^{4/3} \right). \quad (17)$$

Inserting this expansion in the energy, one finds that

$$E_{\text{LLL}}(v) = \frac{\kappa^2}{2\gamma\beta} + \int_{\mathbb{R}} \left(\frac{1}{2}\varepsilon^2 x^2 p(x)^2 + \frac{G}{4} \sqrt{\frac{\gamma\beta}{\pi}} p(x)^4 \right) dx \quad (18)$$

with an error of order $\varepsilon^{4/3}$. This matches our lower bound (15). Let us point out that according to (17), the wave function v has no visible vortices. This is corroborated by the numerical computation displayed in Figure 2. Nevertheless, the inverted parabola profile in the x direction is obtained in the LLL thanks to the existence of invisible vortices.

Conclusion: A striking new feature is the non-existence of visible vortices for the ground state of the energy in the fast rotation regime, that is when $\varepsilon^{1/3} \ll \nu$. Only invisible vortices are needed to recover the profile of the ground state: inverted parabola in the deconfined direction and a fixed Gaussian in the other. When the anisotropy is small compared to how close the rotational velocity is to the critical, that is $\varepsilon^{1/3} \gg \nu$, the behaviour is similar to the isotropic case with an inverted parabola profile and a triangular vortex lattice. The intermediate regime where $\varepsilon^{1/3}$ is of the same order as ν could display rows of vortices as obtained by [11] but the analysis is still open. Our investigation illustrates the changing behaviour and asymmetry effect and opens new prospects for the experiments. A better understanding of what happens when ε gets smaller than ν but of the same order would provide an explanation of the melting of vortices in the mean field regime.

Appendix

As computed in [14] on the basis of ideas of Valatin [17], the eigenvalues of the Hamiltonian $H_{\Omega}^{(1)}$ are $\pm i\mu_1, \pm i\mu_2$, where $\mu_1^2 = 1 + \Omega^2 - \sqrt{\nu^4 + 4\Omega^2}$,

$\mu_2^2 = 1 + \Omega^2 + \sqrt{\nu^4 + 4\Omega^2}$. We define $\alpha = \sqrt{\nu^4 + 4\Omega^2}$, $\beta_1 = (2\Omega\mu_1)/(\alpha - 2\Omega^2 + \nu^2)$, $\beta_2 = (2\Omega\mu_2)/(\alpha + 2\Omega^2 + \nu^2)$, $\gamma = (2\alpha)/\Omega$, $\lambda_1^2 = (\alpha - 2\Omega^2 + \nu^2)/(2\alpha)$, $\lambda_2^2 = (\alpha + 2\Omega^2 + \nu^2)/2\alpha$, $d = (\gamma\lambda_1\lambda_2)/2$, $c = (\lambda_1^2 + \lambda_2^2)/2\lambda_1\lambda_2$. Then $H_{\Omega}^{(1)} = \frac{1}{2} (a_1^\dagger a_1 + a_1 a_1^\dagger) + \frac{1}{2} (a_2^\dagger a_2 + a_2 a_2^\dagger)$ where $a_2 = \frac{\mu_2}{\sqrt{2}} (-i\lambda_1 d^{-1} \partial_x + c\lambda_1 y) + \frac{i}{\sqrt{2}} (-i\lambda_2 \partial_y - (d\lambda_1^{-1} - \lambda_2 c d) x)$, and $a_1 = \frac{\mu_1}{\sqrt{2}} (-i\lambda_2 d^{-1} \partial_y + c\lambda_2 x) + \frac{i}{\sqrt{2}} ((\lambda_1 c d - d\lambda_2^{-1}) y - i\lambda_1 \partial_x)$. We have: $[a_2, a_2^\dagger] = \mu_2$, $[a_1, a_1^\dagger] = \mu_1$, and all other commutators vanish. The LLL is defined by $a_2 \psi = 0$, that is $f(x + i\beta_2 y) e^{[-\frac{1}{8\beta_2} (\frac{2\alpha - \nu^2}{\Omega} x^2 + \frac{2\alpha + \nu^2}{\Omega} (\beta_2 y)^2)] - i\frac{\nu^2}{4\Omega} x y}$, with f analytic. It is always possible to change the analytic function $f(\xi)$ into $f(\xi) \exp(-\delta \xi^2)$ in the above definition, since $\exp(-\delta \xi^2)$ is an analytic function of ξ . Hence, for $\delta = \nu^2/(8\Omega\beta_2)$, we find the alternative definition of the LLL (3). This definition is equivalent to the one given by Fetter in [14]. However, contrary to [14], the coefficients in (3) are not singular in the limit $\varepsilon \rightarrow 0$. Indeed, in this limit, $\beta_2 \sim \sqrt{(1 - \nu^2)/(1 - \nu^2/2)}$ and $\gamma \sim (4 - 2\nu^2)/\sqrt{1 - \nu^2}$. This is due to the addition of the above-mentioned complex Gaussian in the definition of the LLL. In the LLL, we have $\langle H_{\Omega}^{(1)} \psi, \psi \rangle = \frac{1}{2} \langle (a_1^\dagger a_1 + a_1 a_1^\dagger) \psi, \psi \rangle + \frac{\mu_2}{2} \langle \psi, \psi \rangle$. We then express x and y as linear combinations of $a_1, a_2, a_1^\dagger, a_2^\dagger$ [13, 14] and get, if $\psi \in \text{LLL}$,

$$\begin{aligned} \langle H_{\Omega}^{(1)} \psi, \psi \rangle &= \frac{\mu_2}{2} - \frac{\mu_1}{4} \left(\beta_1 \beta_2 + \frac{1}{\beta_1 \beta_2} \right) \\ &+ \frac{\gamma}{4} \int \left(\mu_1 \beta_1 x^2 + \frac{\mu_1}{\beta_1} y^2 \right) |\psi|^2 dx dy \quad (19) \end{aligned}$$

which provides (4) with $\kappa^2 = \gamma\mu_1/2\beta_1$ since $\gamma\mu_1\beta_1 \sim 2\varepsilon^2$.

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