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Representation of continuous linear forms on the set of ladrags processes and the pricing of American claims under proportional costs

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Abstract

We discuss a d -dimensional version (for $\text{l\`a}d\text{l}\grave{a}g$ optional processes) of a duality result by Meyer (1976) between bounded $\text{c\`a}d\text{l}\grave{a}g$ adapted processes and random measures. We show that it allows to establish, in a very natural way, a dual representation for the set of initial endowments which allow to super-hedge a given American claim in a continuous time model with proportional transaction costs. It generalizes a previous result of Bouchard and Temam (2005) who considered a discrete time setting. It also completes the very recent work of Denis, De Vallière and Kabanov (2008) who restricted to $\text{c\`a}d\text{l}\grave{a}g$ American claims and used a completely different approach.

Key words: Randomized stopping times, American options, transaction costs.

MSC Classification (2000): 91B28, 60G42.

1 Introduction and definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t \leq T}$ satisfying the usual assumptions. Here $T < \infty$ is some fixed time horizon and we shall assume all over this paper that $\mathcal{F}_{T-} = \mathcal{F}_T$.

1.1 Model specifications

A d -dimensional market with proportional transaction costs can be described by the exchange rates between the different assets. They are modeled as an adapted càdlàg d -dimensional matrix valued process $\Pi = (\pi_t^{ij})_{i,j \leq d}$. Each entry π_t^{ij} denotes the number of units of assets i which is required to obtain one unit of asset j at time t . In this paper, we shall assume that it satisfies the following natural conditions:

- (i) $\pi_t^{ii} = 1, \pi_t^{ij} > 0$ for all $t \leq T$ \mathbb{P} - a.s.
- (ii) $\pi_t^{ij} \leq \pi_t^{ik} \pi_t^{kj}$ for all $t \leq T$ and $1 \leq i, j, k \leq d$ \mathbb{P} - a.s.

Following [3], we also assume that the technical condition $\Pi_{T-} = \Pi_T$ \mathbb{P} - a.s. is satisfied.

A position in the market at time t is described as a d -dimensional vector \widehat{V}_t whose i -th component coincides with the number of units of asset i held at time t . Such a position is *solvent* if an immediate exchange in the market allows to turn each of its components into non-negative ones. The superscript $\widehat{}$ is used to insist on the fact that we are dealing with quantities. In mathematical terms, this means that it belongs to the closed convex cone $\widehat{K}_t(\omega)$ generated by the unit vectors¹ $e_i, i \leq d$, of \mathbb{R}^d and the vectors $\pi_t^{ij}(\omega)e^i - e^j, i, j \leq d$.

Observe that the above conditions (i)-(ii) imply that $\mathbb{R}_+^d \subset \widehat{K}_t$, where the inclusion has to be understood for all $t \leq T$ \mathbb{P} - a.s.

Noting that an immediate transaction on the market changes the portfolio by a vector of quantities of the form $\xi_t(\omega) \in -\partial\widehat{K}_t(\omega)$, the boundary of $-\widehat{K}_t(\omega)$, it is thus natural to define *self-financing strategies* as vector processes \widehat{V} such that $d\widehat{V}_t(\omega)$ belongs in some sense to $-\widehat{K}_t(\omega)$, the passage from $-\partial\widehat{K}_t(\omega)$ to $-\widehat{K}_t(\omega)$ reflecting the idea that one can always “throw away” some (non-negative) quantities of assets.

Such a modelization was introduced and studied at different levels of generality in [10], [11] and [3] among others, and it is now known from the work of [16] and [3] that a good definition of self-financing wealth processes is the following:

Definition 1.1. *We say that a \mathbb{R}^d -valued làdlàg predictable process \widehat{V} is a self-financing strategy if it has \mathbb{P} - a.s. finite total variation and:*

¹ e_i is the vector of \mathbb{R}^d whose i -th component equals 1 and the others equal 0.

- (i) $\widehat{V}^c := d\widehat{V}^c/d\text{Var}(\widehat{V}^c) \in -\widehat{K} \text{ dVar}(\widehat{V}^c)\text{-a.e. } \mathbb{P} - \text{a.s.}$, where \widehat{V}^c denotes the continuous part of \widehat{V} and $\text{Var}(\widehat{V}^c)$ its total variation,
- (ii) $\Delta^+\widehat{V}_\tau := \widehat{V}_{\tau+} - \widehat{V}_\tau \in -\widehat{K}_\tau \mathbb{P} - \text{a.s.}$ for all stopping time $\tau \leq T \mathbb{P} - \text{a.s.}$,
- (iii) $\Delta\widehat{V}_\tau := \widehat{V}_\tau - \widehat{V}_{\tau-} \in -\widehat{K}_{\tau-} \mathbb{P} - \text{a.s.}$ for all predictable stopping time $\tau \leq T \mathbb{P} - \text{a.s.}$

Given $v \in \mathbb{R}^d$, we denote by $\widehat{\mathcal{V}}^v$ the set of self-financing strategies \widehat{V} such that $\widehat{V}_0 = v$.

Here and below, we use the convention $X_{T+} = X_T$ and $X_{0-} = 0$ for any *l\`a*dl\`ag process X on $[0, T]$.

In order to avoid arbitrage opportunities, we shall assume all over this paper that the following standing assumption holds.

Standing assumption: There exists at least one *c\`a*dl\`ag martingale Z such that

- (i) $Z_t \in \widehat{K}_t^*$ for all $t \leq T$, $\mathbb{P} - \text{a.s.}$
- (ii) for every $[0, T] \cup \{\infty\}$ -valued stopping times $Z_\tau \in \text{Int}(\widehat{K}_\tau^*) \mathbb{P} - \text{a.s.}$ on $\{\tau < \infty\}$ and for every predictable $[0, T] \cup \{\infty\}$ -valued stopping times $Z_{\tau-} \in \text{Int}(\widehat{K}_{\tau-}^*) \mathbb{P} - \text{a.s.}$ on $\{\tau < \infty\}$.

Here, $\widehat{K}_t^*(\omega) := \{y \in \mathbb{R}^d : xy := \sum_{i \leq d} x^i y^i \geq 0 \ \forall x \in \widehat{K}_t(\omega)\}$ is the positive polar of $\widehat{K}_t(\omega)$. We denote by \mathcal{Z}^s the set of processes satisfying the above conditions. Such elements were called *strictly consistent price processes* by [3], see also [17], because they allow to price contingent claims in a strictly consistent way, see the discussion in [3]. Note that $\widehat{K}^* \subset \mathbb{R}_+^d$ since $\mathbb{R}_+^d \subset \widehat{K}$.

As pointed out in Lemma 8 in [3], the set $\widehat{\mathcal{V}}^0$ admits the following alternative representation under the assumption $\mathcal{Z}^s \neq \emptyset$.

Proposition 1.1. *Let \widehat{V} be a \mathbb{R}^d -valued predictable process with \mathbb{P} -a.s. finite total variation such that $\widehat{V}_0 = 0$. Then, $\widehat{V} \in \widehat{\mathcal{V}}^0$ if and only if*

$$\widehat{V}_\tau - \widehat{V}_\sigma \in -\widehat{K}_{\sigma, \tau} \mathbb{P} - \text{a.s.} \text{ for all stopping times } \sigma \leq \tau \leq T, \quad (1.1)$$

with

$$\widehat{K}_{\sigma, \tau}(\omega) := \overline{\text{conv}} \left(\bigcup_{\sigma(\omega) \leq t \leq \tau(\omega)} \widehat{K}_t(\omega), 0 \right)$$

where $\overline{\text{conv}}$ denotes the closure in \mathbb{R}^d of the convex envelope.

Remark 1.1. *The technical conditions $\mathcal{F}_{T-} = \mathcal{F}_T$ and $\Pi_{T-} = \Pi_T$ are used to simplify the presentation. Note that we can always reduce to this case by considering a larger time horizon $T^* > T$ and by considering a model where $\mathcal{F}_t = \mathcal{F}_{T^*}$ and $\Pi_t = \Pi_{T^*}$ for $t \in [T, T^*]$.*

1.2 The super-hedging problem

The super-hedging problem of an European contingent claim was studied at different levels of generality by [10], [11] and [3], see also the references therein. It can be stated as follows.

Given a random variable \widehat{C}_T , we want to characterize the set of initial endowments $v \in \mathbb{R}^d$ such that $\widehat{V}_T - \widehat{C}_T \in \widehat{K}_T \mathbb{P} - \text{a.s.}$ for some $\widehat{V} \in \widehat{\mathcal{V}}^v$. This means that, up to an immediate trade, the position \widehat{V}_T can be turned into a new one such that its i -th component is greater than \widehat{C}_T^i , which should be interpreted as a number of units of asset i to be delivered at time T to the buyer of the European option.

Set $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ and let us consider the set $\widehat{\mathcal{V}}_{\mathcal{Z}^s}^v$ of elements $\widehat{V} \in \widehat{\mathcal{V}}^v$ such that, for some real $c > 0$, $\widehat{V}_T + c\mathbf{1} \in \widehat{K}_T$ and $Z_\tau \widehat{V}_\tau \geq -Z_\tau c\mathbf{1}$ for all $[0, T]$ -valued stopping times τ and $Z \in \mathcal{Z}^s$. In the case where $\widehat{C}_T + a\mathbf{1} \in \widehat{K}_T \mathbb{P} - \text{a.s.}$ for some $a > 0$, it was shown in [3] that there exists $\widehat{V} \in \widehat{\mathcal{V}}_{\mathcal{Z}^s}^v$ such that $\widehat{V}_T - \widehat{C}_T \in \widehat{K}_T \mathbb{P} - \text{a.s.}$ if and only if $\mathbb{E} \left[Z_T (\widehat{C}_T - v) \right] \leq 0$ for all $Z \in \mathcal{Z}^s$. Here and below, we use the notation xy to denote the scalar product $\sum_{i \leq d} x^i y^i$. Comparing this result to well-known results for frictionless market, see [5], we see that \mathcal{Z}^s plays a similar role as the set of equivalent local (sigma) martingale measures in markets without frictions. This generalizes a similar result obtained previously in [11] for the more natural set of strategies $\widehat{\mathcal{V}}_b^v$ made of càdlàg elements $\widehat{V} \in \widehat{\mathcal{V}}^v$ such that $\widehat{V}_t + c\mathbf{1} \in \widehat{K}_t$ for all $t \leq T \mathbb{P} - \text{a.s.}$ (in short $\widehat{V} \succeq -c\mathbf{1}$), for some real $c > 0$. However, it requires additional assumptions, in particular the continuity of Π .

The aim of this paper is to do a similar analysis for “American options”. Namely, we want to characterize the set $\widehat{\mathcal{C}}^v$ of optional *làdlàg* processes \widehat{C} such that $\widehat{V} \succeq \widehat{C}$ for some $\widehat{V} \in \widehat{\mathcal{V}}^v$. Here, the i -th component of \widehat{C} at time t should be interpreted as the number of units of asset i to be delivered to the buyer of the American option if it is exercised at a time t before the maturity time T .

The solution to such a problem is well-known in frictionless financial models. It is related to the optimal stopping of the process \widehat{C} between 0 and T , see [13] and Section 3 below. Since \mathcal{Z}^s plays the same role as the set of equivalent local (sigma) martingale measures in frictionless markets, one could expect that $\widehat{C} \in \widehat{\mathcal{C}}^v$ if and only if $\mathbb{E} \left[Z_\tau (\widehat{C}_\tau - v) \right] \leq 0$ for all $Z \in \mathcal{Z}^s$ and all stopping times $\tau \leq T \mathbb{P} - \text{a.s.}$ However, it was already shown in [4] and [2], for discrete time models, that such a dual formulation does not hold and that one has to replace the notion of stopping times by the notion of randomized stopping times. Their result is of the form: if there exists $a > 0$ such that $\widehat{C} \succeq -a\mathbf{1}$ then

$$C \in \widehat{\mathcal{C}}^v \iff \sup_{A \in \widetilde{\mathcal{D}}} \mathbb{E} \left[\int_0^T (\widehat{C}_t - v) dA_t \right] \leq 0, \quad (1.2)$$

where $\widetilde{\mathcal{D}}$ is a family of càdlàg adapted processes A with integrable total variation

such that

1. $A_{0-} = 0$
2. $\dot{A} \in \widehat{K}^* d\text{Var}(A)$ a.e. $\mathbb{P} - \text{a.s.}$
3. The optional projection \bar{A} of $(A_T - A_t)_{t \leq T}$ satisfies $\bar{A}_t \in \widehat{K}_t^*$ for all $t \leq T$ $\mathbb{P} - \text{a.s.}$
4. There is a deterministic finite non-negative measure ν on $[0, T]$ and an adapted process Z such that $A = \int_0^\cdot Z_t \nu(dt)$.

Here, \dot{A} denotes the density of A with respect to the associated total variation process $\text{Var}(A)$. In the very recent paper [7]², this relation was also proved for continuous time models.

The approach of [7] relies on the approximation of American claims by Bermudan claims. Namely, they first prove that the result holds if we only impose $\widehat{V}_t - \widehat{C}_t \in \widehat{K}_t$ on a finite number of times $t \leq T$, and then pass to the limit. This result is very nice since it provides a direct and simple extension of [2]. However, their approximation requires some regularity and they have to impose a right-continuity condition on \widehat{C} . At first glance, this restriction does not seem important. However, it does not apply to admissible self-financing portfolios of the set $\widehat{\mathcal{V}}^v$, since they are only assumed to be *l\`adl\`ag*, except when Π is continuous in which case the portfolios can be taken to be continuous, see the final discussion in [7].

2 A strong duality approach

In this paper, we use a completely different approach which relies on a direct application of duality results developed by [15], see its Theorem 27 Chapter V, and [1].

Given $\mathbb{Q} \sim \mathbb{P}$, we now denote by $\mathcal{S}^1(\mathbb{Q})$ the set of adapted *l\`adl\`ag* processes \widehat{C} such that $\|\widehat{C}\|_{\mathcal{S}^1(\mathbb{Q})} := \mathbb{E}^{\mathbb{Q}} \left[\|\widehat{C}\|_* \right] < \infty$ where $\|\widehat{C}\|_* := \sup_{t \leq T} \|\widehat{C}_t\|$ and $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d . Given $Z \in \mathcal{Z}^s$, we also define the probability measure \mathbb{Q}_Z by $d\mathbb{Q}_Z/d\mathbb{P} := (\sum_{i \leq d} Z_T^i)/c_Z$ where $c_Z := \mathbb{E} \left[\sum_{i \leq d} Z_T^i \right]$. Note that $\text{Int}(\widehat{K}^*) \subset (0, \infty)^d$ so that $\mathbb{Q}_Z \sim \mathbb{P}$. In the following, we shall also use the notation \mathcal{S}^∞ to denote the set of optional *l\`adl\`ag* processes \widehat{C} such that $\|\widehat{C}\|_*$ is essentially bounded.

Our main result relies on two key ingredients:

- 1- we first observe that the set $\widehat{\mathcal{C}}^0 \cap \mathcal{S}^1(\mathbb{Q}_Z)$ is closed in $\mathcal{S}^1(\mathbb{Q}_Z)$ for all $Z \in \mathcal{Z}^s$,
- 2- we then provide a representation of the dual of $\mathcal{S}^1(\mathbb{Q}_Z)$ in terms of random measures which can be interpreted in terms of randomized quasi stopping times.

The dual formulation is then obtained by applying an usual Hahn-Banach type argument.

²We received this paper while preparing this manuscript. We are grateful to the authors for discussions we had on the subject at the Bachelier Workshop in Métabief, 2008.

2.1 Closure property

We start with the closure property. It can be compared to the Fatou closure property used in [7] and [3], among others. The main difference is that we consider a convergence in $\mathcal{S}^1(\mathbb{Q}_Z)$ for $Z \in \mathcal{Z}^s$.

Proposition 2.1. *For all $Z \in \mathcal{Z}^s$, $\widehat{\mathcal{C}}^0 \cap \mathcal{S}^1(\mathbb{Q}_Z)$ is closed in $\mathcal{S}^1(\mathbb{Q}_Z)$. Moreover, if $a > 0$ and $(\widehat{C}^n)_{n \geq 1}$ is a sequence in $\widehat{\mathcal{C}}^0$ such that $\widehat{C}^n \succeq -a\mathbf{1}$ for all $n \geq 1$ and $\|\widehat{C}^n - \widehat{C}\|_* \rightarrow 0$ in probability for some l\`a\`a\`a\`a optional process \widehat{C} with values in \mathbb{R}^d , then $\widehat{C} \in \widehat{\mathcal{C}}^0$.*

The last assertion is an immediate consequence of Lemma 8, Lemma 12 and Proposition 14 of [3], see also the proof of their Theorem 14. The first one is proved similarly. The only difference is that their admissibility condition $\widehat{V} \in \widehat{\mathcal{V}}_{Z^s}^0$ is replaced by the fact that we restrict to strategies such that $\widehat{V} \succeq \widehat{C}$ with $\widehat{C} \in \mathcal{S}^1(\mathbb{Q}_Z)$. We only explain the main arguments. We start with an easy Lemma which essentially follows from arguments used in the proof of Lemma 8 in [3].

Lemma 2.1. *Fix $\widehat{C} \in \widehat{\mathcal{C}}^0 \cap \mathcal{S}^1(\mathbb{Q}_Z)$ for some $Z \in \mathcal{Z}^s$ and $\widehat{V} \in \widehat{\mathcal{V}}^0$ such that $\widehat{V} \succeq \widehat{C}$. Then, $Z\widehat{V}$ is a supermartingale. Moreover,*

$$\mathbb{E} \left[\int_0^T Z_s \widehat{V}_s^c d\text{Var}_s(\widehat{V}^c) + \sum_{s \leq T} Z_{s-} \Delta \widehat{V}_s + \sum_{s < T} Z_s \Delta^+ \widehat{V}_s \right] \geq \mathbb{E} [Z_T \widehat{V}_T] .$$

Proof. Since $Z_t \in \widehat{K}_t^*$ and $\widehat{V}_t - \widehat{C}_t \in \widehat{K}_t$ for all $t \leq T$ \mathbb{P} -a.s., it follows that $Z_t \widehat{V}_t \geq Z_t \widehat{C}_t$ for all $t \leq T$ \mathbb{P} -a.s. and therefore, by the martingale property of Z ,

$$Z_t \widehat{V}_t \geq \mathbb{E} [Z_T \widehat{C}_T | \mathcal{F}_t] \geq - \mathbb{E} \left[\|Z_T\| \sup_{s \in [0, T]} \|\widehat{C}_s\| | \mathcal{F}_t \right] \text{ for all } t \leq T \text{ } \mathbb{P}\text{-a.s.} \quad (2.1)$$

Since $\widehat{C} \in \mathcal{S}^1(\mathbb{Q}_Z)$, the right-hand side term is a martingale. Moreover, a direct application of the integration by parts formula yields

$$Z_t \widehat{V}_t = \int_0^t \widehat{V}_s dZ_s + \int_0^t Z_s \widehat{V}_s^c d\text{Var}_s(\widehat{V}^c) + \sum_{s \leq t} Z_{s-} \Delta \widehat{V}_s + \sum_{s < t} Z_s \Delta^+ \widehat{V}_s .$$

We now observe that the definitions of \mathcal{Z}^s and $\widehat{\mathcal{V}}^0$ imply that the three last integrals on the right-hand side are equal to non-increasing processes. In view of (2.1), this implies that the local martingale $(\int_0^t \widehat{V}_s dZ_s)_{t \leq T}$ is bounded from below by a martingale and is therefore a super-martingale. Similarly, $Z\widehat{V}$ is a local super-martingale which is bounded from below by a martingale and is therefore a super-martingale. The proof is concluded by taking the expectation in both sides of the previous inequality applied to $t = T$. \square

Proof of Proposition 2.1. As already mentioned, the last assertion is an immediate consequence of Lemma 8, Lemma 12 and Proposition 14 of [3], see also the proof of their Theorem 15. We now prove the first one which is obtained by very similar arguments. Let $(\widehat{C}^n)_n$ be a sequence in $\widehat{\mathcal{C}}^0 \cap \mathcal{S}^1(\mathbb{Q}_Z)$ that converges to some \widehat{C} in $\mathcal{S}^1(\mathbb{Q}_Z)$. After possibly passing to a subsequence, we may assume that the convergence holds a.s. uniformly in $t \leq T$. Let $(\widehat{V}^n)_n$ be a sequence in $\widehat{\mathcal{V}}^0$ such that $\widehat{V}^n \succeq \widehat{C}^n$ for all $n \geq 1$. It follows from the same arguments as in the proof of Lemma 12 in [3] that there is $\tilde{\mathbb{Q}}_Z \sim \mathbb{P}$, which depends only on Z , such that

$$\begin{aligned} & \mathbb{E} \left[- \int_0^T Z_s \widehat{V}_s^{nc} d\text{Var}_s(\widehat{V}^{nc}) - \sum_{s \leq T} Z_{s-} \Delta \widehat{V}_s^n - \sum_{s < T} Z_s \Delta^+ \widehat{V}_s^n \right] \\ & \geq \mathbb{E}^{\tilde{\mathbb{Q}}_Z} \left[\text{Var}_T(\widehat{V}^n) \right]. \end{aligned}$$

In view of Lemma 2.1, this implies that $\mathbb{E}^{\tilde{\mathbb{Q}}_Z} \left[\text{Var}_T(\widehat{V}^n) \right] \leq -\mathbb{E} \left[Z_T \widehat{V}_T^n \right]$ for all $n \geq 1$. We now observe that $\widehat{V}_T^n - \widehat{C}_T \in \widehat{K}_T$ \mathbb{P} -a.s. implies that

$$-\mathbb{E} \left[Z_T \widehat{V}_T^n \right] \leq -\mathbb{E} \left[Z_T \widehat{C}_T^n \right] \leq c_Z \mathbb{E}^{\mathbb{Q}_Z} \left[\|\widehat{C}_T^n\| \right].$$

Since \widehat{C}_T^n converges to \widehat{C}_T in $L^1(\mathbb{Q}_Z)$, the right-hand side of the latter inequality is uniformly bounded and so is the quantity $\mathbb{E}^{\tilde{\mathbb{Q}}_Z} \left[\text{Var}_T(\widehat{V}^n) \right]$. It thus follows from Proposition 14 in [3] that, after possibly passing to convex combinations, we can assume that, \mathbb{P} -a.s., $(\widehat{V}^n)_n$ converges pointwise on $[0, T]$ to a predictable process \widehat{V} with finite variations. The pointwise convergence ensures that $\widehat{V} \succeq \widehat{C}$. By Proposition 1.1, \widehat{V}^n satisfies (1.1) for all $n \geq 1$, and it follows from the pointwise convergence that \widehat{V} satisfies (1.1) too. We can then conclude from Proposition 1.1 that $\widehat{V} \in \widehat{\mathcal{V}}^0$. \square

2.2 Representation of continuous linear forms on $\mathcal{S}^1(\mathbb{Q}_Z)$

Following the approach of [15], [1] and [8], we now characterize the dual of $\mathcal{S}^1(\mathbb{Q}_Z)$ in terms of random measures.

For this purpose, we introduce the set \mathcal{R} of \mathbb{R}^{3d} -valued adapted càdlàg processes $A := (A^-, A^o, A^+)$ with \mathbb{P} -integrable total variation such that

- (i) A^- is predictable,
- (ii) A^+ and A^- are pure jump processes,
- (iii) $A_0^- = 0$ and $A_T^+ = A_{T-}^+$.

Theorem 2.1. *Fix $Z \in \mathcal{Z}^s$ and let μ be a continuous linear form on $\mathcal{S}^1(\mathbb{Q}_Z)$. Then, there exists $A := (A^-, A^o, A^+) \in \mathcal{R}$ such that:*

$$\mu(\widehat{C}) = \mathbb{E} \left[\int_0^T \widehat{C}_{t-} dA_t^- + \int_0^T \widehat{C}_t dA_t^o + \int_0^T \widehat{C}_{t+} dA_t^+ \right], \quad \text{for all } \widehat{C} \in \mathcal{S}^\infty.$$

The proof of this result was provided in [15] and [1] for optional càdlàg processes. A similar result for predictable làdcàd processes can also be found in [15], see Chapter V. A one dimensional version of Theorem 2.1 is given in [8]. We provide a complete proof in the Appendix for seek of completeness.

In the case, where $\mu(\xi) \leq 0$ for all essentially bounded optional làdlàg process ξ such that $-\xi \succeq 0$, the associated elements $A \in \mathcal{R}$ can be further characterized in terms of the polar cone process \widehat{K}^* . In the one dimensional setting, it should be interpreted as follows: if $\mu(\xi) \leq 0$ for all non-positive essentially bounded optional làdlàg process ξ , then each component of A is non-decreasing. This result will be of important use in the proof of our main result, Theorem 2.2 below.

Lemma 2.2. *Let $A = (A^-, A^o, A^+)$ be an element of \mathcal{R} such that*

$$\mathbb{E} \left[\int_0^T \xi_{t-} dA_t^- + \int_0^T \xi_t dA_t^o + \int_0^T \xi_{t+} dA_t^+ \right] \leq 0 \quad (2.2)$$

for all process ξ in \mathcal{S}^∞ such that $-\xi \succeq 0$. Then,

- (i) $\dot{A}^- \in \widehat{K}_-^* d\text{Var}(A^-)$ a.e. $\mathbb{P} - a.s.$
- (ii) $\dot{A}^{oc} \in \widehat{K}^* d\text{Var}(A^{oc})$ a.e. $\mathbb{P} - a.s.$ and $\dot{A}^{od} \in \widehat{K}^* d\text{Var}(A^{od})$ a.e. $\mathbb{P} - a.s.$
- (iii) $\dot{A}^+ \in \widehat{K}^* d\text{Var}(A^+)$ a.e. $\mathbb{P} - a.s.$

where A^{oc} and A^{od} denote the continuous and the purely discontinuous parts of A^o . We denote by $\mathcal{R}_{\widehat{K}}$ the set of processes in \mathcal{R} satisfying (i)-(ii)-(iii) above.

Proof. Let ξ be any bounded optional làdlàg process such that $-\xi \succeq 0$. Given $B \in \mathcal{F}$, let λ be the optional projection of $\mathbf{1}_B$. Note that it is càdlàg, since $\mathbf{1}_B(\omega)$ is constant for each ω , and that the process λ_- coincides with the predictable projection of $\mathbf{1}_B$, see Chapter V in [6]. We then set $\tilde{\xi} := \lambda\xi$. We remark that $\tilde{\xi}$ is the optional projection of $\mathbf{1}_B\xi$, since ξ is optional, and that $\tilde{\xi}_-$ is the predictable projection of $\mathbf{1}_B\xi_-$, since ξ_- is predictable. Since the set valued process \widehat{K} is a cone and λ takes values in $[0, 1]$, we have $-\tilde{\xi} \succeq 0$. Moreover, since A^- is predictable (resp. A^o, A^+ are optional), it follows that the induced measure commutes with the predictable projection (resp. the optional projection), see e.g. Theorem 3 Chapter I in [15]. Applying (2.2) to $\tilde{\xi}$ thus implies that

$$\begin{aligned} 0 &\geq \mathbb{E} \left[\int_0^T \lambda_{t-} \xi_{t-} dA_t^- + \int_0^T \lambda_t \xi_t dA_t^o + \int_0^T \lambda_t \xi_{t+} dA_t^+ \right] \\ &= \mathbb{E} \left[\mathbf{1}_B \left(\int_0^T \xi_{t-} dA_t^- + \int_0^T \xi_t dA_t^o + \int_0^T \xi_{t+} dA_t^+ \right) \right]. \end{aligned}$$

By arbitrariness of B , this shows that the càdlàg process X defined by

$$X := \int_0^\cdot \xi_{t-} dA_t^- + \int_0^\cdot \xi_t dA_t^o + \int_0^\cdot \xi_{t+} dA_t^+$$

satisfies $X_T \leq 0$ \mathbb{P} -a.s. Moreover, replacing ξ by $\xi \mathbf{1}_{(s+\varepsilon, t+\varepsilon \wedge T]}$ for $s < t \leq T$, $\varepsilon > 0$, and sending $\varepsilon \rightarrow 0$ shows that X is non-decreasing (recall (iii) of the definition of \mathcal{R}). In particular, its continuous part is non-decreasing, see e.g. Chapter VII in [14]. Since A^- and A^+ are purely discontinuous, this implies that the continuous part of $\int_0^\cdot \xi_t dA_t^o$ is non-decreasing. Letting A^{oc} denote the continuous part of A^o , we thus deduce that

$$\xi A^{oc} \leq 0 \quad d\text{Var}(A^{oc}) \text{ a.e } \mathbb{P} - \text{a.s.} \quad (2.3)$$

We now replace ξ by $\tilde{\xi} := \xi \mathbf{1}_{(\tau, \tau_h)}$ where τ is some stopping time with values in $[0, T]$ and $\tau_h := (\tau + h) \wedge T$ for some $h > 0$. The same argument as above shows that

$$\int_{\tau+}^{\tau_h} \xi_{t-} dA_t^- + \int_{\tau+}^{\tau_h-} \xi_t dA_t^o + \int_{\tau}^{\tau_h-} \xi_{t+} dA_t^+ \leq 0 \quad \mathbb{P} - \text{a.s.}$$

For $h \rightarrow 0$, this leads to

$$\xi_{\tau+} \Delta A_{\tau}^+ \leq 0 \quad \mathbb{P} - \text{a.s.} \quad (2.4)$$

for all stopping time τ with values in $[0, T]$. Arguing as above with ξ replaced by $\tilde{\xi} := \xi \mathbf{1}_{(\tau_n, \tau)}$ where τ is a predictable stopping time with values in $(0, T]$ and $(\tau_n)_n$ is an announcing sequence for τ , leads to

$$\xi_{\tau-} \Delta A_{\tau}^- \leq 0 \quad \mathbb{P} - \text{a.s.} \quad (2.5)$$

Finally, we replace ξ by $\tilde{\xi} := \xi \mathbf{1}_{\{\tau\}}$ to obtain

$$\xi_{\tau} \Delta A_{\tau}^o \leq 0 \quad \mathbb{P} - \text{a.s.} \quad (2.6)$$

for all stopping time τ with values in $[0, T]$. Since the cone valued process \widehat{K} is generated by a family of *càdlàg* adapted processes, which we can always assume to be bounded, (2.3), (2.4), (2.5), (2.6) and (iii) of the definition of \mathcal{R} imply the required result. \square

2.3 The super-hedging theorem

In this section, we show that a dual formulation for the set $\widehat{\mathcal{C}}^0$ can be deduced from the closure property stated in Proposition 2.1 and the representation of the dual of $\mathcal{S}^1(\mathbb{Q}_Z)$ derived in Theorem 2.1. To this purpose we need to introduce a suitable subset of $\mathcal{R}_{\widehat{K}}$.

Given $A := (A^-, A^o, A^+) \in \mathcal{R}_{\widehat{K}}$, we now define

$$\delta A_t^- := A_T^- - A_t^- + A_T^o - A_{t-}^o + A_T^+ - A_{t-}^+, \quad \delta A_t^+ := A_T^- - A_t^- + A_T^o - A_t^o + A_T^+ - A_{t-}^+$$

and denote by \bar{A}^- (resp. \bar{A}^+) the predictable projection (resp. optional) of $(\delta A_t^-)_{t \leq T}$ (resp. $(\delta A_t^+)_{t \leq T}$). Note that, since $\widehat{K}^* \subset \mathbb{R}_+^d$, the former processes have non-negative components so that their projections are well defined.

Definition 2.1. We say that $A := (A^-, A^o, A^+) \in \mathcal{R}_{\widehat{K}}$ belongs to \mathcal{D} if

- (i) $\bar{A}_\tau^- \in \widehat{K}_{\tau-}^* \mathbb{P}$ - a.s. for all predictable stopping time $\tau \leq T$,
- (ii) $\bar{A}_\tau^+ \in \widehat{K}_\tau^* \mathbb{P}$ - a.s. for all stopping time $\tau \leq T$.

We can now state our main result. Since $\widehat{V}^v = v + \widehat{V}^0$ and therefore $\widehat{C}^v = v + \widehat{C}^0$, we only consider the case $\widehat{C} \in \widehat{C}^0$.

Theorem 2.2. Let \widehat{C} be a *l\`adl\`ag* optional process such that $\widehat{C} \succeq -a\mathbf{1}$ for some $a > 0$. Then, $\widehat{C} \in \widehat{C}^0$ if and only if

$$\mathbb{E} \left[\int_0^T \widehat{C}_{t-} dA_t^- + \int_0^T \widehat{C}_t dA_t^o + \int_0^T \widehat{C}_{t+} dA_t^+ \right] \leq 0 \quad \forall (A^-, A^o, A^+) \in \mathcal{D}. \quad (2.7)$$

Proof. We fix a *l\`adl\`ag* optional process \widehat{C} such that $\widehat{C} \succeq -a\mathbf{1}$, for some $a > 0$.

Step 1. We first show that $C \in \widehat{C}^0$ implies (2.7). Let $\widehat{V} \in \widehat{V}^0$ be such that $\widehat{V} \succeq \widehat{C}$. Then, by definition of $A = (A^-, A^o, A^+) \in \mathcal{D} \subset \mathcal{R}_{\widehat{K}}$

$$\int_0^T (\widehat{C}_{t-} - \widehat{V}_{t-}) dA_t^- + \int_0^T (\widehat{C}_t - \widehat{V}_t) dA_t^o + \int_0^T (\widehat{C}_{t+} - \widehat{V}_{t+}) dA_t^+ \leq 0 \quad \mathbb{P} - \text{a.s.}$$

Thus, it suffices to show that (2.7) holds for \widehat{V} in place of \widehat{C} .

1.a. We first assume that \widehat{V} has an essentially bounded total variation. By Fubini's theorem and the continuity of \widehat{V}^c ,

$$\begin{aligned} \int_0^T \widehat{V}_{t-} dA_t^- &= \int_0^T \left(\int_0^t d\widehat{V}_s^c + \sum_{s < t} (\Delta \widehat{V}_s + \Delta^+ \widehat{V}_s) \right) dA_t^- \\ &= \int_0^T (A_T^- - A_t^-) d\widehat{V}_t^c + \sum_{t < T} (A_T^- - A_t^-) (\Delta \widehat{V}_t + \Delta^+ \widehat{V}_t). \end{aligned}$$

Similarly,

$$\int_0^T \widehat{V}_t dA_t^o = \int_0^T (A_T^o - A_t^o) d\widehat{V}_t^c + \sum_{t \leq T} (A_T^o - A_{t-}^o) \Delta \widehat{V}_t + \sum_{t < T} (A_T^o - A_t^o) \Delta^+ \widehat{V}_t$$

and

$$\int_0^T \widehat{V}_{t+} dA_t^+ = \int_0^T (A_T^+ - A_{t-}^+) d\widehat{V}_t^c + \sum_{t \leq T} (A_T^+ - A_{t-}^+) (\Delta \widehat{V}_t + \Delta^+ \widehat{V}_t).$$

This shows that

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \widehat{V}_{t-} dA_t^- + \int_0^T \widehat{V}_t dA_t^o + \int_0^T \widehat{V}_{t+} dA_t^+ \right] \\ &= \mathbb{E} \left[\sum_{t \leq T} \delta A_t^- \Delta \widehat{V}_t + \int_0^T \delta A_t^+ d\widehat{V}_t^c + \sum_{t < T} \delta A_t^+ \Delta^+ \widehat{V}_t \right] \\ &= \mathbb{E} \left[\sum_{t \leq T} \bar{A}_t^- \Delta \widehat{V}_t + \int_0^T \bar{A}_t^+ d\widehat{V}_t^c + \sum_{t < T} \bar{A}_t^+ \Delta^+ \widehat{V}_t \right], \end{aligned}$$

and the required result follows from the definitions of \mathcal{D} and $\widehat{\mathcal{V}}^0$.

1.b. We now consider the case where the total variation of \widehat{V} is not essentially bounded. Recall that $\widehat{V} \succeq \widehat{C} \succeq -a\mathbf{1}$. We can then approximate \widehat{V} from below (in the sense of \succeq) by the sequence $(\widehat{V}^n)_n$ defined by $\widehat{V}^n = \widehat{V} \mathbb{1}_{[0, \tau_n]} - a\mathbf{1} \mathbb{1}_{(\tau_n, T]}$ where τ_n is a localizing sequence of stopping times for $\text{Var}(\widehat{V})$, so that $\tau_n \rightarrow \infty$. It follows from the previous step that

$$\mathbb{E} \left[\int_0^T \widehat{V}_{t-}^n dA_t^- + \int_0^T \widehat{V}_t^n dA_t^o + \int_0^T \widehat{V}_{t+}^n dA_t^+ \right] \leq 0 .$$

Since $\widehat{V}^n \succeq -a\mathbf{1}$ for all $n \geq 1$ and $\mathcal{D} \subset \mathcal{R}_{\widehat{K}}$, each integral in the expectation is bounded from below, uniformly in n , by an integrable random variable which depends only on A and a . Since $\widehat{V}^n \rightarrow \widehat{V}$ uniformly on compact sets, \mathbb{P} -a.s., we conclude by appealing to Fatou's Lemma.

Step 2. We now prove that (2.7) implies $\widehat{C} \in \widehat{\mathcal{C}}^0$.

2.a. We first consider the case where $\widehat{C} \in \mathcal{S}^\infty$. Assume that \widehat{C} does not belong to the convex cone $\widehat{\mathcal{C}}^0$. Fix $Z \in \mathcal{Z}^s$ and observe that $\widehat{C} \notin \widehat{\mathcal{C}}^0 \cap \mathcal{S}^1(\mathbb{Q}_Z)$. The latter being closed in $\mathcal{S}^1(\mathbb{Q}_Z)$, see Proposition 2.1, it follows from the Hahn-Banach separation theorem that we can find μ in the dual of $\mathcal{S}^1(\mathbb{Q}_Z)$ such that $\mu(X) \leq c < \mu(C)$ for all $X \in \widehat{\mathcal{C}}^0 \cap \mathcal{S}^\infty$, for some real c . Since $\widehat{\mathcal{C}}^0$ is a cone, it is clear that $c = 0$. Thus,

$$\sup_{X \in \widehat{\mathcal{C}}^0 \cap \mathcal{S}^\infty} \mu(X) \leq 0 < \mu(C) . \quad (2.8)$$

Moreover, by Theorem 2.1, there is a process $A := (A^-, A^o, A^+) \in \mathcal{R}$ such that

$$\mu(X) = \mathbb{E} \left[\int_0^T X_{t-} dA_t^- + \int_0^T X_t dA_t^o + \int_0^T X_{t+} dA_t^+ \right] \quad \text{for all } X \in \mathcal{S}^\infty . \quad (2.9)$$

Since $\widehat{C} \in \mathcal{S}^\infty$, it thus suffices to show that $A \in \mathcal{D}$ to obtain a contradiction to (2.7). We first note that Lemma 2.2 implies that $A \in \mathcal{R}_{\widehat{K}}$ since $X = -\xi$ belongs to $\widehat{\mathcal{C}}^0 \cap \mathcal{S}^\infty$ for all bounded optional *l\`a*d*l\`a*g process ξ satisfying $\xi \succeq 0$. It remains to prove the condition (i)-(ii) of the definition of \mathcal{D} . Using the same integration by parts argument as in step 1. above, we deduce from (2.8) that:

$$\mathbb{E} \left[\sum_{t \leq T} \bar{A}_t^- \Delta \widehat{V}_t + \int_0^T \bar{A}_t^+ d\widehat{V}_t^c + \sum_{t < T} \bar{A}_t^+ \Delta^+ \widehat{V}_t \right] \leq 0 ,$$

for all $\widehat{V} \in \widehat{\mathcal{V}}^0 \cap \mathcal{S}^\infty$ with essentially bounded total variation. It thus follows from Definition 1.1 that $\mathbb{E} [\bar{A}_\tau^- \xi] \leq 0$ for all predictable stopping time $\tau \leq T$ \mathbb{P} -a.s. and bounded $\mathcal{F}_{\tau-}$ -measurable ξ taking values in $-\widehat{K}_{\tau-}$ \mathbb{P} -a.s. Similarly, $\mathbb{E} [\bar{A}_\tau^+ \xi] \leq 0$ for all stopping time $\tau < T$ \mathbb{P} -a.s. and bounded \mathcal{F}_τ -measurable ξ taking values in $-\widehat{K}_\tau$ \mathbb{P} -a.s. Observe that $\bar{A}_T^+ = 0 \in \widehat{K}_T^*$ since $\Delta A_T^+ = 0$. Recalling the definition of

\widehat{K} in terms of its generating family based on Π , this implies (i)-(ii) of the definition of \mathcal{D} .

2.b. We conclude the proof by considering the case where \widehat{C} is not bounded but satisfies $\widehat{C} \succeq -a\mathbf{1}$ for some $a > 0$. Define the bounded process $\widehat{C}^n := \widehat{C} \mathbf{1}_{\{\|\widehat{C}\| \leq n\}} - a\mathbf{1}_{\{\|\widehat{C}\| > n\}}$ for $n \geq 1$. Observing that $\widehat{C} \succeq \widehat{C}^n$ for all $n \geq 1$ and recalling that \widehat{C} satisfies (2.7), we deduce from the definition of $\mathcal{D} \subset \mathcal{R}_{\widehat{K}}$ that

$$\mathbb{E} \left[\int_0^T \widehat{C}_{t-}^n dA_t^- + \int_0^T \widehat{C}_t^n dA_t^o + \int_0^T \widehat{C}_{t+}^n dA_t^+ \right] \leq 0$$

for all $A = (A^-, A^o, A^+) \in \mathcal{D}$ and $n \geq 1$. It follows from 2.a that $\widehat{C}^n \in \widehat{\mathcal{C}}^0$ for all $n \geq 1$. Since $\widehat{C}^n \succeq -a\mathbf{1}$ for all $n \geq 1$ and $\|\widehat{C}^n - \widehat{C}\|_* \rightarrow 0$ \mathbb{P} -a.s., it follows from Proposition 2.1 that $\widehat{C} \in \widehat{\mathcal{C}}^0$ too. \square

3 Comments and additional properties

In this section, we discuss additional properties of the set of dual variables \mathcal{D} and provide an alternative to the dual formulation of Theorem 2.2 in the spirit of the one proposed by [7] for càdlàg processes. We also discuss the links between (2.7) and the well-known dual formulation in terms of optimal stopping in frictionless markets.

3.1 Reformulation of the duality result

We first provide an alternative formulation for \mathcal{D} . To this purpose, we need to introduce additional notations.

Let \mathcal{N} denote the set of triplets of non-negative random measures $\nu := (\nu^-, \nu^o, \nu^+)$ such that ν^- is predictable, ν^o and ν^+ are optional, and $(\nu^- + \nu^o + \nu^+)([0, T]) = 1$ \mathbb{P} -a.s.

Given $\nu \in \mathcal{N}$, we define $\tilde{Z}(\nu)$ as the set of \mathbb{R}^{3d} -valued processes $Z := (Z^-, Z^o, Z^+)$ such that:

- (i) Z^i is $\nu^i(dt, \omega)d\mathbb{P}(\omega)$ integrable for $i \in \{-, o, +\}$, Z^- is predictable and Z^o, Z^+ are optional.
- (ii) $A = (A^-, A^o, A^+)$ defined by $A^i = \int_0^\cdot Z_t^i \nu^i(dt)$ for $i \in \{-, o, +\}$ belongs to \mathcal{D} .

Proposition 3.1. *Let $A = (A^-, A^o, A^+)$ be a \mathbb{R}^{3d} -valued process with integrable total variation. Then, $A \in \mathcal{D}$ if and only if there exists $\nu := (\nu^-, \nu^o, \nu^+) \in \mathcal{N}$ and $Z := (Z^-, Z^o, Z^+) \in \tilde{Z}(\nu)$ such that*

$$A^i = \int_0^\cdot Z_t^i \nu^i(dt), \quad i \in \{-, o, +\}. \quad (3.1)$$

Proof. It is clear that given $(\nu^-, \nu^\circ, \nu^+) \in \mathcal{N}$ and $(Z^-, Z^\circ, Z^+) \in \tilde{\mathcal{Z}}(\nu)$, the process defined in (3.1) belong to \mathcal{D} . We now prove the converse assertion.

1. We first observe that, given $A = (A^-, A^\circ, A^+) \in \mathcal{R}$, we can find a \mathbb{R}^{3d} -adapted process $Z := (Z^-, Z^\circ, Z^+)$ and a triplet of real positive random measures $\nu := (\nu^-, \nu^\circ, \nu^+)$ on $[0, T]$ such that Z^- and ν^- are predictable, (Z°, Z^+) and (ν°, ν^+) are optional, and $A^i = \int_0^\cdot Z_t^i \nu^i(dt)$ for $i \in \{-, \circ, +\}$.

2. We can then always assume that $\bar{\nu} := \nu^- + \nu^\circ + \nu^+$ satisfies $\bar{\nu}([0, T]) \leq 1 \mathbb{P} - \text{a.s.}$ Indeed, let f be some strictly increasing function mapping $[0, \infty)$ into $[0, 1/3)$. Then, for $i \in \{-, \circ, +\}$, ν^i is absolutely continuous with respect to $\tilde{\nu}^i := f(\nu^i)$ and thus admits a density. Replacing ν^i by $\tilde{\nu}^i$ and multiplying Z^i by the optional (resp. predictable) projection of the associated density leads to the required representation for $i \in \{\circ, +\}$ (resp. $i = -$).

3. Finally, we can reduce to the case where $\bar{\nu}([0, T]) = 1 \mathbb{P} - \text{a.s.}$ Indeed, since ν^- is only supported by graphs of $[0, T]$ -valued random variables (recall that A^- is a pure jump process), we know that it has no continuous part at $\{T\}$. We can thus replace ν^- by $\tilde{\nu}^- := \nu^- + \delta_{\{T\}}(1 - \bar{\nu}([0, T]))$ where $\delta_{\{T\}}$ denotes the Dirac mass at T . We then also replace Z^- by

$$\tilde{Z}^- := Z^- [\mathbf{1}_{\{t < T\}} + \mathbf{1}_{\{t=T\}} \mathbf{1}_{\{\bar{\nu}([0, T]) < 1\}} \nu^-(\{T\})(\nu^-(\{T\}) + 1 - \bar{\nu}([0, T]))^{-1}]$$

so that $A^- = \int_0^\cdot \tilde{Z}_t^- \tilde{\nu}^-(dt)$. Observe that the assumption $\mathcal{F}_{T-} = \mathcal{F}_T$ ensures that $\tilde{\nu}^-$ and \tilde{Z}^- are still predictable. \square

Remark 3.1. It follows from the above arguments that the representation given in Theorem 2.1 can be alternatively written

$$\mu(\hat{C}) = \mathbb{E} \left[\int_0^T \hat{C}_{t-} Z_t^- \nu^-(dt) + \int_0^T \hat{C}_t Z_t^\circ \nu^\circ(dt) + \int_0^T \hat{C}_{t+} Z_t^+ \nu^+(dt) \right]$$

for some $(\nu^-, \nu^\circ, \nu^+) \in \mathcal{N}$ and some \mathbb{R}^{3d} -valued processes $Z := (Z^-, Z^\circ, Z^+)$ which satisfies the assertion (i) of the definition of $\tilde{\mathcal{Z}}(\nu)$.

In view of Proposition 3.1, the dual formulation of Theorem 2.2 can be written as follows.

Corollary 3.1. *Let \hat{C} be a l\`a d\`a g optional process such that $\hat{C} \succeq -a\mathbf{1}$ for some $a > 0$. Then, $\hat{C} \in \hat{\mathcal{C}}^0$ if and only if*

$$\mathbb{E} \left[\int_0^T \hat{C}_{t-} Z_t^- \nu^-(dt) + \int_0^T \hat{C}_t Z_t^\circ \nu^\circ(dt) + \int_0^T \hat{C}_{t+} Z_t^+ \nu^+(dt) \right] \leq 0 \quad (3.2)$$

for all $\nu \in \mathcal{N}$ and $Z \in \tilde{\mathcal{Z}}(\nu)$.

This formulation is very close to the formulation (1.2) of [7] up to two differences. Only the measure ν° appears in their formulation and in their case it is deterministic.

In this sense our result is less tractable. However, as already mentioned, their approach requires to impose a right-continuity assumption on \widehat{C} , while ours allows to consider general $\text{l\`a}d\text{l\`a}g$ processes.

3.2 Comparison with frictionless markets

Let us first recall that the frictionless markets case corresponds to the situation where $\pi^{ij} = 1/\pi^{ji}$ for all $i, j \leq d$. In this case, the price process (in terms of the first asset) is $S^i := \pi^{1i}$ and is a $\text{c\`a}d\text{l\`a}g$ semimartingale, see [5]. In order to avoid technicalities, it is usually assumed to be locally bounded. The no-arbitrage condition, more precisely no free lunch with vanishing risk, implies that the set \mathcal{M} of equivalent measures \mathbb{Q} under which $S = (S^i)_{i \leq d}$ is a local martingale is non-empty. Such measures should be compared to the strictly consistent price processes Z of \mathcal{Z}^s . Indeed, if H denotes the density process associated to \mathbb{Q} , then HS is “essentially” an element of \mathcal{Z}^s , and conversely, up to an obvious normalization. The term “essentially” is used here because in this case the interior of \widehat{K}^* is empty and the notion of interior as to be replaced by that of relative interior.

In such models, the wealth process is a real valued process which describes the value (in terms of the first asset) of the portfolio. It corresponds to $V = S\widehat{V}$. The main difference is that the set of admissible strategies is no more described by $\widehat{\mathcal{V}}^0$ but in terms of stochastic integrals with respect to S .

In the case where $\mathcal{M} = \{\mathbb{Q}\}$, the so-called complete market case, the super-hedging price of an American claim \widehat{C} , such that $C := S\widehat{C}$ is bounded from below, coincides with the value at time 0 of the Snell envelope of C computed under \mathbb{Q} , see e.g. [13] and the references therein. Equivalently, the American claim \widehat{C} can be super-hedged from a zero initial endowment if and only if the \mathbb{Q} -Snell envelope of C at time 0 is non-positive.

In the case where C is $\text{l\`a}d\text{l\`a}g$ and of class (D), the \mathbb{Q} -Snell envelope $J^\mathbb{Q}$ of C satisfies, see [8] and [9],

$$J_0^\mathbb{Q} = \sup_{\tau \in \mathcal{T}} \mathbb{E}^\mathbb{Q}[C_\tau] = \sup_{(\tau^-, \tau^o, \tau^+) \in \widetilde{\mathcal{T}}} \mathbb{E}^\mathbb{Q}[C_{\tau^-} + C_{\tau^o} + C_{\tau^+}] \quad (3.3)$$

where \mathcal{T} is the set of all $[0, T]$ -valued stopping times, $\widetilde{\mathcal{T}}$ is the set of triplets of $[0, T] \cup \{\infty\}$ -valued stopping times (τ^-, τ^o, τ^+) such that τ^- is predictable and, a.s., only one of them is finite. Here, we use the convention $C_{\infty^-} = C_\infty = C_{\infty^+} = 0$. The first formulation is simple but does not allow to provide an existence result, while the second does. Indeed, it is shown in [9] that

$$J_0^\mathbb{Q} = \mathbb{E}^\mathbb{Q}[C_{\widehat{\tau}^-} \mathbb{1}_{A^-} + C_{\widehat{\tau}} \mathbb{1}_{A^o} + C_{\widehat{\tau}^+} \mathbb{1}_{A^+}]$$

where

$$\widehat{\tau} := \inf\{t \in [0, T] : J_{t^-}^\mathbb{Q} = C_{t^-} \text{ or } J_t^\mathbb{Q} = C_t \text{ or } J_{t^+}^\mathbb{Q} = C_{t^+}\}$$

and

$$A^- := \{J_{t-}^{\mathbb{Q}} = C_{t-}\}, \quad A^o := \{J_t^{\mathbb{Q}} = C_t\} \cap (A^-)^c, \quad A^+ := (A^- \cup A^o)^c.$$

It thus suffices to set $\tau^i := \hat{\tau} \mathbb{1}_{A^i} + \infty \mathbb{1}_{(A^i)^c}$ for $i \in \{-, o, +\}$ to obtain

$$J_0^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}} [C_{\hat{\tau}--} + C_{\hat{\tau}^o} + C_{\hat{\tau}++}] .$$

This shows that, in general, one needs to consider quasi stopping times instead of stopping times if one wants to establish an existence result, see also [1] for the case of *càdlàg* processes.

In the case of incomplete markets, the super-hedging price is given by the sup over all $\mathbb{Q} \in \mathcal{M}$ of $J_0^{\mathbb{Q}}$, see [13].

In our framework, the measure $\nu \in \mathcal{N}$ that appears in (3.2) can be interpreted as a randomized version of the quasi stopping times while the result of [7] should be interpreted as a formulation in terms of randomized stopping times. Both are consistent with the results of [2] and [4] that show that the duality does not work in discrete time models if we restrict to (non-randomized) stopping times. In both cases the process $Z \in \tilde{Z}(\nu)$ plays the role of $H^{\mathbb{Q}}S$ where $H^{\mathbb{Q}}$ is the density process associated to the equivalent martingale measures \mathbb{Q} mentioned above. These two formulations thus corresponds to the two representations of the Snell envelope in (3.3). As in frictionless markets, the formulation of [7] is simpler while ours should allow to find the optimal randomized quasi stopping time, at least when Z is fixed. We leave this point for further research.

A Appendix: Proof of the representation result for the dual of $\mathcal{S}^1(\mathbb{Q}_Z)$

We provide here the proof of Theorem 2.1. It is obtained by following almost line by line the proof of Theorem 27 in Chapter V of [15], see also [1]. We split the proof in different Lemmata. It is clear that we can always reduce to the one dimensional case since μ is linear. From now on, we shall therefore only consider the case $d = 1$.

We first introduce some notations. Let W be the subset of $[0, T] \times \Omega \times \{-, o, +\}$ defined by

$$W := ((0, T] \times \Omega \times \{-\}) \cup ([0, T] \times \Omega \times \{o\}) \cup ([0, T] \times \Omega \times \{+\}) .$$

Given a subset C of $[0, T] \times \Omega$, we set

$$\begin{aligned} C_- &= \{(t, \omega, -) \in W \mid (t, \omega) \in C, t > 0\} \\ C_o &= \{(t, \omega, o) \in W \mid (t, \omega) \in C\} \\ C_+ &= \{(t, \omega, +) \in W \mid (t, \omega) \in C, t < T\} . \end{aligned}$$

If c is a function on $[0, T] \times \Omega$, we also introduce the three functions c_- , c_\circ and c_+ on W

$$\begin{aligned} c_-(t, w, +) &= c_-(t, w, \circ) = 0 \quad \text{and} \quad c_-(t, w, -) = c(t, w), \\ c_\circ(t, w, +) &= c_\circ(t, w, -) = 0 \quad \text{and} \quad c_\circ(t, w, \circ) = c(t, w), \\ c_+(t, w, -) &= c_+(t, w, \circ) = 0 \quad \text{and} \quad c_+(t, w, +) = c(t, w). \end{aligned}$$

We denote by $\tilde{\mathcal{S}}^\infty$ the set of *l\`adl\`ag* $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable \mathbb{P} -essentially bounded processes. For a process $X \in \tilde{\mathcal{S}}^\infty$, we define \bar{X} as follows

$$\bar{X}(t, \omega, -) := X_{t-}(\omega), \quad \bar{X}(t, \omega, \circ) := X_t \quad \text{and} \quad \bar{X}(t, \omega, +) := X_{t+}(\omega).$$

Finally, we set $\bar{\mathcal{S}}^\infty := \{\bar{X} \mid X \in \tilde{\mathcal{S}}^\infty\}$ and $\mathcal{W} := \sigma(\bar{X}, \bar{X} \in \bar{\mathcal{S}}^\infty)$.

Note that $\bar{\mathcal{S}}^\infty$ is a lattice and $X \mapsto \bar{X}$ is a bijection. Thus, for a linear form $\tilde{\mu}$ on $\tilde{\mathcal{S}}^\infty$, we can always define the linear form $\bar{\mu}$ on $\bar{\mathcal{S}}^\infty$ by $\bar{\mu}(\bar{X}) := \tilde{\mu}(X)$.

Lemma A.1. *Let $\tilde{\mu}$ be a linear form on $\tilde{\mathcal{S}}^\infty$ such that:*

(C1) $\tilde{\mu}(X^n) \rightarrow 0$ for all sequence $(X^n)_n$ of positive elements of $\tilde{\mathcal{S}}^\infty$ such that $\sup_n \|X^n\|_{\mathcal{S}^\infty} \leq M$ for some $M > 0$ and satisfying $\|X^n\|_* \rightarrow 0$ \mathbb{P} -a.s.

Then, there exists a signed bounded measure $\bar{\nu}$ on (W, \mathcal{W}) such that $\tilde{\mu}(X) = \bar{\mu}(\bar{X}) = \bar{\nu}(\bar{X})$ and $|\tilde{\mu}|(X) = |\bar{\mu}|(\bar{X}) = |\bar{\nu}|(\bar{X})$ for all $X \in \tilde{\mathcal{S}}^\infty$.

Proof. Using the standard decomposition argument $\tilde{\mu} = \tilde{\mu}^+ - \tilde{\mu}^-$, one can assume (and do) that the linear form $\tilde{\mu}$ is non-negative. We have to prove that $\bar{\mu}$ satisfies the Daniell's condition:

(C2) If $(\bar{X}^n)_{n \geq 0}$ decreases to zero then $\bar{\mu}(\bar{X}^n) \rightarrow 0$.

Let $(\bar{X}^n)_{n \geq 0}$ be a sequence of non-negative elements of $\bar{\mathcal{S}}^\infty$ that decreases to 0. For $\epsilon > 0$, we introduce the sets

$$\begin{aligned} A_n(\omega) &:= \{t \in [0, T] \mid X_{t+}^n(\omega) \geq \epsilon \text{ or } X_{t-}^n(\omega) \geq \epsilon\}, \\ B_n(\omega) &:= \{t \in [0, T] \mid X_t^n(\omega) \geq \epsilon\}, \\ K_n(\omega) &:= A_n(\omega) \cup B_n(\omega). \end{aligned} \tag{A.1}$$

Obviously, $K_{n+1}(\omega) \subset K_n(\omega)$, $\bigcap_{n \geq 0} K_n(\omega) = \emptyset$ and $A_n(\omega)$ is closed. Let $(t_k)_{k \geq 1}$ be a sequence of $K_n(\omega)$ converging to $s \in [0, T]$. If there is a subsequence $(t_{\phi(k)})_{k \geq 1}$ such that $X_{t_{\phi(k)}} \in A_n(\omega)$ for all $k \geq 0$, then $s \in K_n(\omega)$, since $A_n(\omega)$ is closed. If not, we can suppose that t_k belongs to $B_n(\omega)$ for all $k \geq 1$, after possibly passing to a subsequence. Since $X(\omega)$ is *l\`adl\`ag* and bounded, we can extract a subsequence $(t_{\phi(k)})_{k \geq 1}$ such that $\lim X_{t_{\phi(k)}}(\omega) \in \{X_{s-}(\omega), X_s(\omega), X_{s+}(\omega)\}$. Since $X_{t_{\phi(k)}}(\omega) \geq \epsilon$, we deduce that $s \in K_n(\omega)$. This proves that $K_n(\omega)$ is closed. Using the compactness

of $[0, T]$, we then obtain that there exists some $N_\epsilon > 0$ for which $\cap_{n \geq N_\epsilon} K_n(\omega) = \emptyset$. Thus, $\|X^n(\omega)\|_* < \epsilon$ for $n \geq N_\epsilon$. Since $\tilde{\mu}$ satisfies **(C1)**, this implies that $\bar{\mu}$ satisfies Daniell's condition **(C2)**, which provides the existence of the measure $\bar{\nu}$. \square

Lemma A.2. *If S is a \mathcal{F} -measurable $[0, T]$ -valued random variable, then $\llbracket S \rrbracket_+, \llbracket S \rrbracket_\circ$ and $\llbracket S \rrbracket_-$ belongs to \mathcal{W} .*

Proof. For $\epsilon > 0$, we set $X^\epsilon := \mathbf{1}_{(S, (S+\epsilon) \wedge T)}$ which belongs to $\tilde{\mathcal{S}}^\infty$. The associated process \bar{X}^ϵ is the indicator function of the set $I^\epsilon := (S, (S + \epsilon) \wedge T]_- \cup (S, (S + \epsilon) \wedge T)_\circ \cup [S, (S + \epsilon) \wedge T)_+$ which belongs \mathcal{W} . Taking $\epsilon_n := 1/n$ with $n \geq 1$, we thus obtain $\cap_{n \geq 1} I^{\epsilon_n} = \llbracket S \rrbracket_+ \in \mathcal{W}$. Using the same arguments with $X^\epsilon := \mathbf{1}_{(0 \vee (S-\epsilon), S)}$, we get that $\llbracket S \rrbracket_- \in \mathcal{W}$. Finally working with $X^\epsilon := \mathbf{1}_{[S, (S+\epsilon) \wedge T)}$, we also obtain that $\llbracket S \rrbracket_+ \cup \llbracket S \rrbracket_\circ \in \mathcal{W}$. Since $\llbracket S \rrbracket_\circ = (\llbracket S \rrbracket_+ \cup \llbracket S \rrbracket_\circ) \cap (\llbracket S \rrbracket_+)^c$, this shows that $\llbracket S \rrbracket_\circ \in \mathcal{W}$. \square

Lemma A.3. *If C is a measurable set of $[0, T] \times \Omega$, then $C_+ \cup C_\circ \cup C_- \in \mathcal{W}$.*

Proof. Since $\mathcal{B}([0, T]) \otimes \mathcal{F}$ is generated by continuous adapted processes, it suffices to check that $X_- + X + X_+$ is \mathcal{W} -measurable whenever X is continuous and measurable. This is obvious since $\bar{X} = X_- + X_\circ + X_+$ in this case. \square

Lemma A.4. *There exists four measures $\alpha_-, \alpha_\circ^\delta, \alpha_\circ^c$ and α_+ on $[0, T] \times \Omega$ such that*

1. α_- is supported by $(0, T] \times \Omega$ and by a countable union of $[0, T]$ -valued \mathcal{F} -measurable random variable S such that $\alpha_-(\llbracket S \rrbracket) = \bar{\nu}(\llbracket S \rrbracket_-)$.
2. α_+ is supported by $[0, T] \times \Omega$ and by a countable union of graphs of $[0, T]$ -valued \mathcal{F} -measurable random variable such that $\alpha_+(\llbracket S \rrbracket) = \bar{\nu}(\llbracket S \rrbracket_+)$.
3. α_\circ^δ is supported by $[0, T] \times \Omega$ and by a countable union of graphs of $[0, T]$ -valued \mathcal{F} -measurable random variable such that $\alpha_\circ^\delta(\llbracket S \rrbracket) = \bar{\nu}(\llbracket S \rrbracket_\circ)$.
4. α_\circ^c is supported by $[0, T] \times \Omega$ and does not charge any graph of $[0, T]$ -valued \mathcal{F} -measurable random variable.
5. For all $X \in \tilde{\mathcal{S}}^\infty$, we have

$$\tilde{\mu}(X) = \int_\Omega \int_0^T X_{t-}(\omega) \alpha_-(dt, d\omega) + \int_\Omega \int_0^T X_t(\omega) \alpha_\circ(dt, d\omega) + \int_\Omega \int_0^T X_{t+}(\omega) \alpha_+(dt, d\omega),$$

where $\alpha_\circ = \alpha_\circ^c + \alpha_\circ^\delta$.

Proof. We first define \mathcal{H} as the collection of sets of the form $A = \bigcup_{n \geq 0} \llbracket S_n \rrbracket_+$ for a given sequence $(S_n)_{n \geq 0}$ of $[0, T]$ -valued \mathcal{F} -measurable random variables. This set is closed under countable union. The quantity $\sup_{A \in \mathcal{H}} |\bar{\nu}|(A) =: M$ is well defined since $\bar{\nu}$ is bounded. Let $(A_n)_{n \geq 1}$ be a sequence such that $\lim |\bar{\nu}|(A_n) = M$ and set $G_+ := \bigcup_{n \geq 0} A_n$, so that $|\bar{\nu}|(G_+) = M$. Observe that we can easily reduce to

the case where the G_+ is the union of disjoint graphs. We then define the measure $\bar{\nu}_+ := \bar{\nu}(\cdot \cap G_+)$ and, recall Lemma A.3,

$$\alpha_+(C) := \bar{\nu}_+(C_+ \cup C_- \cup C_\circ) = \bar{\nu}_+(C_+)$$

for $C \in \mathcal{B}([0, T]) \otimes \mathcal{F}$. The measure α_+ is supported by graphs of $[0, T]$ -valued \mathcal{F} -measurable random variable. Moreover, for all $[0, T]$ -valued \mathcal{F} -measurable random variable S , we have

$$\alpha_+(\llbracket S \rrbracket) = \bar{\nu}(\llbracket S \rrbracket_+).$$

Indeed, $\bar{\nu}(\llbracket S \rrbracket_+) > \bar{\nu}(\llbracket S \rrbracket_+ \cap G_+)$ implies $\bar{\nu}(\llbracket S \rrbracket_+ \cup G_+) > \bar{\nu}(G_+)$, which contradicts the maximality of G_+ .

We construct G_- , G_\circ and the measures α_- , α_\circ^δ and $\bar{\nu}_-$, $\bar{\nu}_\circ^\delta$ similarly. We then set $\bar{\nu}_\circ^c := \bar{\nu} - \bar{\nu}_+ - \bar{\nu}_- - \bar{\nu}_\circ^\delta$ and define α_\circ^c by $\alpha_\circ^c(C) := \bar{\nu}_\circ^c(C_+ \cup C_\circ \cup C_-)$ for $C \in \mathcal{B}([0, T]) \times \mathcal{F}$, recall Lemma A.3 again. Observe that $\bar{\nu}_\circ^\delta$, $\bar{\nu}_\circ^c$ and $\bar{\nu}_-$ do not charge any element of the form $\llbracket S \rrbracket_+$ with S a $[0, T]$ -valued \mathcal{F} -measurable random variable. This follows from the maximal property of G_+ . Similarly, $\bar{\nu}_\circ^c$, $\bar{\nu}_\circ^\delta$ and $\bar{\nu}_+$ do not charge any element of the form $\llbracket S \rrbracket_-$ and $\bar{\nu}_\circ^c$, $\bar{\nu}_-$ and $\bar{\nu}_+$ do not charge any element of the form $\llbracket S \rrbracket_\circ$.

We now fix $X \in \tilde{\mathcal{S}}^\infty$ and set $u : (t, \omega) \mapsto X_{t-}(\omega)$, $v : (t, \omega) \mapsto X_t(\omega)$ and $w : (t, \omega) \mapsto X_{t+}(\omega)$. Then, $\bar{X} = u_- + v_\circ + w_+$ and, by Lemma A.1,

$$\tilde{\mu}(X) = \bar{\nu}(\bar{X}) = (\bar{\nu}_- + \bar{\nu}_\circ^\delta + \bar{\nu}_\circ^c + \bar{\nu}_+)(u_- + v_\circ + w_+).$$

Since $\bar{\nu}_-$ is supported by G_- , $\bar{\nu}_+$ by G_+ , $\bar{\nu}_\circ^\delta$ by G_\circ and $\bar{\nu}_\circ^c$ does not charge any graph of $[0, T]$ -valued \mathcal{F} -measurable random variable, we deduce that

$$\bar{\nu}_+(u_- + v_\circ + w_+) = \bar{\nu}_+(w_+) = \alpha_+(w),$$

where the last equality comes from the definition of α_+ and w . Similarly, we have

$$\begin{aligned} \bar{\nu}_-(u_- + v_\circ + w_+) &= \bar{\nu}_-(u_-) = \alpha_-(u), \\ \bar{\nu}_\circ^\delta(u_- + v_\circ + w_+) &= \bar{\nu}_\circ^\delta(v_\circ) = \alpha_\circ^\delta(v). \end{aligned}$$

Since u , v and w differs only on a countable union of graphs, it also follows that $\bar{\nu}_\circ^c(u_- + v_\circ + w_+) = \bar{\nu}_\circ^c(v_\circ) = \alpha_\circ^c(v)$. Hence

$$\mu(X) = \alpha_-(u) + \alpha_\circ^c(v) + \alpha_\circ^\delta(v) + \alpha_+(w)$$

which is assertion 5. of the claim. \square

Conclusion of the proof of Theorem 2.1. Observe that $\mathcal{S}^1(\mathbb{Q}_Z)$ is closed in the set $\tilde{\mathcal{S}}^1(\mathbb{Q}_Z)$ of all *l\`a*d*l\`a*g $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable processes X satisfying

$\mathbb{E}^{\mathbb{Q}_Z}[\|X\|_*] < \infty$. Using the Hahn-Banach theorem, we can find an extension $\tilde{\mu}$ of μ defined on $\tilde{\mathcal{S}}^1(\mathbb{Q}_Z)$, i.e. $\tilde{\mu}(X) = \mu(X)$ for $X \in \mathcal{S}^1(\mathbb{Q}_Z)$. Obviously, $\tilde{\mu}$ satisfies condition (C1) of Lemma A.1. Thus it satisfies the representation of 5. of Lemma A.4, and so does μ on \mathcal{S}^∞ .

Since $\mu(X) = 0$ for all *l\`a*d*l\`a*g processes X such that $X = 0$ \mathbb{Q}_Z -a.s., the measures α_- , α_\circ and α_+ admit transition kernels with respect to $\mathbb{P} \sim \mathbb{Q}_Z$. We can thus find three \mathbb{R}^d -valued processes \tilde{A}^- , \tilde{A}° and \tilde{A}^+ with essentially bounded total variation satisfying for $X \in \mathcal{S}^\infty$:

$$\mu(X) = \mathbb{E} \left[\int_0^T X_{t-} d\tilde{A}_t^- + \int_0^T X_t d\tilde{A}_t^\circ + \int_0^T X_{t+} d\tilde{A}_t^+ \right],$$

with $\tilde{A}_0^- = 0$ and $\tilde{A}_T^+ = \tilde{A}_{T-}^+$, \tilde{A}^+ and \tilde{A}^- are pure jump processes. To conclude, it suffices to replace \tilde{A}^- by its dual predictable projection A^- , and $\tilde{A}^\circ, \tilde{A}^+$ by their dual optional projections A° and A^+ . One can always add the continuous parts of A^- and A^+ to A° to reduce to the case where A^- and A^+ coincide with pure jumps processes. \square

References

- [1] Bismut J.-M. (1979). Temps d'arrêt optimal, quasi-temps d'arrêt et retournement du temps. *Ann. Probab.* 7, 933-964.
- [2] Bouchard B. and E. Temam (2005). On the Hedging of American Options in Discrete Time Markets with Proportional Transaction Costs. *Electronic Journal of Probability*, 10, 746-760.
- [3] Campi L. and W. Schachermayer (2006). A super-replication theorem in Kabanov's model of transaction costs. *Finance and Stochastics*, 10(4), 579-596.
- [4] Chalasani P. and S. Jha (2001). Randomized stopping times and American option pricing with transaction costs. *Mathematical Finance*, 11(1), 33-77.
- [5] Delbaen F. and W. Schachermayer (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Annalen*, **312**, 215-250.
- [6] Dellacherie C. (1972). *Capacités et processus stochastiques*. Springer-Verlag.
- [7] Denis E., D. De Vallière and Y. Kabanov (2008). Hedging of american options under transaction costs. *preprint*.
- [8] El Karoui N. (1979). Les aspects probabilistes du contrôle stochastique. *Ecole d'Eté de Probabilités de Saint Flour IX*, Lecture Notes in Mathematics 876, Springer Verlag.

- [9] El Karoui N. (1982). Une propriété de domination de l'enveloppe de Snell des semimartingales fortes. *Sém. prob. Strasbourg*, 16, 400-408.
- [10] Kabanov Y. and G. Last (2002). Hedging under transaction costs in currency markets: a continuous time model. *Mathematical Finance*, 12, 63-70.
- [11] Kabanov Y. and C. Stricker (2002). Hedging of contingent claims under transaction costs. *Advances in Finance and Stochastics*. Eds. K. Sandmann and Ph. Schönbucher, Springer, 125-136.
- [12] Karatzas I. and S. E. Shreve (1991). *Brownian motion and stochastic calculus*. Springer Verlag, Berlin.
- [13] Kramkov D. (1996). Optional decomposition of supermartingales and hedging in incomplete security markets. *Probability Theory and Related Fields*, 105 (4), 459-479.
- [14] Meyer P.A. (1966). *Probabilités et potentiel*. Hermann, Paris.
- [15] Meyer P.A. (1976). Un cours sur les intégrales stochastiques. *Sém. prob. Strasbourg*, 10, 245-400.
- [16] Rasonyi M. (2003). A remark on the superhedging theorem under transaction costs. *Séminaire de Probabilités XXXVII*, Lecture Notes in Math., 1832, Springer, Berlin-Heidelberg-New York, 394-398.
- [17] Schachermayer W. (2004). The Fundamental Theorem of Asset Pricing under Proportional Transaction Costs in Finite Discrete Time. *Mathematical Finance*, 14 (1), 19-48.