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Adaptive sequential estimation for ergodic diffusion processes in quadratic metric.

Part 2: Asymptotic efficiency.

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Abstract

Asymptotic efficiency is proved for the constructed in part 1 procedure, i.e. Pinsker's constant is found in the asymptotic lower bound for the minimax quadratic risk. It is shown that the asymptotic minimax quadratic risk of the constructed procedure coincides with this constant. ^{1 2}

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²Key words: adaptive estimation, asymptotic bounds, efficient estimation, nonparametric estimation, non-asymptotic estimation, oracle inequality, Pinsker's constant.

1 Introduction

The paper is a continuation of the investigation carried in [11] and it deals with asymptotic nonparametric estimation of the drift coefficient S in observed diffusion process $(y_t)_{t \geq 0}$ governed by the stochastic differential equation

$$dy_t = S(y_t) dt + dw_t, \quad 0 \leq t \leq T, \quad y_0 = y, \quad (1.1)$$

where $(w_t)_{t \geq 0}$ is a scalar standard Wiener process, $y_0 = y$ is a initial condition.

In the paper [11] we have constructed a non-asymptotic adaptive procedure for which a sharp non-asymptotic oracle inequality is obtained. This oracle inequality gives a upper bound for a quadratic risk. In this paper we analyze asymptotic properties (as $T \rightarrow \infty$) of the above adaptive procedure and state that it is asymptotically efficient. This means that the procedure provides the optimal convergence rate and the best constant (the Pinsker constant).

The problem of asymptotic (as $T \rightarrow \infty$) minimax nonparametric estimation of the drift coefficient S in the model (1.1) has been studied in a number of papers, see for example, [2]–[10]. So the papers [6], [8] and [10] deal with the estimation problem at a fixed point. In [8] and [10] in the case of known smoothness of the function S , efficient procedures were constructed which possess the optimal convergence rate and which provide the sharp minimax constant in asymptotic risks. Further in [8], a adaptive estimation procedure was given when the smoothness of the function S is unknown, the procedure provides the optimal convergence rate. Moreover, for estimation in \mathcal{L}_2 -norme, in [7] a adaptive sequential estimation procedure was constructed. The procedure possesses the optimal convergence rate and

it is based on the model selection (see, [1] and [4]).

The sharp asymptotic bounds and efficient estimators for the drift S in model (1.1) with the known Sobolev smoothness was given in [3] and with unknown one in [2] for local weighted \mathcal{L}_2 -losses, where the weight function is equal to the squared unknown ergodic density. Note that the weighted \mathcal{L}_2 -risk considered in the papers [2]-[3] is restrictive for the following reasons. The ergodic density being exponentially decreasing, the feasible estimation is possible on an finite interval which depends on unknown function S . Moreover, the weighted \mathcal{L}_2 -risk in these papers is local and the centres of vicinities in the local risk should be smoother than the function to be estimated. Since in the local risk the vicinity radius tends to zero, it means really that the proposed procedure estimates the centre of the vicinity which can be estimated with a better convergence rate. So the approach proposed in [2]-[3] permits to calculate the sharp asymptotic constant by losing the optimal convergence rate.

In this paper we consider the global \mathcal{L}_2 -risk and we show how to obtain the optimal convergence rate and to reach the Pinsker constant. We prove that the constructed in [11] procedure provides the both above properties.

The paper is organized as follows. In the next Section we formulate the problem and give the definitions of the functional classes and the global quadratic risk. In Section 3 the sequential adaptive procedure is constructed. The sharp upper bound for the global minimax quadratic risk over all estimates is given in Section 4 (Th. 4.1). In Section 5 we prove that the lower bound of the global risk for the sequential kernel estimate coincides with the sharp lower bound, i.e. this estimate is asymptotically efficient. The

Appendix contains the proofs of auxiliary results.

2 Main results.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions and $(w_t, \mathcal{F}_t)_{t \geq 0}$ be a standard Wiener process.

Suppose that the observed process $(y_t)_{t \geq 0}$ is governed by the stochastic differential equation (1.1), where the unknown function $S(\cdot)$ satisfies the Lipschitz condition, $S \in Lip_L(\mathbb{R})$, with

$$Lip_L(\mathbb{R}) = \left\{ f \in \mathcal{C}(\mathbb{R}) : \sup_{x, y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|} \leq L \right\}.$$

In this case the equation (1.1) admits a strong solution. We denote by $(\mathcal{F}_t^y)_{t \geq 0}$ the natural filtration of the process $(y_t)_{t \geq 0}$ and by \mathbf{E}_S the expectation with respect to the distribution law \mathbf{P}_S of the process $(y_t)_{t \geq 0}$ given the drift S . The problem is to estimate the function S in $\mathcal{L}_2[a, b]$ -risk, for some $a < b, b - a \geq 1$, i.e. for any estimate \hat{S}_T of S based on $(y_t)_{0 \leq t \leq T}$, we consider the following quadratic risk :

$$\mathcal{R}(\hat{S}_T, S) = \mathbf{E}_S \|\hat{S}_T - S\|^2, \quad \|S\|^2 = \int_a^b S^2(x) dx. \quad (2.1)$$

To obtain a good estimate for the function S it is necessary to impose some conditions on the function S which are similar to the periodicity of the deterministic signal in the white noise model. One of conditions which is sufficient for this purpose is the assumption that the process (y_t) in (1.1) returns to any vicinity of each point $x \in [a, b]$ infinite times. The ergodicity of (y_t) provides this property.

Let $L \geq 1$. We define the following functional class :

$$\begin{aligned} \Sigma_L &= \{S \in Lip_L(\mathbb{R}) : \sup_{|z| \leq L} |S(z)| \leq L; \forall |x| \geq L, \\ &\exists \dot{S}(x) \in \mathcal{C} \text{ such that } -L \leq \dot{S}(x) \leq -1/L\}. \end{aligned} \quad (2.2)$$

It is easy to see that

$$\nu^* = \sup_{x \in [a, b]} \sup_{S \in \Sigma_L} S^2(x) < \infty. \quad (2.3)$$

Moreover, if $S \in \Sigma_L$, then there exists the ergodic density

$$q(x) = q_S(x) = \frac{\exp\{2 \int_0^x S(z) dz\}}{\int_{-\infty}^{+\infty} \exp\{2 \int_0^y S(z) dz\} dy} \quad (2.4)$$

(see, e.g., Gihman and Skorohod (1972), Ch.4, 18, Th2). It is easy to see that this density satisfies the following inequalities

$$0 < q_* := \inf_{|x| \leq b_*} \inf_{S \in \Sigma_L} q_S(x) \leq \sup_{x \in \mathbb{R}} \sup_{S \in \Sigma_L} q_S(x) := q^* < \infty, \quad (2.5)$$

where $b_* = 1 + |a| + |b|$. Let S_0 be a known k times differentiable function from Σ_L . We define the following functional class

$$W_r^k = \{S : S - S_0 \in \Sigma_L \cap \mathcal{C}_{per}^k([a, b]), \sum_{j=0}^k \|S^{(j)} - S_0^{(j)}\|^2 \leq r\}, \quad (2.6)$$

where $r > 0$, $k \geq 1$ are some parameters, $\mathcal{C}_{per}^k([a, b])$ is a set of k times differentiable functions $f : [a, b] \rightarrow \mathbb{R}$ such that $f^{(i)}(a) = f^{(i)}(b)$ for all $0 \leq i \leq k$.

Note that, we can represent the functional class W_r^k as the ellipse in $\mathcal{L}_2[a, b]$, i.e.

$$W_r^k = \{S : S - S_0 \in \Sigma_L \cap \mathcal{C}_{per}^k([a, b]), \sum_{j=1}^{\infty} \varpi_j \theta_j^2 \leq r\}, \quad (2.7)$$

where

$$\varpi_j = \sum_{i=0}^k \left(\frac{2\pi[j/2]}{b-a} \right)^{2i} \quad \text{and} \quad \theta_j = \int_a^b (S(x) - S_0(x)) \phi_j(x) dx.$$

Here $(\phi_j)_{j \geq 1}$ is the standard trigonometric basis in $\mathcal{L}_2[a, b]$ (see the definition (4.6) in [11]) and $[a]$ is the integer part of a number a .

Remark 2.1. *Note that the functional class W_r^k is an ellipse with the centre at S_0 . Usually in such kind problems one takes an ellipse with the centre $S_0 \equiv 0$. In the model (1.1) we cannot take $S_0 \equiv 0$ as the centre since this function does not belong to the space Σ_L , i.e. the process (1.1) is not ergodic for this function.*

In [11] we constructed an adaptive sequential estimator \hat{S}_* for which the oracle inequality (Theorem 4.2) holds. In this paper we prove that this inequality is sharp in the asymptotic sense, i.e. we show that the minimax quadratic risk for \hat{S}_* asymptotically equals to the Pinsker constant.

To formulate our asymptotic results we define the following normalizing coefficient

$$\gamma(S) = ((1 + 2k)r)^{1/(2k+1)} \left(\frac{J(S)k}{\pi(k+1)} \right)^{2k/(2k+1)} \quad (2.8)$$

with

$$J(S) = \int_a^b \frac{1}{q_S(u)} du. \quad (2.9)$$

It is well known that for any $S \in W_r^k$ the optimal rate is $T^{2k/(2k+1)}$ (see, for example, [7]). In this paper we show that the adaptive estimator \hat{S}_* , defined by (4.17) in [11], is asymptotically efficient.

Theorem 2.1. *The quadratic risk (2.1) of the sequential estimator \hat{S}_* has the following asymptotic upper bound*

$$\limsup_{T \rightarrow \infty} T^{2k/(2k+1)} \sup_{S \in W_r^k} \frac{\mathcal{R}(\hat{S}_*, S)}{\gamma(S)} \leq 1. \quad (2.10)$$

Moreover, the following result claims that this upper bound is sharp.

Theorem 2.2. *For any estimator \hat{S}_T of S measurable with respect to \mathcal{F}_T^y ,*

$$\liminf_{T \rightarrow \infty} \inf_{\hat{S}_T} T^{2k/(2k+1)} \sup_{S \in W_r^k} \frac{\mathcal{R}(\hat{S}_T, S)}{\gamma(S)} \geq 1. \quad (2.11)$$

Our approach is based on the truncated sequential procedure proposed in [6], [7] and [10] for the diffusion model (1.1). Through this procedure we pass to discrete regression model in which we make use of the adaptive procedure \hat{S}_* proposed in [9] for the family $(\hat{S}_\alpha, \alpha \in \mathcal{A})$, where \hat{S}_α is a weighted least squares estimator with the Pinsker weights. In the next section we describe the discrete regression model.

3 Adaptive procedure

We remind of that in [11] we pass by the sequential method to discrete scheme at the points

$$x_l = a + \frac{l}{n}(b - a), \quad 1 \leq l \leq n, \quad (3.1)$$

with $n = 2[(T - 1)/2] + 1$. At each x_l we use the sequential kernel estimator

$$\begin{cases} S_l^* &= \frac{1}{H_l} \int_{t_0}^{\tau_l} Q\left(\frac{y_s - x_l}{h}\right) dy_s, \\ \tau_l &= \inf\{t \geq t_0 : \int_{t_0}^t Q\left(\frac{y_s - x_l}{h}\right) ds \geq H_l\}, \end{cases} \quad (3.2)$$

where $h = (b - a)/(2n)$, $Q(z) = \mathbf{1}_{\{|z| \leq 1\}}$ and

$$H_l = (T - t_0)(2\tilde{q}_T(x_l) - \epsilon_T^2)h$$

with

$$\tilde{q}_T(x_l) = \max\{\hat{q}(x_l), \epsilon_T\} \quad \text{and} \quad \hat{q}(x_l) = \frac{1}{2t_0h} \int_0^{t_0} Q\left(\frac{y_s - x_l}{h}\right) ds.$$

Note that $\tau_l < \infty$ a.s., for any $S \in \Sigma_L$ and for all $1 \leq l \leq n$ (see, [8]).

Moreover, we assume that the parameters $t_0 = t_0(T)$ and ϵ_T satisfy the following conditions

H₁) For any $T \geq 32$,

$$16 \leq t_0 \leq T/2 \quad \text{and} \quad \sqrt{2}/t_0^{1/8} \leq \epsilon_T \leq 1.$$

H₂) For any $\delta > 0$ and $\nu > 0$,

$$\lim_{T \rightarrow \infty} T^\nu e^{-\delta\sqrt{t_0}} = 0.$$

H₃)

$$\lim_{T \rightarrow \infty} t_0(T) = \infty, \quad \lim_{T \rightarrow \infty} \epsilon_T = 0, \quad \lim_{T \rightarrow \infty} T\epsilon_T/t_0(T) = \infty.$$

From (1.1), (3.1)–(3.2) we obtain the discrete regression model

$$S_l^* = S(x_l) + \zeta_l.$$

The error term ζ_l is represented as the following sum of the approximated term B_l and the stochastic term

$$\zeta_l = B_l + \frac{1}{\sqrt{H_l}} \xi_l,$$

where

$$B_l = \frac{1}{H_l} \int_{t_0}^{\tau_l} Q\left(\frac{y_s - x_l}{h}\right) (S(y_s) - S(x_l)) ds,$$

$$\xi_l = \frac{1}{\sqrt{H_l}} \int_{t_0}^{\tau_l} Q\left(\frac{y_s - x_l}{h}\right) dw_s.$$

Moreover, note that for any function $S \in W_r^k$

$$|B_l| \leq 2Lh = L(b-a)/n. \quad (3.3)$$

It is easy to see that the random variables $(\xi_l)_{1 \leq l \leq n}$ are i.i.d. normal $\mathcal{N}(0, 1)$.

Therefore, by putting

$$\Gamma = \Gamma_T = \left\{ \max_{1 \leq l \leq n} \tau_l \leq T \right\} \quad \text{and} \quad Y_l = S_l^* \mathbf{1}_\Gamma,$$

we obtain on the set Γ the following regression model

$$Y_l = S(x_l) + \zeta_l, \quad \zeta_l = B_l + \sigma_l \xi_l, \quad (3.4)$$

where

$$\sigma_l^2 = \frac{n}{(T - t_0)(\tilde{q}_T(x_l) - \epsilon_T^2/2)(b-a)} \leq \frac{4}{\epsilon_T(b-a)} := \sigma_*.$$

In Appendix A.1 we prove the following result:

Proposition 3.1. *Suppose that the parameters t_0 and ϵ_T satisfy the conditions $\mathbf{H}_1)$ – $\mathbf{H}_3)$. Then, for any $L \geq 1$,*

$$\lim_{T \rightarrow \infty} \sup_{S \in \Sigma_L} \sup_{1 \leq l \leq n} \mathbf{E}_S |\bar{\sigma}_l| = 0,$$

where

$$\bar{\sigma}_l = \sigma_l^2 - \frac{1}{q_S(x_l)(b-a)}.$$

Now we suppose that the parameters k and r of the space W_r^k in (2.7) are unknown. We describe the adaptive procedure from [11]. First we fixe $\varepsilon > 0$ and we define the sieve \mathcal{A}_ε in the space $\mathbb{N} \times \mathbb{R}_+$:

$$\mathcal{A}_\varepsilon = \{1, \dots, k_\varepsilon\} \times \{t_1, \dots, t_m\}, \quad (3.5)$$

where $k_* = \lceil 1/\sqrt{\varepsilon} \rceil$, $t_i = i\varepsilon$, $m = \lceil 1/\varepsilon^2 \rceil$ and we take $\varepsilon = 1/\ln n$. We remind of that $n = 2\lceil (T-1)/2 \rceil + 1 \geq 30$, due to condition \mathbf{H}_1 .

For any $\alpha = (\beta, t) \in \mathcal{A}_\varepsilon$ we define the weight vector $\lambda_\alpha = (\lambda_\alpha(1), \dots, \lambda_\alpha(n))'$ with

$$\lambda_\alpha(j) = \begin{cases} 1, & \text{for } 1 \leq j \leq j_0, \\ (1 - (j/\omega_\alpha)^\beta)_+, & \text{for } j_0 < j \leq n, \end{cases} \quad (3.6)$$

where $j_0 = j_0(\alpha) = \lceil \omega_\alpha / \ln(n+2) \rceil + 1$,

$$\omega_\alpha = (A_\beta t n)^{1/(2\beta+1)} \quad \text{and} \quad A_\beta = \frac{(b-a)^{2\beta+1}(\beta+1)(2\beta+1)}{\pi^{2\beta}\beta}.$$

For any $\alpha \in \mathcal{A}_\varepsilon$, through the weight $\lambda_\alpha = (\lambda_\alpha(1), \dots, \lambda_\alpha(n))'$ we construct the weighted least squares estimator

$$\begin{cases} \hat{S}_\alpha &= S_0 + \sum_{j=1}^n \lambda_\alpha(j) \hat{\theta}_{j,n} \phi_j \mathbf{1}_\Gamma, \\ \hat{\theta}_{j,n} &= ((b-a)/n) \sum_{l=1}^n (Y_l - S_0(x_l)) \phi_j(x_l). \end{cases} \quad (3.7)$$

We remind of (see Section 4 in [11]) that to construct an adaptive procedure one has to minimize the empiric squared error of estimator (3.7) over the weight family $\{\lambda_\alpha, \alpha \in \mathcal{A}_\varepsilon\}$. A difficulty appears since the empiric squared error contains a term which depends on unknown function S . We estimate this term as follows

$$\tilde{\theta}_{j,n} = \hat{\theta}_{j,n}^2 - \frac{(b-a)^2}{n} s_{j,n} \quad \text{with} \quad s_{j,n} = \frac{1}{n} \sum_{l=1}^n \sigma_l^2 \phi_j^2(x_l).$$

For any $\lambda \in \{\lambda_\alpha, \alpha \in \mathcal{A}_\varepsilon\}$ we define the empiric cost function $J_n(\lambda)$ by the following way

$$J_n(\lambda) = \sum_{j=1}^n \lambda^2(j) \hat{\theta}_{j,n}^2 - 2 \sum_{j=1}^n \lambda(j) \tilde{\theta}_{j,n} + \frac{1}{\ln T} P_n(\lambda)$$

with the penalty term defined as

$$P_n(\lambda) = \frac{|\lambda|^2(b-a)^2 s_n}{n},$$

where $|\lambda|^2 = \sum_{j=1}^n \lambda^2(j)$ and $s_n = n^{-1} \sum_{l=1}^n \sigma_l^2$. We set

$$\hat{\alpha} = \operatorname{agrmin}_{\alpha \in \mathcal{A}_\varepsilon} J_n(\lambda_\alpha) \quad \text{and} \quad \hat{S}_* = \hat{S}_{\hat{\alpha}}. \quad (3.8)$$

In [11] we proved the following non-asymptotic oracle inequality.

Theorem 3.2. *Assume that $S \in \Sigma_L$ and \hat{S}_α is defined in (3.7). Then, for any $T \geq 32$, the adaptive estimator (3.8) satisfies the following inequality*

$$\mathcal{R}(\hat{S}_*, S) \leq (1 + D(\rho)) \min_{\alpha \in \mathcal{A}_\varepsilon} \mathcal{R}(\hat{S}_\alpha, S) + \frac{\mathcal{B}_T(\rho)}{n}, \quad (3.9)$$

where

$$\rho = 1/(6 + \ln n) \quad \text{and} \quad n = 2[(T-1)/2] + 1.$$

Moreover, the functions $D(\rho)$ and $\mathcal{B}_T(\rho)$ defined in Theorem 4.2 from [11] are such that $\lim_{\rho \rightarrow 0} D(\rho) = 0$ and, for any $\delta > 0$,

$$\lim_{T \rightarrow \infty} \frac{\mathcal{B}_T(\rho)}{T^\delta} = 0. \quad (3.10)$$

Our principal goal in this paper is to show that the inequality (3.9) is sharp in asymptotic sense, i.e. it yields inequalities (2.10) and (2.11).

4 Upper bound

4.1 Known smoothness

We start with the estimation problem (1.1) under the condition that $S \in W_r^k$ with known parameters k, r and $J(S)$ defined in (2.8). In this case we use

the estimator from family (3.7)

$$\tilde{S} = \hat{S}_{\tilde{\alpha}} \quad \text{with} \quad \tilde{\alpha} = (k, \tilde{t}_n), \quad \tilde{t}_n = \tilde{l}_n \varepsilon, \quad (4.1)$$

where

$$\tilde{l}_n = \inf\{i \geq 1 : i\varepsilon \geq \bar{r}(S)\}, \quad \bar{r}(S) = r/J(S)$$

and $\varepsilon = \varepsilon_n = 1/\ln n$. Note that for sufficiently large T , therefore large $m = \lceil 1/\varepsilon^2 \rceil = \lceil \ln^2 n \rceil$, the parameter $\tilde{\alpha}$ belongs to the set (3.5). In this section we obtain the upper bound for the empiric squared error of the estimator (4.1). We define the empiric squared error of the estimator \tilde{S} as

$$\|\tilde{S} - S\|_n^2 = \frac{b-a}{n} \sum_{l=1}^n (\tilde{S}(x_l) - S(x_l))^2,$$

where the points $(x_l)_{1 \leq l \leq n}$ are defined in (3.1).

Theorem 4.1. *The estimator \tilde{S} satisfies the following asymptotic upper bound*

$$\limsup_{T \rightarrow \infty} T^{2k/(2k+1)} \sup_{S \in W_r^k} \frac{1}{\gamma(S)} \mathbf{E}_S \|\tilde{S} - S\|_n^2 \mathbf{1}_\Gamma \leq 1. \quad (4.2)$$

Proof. We denote $\tilde{\lambda} = \lambda_{\tilde{\alpha}}$ and $\tilde{\omega} = \omega_{\tilde{\alpha}}$. Now we remind of that the sieve Fourier coefficients $(\hat{\theta}_{j,n})$ defined in (3.7) satisfy on the set Γ the following relation (see [11])

$$\hat{\theta}_{j,n} = \theta_{j,n} + \zeta_{j,n} \quad (4.3)$$

with

$$\theta_{j,n} = \frac{b-a}{n} \sum_{l=1}^n (S(x_l) - S_0(x_l)) \phi_j(x_l)$$

and

$$\zeta_{j,n} = (\zeta, \phi_j)_n = \frac{b-a}{\sqrt{n}} \xi_{j,n} + \delta_{j,n},$$

where

$$\xi_{j,n} = \frac{1}{\sqrt{n}} \sum_{l=1}^n \sigma_l \xi_l \phi_j(x_l) \quad \text{and} \quad \delta_{j,n} = \frac{b-a}{n} \sum_{l=1}^n B_l \phi_j(x_l). \quad (4.4)$$

The inequality (3.3) implies that

$$|\delta_{j,n}| \leq L(b-a)^{3/2}/n. \quad (4.5)$$

On the set Γ we can represent the empiric squared error as follows

$$\begin{aligned} \|\tilde{S} - S\|_n^2 &= \sum_{j=1}^n (1 - \tilde{\lambda}(j))^2 \theta_{j,n}^2 + 2(b-a)M_n \\ &\quad + 2 \sum_{j=1}^n (1 - \tilde{\lambda}(j)) \tilde{\lambda}(j) \theta_{j,n} \delta_{j,n} + \sum_{j=1}^n \tilde{\lambda}^2(j) \zeta_{j,n}^2, \end{aligned} \quad (4.6)$$

where

$$M_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n (1 - \tilde{\lambda}(j)) \tilde{\lambda}(j) \theta_{j,n} \xi_{j,n}.$$

Note that, for any $\rho > 0$,

$$2 \sum_{j=1}^n (1 - \tilde{\lambda}(j)) \tilde{\lambda}(j) \theta_{j,n} \delta_{j,n} \leq \rho \sum_{j=1}^n (1 - \tilde{\lambda}(j))^2 \theta_{j,n}^2 + \rho^{-1} \sum_{j=1}^n \tilde{\lambda}^2(j) \delta_{j,n}^2.$$

Therefore by (4.4)-(4.5) we obtain that

$$\begin{aligned} \|\tilde{S} - S\|_n^2 &\leq (1 + \rho) \sum_{j=1}^n (1 - \tilde{\lambda}(j))^2 \theta_{j,n}^2 + 2(b-a)M_n \\ &\quad + \frac{L^2(b-a)^3}{\rho n} + \sum_{j=1}^n \tilde{\lambda}^2(j) \zeta_{j,n}^2. \end{aligned}$$

By the same way we estimate the last term in the right-hand part as

$$\begin{aligned} \sum_{j=1}^n \tilde{\lambda}^2(j) \zeta_{j,n}^2 &\leq \frac{(1 + \rho)(b-a)^2}{n} \sum_{j=1}^n \tilde{\lambda}^2(j) \xi_{j,n}^2 \\ &\quad + (1 + \rho^{-1}) \frac{L^2(b-a)^3}{n}. \end{aligned}$$

Therefore on the set Γ we find that

$$\begin{aligned} \|\tilde{S}_n - S\|_n^2 &\leq (1 + \rho)\hat{\gamma}_n(S) + 2(b - a)M_n \\ &\quad + (1 + \rho)\Delta_n + \frac{L^2(b - a)^3(\rho + 2)}{\rho n}, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} \hat{\gamma}_n(S) &= \sum_{j=1}^n (1 - \tilde{\lambda}(j))^2 \theta_{j,n}^2 + \frac{J(S)}{(b - a)n} \sum_{j=1}^n \tilde{\lambda}^2(j), \\ \Delta_n &= \frac{1}{n} \sum_{j=1}^n \tilde{\lambda}^2(j) \left((b - a)^2 \xi_{j,n}^2 - \frac{J(S)}{b - a} \right). \end{aligned}$$

Let us estimate the first term in the right-hand part of (4.7). Note that the bounds (2.5) imply the corresponding bounds for the function $J(S)$, i.e.

$$0 < \frac{b - a}{q^*} \leq \inf_{S \in \Sigma_L} J(S) \leq \sup_{S \in \Sigma_L} J(S) \leq \frac{b - a}{q_*} < \infty. \quad (4.8)$$

This implies directly that

$$\lim_{n \rightarrow \infty} \sup_{S \in \Sigma_L} \left| \frac{\tilde{t}_n}{\bar{r}(S)} - 1 \right| = 0. \quad (4.9)$$

Moreover, from the definition (3.6) we get that

$$\lim_{n \rightarrow \infty} \sup_{S \in \Sigma_L} \left| \frac{\sum_{j=1}^n \tilde{\lambda}(j)^2}{n^{1/(2k+1)}} - \frac{(A_k \bar{r}(S))^{1/(2k+1)} k^2}{(k + 1)(2k + 1)} \right| = 0. \quad (4.10)$$

Taking this into account, in Appendix A.2 we show that

$$\limsup_{T \rightarrow \infty} \sup_{S \in W_T^k} T^{2k/(2k+1)} \frac{\hat{\gamma}_n(S)}{\gamma(S)} \leq 1. \quad (4.11)$$

To estimate the second term in the right-hand part of inequality (4.7) we use Lemma A.1. We get

$$\mathbf{E}_S M_n^2 \leq \frac{\sigma_*}{(b - a)n} \sum_{j=1}^n \theta_{j,n}^2 = \frac{\sigma_*}{(b - a)n} \|S\|_n^2 \leq \frac{\sigma_* \mathcal{V}^*}{(b - a)n},$$

where the constants σ_* and ν^* are defined in (3.4) and (2.3), respectively.

Taking into account that $\mathbf{E}_S M_n = 0$ and making use of Proposition 3.1 from [11] we obtain that

$$|\mathbf{E}_S M_n \mathbf{1}_\Gamma| = |\mathbf{E}_S M_n \mathbf{1}_{\Gamma^c}| \leq \sqrt{\sigma_* \Pi_T} \frac{L}{\sqrt{n}}.$$

Therefore

$$\lim_{T \rightarrow \infty} T^{2k/(2k+1)} \sup_{S \in W_r^k} |\mathbf{E}_S M_n \mathbf{1}_\Gamma| = 0. \quad (4.12)$$

Now we show that

$$\lim_{T \rightarrow \infty} T^{2k/(2k+1)} \sup_{S \in W_r^k} |\mathbf{E}_S \Delta_n| = 0. \quad (4.13)$$

First of all, note that, for $j \geq 2$,

$$\begin{aligned} (b-a)^2 \mathbf{E}_S \xi_{j,n}^2 &= \frac{(b-a)^2}{n} \mathbf{E}_S \sum_{l=1}^n \sigma_l^2 \phi_j^2(x_l) \\ &= (b-a) \mathbf{E}_S s_n + (b-a) \mathbf{E}_S \bar{\varsigma}_{j,n}, \end{aligned} \quad (4.14)$$

where

$$\bar{\varsigma}_{j,n} = \frac{1}{n} \sum_{l=1}^n \sigma_l^2 \bar{\phi}_j(x_l) \quad \text{with} \quad \bar{\phi}_j(z) = (b-a) \phi_j^2(z) - 1.$$

Moreover,

$$\begin{aligned} \sup_{S \in W_r^k} \left| (b-a) \mathbf{E}_S s_n - \frac{J(S)}{b-a} \right| &\leq \frac{b-a}{n} \sum_{l=1}^n \sup_{S \in W_r^k} \mathbf{E}_S |\bar{\sigma}_l| \\ &\quad + \sup_{S \in W_r^k} \left| \frac{1}{b-a} \int_a^b q_S^{-1}(x) dx - \frac{1}{n} \sum_{l=1}^n q_S^{-1}(x_l) \right| \\ &\leq (b-a) \sup_{S \in W_r^k} \max_{1 \leq l \leq n} \mathbf{E}_S |\bar{\sigma}_l| + \frac{q'_*(b-a)}{q_* n}, \end{aligned}$$

where $q'_* = \max_{a \leq x \leq b} \sup_{S \in \Sigma_L} |q'_S(x)|$. Therefore Proposition 3.1 implies that

$$\lim_{n \rightarrow \infty} T^{2k/(2k+1)} \sup_{S \in W_r^k} \left| (b-a) \mathbf{E}_S s_n - \frac{J(S)}{b-a} \right| = 0.$$

To estimate the second term in (4.14) we make use of Lemma 6.2 from [9].

We have

$$\begin{aligned} \left| \sum_{j=1}^n \tilde{\lambda}^2(j) \bar{\varsigma}_{j,n} \right| &= \frac{1}{n} \left| \sum_{l=1}^n \sigma_l^2 \sum_{j=1}^n \tilde{\lambda}^2(j) \bar{\phi}_j(x_l) \right| \\ &\leq \frac{1}{n} \sum_{l=1}^n \sigma_l^2 \left| \sum_{j=1}^n \tilde{\lambda}^2(j) \bar{\phi}_j(x_l) \right| \\ &\leq \sigma_* (2^{2k+1} + 2^{k+2} + 1) \leq 5\sigma_* 2^{2k} \quad \text{a.s.} \end{aligned}$$

Thus from (4.10) we obtain (4.13). Moreover, we can calculate that

$$\mathbf{E}_S \xi_{j,n}^4 \leq \frac{3\sigma_*^2}{(b-a)^2}.$$

Due to Proposition 3.1 from [11], we obtain that

$$\begin{aligned} \mathbf{E}_S |\Delta_n| \mathbf{1}_{\Gamma^c} &\leq \frac{(b-a)^2}{n} \sum_{j=1}^n \mathbf{E}_S \xi_{j,n}^2 \mathbf{1}_{\Gamma^c} + J(S) \Pi_T \\ &\leq \frac{\sqrt{3}\sigma_*}{b-a} \sqrt{\Pi_T} + \frac{1}{q_*} \Pi_T. \end{aligned}$$

This means that

$$\lim_{n \rightarrow \infty} T^{2k/(2k+1)} \sup_{S \in W_r^k} \mathbf{E}_S |\Delta_n| \mathbf{1}_{\Gamma^c} = 0.$$

Therefore by (4.13) we get that

$$\lim_{n \rightarrow \infty} T^{2k/(2k+1)} \sup_{S \in W_r^k} |\mathbf{E}_S \mathbf{1}_\Gamma \Delta_n| = 0.$$

Hence Theorem 4.1. \square

4.2 Unknown smoothness

In this subsection we prove Theorem 2.1. First of all notice that inequalities (4.8) yield

$$\inf_{S \in W_r^k} \gamma(S) > 0.$$

Therefore Theorem 4.1, upper bound (2.3) and Proposition 3.1 from [11] imply that

$$\limsup_{T \rightarrow \infty} T^{2k/(2k+1)} \sup_{S \in W_r^k} \frac{1}{\gamma(S)} \mathbf{E}_S \|\tilde{S} - S\|_n^2 \leq 1. \quad (4.15)$$

Let us remind of of that we define the estimator \tilde{S} from the sieve (3.1) onto all interval $[a, b]$ by the standard method as

$$\tilde{S}(x) = \tilde{S}(x_1) \mathbf{1}_{\{a \leq x \leq x_1\}} + \sum_{l=2}^n \tilde{S}(x_l) \mathbf{1}_{\{x_{l-1} < x \leq x_l\}}, \quad (4.16)$$

where $\mathbf{1}_A$ is the indicator of a set A . Putting $\varrho(x) = \tilde{S}(x) - S(x)$ we find that

$$\begin{aligned} \|\varrho\|^2 &= \|\varrho\|_n^2 + 2 \sum_{l=1}^n \int_{x_{l-1}}^{x_l} \varrho(x_l) (S(x_l) - S(x)) dx \\ &\quad + \sum_{l=1}^n \int_{x_{l-1}}^{x_l} (S(x_l) - S(x))^2 dx. \end{aligned}$$

For any $0 < \epsilon < 1$, we estimate the norm $\|\varrho\|^2$ as

$$\|\varrho\|^2 \leq (1 + \epsilon) \|\varrho\|_n^2 + (1 + \epsilon^{-1}) \sum_{l=1}^n \int_{x_{l-1}}^{x_l} (S(x_l) - S(x))^2 dx.$$

This means that, for any $S \in \Sigma_L$,

$$\mathcal{R}(\tilde{S}, S) \leq (1 + \epsilon) \mathbf{E}_S \|\varrho\|_n^2 + (1 + \epsilon^{-1}) \frac{L^2(b-a)^3}{n^2}. \quad (4.17)$$

We recall that $\tilde{S} = \hat{S}_{\tilde{\alpha}}$ with $\tilde{\alpha} \in \mathcal{A}_\epsilon$. Therefore, Theorem 3.2 with inequalities (4.15)–(4.17) imply Theorem 2.1 \square

5 Lower bound

In this section we prove Theorem 2.2. We follow the proof of Theorem 4.2 from [13]. Similarly, we start with the approximation for an indicator function,

i.e. for any $\eta > 0$, we set

$$I_\eta(x) = \eta^{-1} \int_{\mathbb{R}} \mathbf{1}_{(|u| \leq 1-\eta)} G\left(\frac{u-x}{\eta}\right) du, \quad (5.1)$$

where the kernel $V \in \mathbf{C}^\infty(\mathbb{R})$ is a probability density on $[-1, 1]$. It is easy to see that $I_\eta \in \mathcal{C}^\infty$ and for any $m \geq 1$ and any integrable function $f(x)$

$$\lim_{\eta \rightarrow 0} \int_{\mathbb{R}} f(x) I_\eta^m(x) dx = \int_{-1}^1 f(x) dx.$$

Further, we will make use of the following trigonometric basis $\{e_j, j \geq 1\}$ in $\mathcal{L}_2[-1, 1]$ with

$$e_1(x) = 1/\sqrt{2}, \quad e_j(x) = \text{Tr}_j(\pi[j/2]x), \quad j \geq 2. \quad (5.2)$$

Here $\text{Tr}_l(x) = \cos(x)$ for even l and $\text{Tr}_l(x) = \sin(x)$ for odd l .

Moreover, we denote

$$J_0 = J(S_0), \quad q_0 = q_{S_0}, \quad \gamma_0 = \gamma(S_0),$$

where the function S_0 is defined in (2.6).

Let us now fix some arbitrary $0 < \varepsilon < 1$ and according to [13] we put

$$h = (v_\varepsilon^*)^{\frac{1}{2k+1}} N_T T^{-\frac{1}{2k+1}} \quad (5.3)$$

with

$$v_\varepsilon^* = \frac{k\pi^{2k} J_0}{(1-\varepsilon)r^{2k+1}(k+1)(2k+1)} \quad \text{and} \quad N_T = \ln^4 T.$$

To construct a parametric family we divide the interval $[a, b]$ by the intervals $[\tilde{x}_m - h, \tilde{x}_m + h]$ with $\tilde{x}_m = a + 2hm$. The maximal number of such intervals is equal to

$$M = [(b-a)/(2h)] - 1.$$

Onto each interval $[\tilde{x}_m - h, \tilde{x}_m + h]$, we approximate any unknown function by a trigonometric series with N terms, i.e. for any array $z = (z_{m,j})_{1 \leq m \leq M, 1 \leq j \leq N}$, we set

$$S_{z,T}(x) = S_0(x) + \sum_{m=1}^M \sum_{j=1}^N z_{m,j} D_{m,j}(x) \quad (5.4)$$

with $D_{m,j}(x) = e_j(v_m(x)) I_\eta(v_m(x))$ and $v_m(x) = (x - \tilde{x}_m)/h$.

Now to obtain the bayesian risk we choose a prior distribution on \mathbb{R}^{MN} by making use of the random array $\theta = (\theta_{m,j})_{1 \leq m \leq M, 1 \leq j \leq N}$ defined as

$$\theta_{m,j} = t_{m,j} \zeta_{m,j}, \quad (5.5)$$

where $\zeta_{m,j}$ are i.i.d. gaussian $\mathcal{N}(0, 1)$ random variables and the coefficients

$$t_{m,j} = \frac{\sqrt{y_j^*}}{\sqrt{Thq_0(\tilde{x}_m)}}.$$

We chose the sequence $(y_j^*)_{1 \leq j \leq N}$ by thre samle way as in (8.11) in [13], i.e.

$$y_j^* = \Omega_T j^{-k} - 1 \quad \text{with} \quad \Omega_T = \frac{R_T^* + \sum^N j^{2k}}{\sum^N j^k},$$

where

$$R_T^* = \frac{J_0 k}{\hat{J}_0(k+1)(2k+1)} N^{2k+1},$$

and

$$\hat{J}_0 = 2h \sum_{m=1}^M \frac{1}{q_0(\tilde{x}_m)}. \quad (5.6)$$

In the sequel we make use of the following set

$$\Xi_T = \left\{ \max_{1 \leq m \leq M} \max_{1 \leq j \leq N} |\zeta_{m,j}| \leq \ln T \right\}. \quad (5.7)$$

Obviously, that for any $p > 0$

$$\lim_{T \rightarrow \infty} T^p \mathbf{P}(\Xi_T^c) = 0. \quad (5.8)$$

Note that on the set Ξ_T the uniform norm

$$|S_{\theta,T} - S_0|_* = \sup_{a \leq x \leq b} |S_{\theta,T}(x) - S_0(x)|$$

is bounded

$$|S_{\theta,T} - S_0|_* \leq \frac{\ln T}{\sqrt{q_* T h}} \sum_{j=1}^N \sqrt{y_j^*} := \epsilon_T. \quad (5.9)$$

Taking into account here that

$$\lim_{T \rightarrow \infty} \hat{J}_0 = J_0 \quad (5.10)$$

it is easy to deduce that $\epsilon_T \rightarrow 0$ as $T \rightarrow \infty$.

For any estimator \hat{S}_T , we denote by \hat{S}_T^0 its projection on W_r^k , i.e. $\hat{S}_T^0 = \text{Pr}_{W_r^k}(\hat{S}_T)$. Since W_r^k is a convex set, we get that

$$\|\hat{S}_T - S\|^2 \geq \|\hat{S}_T^0 - S\|^2.$$

Therefore, denoting by μ_θ the distribution of θ in \mathbb{R}^d with $d = MN$ and taking into account (5.9) we can write that

$$\sup_{S \in W_r^k} \frac{\mathcal{R}(\hat{S}_T, S)}{\gamma(S)} \geq \frac{1}{\gamma_T^*} \int_{\{z \in \mathbb{R}^d : S_{z,n} \in W_r^k\} \cap \Xi_T} \mathbf{E}_{S_{z,T}} \|\hat{S}_T^0 - S_{z,T}\|^2 \mu_\theta(dz)$$

with

$$\gamma_T^* = \sup_{S \in \mathcal{U}_T} \gamma(S),$$

where

$$\mathcal{U}_T = \{S : |S - S_0|_* \leq \epsilon_T, S(x) = S_0(x) \text{ for } x \notin [a, b]\}.$$

Since function (2.4) is continuous with respect to S , then

$$\lim_{T \rightarrow \infty} \gamma_T^* = \gamma_0. \quad (5.11)$$

Making use of the distribution μ_θ we introduce the following Bayes risk

$$\tilde{\mathcal{R}}(\hat{S}_T) = \int_{\mathbb{R}^d} \mathcal{R}(\hat{S}_T, S_{z,T}) \mu_\theta(dz)$$

Now noting that $\|\hat{S}_T^0\|^2 \leq r$ through this risk we can write that

$$\sup_{S \in W_r^k} \frac{\mathcal{R}(\hat{S}_T, S)}{\gamma(S)} \geq \frac{1}{\gamma_T^*} \tilde{\mathcal{R}}(\hat{S}_T^0) - \frac{2}{\gamma_T^*} \varpi_T, \quad (5.12)$$

with

$$\varpi_T = \mathbf{E}(\mathbf{1}_{\{S_{\theta,T} \notin W_r^k\}} + \mathbf{1}_{\Xi_T^c})(r + \|S_{\theta,T}\|^2).$$

Propositions 7.2–7.3 from [13] imply that for any $p > 0$

$$\lim_{T \rightarrow \infty} T^p \varpi_T = 0.$$

Let us consider the first term in the right-hand side of (5.12). To obtain a lower bound for this term we use the $\mathcal{L}_2[a, b]$ -orthonormal function family $(\tilde{e}_{m,j})_{1 \leq m \leq M, 1 \leq j \leq N}$ which is defined as

$$\tilde{e}_{m,j}(x) = \frac{1}{\sqrt{h}} e_j(v_m(x)) \mathbf{1}_{(|v_m(x)| \leq 1)}.$$

We denote by $\hat{\lambda}_{m,j}$ and $\lambda_{m,j}(z)$ the Fourier coefficients for the functions \hat{S}_T^0 and S_z , respectively, i.e.

$$\hat{\lambda}_{m,j} = \int_a^b \hat{S}_T^0(x) \tilde{e}_{m,j}(x) dx \quad \text{and} \quad \lambda_{m,j}(z) = \int_a^b S_z(x) \tilde{e}_{m,j}(x) dx.$$

Now it is easy to see that

$$\|\hat{S}_T^0 - S_z\|^2 \geq \sum_{m=1}^M \sum_{j=1}^N (\hat{\lambda}_{m,j} - \lambda_{m,j}(z))^2.$$

Let us introduce the following $\mathcal{L}_1 \rightarrow \mathbb{R}$ functional

$$\bar{e}_j(f) = \int_{-1}^1 e_j^2(v) f(v) dv$$

Therefore from definition (5.4) we obtain that

$$\frac{\partial}{\partial z_{m,j}} \lambda_{m,j}(z) = \sqrt{h} \bar{e}_j(I_\eta).$$

Now Lemma A.2 implies that

$$\tilde{\mathcal{R}}(\hat{S}_T^0) \geq h \sum_{m=1}^M \sum_{j=1}^N \frac{\bar{e}_j^2(I_\eta)}{(1 + \varsigma_{m,j}(T)) \bar{e}_j(I_\eta^2) q_0(\tilde{x}_m) Th + t_{m,j}^{-2}}, \quad (5.13)$$

where

$$\varsigma_{m,j}(T) = \mathbf{E} \frac{\mathbf{E}_{S_{\theta,T}} \int_0^T D_{m,j}^2(y_t) dt}{Th \bar{e}_j(I_\eta^2) q_0(\tilde{x}_m)} - 1.$$

In Appendix we show that

$$\lim_{T \rightarrow \infty} \max_{1 \leq m \leq M} \max_{1 \leq j \leq N} |\varsigma_{m,j}(T)| = 0. \quad (5.14)$$

Therefore taking this into account in inequality (5.13) we obtain that for sufficiently large T and for arbitrary $\nu > 0$

$$\tilde{\mathcal{R}}(\hat{S}_T^0) \geq \frac{\hat{J}_0}{2Th(1+\nu)} \sum_{j=1}^N \tau_j(\eta, y_j^*),$$

where

$$\tau_j(\eta, y) = \frac{\bar{e}_j^2(I_\eta) y}{\bar{e}_j(I_\eta^2) y + 1}.$$

By making use of limit equality (8.9) from [13] we obtain that for sufficiently small η and sufficiently large T

$$\tilde{\mathcal{R}}(\hat{S}_T^0) \geq \frac{1}{(1+\nu)^2} \frac{\hat{J}_0}{2Th} \sum_{j=1}^N \frac{y_j^*}{y_j^* + 1},$$

where \hat{J}_0 is defined in (5.7). Thus making use of (5.10) this implies that

$$\liminf_{T \rightarrow \infty} \inf_{\hat{S}_T} T^{\frac{2k}{2k+1}} \tilde{\mathcal{R}}(\hat{S}_T) \geq (1 - \varepsilon)^{\frac{1}{2k+1}} \gamma_0.$$

Taking into account this inequality in (5.12) and limit equality (5.11) we obtain that for any $0 < \varepsilon < 1$

$$\liminf_{T \rightarrow \infty} \inf_{\hat{S}_T} T^{\frac{2k}{2k+1}} \sup_{S \in W_r^k} \frac{\mathcal{R}(\hat{S}_T, S)}{\gamma(S)} \geq (1 - \varepsilon)^{\frac{1}{2k+1}}.$$

Taking here limit as $\varepsilon \rightarrow 0$ implies Theorem 2.2. \square

A Appendix

A.1 Proof of Proposition 3.1

We use all notations from [11]. For any function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\sup_{y \in \mathbb{R}} |\psi(y)| < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} |\psi(y)| dy \leq c^* < \infty \quad (\text{A.1})$$

we set

$$\mathbf{M}_S(\psi) = \int_{-\infty}^{+\infty} \psi(y) q_S(y) dy \quad \text{and} \quad \Delta_T(\psi) = \frac{1}{\sqrt{T}} \int_0^T (\psi(y_t) - \mathbf{M}_S(\psi)) dt.$$

In [12] (see Theorem 3.2) we show that, for any $\nu > 0$ and for any ψ satisfying (A.1), there exists $\gamma = \gamma(c^*, L) > 0$ such that the following inequality holds

$$\sup_{S \in \Sigma_L} \mathbf{P}_S(|\Delta_T(\psi)| \geq \nu) \leq 8 e^{-\gamma \nu^2}. \quad (\text{A.2})$$

We shall apply this inequality to the function

$$\psi_{h,k}(y) = \frac{1}{h} Q\left(\frac{y - x_k}{h}\right),$$

for which $\int_{-\infty}^{\infty} \psi_{h,k}(y) dy = 2$. Note now that

$$2\hat{q}(x_k) - \mathbf{M}_S(\psi_{h,k}) = \frac{1}{\sqrt{t_0}} \Delta_{t_0}(\psi_{h,k}). \quad (\text{A.3})$$

Moreover,

$$\mathbf{M}_S(\psi_{h,k}) = \int_{-1}^1 q_S(x_k + hz) dz \geq 2q_*,$$

where q_* is defined in (2.5). Therefore we get that

$$\begin{aligned} \mathbf{P}_S(\hat{q}(x_k) < \epsilon_T) &= \mathbf{P}_S\left(\frac{1}{t_0} \int_0^{t_0} \psi_{h,k}(y_t) dt < 2\epsilon_T\right) \\ &= \mathbf{P}_S(\Delta_{t_0}(\psi_{h,k}) < (2\epsilon_T - \mathbf{M}_S(\psi_{h,k}))\sqrt{t_0}) \\ &\leq \mathbf{P}_S(\Delta_{t_0}(\psi_{h,k}) < 2(\epsilon_T - q_*)\sqrt{t_0}). \end{aligned}$$

Note that for $\epsilon_T \leq q_*/2$ the inequality (A.2) implies the following exponential upper bound

$$\mathbf{P}_S(\hat{q}(x_k) < \epsilon_T) \leq 8e^{-\gamma q_*^2 t_0}. \quad (\text{A.4})$$

Now we show that

$$\lim_{T \rightarrow \infty} \sup_{1 \leq l \leq n} \sup_{S \in \Sigma_L} \frac{1}{\epsilon_T} \mathbf{E}_S |\tilde{q}_T(x_l) - q_S(x_l)| = 0. \quad (\text{A.5})$$

To end this we have to prove that

$$\lim_{T \rightarrow \infty} \sup_{1 \leq l \leq n} \sup_{S \in \Sigma_L} \frac{1}{\epsilon_T} \mathbf{E}_S |\hat{q}(x_l) - \mathbf{M}_S(\psi_{h,l})/2| = 0. \quad (\text{A.6})$$

Indeed, from (A.2)–(A.3) we find

$$\begin{aligned} \mathbf{E}_S |\hat{q}(x_l) - \mathbf{M}_S(\psi_{h,l})/2| &= \frac{1}{\sqrt{t_0}} \mathbf{E}_S |\Delta_{t_0}(\psi_{h,k})| \\ &= \frac{1}{\sqrt{t_0}} \int_0^\infty \mathbf{P}_S(|\Delta_{t_0}(\psi_{h,k})| \geq z) dz \\ &\leq \frac{8}{\sqrt{t_0}} \int_0^\infty e^{-\gamma z^2} dz. \end{aligned}$$

The condition \mathbf{H}_1) implies that $\epsilon_T \sqrt{t_0} \rightarrow \infty$ as $T \rightarrow \infty$. Therefore this inequality implies (A.6). Moreover, taking into account that $h/\epsilon_T \rightarrow 0$ as

$T \rightarrow \infty$ we obtain, for sufficiently large T , the following bound

$$\begin{aligned} |\mathbf{M}_S(\psi_{h,l})/2 - q_S(x_l)| &\leq \int_{-1}^1 |q_S(x_l + vh) - q_S(x_l)| dv \\ &\leq q_*'' h^2 \leq \epsilon_T^2, \end{aligned}$$

where $q_*'' = \sup_{|x| \leq R} \sup_{S \in \Sigma_L} |q_S''(x)|$. From this inequality, taking into account inequality (A.6) and the condition \mathbf{H}_3 , we obtain (A.5).

Since $T - 2 \leq n \leq T$, we find that, for sufficiently large T providing $\epsilon_T \leq 1$,

$$\begin{aligned} \mathbf{E}_S \left| \sigma_l^2 - \frac{1}{q_S(x_l)(b-a)} \right| &= \frac{1}{b-a} \mathbf{E}_S \left| \frac{2n}{(T-t_0)(2\tilde{q}(x_l) - \epsilon_T^2)} - \frac{1}{q_S(x_l)} \right| \\ &\leq 2 \mathbf{E}_S \frac{|\tilde{q}(x_l) - q_S(x_l)|}{\epsilon_T q_*(b-a)} + \frac{\epsilon_T}{q_*(b-a)} \\ &\quad + \frac{4}{(T-t_0)\epsilon_T(b-a)} + \frac{2t_0}{(T-t_0)\epsilon_T(b-a)}. \end{aligned}$$

The condition \mathbf{H}_3) and (A.5) imply directly Proposition 3.1. \square

A.2 Proof of the limiting inequality (4.11)

We set $\tilde{t}_0 = j_0(\tilde{\alpha})$ and $\tilde{t}_1 = [\tilde{\omega} \ln(n+1)]$. Then we can represent $\hat{\gamma}_n(S)$ by the following way

$$\hat{\gamma}_n(S) = \sum_{j=\tilde{t}_0}^{\tilde{t}_1} (1 - \tilde{\lambda}(j))^2 \theta_{j,n}^2 + \frac{J(S)}{(b-a)n} \sum_{j=1}^n \tilde{\lambda}^2(j) + \Delta_{1,n}$$

with $\Delta_{1,n} = \sum_{j=\tilde{t}_1}^n \theta_{j,n}^2$. Note now that, for any $0 < \delta < 1$,

$$\begin{aligned} \hat{\gamma}_n(S) &\leq (1 + \delta) \sum_{j=\tilde{t}_0}^{\tilde{t}_1-1} (1 - \tilde{\lambda}(j))^2 \theta_j^2 + \frac{J(S)}{(b-a)n} \sum_{j=1}^n \tilde{\lambda}(j)^2 \\ &\quad + \Delta_{1,n} + (1 + 1/\delta) \Delta_{2,n}, \end{aligned} \tag{A.7}$$

where $\Delta_{2,n} = \sum_{j=\tilde{t}_0}^{\tilde{t}_1-1} (\theta_{j,n} - \theta_j)^2$.

Due to the uniform convergence (4.9), Lemmas 6.1 and 6.3 from [9] yield

$$\lim_{n \rightarrow \infty} \sup_{S \in W_r^k} n^{2k/(2k+1)} \sum_{l=1}^2 |\Delta_{l,n}| = 0.$$

Now we set

$$v_n(S) = n^{2k/(2k+1)} \sup_{j \geq \tilde{\iota}_0} (1 - \tilde{\lambda}(j))^2 / \varpi_j,$$

with the sequence ϖ_j defined in (2.7) and

$$v^*(S) = \left(\frac{b-a}{\pi} \right)^{2k} \frac{1}{(A_k \bar{r}(S))^{2k/(2k+1)}},$$

where the coefficient A_k is defined in (3.6). Moreover, one can calculate directly that

$$\limsup_{T \rightarrow \infty} \sup_{S \in \Sigma_L} \frac{v_n(S)}{v^*(S)} \leq 1. \quad (\text{A.8})$$

Therefore, due to the definition (2.7) and to the fact that

$$\gamma(S) = v^*(S)r + \frac{J(S)}{b-a} (A_k \bar{r}(S))^{1/(2k+1)} \frac{2k^2}{(k+1)(2k+1)},$$

the inequality (A.7) and the limits (A.8) and (4.10) imply (4.11). \square

A.3 Moment bounds

Lemma A.1. *Let $\xi_{j,n}$ be defined in (4.4). Then, for any real numbers*

$v_1, \dots, v_n,$

$$\mathbf{E} \left(\sum_{j=1}^n v_j \xi_{j,n} \right)^2 \leq \frac{\sigma_*}{b-a} V_n, \quad \mathbf{E} \left(\sum_{j=1}^n v_j \xi_{j,n} \right)^4 \leq \frac{3\sigma_*^2}{(b-a)^2} V_n^2,$$

where $\sigma_* = \max_{1 \leq j \leq n} \sigma_j^2$ and $V_n = \sum_{j=1}^n v_j^2$.

The proof of this Lemma is similar to the proof of Lemma 6.4 in [9]. \square

A.4 Application of the van Trees inequality to diffusion processes.

Let $(\mathcal{C}[0, T], \mathcal{B}, (\mathcal{B}_t)_{0 \leq t \leq T}, (\mathbf{P}_\theta, \theta \in \mathbb{R}^d))$ be a filtered statistical model with cylindrical σ -fields \mathcal{B}_t on $\mathcal{C}[0, t]$ and $\mathcal{B} = \cup_{0 \leq t \leq T} \mathcal{B}_t$. As to the distributions \mathbf{P}_θ we assume that it is distribution in $\mathcal{C}[0, T]$ of the stochastic process $(y_t)_{0 \leq t \leq T}$ governed by the stochastic differential equation

$$dy_t = S(y_t, \theta)dt + dw_t, \quad 0 \leq t \leq T, \quad (\text{A.9})$$

where $\theta = (\theta_1, \dots, \theta_d)'$ is vector of unknown parameters, $w = (w_t)_{0 \leq t \leq T}$ is a standart Wiener process. Moreover, we assume also that S is a linear function with respect to θ , i.e.

$$S(y, \theta) = \sum_{i=1}^d \theta_i S_i(y), \quad (\text{A.10})$$

where the functions $(S_i)_{1 \leq i \leq d}$ are bound and satisfy the Lipschitz condition, i.e. for some constant $0 < L < \infty$

$$\max_{1 \leq i \leq d} \sup_{x \in \mathbb{R}} |S_i(x)| \leq L \quad \text{and} \quad \max_{1 \leq i \leq d} \sup_{x, y \in \mathbb{R}} \frac{|S_i(y) - S_i(x)|}{|y - x|} \leq L.$$

In this case (see, for example, [5]) stochastic equation (A.9) has the unique strong solution $(y_t)_{0 \leq t \leq T}$ for any random variable θ with values in \mathbb{R}^d .

Moreover (see, for example [17]), for any $\theta \in \mathbb{R}^d$ the distribution \mathbf{P}_θ is absalutly continuous with respect to the Wiener measure ν_w in $\mathcal{C}[0, T]$ and the corresponding Radon-Nikodym derivative for any function $x = (x_t)_{0 \leq t \leq T}$ from $\mathcal{C}[0, T]$ is defined as

$$\frac{d\mathbf{P}_\theta}{d\nu_w} = f(x, \theta) = \exp \left\{ \int_0^T S(x_t, \theta) dx_t - \frac{1}{2} \int_0^T S^2(x_t, \theta) dt \right\}. \quad (\text{A.11})$$

Let Φ be a prior density in \mathbb{R}^d having the following form:

$$\Phi(\theta) = \Phi(\theta_1, \dots, \theta_d) = \prod_{j=1}^d \varphi_j(\theta_j),$$

where φ_j is some continuously differentiable density in \mathbb{R} . Moreover, let $\lambda(\theta)$ be a continuously differentiable $\mathbb{R}^d \rightarrow \mathbb{R}$ function such that for each $1 \leq j \leq d$

$$\lim_{|\theta_j| \rightarrow \infty} \lambda(\theta) \varphi_j(\theta_j) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |\lambda'_j(\theta)| \Phi(\theta) d\theta < \infty, \quad (\text{A.12})$$

where

$$\lambda'_j(\theta) = \frac{\partial \lambda(\theta)}{\partial \theta_j}.$$

For any $\mathcal{B}(\mathcal{X}) \times \mathcal{B}(\mathbb{R}^d)$ -measurable integrable function $\xi = \xi(x, \theta)$ we denote

$$\tilde{\mathbf{E}}\xi = \int_{\mathbb{R}^d} \int_{\mathcal{X}} \xi(x, \theta) d\mathbf{P}_\theta \Phi(\theta) d\theta = \int_{\mathbb{R}^d} \int_{\mathcal{X}} \xi(x, \theta) f(x, \theta) \Phi(\theta) d\nu_w(x) d\theta,$$

where $\mathcal{X} = \mathcal{C}[0, T]$.

Lemma A.2. *For any square integrable function $\hat{\lambda}_T$ measurable with respect to $(Y_t)_{0 \leq t \leq T}$ and for any $1 \leq j \leq d$ the following inequality holds*

$$\tilde{\mathbf{E}}(\hat{\lambda}_T - \lambda(\theta))^2 \geq \frac{\Lambda_j^2}{\tilde{\mathbf{E}} \int_0^T S_j^2(Y_t) dt + I_j},$$

where

$$\Lambda_j = \int_{\mathbb{R}^d} \lambda'_j(\theta) \Phi(\theta) d\theta \quad \text{and} \quad I_j = \int_{\mathbb{R}} \frac{\dot{\varphi}_j^2(z)}{\varphi_j(z)} dz.$$

Proof. First of all note that for the function (A.10) and for the Wiener process $w = (w_t)_{0 \leq t \leq T}$ density (A.11) is bounded with respect to $\theta_j \in \mathbb{R}$ for any $1 \leq j \leq d$, i.e.

$$\limsup_{|\theta_j| \rightarrow \infty} f(w, \theta) < \infty \quad \text{a.s.}$$

Therefore taking into account condition (A.12) by integration by parts one gets

$$\begin{aligned}\tilde{\mathbf{E}}\left((\hat{\lambda}_T - \lambda(\theta))\Psi_j\right) &= \int_{\mathcal{X} \times \mathbb{R}^d} (\hat{\lambda}_T(x) - \lambda(\theta)) \frac{\partial}{\partial \theta_j} (f(x, \theta)\Phi(\theta)) \, d\theta \nu_w(dx) \\ &= \int_{\mathcal{X} \times \mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \lambda'_j(\theta) f(x, \theta)\Phi(\theta) \, d\theta_j \right) \prod_{i \neq j} d\theta_i \nu_w(dx) \\ &= \Lambda_j.\end{aligned}$$

Now by the Bouniakovskii-Cauchy-Schwarz inequality we obtain the following lower bound for the quadratic risk

$$\tilde{\mathbf{E}}(\hat{\lambda}_T - \lambda(\theta))^2 \geq \frac{\Lambda_j^2}{\tilde{\mathbf{E}}\Psi_j^2},$$

where

$$\begin{aligned}\Psi_j &= \Psi_j(x, \theta) = \frac{\partial}{\partial \theta_j} \ln(f(x, \theta)\Phi(\theta)) \\ &= \frac{\partial}{\partial \theta_j} \ln f(x, \theta) + \frac{\partial}{\partial \theta_j} \ln \Phi(\theta).\end{aligned}$$

Note that from (A.11) it is easy to deduce that

$$\frac{\partial}{\partial \theta_j} \ln f(y, \theta) = \int_0^T S_j(y_t) \, dw_t.$$

Therefore, due to the boundness of the functions S_j we find that for each $\theta \in \mathbb{R}^d$

$$\mathbf{E}_\theta \frac{\partial}{\partial \theta_j} \ln f(y, \theta) = 0 \quad \text{and} \quad \mathbf{E}_\theta \left(\frac{\partial}{\partial \theta_j} \ln f(y, \theta) \right)^2 = \mathbf{E}_\theta \int_0^T S_j^2(y_t) \, dt.$$

Taking this into account we can calculate now $\tilde{\mathbf{E}}\Psi_j^2$, i.e.

$$\tilde{\mathbf{E}}\Psi_j^2 = \tilde{\mathbf{E}} \int_0^T S_j^2(y_t) \, dt + I_j.$$

Hence Lemma A.2. \square

A.5 Proof of (5.14)

We set

$$\psi_{m,j}(y) = \frac{1}{h} D_{m,j}^2(y).$$

Then by making use of definitions in (A.1) we can estimate the term $\varsigma_{m,j}(T)$

as

$$|\varsigma_{m,j}(T)| \leq \frac{\mathbf{E} \left(\mathbf{E}_{S_{\theta,T}} |\Delta_T(\psi_{m,j})| \right)}{\bar{e}_j(I_\eta^2) q_0(\tilde{x}_m) \sqrt{T}} + \mathbf{E} \left| \frac{M_{S_{\theta,T}}(\psi_{m,j})}{\bar{e}_j(I_\eta^2) q_0(\tilde{x}_m)} - 1 \right|.$$

Moreover, taking into account that

$$\lim_{\eta \rightarrow 0} \sup_{j \geq 1} |\bar{e}_j(I_\eta^2) - 1| = 0$$

we chose $\eta > 0$ for which

$$\inf_{j \geq 1} \bar{e}_j(I_\eta^2) \geq 1/2.$$

Therefore we can write that

$$\begin{aligned} |\varsigma_{m,j}(T)| &\leq \frac{2}{q_* \sqrt{T}} \mathbf{E} \left(\mathbf{E}_{S_{\theta,T}} |\Delta_T(\psi_{m,j})| \right) + \frac{2}{q_*} \mathbf{E} \left| M_{S_{\theta,T}}(\psi_{m,j}) - M_{S_0}(\psi_{m,j}) \right| \\ &\quad + \frac{2}{q_*} \left| M_{S_0}(\psi_{m,j}) - \bar{e}_j(I_\eta^2) q_0(\tilde{x}_m) \right|. \end{aligned} \quad (\text{A.13})$$

We remind that on the set (5.7) for sufficiently large T the function $S_{\theta,T} \in \Sigma_L$ therefore we estimate the first term in the right side of the last inequality as

$$\mathbf{E} \left(\mathbf{E}_{S_{\theta,T}} |\Delta_T(\psi_{m,j})| \right) \leq \frac{2}{h} \mathbf{P}(\Xi_T^c) + \sup_{S \in \Sigma_L} \mathbf{E}_S |\Delta_T(\psi_{m,j})|.$$

Moreover, taking into account that

$$\int_{-\infty}^{\infty} |\psi_{m,j}(y)| dy = \int_{-1}^1 e_j^2(v) I_\eta^2(v) dv \leq 2$$

we obtain that

$$\limsup_{T \rightarrow \infty} \max_{1 \leq j \leq M} \max_{1 \leq j \leq N} \mathbf{E} \left(\mathbf{E}_{S_{\theta,T}} |\Delta_T(\psi_{m,j})| \right) < \infty.$$

To estimate the next term in (A.13) we make use of the fact that on the set Ξ_T the function $S_{\theta,T}$ satisfies inequality (5.9) and one can check directly that on this set

$$|q_{S_{\theta,T}} - q_0|_* \leq C^* (e^{2(b-a)\epsilon_T} - 1).$$

Therefore, with the help of this inequality we obtain that

$$\begin{aligned} \mathbf{E} \left| M_{S_{\theta,T}}(\psi_{m,j}) - M_{S_0}(\psi_{m,j}) \right| &\leq \int_{-1}^1 e_j^2(v) I_\eta^2(v) \mathbf{E} |q_{S_{\theta,T}}(\tilde{x}_m + vh) - q_0(\tilde{x}_m + vh)| dv \\ &\leq 2\mathbf{P}(\Xi_T^c) q^* + 2C^* (e^{2(b-a)\epsilon_T} - 1). \end{aligned}$$

Finally it is easy to see that

$$\begin{aligned} \left| M_{S_0}(\psi_{m,j}) - \bar{e}_j(I_\eta^2) q_0(\tilde{x}_m) \right| &\leq \int_{-1}^1 e_j^2(v) I_\eta^2(v) |q_0(\tilde{x}_m + vh) - q_0(\tilde{x}_m)| dv \\ &\leq 2 \sup_{a \leq u, v \leq b, |u-v| \leq h} |q_0(u) - q_0(v)|, \end{aligned}$$

i.e. this term goes to zero as $h \rightarrow 0$ uniformly over $1 \leq m \leq M$ and $1 \leq j \leq N$. Hence limit equality (5.14)

□

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