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Extremality and designs in spaces of quadratic forms

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Abstract

A well known theorem of Voronoi characterizes extreme quadratic forms and Euclidean lattices, that is those which are local maxima for the Hermite function, as perfect and eutactic. This characterization has been extended in various cases, such that family of lattices, sections of lattices, Humbert forms, etc. Moreover, there is a criterion for extreme lattices, discovered by Venkov, formulated in terms of spherical designs which has been extended in the case of Grassmannians and sections of lattices.

In this article, we define a general frame, in which there is a “Voronoi characterization”, and a “Venkov criterion” through an appropriate notion of design. This frame encompasses many interesting situations in which a “Voronoi characterization” has been proved.

We also discuss the question of extremality relatively to the Epstein zeta function, and we extend to our frame a characterization of final zeta-extremality formulated by Delone and Ryshkov and a criterion in terms of designs found by Coulangeon.

Consider the following function on positive definite quadratic forms on \mathbb{R}^n of determinant 1,

$$\gamma_n(Q) = \min_{z \in \mathbb{Z}^n \setminus \{0\}} Q(z),$$

which is called the *Hermite function*, and its maximal value, γ_n , which is the *Hermite constant*. The problem of estimating γ_n is an important subject of research, and the systematic investigation of *extreme* forms—the forms which are local maxima for the Hermite function—comes back to Korkine and Zolotarev in the nineteenth century.

The problem has an equivalent formulation in terms of Euclidean lattices: For a lattice $\Lambda \subseteq \mathbb{R}^n$ of determinant 1, we define:

$$\gamma_n(\Lambda) = \min_{z \in \Lambda \setminus \{0\}} \langle z | z \rangle.$$

The Hermite constant γ_n is equal to the maximal value of this function among all Euclidean lattices of dimension n and determinant 1. Both formulations have their own advantages and drawbacks; the latter one provides a more graphical representation of the problem and is naturally related to the problem of finding densest ball packings in Euclidean space.

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A famous theorem of Voronoi (1908) [Voro08] characterizes extreme forms as perfect and eutactic (the definition of these notions is given later). This theorem has been extended in various situation, as dual-extremality of lattices (Bergé, Martinet, 1989 [BerMar89]), extremality in families of lattices (Bergé, Martinet, 1991 [BerMar91], [BerMar95]), extremality for the Rankin function of lattices (Coulangeon, 1996 [Coul96]), extremality of Humbert forms (Coulangeon, 2001 [Coul01]), and extremality in a more general frame, which includes systoles on a Riemannian manifold (Bavard, 1997 [Bava97, Theorem 2.1]).

A relation between extreme lattices and so-called *spherical designs* has been discovered by Boris Venkov (1998), who has shown that a lattice whose minimal layer forms a 4-design is perfect and eutactic [VenMar01]. Recall that the *minimal layer* of a Euclidean lattice Λ is the set of vectors x of Λ such that $\langle x | x \rangle$ is minimal among all nonzero vectors of Λ , and that a *spherical τ -design* is a finite subset X of the Euclidean sphere \mathbb{S}^{n-1} such that

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{\text{vol}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} f(x)$$

for all polynomial functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree at most τ . The criterion of Venkov has been extended to extremality relatively to the Rankin function through an appropriate notion of design in Grassmannian spaces (Bachoc, Coulangeon, Nebe, 2002 [BaCoNe02]).

There is another kind of extremality for quadratic forms and lattices, namely the extremality of the so-called Epstein zeta function : To a quadratic form Q on \mathbb{R}^n , and a complex number s , we associate the series

$$\zeta(Q, s) = \sum_{x \in \mathbb{Z}^n \setminus \{0\}} Q(x)^{-s},$$

which converges for $\Re s > n/2$. The corresponding series for a lattice $\Lambda \subseteq \mathbb{R}^n$ is

$$\zeta(\Lambda, s) = \sum_{x \in \Lambda \setminus \{0\}} \langle x | x \rangle^{-s}.$$

A quadratic form or a lattice is ζ -*extreme* at $s > n/2$ when the value of its zeta function at s is locally minimal among quadratic forms or lattices of same determinant. A characterization of finally ζ -extreme lattices—that is lattices which are ζ -extreme at s for all s large enough—has been formulated by Delone and Ryshkov (1967) [DelRyš67]: a Euclidean lattice is finally ζ -extreme if and only if it is perfect and all its layers are strongly eutactic. This result may be seen as a version for final ζ -extremality of Voronoi’s characterization of extremality. Such a criterion has also been recently proved for Humbert forms by Coulangeon [Coul08].

As for classic extremality, there is a criterion for ζ -extremality in terms of designs, discovered by Coulangeon (2006) [Coul06]: if all layers of a lattice of dimension n are spherical 4-designs, then it is ζ -extreme for all $s > n/2$.

The goal of this paper is to provide a theoretical frame in which the four results on extremality mentioned above hold. Our paper is organized as follows:

- In Section 1, we give the definitions and main properties of our frame, that we call *Voronoi space*, and we give examples of Voronoi spaces, showing how to integrate in our frame most of the cases mentioned in the introduction;
- In Section 2, we use some theory of linear Lie groups in order to establish the structure of the group of a Voronoi space, which is a fundamental object in our theory;

- In Section 3, we prove a characterization à la Voronoi of extremality and strict extremality in Voronoi spaces (Theorem 18). We also establish a property which, in many cases, may be used to show easily the equivalence of extremality and strict extremality (Proposition 20);
- In Section 4, we introduce *alternative Voronoi spaces*, which are essentially equivalent to Voronoi spaces with respect to our problem of extremality, just as quadratic forms are equivalent to lattices with respect to the Hermite function;
- In Section 5, we define a notion of design for appropriate alternative Voronoi spaces, and we prove the criterion corresponding to the criterion of Venkov (Theorem 33 and Corollary 36);
- Finally, the problem of ζ -extremality is introduced in Section 6, and the corresponding characterization of final extremality (Theorem 44) and the criterion using designs (Theorem 46) are proved.

The theory developed in this paper is not the first attempt to provide a unification of the problem of extremality introduced above. In particular, the reader may check that our notion of Voronoi space is a particular case of a more general frame introduced by Christophe Bavard, in which there is also a “Voronoi characterization” under a condition of convexity, called “Condition C” [Bava97, §2.2]. In our frame, we do not need such a condition, but instead we use the structure of the Lie group acting on the situation to obtain the convexity property given by Lemma 15 and the reducibility criterion given by Proposition 20. Moreover, there is no natural notion of design in the frame of Bavard. On the other hand, our frame does not encompass all interesting cases; in particular, we have not succeeded to apply it to the problem of systoles on Riemannian manifolds, that Bavard has studied in the cited paper.

Note also that the characterization of ζ -extremality of Delone and Ryshkov (Theorem 44) may also be proved in the frame of Bavard.

1 Voronoi spaces

As stated in the introduction, the goal of this paper is to provide a practical frame in which there is an equivalent of the characterization of extremality of Voronoi [Voro08] and some other results on extremality that have been extended in various cases. The notion of *Voronoi space* we introduce here is based on the observation of features which often appear in the frames where those different results of extremality have been proved. There are: a ground space that can be expressed as real submanifold of the cone of positive definite quadratic forms on a finite-dimensional real vector space, and the high symmetry of that ground space which is expressed by the existence of a Lie group acting naturally and transitively on it.

In the last paragraph of this section (Paragraph 1.5), we give some examples of Voronoi spaces corresponding to situations studied before by others.

1.1 General conventions and notations

Let V be a finite-dimensional real vector space. If Q is a quadratic form on V , we denote by the same symbol Q the associated symmetric bilinear form. The relation

between the two is:

$$Q(x, y) = \frac{1}{2} \left(Q(x + y) - Q(x) - Q(y) \right),$$

$$Q(x) = Q(x, x).$$

For a nondegenerate quadratic form (or symmetric bilinear form) Q on V , and a linear transformation H of V , we denote by H_Q^* the Q -adjoint of H , that is the unique linear transformation of V such that $Q(x, Hy) = Q(H_Q^*x, y)$ for every $x, y \in V$. The linear transformation H is Q -selfadjoint, respectively Q -antiselfadjoint, respectively Q -orthogonal, when $H_Q^* = H$, respectively $H_Q^* = -H$, respectively $H_Q^* = H^{-1}$.

Let Q be a positive definite quadratic form on V . We define the norm $\|\cdot\|_Q$ on the associative algebra $\text{End}(V)$ of linear endomorphisms of V by:

$$\|H\|_Q^2 = \sup_{x \in V \setminus \{0\}} \frac{Q(Hx)}{Q(x)}, \quad H \in \text{End}(V).$$

If x is an element or a subset in $V \setminus \{0\}$, we denote by $[x]$ the corresponding element or subset in the projective space $\mathbb{P}(V)$, that is the image of x through the projection $V \setminus \{0\} \rightarrow \mathbb{P}(V)$.

1.2 Voronoi space

Here is the frame in which we do our theory:

1. DEFINITION. A *Voronoi space* $(V, \mathcal{Q}, Z, \mathbf{G})$ is the data of

- (a) a finite-dimensional real vector space V ,
- (b) a connected submanifold \mathcal{Q} in the half-cone of positive definite quadratic forms on V ,
- (c) a closed discrete subset Z of V not containing 0 and spanning V ,
- (d) a closed linear Lie group $\mathbf{G} < \mathbf{GL}(V)$,

satisfying the following properties:

- (i) For $Q \in \mathcal{Q}$ and $c > 0$, we have $cQ \in \mathcal{Q}$ only if $c = 1$;
- (ii) for all $g \in \mathbf{G}$ and $Q \in \mathcal{Q}$, we have $g_Q^* \in \mathbf{G}$;
- (iii) for all $g \in \mathbf{G}$ and $Q \in \mathcal{Q}$, we have $Q \circ g \in \mathcal{Q}$, and \mathbf{G} is transitive on \mathcal{Q} .

The point of restriction (i) is to avoid to deal with some special cases which would be anyway trivially uninteresting for the problems of extremality which occupy us in this paper.

The standard example of Voronoi space is $V = \mathbb{R}^n$ with $n \geq 2$, and \mathcal{Q} the set of positive definite quadratic forms of determinant 1, together with $Z = \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{G} = \mathbf{SL}^\pm(n, \mathbb{R})$, where

$$\mathbf{SL}^\pm(n, \mathbb{R}) = \{g \in \mathbf{GL}(n, \mathbb{R}) \mid \det g = \pm 1\}. \quad (1)$$

We refer to this example as the “classic Voronoi space of rank n ”. Before we give other examples, we fix some general notations and properties of Voronoi spaces.

The group \mathbf{G} of the definition is the *group* of the Voronoi space. Given a Voronoi space $(V, \mathcal{Q}, Z, \mathbf{G})$, there exist two linear groups \mathbf{G}_{\min} and \mathbf{G}_{\max} such that $(V, \mathcal{Q}, Z, \tilde{\mathbf{G}})$ is a Voronoi space if and only if we have $\mathbf{G}_{\min} < \tilde{\mathbf{G}} < \mathbf{G}_{\max}$. Moreover, $(V, \mathcal{Q}, Z, \tilde{\mathbf{G}})$ is equivalent to $(V, \mathcal{Q}, Z, \mathbf{G})$ with respect to the problems of extremality that occupies us in this paper. For the classic Voronoi space of rank n , we have $\mathbf{G}_{\min} = \mathbf{SL}(n, \mathbb{R})$ and $\mathbf{G}_{\max} = \mathbf{SL}^\pm(n, \mathbb{R})$.

1.3 Group of a Voronoi space

The group of a Voronoi space and the related objects play an important role in our theory, as they assure the ‘‘convexity’’ of the functions $Q \mapsto Q(x)$, on \mathcal{Q} (Lemma 15).

2. DEFINITION. Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, and let $Q \in \mathcal{Q}$. Then \mathbf{G} is the *group* of the Voronoi space. Moreover, we define the following objects:

- $\mathbf{K}_Q = \{g \in \mathbf{G} \mid g_Q^* = g^{-1}\}$, the (compact) group of Q -selfadjoint elements of \mathbf{G} ;
- \mathfrak{g} , the Lie algebra of \mathbf{G} ;
- $\mathfrak{k}_Q = \{H \in \mathfrak{g} \mid H + H_Q^* = 0\}$, the Lie subalgebra of Q -antiselfadjoint elements of \mathfrak{g} , which is the Lie algebra of \mathbf{K}_Q ;
- $\mathfrak{p}_Q = \{H \in \mathfrak{g} \mid H - H_Q^* = 0\}$, the set of Q -selfadjoint elements of \mathfrak{g} ;
- \mathbf{G}^0 and \mathbf{K}_Q^0 , the connected component of the identity of \mathbf{G} and \mathbf{K}_Q respectively.

Note that we have $\mathfrak{g} = \mathfrak{k}_Q \oplus \mathfrak{p}_Q$, and, for any $c \in \mathbb{R}$, we have $c \text{id}_V \in \mathbf{G}$ if and only if $c = \pm 1$, and $c \text{id}_V \in \mathfrak{g}$ if and only if $c = 0$.

3. LEMMA. Let $Q, Q' \in \mathcal{Q}$ and let $g \in \mathbf{G}$ such that $Q' = Q \circ g$. Then

$$\mathfrak{p}_{Q'} = g^{-1} \mathfrak{p}_Q g, \quad \text{and} \quad \mathfrak{k}_{Q'} = g^{-1} \mathfrak{k}_Q g.$$

Proof. Let $H \in \mathfrak{g}$. We have, for any linear endomorphism A of V ,

$$A_{Q'}^* = (g_Q^* g)^{-1} A_Q^* (g_Q^* g);$$

in particular, setting $A = g^{-1} H g$, we get

$$(g^{-1} H g)_{Q'}^* = (g_Q^* g)^{-1} (g^{-1} H g)_Q^* (g_Q^* g) = g^{-1} H_Q^* g.$$

Therefore H is Q -selfadjoint, respectively Q -antiselfadjoint, if and only if $g^{-1} H g$ is Q' -selfadjoint, respectively Q' -antiselfadjoint. \square

1.4 Layers, Hermite function and extremality

4. DEFINITION. Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space.

- (i) The *Hermite function* associated to $(V, \mathcal{Q}, Z, \mathbf{G})$ is the function γ on \mathcal{Q} defined by

$$\gamma(Q) = \min_{x \in Z} Q(x), \quad Q \in \mathcal{Q}.$$

- (ii) For $r > 0$, the *layer* of square radius r of a quadratic form $Q \in \mathcal{Q}$ is the set

$$Q_r = \{x \in Z \mid Q(x) = r\}.$$

- (iii) The set of *minimal vectors* of a quadratic form $Q \in \mathcal{Q}$ is the nonempty layer of minimal radius, that is

$$Q_{\min} = Q_{\gamma(Q)} = \{x \in Z \mid Q(x) = \gamma(Q)\}.$$

- (iv) A quadratic form $Q \in \mathcal{Q}$ is *extreme*, respectively *strictly extreme*, when it is a local maximum, respectively a strict local maximum, of γ .

In the case of the classic Voronoi space of rank n , the function γ is the “classic” Hermite function, and its maximal value is the Hermite constant γ_n .

The main goal of this paper is to find criteria for extreme forms in Voronoi spaces. The first question to consider is if the supremum of γ on \mathcal{Q} is indeed a local maximum. Sadly, we have not found any easy criterion to check this property; nevertheless, the following definition is justified by many interesting examples of Voronoi spaces.

Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, and consider the discrete subgroup

$$\mathbf{H} = \{g \in \mathbf{G} \mid gZ = Z\}.$$

The quotient \mathcal{Q}/\mathbf{H} is separated, and we have a map

$$\begin{aligned} \bar{\gamma} : \mathcal{Q}/\mathbf{H} &\longrightarrow]0, \infty[\\ Q\mathbf{H} &\longmapsto \gamma(Q), \end{aligned}$$

5. DEFINITION. The Voronoi space $(V, \mathcal{Q}, Z, \mathbf{G})$ is *compact* when $\bar{\gamma}$ is bounded and proper.

In a compact Voronoi space, the supremum of the Hermite function is also a local maximum.

The statement that classic Voronoi spaces are compact is essentially equivalent to Mahler’s Compactness Theorem (see, e.g., [Mart03, §2.4]).

1.5 Examples

The examples of Voronoi space we give here show how to integrate most cases cited in the introduction in our frame. All these examples have an alternative formulation, which is made explicit in Section 4.

1. *Classic spaces.* Let $V = \mathbb{R}^n$ with $n \geq 2$, let \mathcal{Q} be the set of positive definite quadratic forms on \mathbb{R}^n of determinant 1, let $Z = \mathbb{Z}^n \setminus \{0\}$, and let $\mathbf{G} = \mathbf{SL}^\pm(n, \mathbb{R}) = \{g \in \mathbf{GL}(n, \mathbb{R}) \mid \det g = \pm 1\}$. Then $(V, \mathcal{Q}, Z, \mathbf{G})$ is a compact Voronoi space with the following parameters, where Q_0 is the canonical quadratic form on \mathbb{R}^n :

$$\begin{aligned} \mathbf{K}_{Q_0} &= \{g \in \text{Mat}(n, \mathbb{R}) \mid g^* = g^{-1}\} = \mathbf{O}(n), \\ \mathfrak{g} &= \{H \in \text{Mat}(n, \mathbb{R}) \mid \text{tr } H = 0\} = \mathfrak{sl}(n, \mathbb{R}), \\ \mathfrak{k}_{Q_0} &= \{H \in \text{Mat}(n, \mathbb{R}) \mid H + H^* = 0 \text{ and } \text{tr } H = 0\} = \mathfrak{so}(n, \mathbb{R}), \\ \mathfrak{p}_{Q_0} &= \{H \in \text{Mat}(n, \mathbb{R}) \mid H^* = H \text{ and } \text{tr } H = 0\}. \end{aligned}$$

For this space, the Hermite function is the classic Hermite function (restricted to quadratic forms of determinant 1), and its maximal value is the classic Hermite constant γ_n .

2. *Duality.* Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space. Let $Q \mapsto Q^\alpha$ be an involution of \mathcal{Q} , and $g \mapsto g^\alpha$ an involution and automorphism of \mathbf{G} such that $(g^*)^\alpha = (g^\alpha)^*$ and

$$(Q \circ g)^\alpha = Q^\alpha \circ g^\alpha, \quad \forall g \in G, \forall Q \in \mathcal{Q}.$$

Then we define a Voronoi space $(\tilde{V}, \tilde{\mathcal{Q}}, \tilde{Z}, \tilde{\mathbf{G}})$ as follows:

$$\begin{aligned}\tilde{V} &= V \times V \\ \tilde{\mathcal{Q}} &= \{\tilde{Q} \mid \tilde{Q}((x_1, x_2)) = Q(cx_1) + Q^\alpha(c^{-1}x_2), \text{ for some } Q \in \mathcal{Q} \text{ and } c \in \mathbb{R}^*\}, \\ \tilde{Z} &= (Z \times \{0\}) \cup (\{0\} \times Z), \\ \tilde{\mathbf{G}} &= \left\{ \tilde{g} \in \mathbf{GL}(V \times V) \mid \tilde{g} = \begin{pmatrix} cg & 0 \\ 0 & c^{-1}g^\alpha \end{pmatrix} \text{ or } \begin{pmatrix} 0 & cg \\ c^{-1}g^\alpha & 0 \end{pmatrix} \right. \\ &\quad \left. \text{for some } g \in \mathbf{G} \text{ and } c \in \mathbb{R}^* \right\}.\end{aligned}$$

For $Q \in \mathcal{Q}$ and $c \in \mathbb{R}^*$ let $\tilde{Q}_c \in \tilde{\mathcal{Q}}$ such that $\tilde{Q}_c((x_1, x_2)) = Q(cx_1) + Q^\alpha(c^{-1}x_2)$. The Hermite function $\tilde{\gamma}$ of $(\tilde{V}, \tilde{\mathcal{Q}}, \tilde{Z}, \tilde{\mathbf{G}})$ is equal to

$$\tilde{\gamma}(\tilde{Q}_c) = \min(c^2\gamma(Q), c^{-2}\gamma(Q^\alpha)).$$

For a given $Q \in \mathcal{Q}$, the maximum of $\gamma(\tilde{Q}_c)$ among all values of c is attained when $c^2\gamma(Q) = c^{-2}\gamma(Q^\alpha)$, that is when $c^2 = \gamma(Q)^{-1/2}\gamma(Q^\alpha)^{1/2}$, and, for this value of c , we have

$$\tilde{\gamma}(\tilde{Q}_c) = (\gamma(Q)\gamma(Q^\alpha))^{1/2}.$$

Therefore, the local maxima of the Hermite function $\tilde{\gamma}$ on $\tilde{\mathcal{Q}}$ correspond to the maxima of the function $\gamma'(Q) = (\gamma(Q)\gamma(Q^\alpha))^{1/2}$ on \mathcal{Q} . A quadratic form $Q \in \mathcal{Q}$ such that the corresponding form \tilde{Q}_c is extreme is called *dual-extreme*.

If $(V, \mathcal{Q}, Z, \mathbf{G})$ is compact and if we have $g^\alpha Z = Z$ for every $g \in \mathbf{G}$ satisfying $gZ = Z$, then $(\tilde{V}, \tilde{\mathcal{Q}}, \tilde{Z}, \tilde{\mathbf{G}})$ is compact.

As a particular case, we take for $(V, \mathcal{Q}, Z, \mathbf{G})$ the classic Voronoi space of rank n , with $g^\alpha = g^{*-1}$ and $Q^\alpha = Q^{-1}$, where g^* denotes the adjoint of g for the canonical scalar product on $V = \mathbb{R}^n$, and where $Q^{-1}(x) = \langle x \mid A^{-1}x \rangle$ if $Q(x) = \langle x \mid Ax \rangle$. The maximal value γ'_n of the Hermite function on the corresponding space $(\tilde{V}, \tilde{\mathcal{Q}}, \tilde{Z}, \tilde{\mathbf{G}})$ is known as *Bergé-Martinet constant*, and has been first studied in [BerMar89].

3. *Family of quadratic forms.* Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, let Q be an element of \mathcal{Q} , and let \mathbf{H} be a connected closed subgroup of \mathbf{G} such that $g_Q^* \in \mathbf{H}$ when $g \in \mathbf{H}$. Let $Q\mathbf{H}$ be the orbit of Q in \mathcal{Q} . Then $(V, Q\mathbf{H}, Z, \mathbf{H})$ is a Voronoi space, which is compact if $(V, \mathcal{Q}, Z, \mathbf{G})$ is compact.

When $(V, \mathcal{Q}, Z, \mathbf{G})$ is a classic Voronoi space, the situation (but formulated in terms of lattices instead of quadratic forms, see Paragraph 4.4) has been studied by Anne-Marie Bergé and Jacques Martinet in [BerMar95]. As special cases, we have:

- (a) Let Γ be a finite subgroup of \mathbf{G} such that $\gamma Z = Z$ for every $\gamma \in \Gamma$. An element $Q \in \mathcal{Q}$ such that the elements of Γ are Q -orthogonal is called *Γ -invariant*.

Let $Q \in \mathcal{Q}$ be a Γ -invariant quadratic form, and let \mathbf{H} be the connected component of the identity of the centralizer of Γ in \mathbf{G} . The group \mathbf{H} is Q -selfadjoint, because so is Γ . However, the following fact is less evident:

6. LEMMA. *$Q\mathbf{H}$ is the set of Γ -invariant elements of \mathcal{Q} .*

Proof. The nontrivial fact to show is that any Γ -invariant element of \mathcal{Q} is of the form $Q \circ h$ with $h \in \mathbf{H}$. Let Q' be such an element; according to Proposition 14, we have $Q'(x, y) = Q(x, \exp(H)y)$ for some $H \in \mathfrak{p}_Q$.

Then, for $\gamma \in \Gamma$, we have $Q'(\gamma x, \gamma y) = Q(x, \exp(\gamma^{-1}H\gamma)y)$; therefore the condition $Q' \circ \gamma = Q'$ is equivalent to

$$\exp(\gamma^{-1}H\gamma) = \exp(H).$$

Since \exp is injective on \mathfrak{p}_Q , the last condition is equivalent to $\gamma^{-1}H\gamma = H$. Then $h = \exp(H/2)$ is an element of \mathbf{H} , and we have $Q' = Q \circ h$. \square

So, $(V, Q\mathbf{H}, Z, \mathbf{H})$ is a Voronoi space, and their extreme forms are local maxima of the Hermite function restricted to the Γ -invariant elements of \mathcal{Q} . The case where $(V, \mathcal{Q}, Z, \mathbf{G})$ is a classic Voronoi space has been studied by Bergé and Martinet in [BerMar91].

- (b) Let $Q \mapsto Q^\alpha$ and $g \mapsto g^\alpha$ as in Example 2; let $\sigma \in \mathbf{G}$ such that $\sigma Z = Z$. An element $Q \in \mathcal{Q}$ such that $Q = Q^\alpha \circ \sigma$, is called σ -isodual. Note that, if $Q \in \mathcal{Q}$ is σ -isodual, we have $\gamma(Q) = \gamma(Q_\alpha)$.

Let $Q \in \mathcal{Q}$ be a σ -isodual quadratic form, Let \mathbf{H} be the connected component of the identity of set of the elements $g \in \mathbf{G}$ satisfying $\sigma g = g^\alpha \sigma$.

7. LEMMA. $Q\mathbf{H}$ is the set of σ -isodual elements of \mathcal{Q} .

Proof. The general line of the proof is the same as the one of Lemma 6: The nontrivial fact to show is that any σ -isodual element of \mathcal{Q} is of the form $Q \circ h$ with $h \in \mathbf{H}$. Let Q' be such an element; according to Proposition 14, we have $Q'(x, y) = Q(x, \exp(H)y)$ for some $H \in \mathfrak{p}_Q$. Let $H \mapsto H^A$ be the derivative of $g \mapsto g^\alpha$; we have $\exp(H^A) = \exp(H)^\alpha$ and $(H_Q^*)^A = (H^A)_{Q^\alpha}^*$. The property $(Q \circ h)^\alpha = Q^\alpha \circ h^\alpha$ applied to $h = \exp(H/2)$ gives $Q'^\alpha(x, y) = Q^\alpha(x, \exp(H^A)y)$. Therefore, we have

$$\begin{aligned} Q(x, \exp(H)y) &= Q'(x, y) = Q'^\alpha(\sigma x, \sigma y) \\ &= Q^\alpha(\sigma x, \exp(H^A)\sigma y) = Q^\alpha(\sigma x, \sigma \exp(\sigma^{-1}H^A\sigma)y) \\ &= Q(x, \exp(\sigma^{-1}H^A\sigma)y). \end{aligned}$$

We infer the following equality:

$$\exp(H) = \exp(\sigma^{-1}H^A\sigma).$$

If we have $\sigma^{-1}H^A\sigma \in \mathfrak{p}_Q$, the injectivity of \exp on \mathfrak{p}_Q implies $H = \sigma^{-1}H^A\sigma$, and we have $Q' = Q \circ h$ with $h = \exp(H/2) \in \mathbf{H}$. Therefore, it suffices to show the equality

$$(\sigma^{-1}H^A\sigma)_Q^* = \sigma^{-1}H^A\sigma.$$

Since $Q^\alpha = Q \circ \sigma^{-1}$, we have, for any linear endomorphism A of V ,

$$A_{Q^\alpha}^* = ((\sigma^{-1})_Q^* \sigma^{-1})^{-1} A_Q^* ((\sigma^{-1})_Q^* \sigma^{-1}) = (\sigma \sigma_Q^*) A_Q^* (\sigma \sigma_Q^*)^{-1}.$$

In particular, we get

$$(H^A)_Q^* = (\sigma \sigma_Q^*)^{-1} (H^A)_{Q^\alpha}^* (\sigma \sigma_Q^*) = (\sigma \sigma_Q^*)^{-1} (H_Q^*)^A (\sigma \sigma_Q^*).$$

Therefore, we have

$$(\sigma^{-1}H^A\sigma)_Q^* = \sigma_Q^* (\sigma \sigma_Q^*)^{-1} (H_Q^*)^A (\sigma \sigma_Q^*) (\sigma^{-1})_Q^* = \sigma^{-1}H^A\sigma,$$

which is what remained to prove. \square

8. LEMMA. For any $g \in \mathbf{H}$, and any $Q' \in Q\mathbf{H}$, we have $g_{Q'}^* \in \mathbf{H}$.

Proof. Let $h \in \mathbf{H}$, and let $Q' = Q \circ j$ with $j \in \mathbf{H}$. since we have $g_{Q'}^* = (j_Q^* j)^{-1} h_Q^* (j_Q^* j)$, it suffices to show the lemma for $Q' = Q$. By the same argument given in the previous proof for the calculation of $(H^A)_Q^*$, we have

$$(h^\alpha)_Q^* = (\sigma \sigma_Q^*)^{-1} (h_Q^*)^\alpha (\sigma \sigma_Q^*),$$

and therefore

$$\sigma h_Q^* = (\sigma^{-1} \sigma h \sigma_Q^*)^* = (\sigma^{-1} h^\alpha \sigma \sigma_Q^*)^* = (\sigma \sigma_Q^*) (h^\alpha)_Q^* (\sigma \sigma_Q^*)^{-1} \sigma = (h_Q^*)^\alpha \sigma,$$

therefore, we have $h_Q^* \in \mathbf{H}$. \square

So, $(V, \mathcal{Q}, Z, \mathbf{H})$ is a Voronoi space, and their extreme forms are local maxima of the Hermite function restricted to the σ -isodual elements of \mathcal{Q} . The dual case of this situation (see Paragraph 4.4, Example 3) with the classic Voronoi space, has been studied by Bergé and Martinet in [BerMar95]. The case where $\sigma^2 = -\text{id}_V$ is also interesting, see [BusSar94].

4. *Exterior powers.* Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space and let $1 \leq m \leq \dim(V)/2$. We define a space $(V^{\wedge m}, \mathcal{Q}^{\wedge m}, Z^{\wedge m}, \mathbf{G}^{\wedge m})$ as follows: $V^{\wedge m}$ is the m th exterior power of V , and

$$\begin{aligned} \mathcal{Q}^{\wedge m} &= \{Q^{\wedge m} \mid Q \in \mathcal{Q}\}, \text{ where } Q^{\wedge m}(x_1 \wedge \cdots \wedge x_m) = \det(Q(x_i, x_j)_{i,j=1}^m), \\ Z^{\wedge m} &= \{x_1 \wedge \cdots \wedge x_m \mid x_1, \dots, x_m \text{ are linearly independent elements of } Z\}, \\ \mathbf{G}^{\wedge m} &= \{\tilde{g} \mid g \in \mathbf{G}\}, \text{ where } \tilde{g}(x_1 \wedge \cdots \wedge x_m) = g x_1 \wedge \cdots \wedge g x_m. \end{aligned}$$

The elements of its Lie algebra $\mathfrak{g}^{\wedge m}$ are of the form

$$\begin{aligned} \tilde{H}(x_1 \wedge \cdots \wedge x_m) &= (H x_1 \wedge x_2 \wedge \cdots \wedge x_m) \\ &\quad + (x_1 \wedge H x_2 \wedge x_3 \wedge \cdots \wedge x_m) + \cdots + (x_1 \wedge \cdots \wedge x_{m-1} \wedge H x_m), \end{aligned}$$

with $H \in \mathfrak{g}$.

If $(V, \mathcal{Q}, Z, \mathbf{G})$ is the classic Voronoi space of rank n , then $(V^{\wedge m}, \mathcal{Q}^{\wedge m}, Z^{\wedge m}, \mathbf{G}^{\wedge m})$ is a compact Voronoi space. The maximal value of the corresponding Hermite function is the maximal value of

$$\det\left(\left(\langle x_i \mid x_j \rangle\right)_{i,j=1}^m\right)$$

among all sets of m linear independent n -dimensional vectors x_1, \dots, x_m with integral entries. It is known as the *Rankin constant* $\gamma_{n,m}$, and it has been first considered by Rankin in [Rank53].

5. *Humbert forms.* Let K be a number field of signature (r, s) of integral ring \mathcal{O}_K , let $\sigma_1, \dots, \sigma_r$ be the real embeddings of K , and $\sigma_{r+1}, \dots, \sigma_{r+2s}$ be its complex embeddings, with $\sigma_{r+s+i} = \overline{\sigma_{r+i}}$, and let n be a positive integer. In what follows, “ \otimes ” denotes always the tensor product as *real* vector space. Let V be the linear subspace of $(\mathbb{R}^n)^{\otimes r} \otimes (\mathbb{C}^n)^{\otimes 2s}$ generated by $x_1 \otimes \cdots \otimes x_{r+2s}$, where

$$\begin{aligned} x_i &\in \mathbb{R}^n \quad \text{for } 1 \leq i \leq r, \\ x_{r+i} &\in \mathbb{C}^n \text{ and } x_{r+s+i} = \overline{x_{r+i}} \quad \text{for } 1 \leq i \leq s. \end{aligned}$$

Let Z be the subset of V given by

$$Z = \{\sigma_1(x) \otimes \cdots \otimes \sigma_{r+s}(x) \mid x \in \mathcal{O}_K^n\}.$$

Let \mathcal{Q} be the space of quadratic forms Q on V satisfying

$$Q(x_1 \otimes \cdots \otimes x_{r+2s}) = \prod_{j=1}^{r+2s} \langle x_j | A_j x_j \rangle,$$

where A_i is a positive definite symmetric operator on \mathbb{R}^n for $1 \leq i \leq r$, and $\overline{A_{r+i}}$ is a positive definite hermitian operator on \mathbb{C}^n for $1 \leq i \leq s$, and $A_{r+s+i} = \overline{A_{r+i}}$ for $1 \leq i \leq s$, and

$$\prod_{j=1}^{r+2s} \det A_j = 1.$$

Finally, let \mathbf{G} be the linear group acting on V and whose elements g satisfy

$$g(x_1 \otimes \cdots \otimes x_{r+2s}) = g_1 x_1 \otimes \cdots \otimes g_{r+2s} x_{r+2s},$$

for some $g_i \in \mathbf{GL}(n, \mathbb{R})$ for $1 \leq i \leq r$, and some $g_{r+i} \in \mathbf{GL}(n, \mathbb{C})$ and $g_{r+s+i} = \overline{g_{r+i}}$ for $1 \leq i \leq r$, such that

$$\prod_{j=1}^{r+2s} \det(g_j) = 1.$$

The elements H of its Lie algebra \mathfrak{g} are of the form

$$\begin{aligned} H(x_1 \otimes \cdots \otimes x_m) &= (H_1 x_1 \otimes x_2 \otimes \cdots \otimes x_m) \\ &+ (x_1 \otimes H_2 x_2 \otimes x_3 \otimes \cdots \otimes x_m) + \cdots + (x_1 \otimes \cdots \otimes x_{r+2s-1} \otimes H_{r+2s} x_{r+2s}), \end{aligned}$$

where $H_i \in \mathfrak{gl}(n, \mathbb{R})$ for $1 \leq i \leq r$, and $H_{r+i} \in \mathfrak{gl}(n, \mathbb{C})$ and $H_{r+s+i} = \overline{H_{r+i}}$ for $1 \leq i \leq r$, and

$$\sum_{j=1}^{r+2s} \operatorname{tr}(H_j) = 0.$$

Then $(V, \mathcal{Q}, Z, \mathbf{G})$ is a Voronoi space. The maximal values of the associated Hermite function has been in particular studied by R. Baeza and M. I. Icaza in [Icaz97] and [BaeIca97].

2 Structure of the group of Voronoi space

The theoretical results of our paper rely strongly on the structure of the group of Voronoi space, which is a selfdual real Lie group with some property of connectedness. The main features we will extract from this structure are the convexity property of Lemma 15, the reducibility criterion of Proposition 20, the structure allowing a natural definition of design (Definition 31), and the connection expressed in Theorem 33 between designs and notions related to the extremality problem.

In this section, we give details concerning the important aspects of the structure of the group of Voronoi space, relying on well-known structure results of the theory of real Lie groups. The statements of this section may also be readily checked for the particular examples of Paragraph 1.5, but we give here the proofs for the general case.

2.1 Structure of selfdual linear Lie groups

Let E be a Euclidean space; we denote by $\mathbf{P}(E)$ the cone of positive definite selfadjoint linear transformations of E , and by $\mathfrak{p}(E)$ the vector space of selfadjoint linear operators of E . Note that $\exp : \mathfrak{p}(E) \rightarrow \mathbf{P}(E)$ is a diffeomorphism. We have

a scalar product on $\text{End}(E)$, the associative algebra of linear endomorphisms of V , given by

$$\langle X | Y \rangle = \text{tr}(X^*Y).$$

For this scalar product, the selfadjoint linear operators on E are orthogonal to the antiselfadjoint linear operators, and we have

$$\langle X | [Y, Z] \rangle = \langle [Y^*, X] | Z \rangle, \quad \forall X, Y, Z \in \text{End}(E)$$

9. THEOREM. *Let E be a Euclidean space, and let $\mathbf{G} \subseteq \mathbf{GL}(E)$ be a linear Lie group such that*

$$g \in \mathbf{G} \quad \Rightarrow \quad g^* \in \mathbf{G} \quad \text{and} \quad g^*g \in \mathbf{G}^0.$$

Let moreover

- $\mathbf{K} = \mathbf{G} \cap \mathbf{O}(E)$ be the subgroup of orthogonal elements of \mathbf{G} ;
- $\mathbf{P} = \mathbf{G} \cap \mathbf{P}(E)$ be the set of positive definite selfadjoint elements of \mathbf{G} ;
- $\mathfrak{g} \subseteq \mathfrak{gl}(E)$ be the Lie algebra of \mathbf{G} ;
- $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{so}(E)$ be the subalgebra of antiselfadjoint elements of \mathfrak{g} , or the Lie algebra of \mathbf{K} ;
- $\mathfrak{p} = \mathfrak{g} \cap \mathfrak{p}(E)$ be the linear subspace of selfadjoint elements of \mathfrak{g} , or the tangent space of \mathbf{P} .

Then

- (i) we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple;
- (ii) $\mathbf{P} = \exp(\mathfrak{p})$;
- (iii) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathbf{G} = \mathbf{KP}$; moreover, for any $g \in \mathbf{G}$, there exist unique $k \in \mathbf{K}$ and $p \in \mathbf{P}$ such that $g = kp$.
- (iv) there exists a triangular connected Lie subgroup $\mathbf{D} \subseteq \mathbf{G}$ (i.e., a connected Lie subgroup whose elements are expressed by upper triangular matrices with respect to some basis of E) such that $\mathbf{G} = \mathbf{KD}$.

If \mathbf{G} were a connected semisimple group, Claims (iii) and (iv) would be respectively the classic ‘‘Cartan decomposition’’ and ‘‘Iwasawa decomposition’’ of \mathbf{G} , that we recall here:

10. THEOREM. *Let \mathbf{G} be a connected semisimple group, and let \mathfrak{g} be its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of \mathfrak{g} , that is a direct sum of vector subspaces such that the map $\theta : X + Y \mapsto X - Y$ ($X \in \mathfrak{k}$, $Y \in \mathfrak{p}$) is an automorphism of \mathfrak{g} , and the bilinear form*

$$b_\theta(X, Y) = -\text{tr}_{\mathfrak{g}}(\text{ad}(\theta X) \text{ad}(Y)), \quad X, Y \in \mathfrak{g},$$

is positive definite. Then:

- (i) (Cartan decomposition.) Let $\mathbf{K} = \exp(\mathfrak{k})$. We have a diffeomorphism

$$\varphi : \mathbf{K} \times \mathfrak{p} \rightarrow \mathbf{G}, \quad \varphi(k, H) = k \exp(H).$$

- (ii) (Iwasawa decomposition.) There exists a connected triangular Lie subgroup $\mathbf{D} \subseteq \mathbf{G}$ such that we have a diffeomorphism

$$\mu : \mathbf{K} \times \mathbf{D} \rightarrow \mathbf{G}, \quad \mu(k, d) = kd.$$

See, e.g., [OniVin94, Chap. 4, Theorem 3.2, p.145, and Theorem 4.6, p. 159]. Our proof of Theorem 9 relies on that result; but since our group \mathbf{G} is not semisimple, we need some work before we can apply it.

11. LEMMA. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(E)$ be a linear Lie algebra such that $X^* \in \mathfrak{g}$ whenever $X \in \mathfrak{g}$. Then we have the decomposition*

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g}),$$

the algebra $[\mathfrak{g}, \mathfrak{g}]$ is semisimple, the spaces $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}(\mathfrak{g})$ are stable under the adjunction $X \mapsto X^$, and they are orthogonal one to the other. Moreover, the decomposition*

$$[\mathfrak{g}, \mathfrak{g}] = ([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{k}) \oplus ([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p})$$

is a Cartan decomposition.

Proof. Let $X \in \mathfrak{g}$; we have $X \in [\mathfrak{g}, \mathfrak{g}]^\perp$ if and only if

$$0 = \langle X \mid [Y, Z] \rangle = \langle [Y^*, X] \mid Z \rangle \quad \forall Y, Z \in \mathfrak{g};$$

thus since \mathfrak{g} is stable under the adjunction, we have $[Y^*, X] = 0$ for all $Y^* \in \mathfrak{g}$, that is $X \in \mathfrak{z}(\mathfrak{g})$. This shows that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}(\mathfrak{g})$ are orthogonal.

Let now \mathfrak{r} be the radical of \mathfrak{g} ; it is clearly stable under $X \mapsto X^*$, so we can apply to \mathfrak{r} what we have proved for \mathfrak{g} , so we have $\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] \oplus \mathfrak{z}(\mathfrak{r})$. If $\mathfrak{r}' = [\mathfrak{r}, \mathfrak{r}]$, we have

$$\mathfrak{r}' = [\mathfrak{r}, \mathfrak{r}] = [\mathfrak{r}' \oplus \mathfrak{z}(\mathfrak{r}), \mathfrak{r}' \oplus \mathfrak{z}(\mathfrak{r})] = [\mathfrak{r}', \mathfrak{r}'];$$

therefore, since \mathfrak{r} is solvable, we have $[\mathfrak{r}, \mathfrak{r}] = \{0\}$. Now, since \mathfrak{r} is an ideal of \mathfrak{g} , we have $[\mathfrak{r}, [\mathfrak{r}, \mathfrak{g}]] \subseteq [\mathfrak{r}, \mathfrak{r}] = \{0\}$, and then

$$\langle [R, X] \mid [R, X] \rangle = \langle [R^*, [R, X]] \mid X \rangle = 0 \quad \forall R \in \mathfrak{r}, X \in \mathfrak{g}.$$

Therefore we have $\mathfrak{r} \subseteq \mathfrak{z}(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.

Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{k}' = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{k}$, and $\mathfrak{p}' = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$, and let us show that $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ is a Cartan decomposition. The corresponding automorphism is given by $\theta(X) = -X^*$, and the bilinear form is

$$b_\theta(X, Y) = \text{tr}_{\mathfrak{g}'}(\text{ad}(X^*) \text{ad}(Y))$$

In order to show that b_θ is positive definite, it suffices to show that $\text{ad}(X^*) = \text{ad}(X)_{\mathfrak{g}'}^*$, where $\alpha_{\mathfrak{g}'}^*$ denotes the adjoint of $\alpha \in \mathfrak{gl}(\mathfrak{g}')$ for the scalar product $\langle X \mid Y \rangle = \text{tr}(X^*Y)$ on \mathfrak{g}' . We have

$$\langle \text{ad}(X^*)Y \mid Z \rangle = \langle [X^*, Y] \mid Z \rangle = \langle Y \mid [X, Z] \rangle = \langle Y \mid \text{ad}(X)Z \rangle, \quad \forall X, Y, Z \in \mathfrak{g}'.$$

Therefore $\text{ad}(X)_{\mathfrak{g}'}^* = \text{ad}(X^*)$. □

12. LEMMA. *Let E be a Euclidean space, and let $k_1, k_2 \in \mathbf{O}(E)$ and $p_1, p_2 \in \mathbf{P}(E)$ such that $k_1 p_1 = k_2 p_2$. Then $k_1 = k_2$ and $p_1 = p_2$.*

Proof. We have $p_1 = k_1^* k_2 p_2$, and also $p_1 = p_1^* = p_2 k_2^* k_1$. Therefore

$$p_1^2 = (p_2 k_2^* k_1)(k_1^* k_2 p_2) = p_2^2.$$

Since p_1 and p_2 are positive definite, we have $p_1 = p_2$. The equality $k_1 = k_2$ follows. □

Proof of Theorem 9. (i) This follows from Lemma 11.

(ii) and (iii) Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{k}' = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{k}$, and $\mathfrak{p}' = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$. According to Theorem 10(i), we have

$$\mathbf{G}'^0 = \mathbf{K}'^0 \exp(\mathfrak{p}'),$$

where $\mathbf{G}'^0 = \exp(\mathfrak{g}')$, $\mathbf{K}'^0 = \exp(\mathfrak{k}')$. It follows from Lemma 11 that we have

$$\begin{aligned} \mathbf{K}^0 &= \mathbf{K}'^0 (\mathbf{Z}(\mathbf{G}')^0 \cap \mathbf{K}), & \exp(\mathfrak{p}) &= \exp(\mathfrak{p}') (\mathbf{Z}(\mathbf{G}')^0 \cap \mathbf{P}), \\ \mathbf{G}^0 &= \mathbf{K}^0 \exp(\mathfrak{p}). \end{aligned}$$

Now, let $g \in \mathbf{G}$; as we have $g^*g \in \mathbf{G}^0$, there exists $\tilde{k} \in \mathbf{K}^0$ and $H \in \mathfrak{p}$ such that $g^*g = \tilde{k} \exp(H)$. Since g^*g is symmetric positive definite, by Lemma 12, we have in fact $g^*g = \exp(H)$. Then we have $g = kp$ with

$$k = g \exp(-H/2) \in \mathbf{K} \quad \text{and} \quad p = \exp(H/2) \in \exp(\mathfrak{p}).$$

In particular, if $p \in \mathbf{P}$, we have a decomposition $p = kp'$ with $p' \in \exp(\mathfrak{p})$. By Lemma 12, we have $p = p'$; therefore, the equality $\mathbf{P} = \exp(\mathfrak{p})$ holds. The same Lemma 12 shows the uniqueness of the decomposition $g = kp$.

(iv) According to Theorem 10(ii), there exists a connected triangular Lie subgroup $\mathbf{D}' \subseteq \mathbf{G}'$ such that $\mathbf{G}'^0 = \mathbf{K}'^0 \mathbf{D}'$. The group

$$\mathbf{D} = \mathbf{D}' (\mathbf{Z}(\mathbf{G}')^0 \cap \mathbf{P})$$

is connected and triangular, and we have $\mathbf{G}^0 = \mathbf{K}^0 \mathbf{D}$. Now, we have

$$\mathbf{K} \mathbf{D} = \mathbf{K} \mathbf{K}^0 \mathbf{D} = \mathbf{K} \mathbf{G}^0 = \mathbf{K} \mathbf{K}^0 \mathbf{P} = \mathbf{K} \mathbf{P} = \mathbf{G}. \quad \square$$

2.2 Structure of the group of Voronoi space

In order to apply Theorem 9 to the group of a Voronoi space, we have to check that the assumption of the theorem is satisfied.

13. LEMMA. *Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, and let $Q \in \mathcal{Q}$. Then, for every $g \in \mathbf{G}$, we have $g_Q^*g \in \mathbf{G}^0$.*

Proof. Let $g \in \mathbf{G}$, and let $Q' = Q \circ g$. As \mathcal{Q} is connected, there exists $h \in \mathbf{G}^0$ such that $Q' = Q \circ h$. Then we have

$$Q'(x, y) = Q(x, g_Q^*gy) = Q(x, h_Q^*hy), \quad \forall x, y \in V.$$

Therefore, we have $g_Q^*g = h_Q^*h \in \mathbf{G}^0$. \square

As a first application of Theorem 9, we have the following description of the set of quadratic forms of a Voronoi space.

14. PROPOSITION. *Let $(V, \mathcal{Q}, Z, \mathbf{G})$ a Voronoi space, and let $Q \in \mathcal{Q}$. For $H \in \mathfrak{p}_Q$, we define Q_H by*

$$Q_H(x) = Q(x, \exp(H)x).$$

Then we have

$$\mathcal{Q} = \{Q_H \mid H \in \mathfrak{p}_Q\}.$$

In particular, if, for $Q \in \mathcal{Q}$ and $\epsilon > 0$, we set

$$V_{Q, \epsilon} = \{Q_H \mid H \in \mathfrak{p}_Q \text{ and } \|H\|_Q < \epsilon\},$$

then the family $\{V_{Q, \epsilon}\}_{\epsilon > 0}$ is a basis of neighborhoods of Q in \mathcal{Q} .

Proof. Let $Q' \in \mathcal{Q}$, and let $g \in \mathbf{G}$ such that $Q' = Q \circ g$. According to Theorem 9, we have $g_Q^*g \in \mathbf{P}_Q = \exp(\mathfrak{p}_Q)$, therefore there exists an $H \in \mathfrak{p}_Q$ such that $g_Q^*g = \exp(H)$, and we have $Q' = Q_H$. Conversely, for $H \in \mathfrak{p}_Q$, we have $Q_H = Q \circ g$ with $g = \exp(H/2)$, so $Q_H \in \mathcal{Q}$. The rest of the lemma is straightforward. \square

3 Characterization of extremality à la Voronoi

The notions of perfection and eutaxy play a key role in theories involving problems of extremality for a Hermite-type function, because they are the main ingredients of the theory of Voronoi and its generalizations.

In this section, we introduce the notion of perfection and eutaxy in a Voronoi space, and we prove a characterization of extremality and strict extremality in the spirit of the one of Voronoi (Theorem 18). Finally, we prove a statement (Proposition 20) which provides an efficient tool to check that extremality implies strict extremality in many Voronoi spaces.

3.1 Neighborhoods in Voronoi space

Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space. Firstly, we need estimates of the value taken by forms in a neighborhood of a given quadratic form Q in \mathcal{Q} . Recall (Proposition 14) that such quadratic forms are of the form $Q_H(x) = Q(x, \exp(H)x)$, with $H \in \mathfrak{p}_Q$ and $\|H\|_Q$ small. Recall that we have defined

$$\|H\|_Q^2 = \sup_{x \in V \setminus \{0\}} \frac{Q(Hx)}{Q(x)},$$

The triangular inequality implies $|Q(x, Hx)| \leq \|H\| \times Q(x)$.

The following lemma states that the functions $t \mapsto Q_{tH}(x)$ are strictly convex near to 0, except when they are constant. The convexity is a key property in all characterizations of extremality by perfection and eutaxy.

15. LEMMA. *Let $Q \in \mathcal{Q}$ and $0 < t \leq 1$, and $H \in \mathfrak{p}_Q$ such that $\|H\|_Q \leq 1$. For every $x \in V$, we have*

$$Q(x) + tQ(x, Hx) \leq Q_{tH}(x) \leq Q(x) + tQ(x, Hx) + t^2Q(Hx),$$

with equality if and only if $Hx = 0$.

Proof. We have

$$\begin{aligned} Q_{tH}(x) &= Q(x, \exp(tH)x) = \sum_{k \geq 0} \frac{t^k}{k!} Q(x, H^k x) \\ &= Q(x) + tQ(x, Hx) + \frac{t^2}{2}Q(Hx) + \sum_{k \geq 3} \frac{t^k}{k!} Q(Hx, H^{k-1}x). \end{aligned}$$

Let S be the sum at the end of that expression. We have $|Q(Hx, H^{k-1}x)| \leq \|H\|_Q^{k-2} Q(Hx)$, and therefore

$$|S| = \left| \sum_{k \geq 3} \frac{t^k}{k!} Q(Hx, H^{k-1}x) \right| \leq t^2 \sum_{k \geq 3} \frac{(t\|H\|_Q)^{k-2}}{k!} Q(Hx)$$

Now, using $t \leq 1$ and $\|H\|_Q \leq 1$, we get

$$|S| \leq t^2 \sum_{k \geq 3} \frac{1}{k!} Q(Hx) = t^2(e - 5/2) Q(Hx) \leq \frac{1}{2}t^2 Q(Hx),$$

with equality if and only if $Q(Hx) = 0$. From this, we deduce

$$Q(x) + tQ(x, Hx) \leq Q(x, \exp(tH)x) \leq Q(x) + tQ(x, Hx) + t^2Q(Hx),$$

with equality if and only if $Hx = 0$. □

The next lemma is standard in all Voronoi theories. It states that, when studying the value of the Hermite function on quadratic forms in a neighborhood of $Q \in \mathcal{Q}$, it suffices to consider the values taken by those quadratic forms on the minimal layer of Q .

16. LEMMA. *Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, and let $Q \in \mathcal{Q}$. Then there exists a neighborhood $U \subseteq \mathcal{Q}$ of Q such that $Q'_{\min} \subseteq Q_{\min}$ for every $Q' \in U$.*

Proof. Let $\gamma = \gamma(Q) = \min_{x \in Z} Q(x)$ and let $\delta = \min_{x \in Z \setminus Q_{\min}} Q(x) - \gamma > 0$. Let $H \in \mathfrak{p}_Q$ with $\|H\|_Q \leq 1$; by Lemma 15, we have, for any $x \in Q_{\min}$,

$$\begin{aligned} Q_H(x) &\leq Q(x) + Q(x, Hx) + Q(Hx) \leq Q(x) + \|H\|_Q Q(x) + \|H\|_Q^2 Q(x) \\ &\leq \gamma(1 + \|H\|_Q + \|H\|_Q^2), \end{aligned}$$

and for any $z \in Z \setminus Q_{\min}$

$$Q_H(z) \geq Q(z) + Q(z, Hz) \geq Q(z) - \|H\|_Q Q(z) \geq (\gamma + \delta)(1 - \|H\|_Q)$$

Let $\epsilon > 0$ such that $\gamma(1 + \eta + \eta^2) < (\gamma + \delta)(1 - \eta)$ for every η with $0 \leq \eta < \epsilon$. Then $Q_H(x) < Q_H(z)$ if $\|H\|_Q < \epsilon$. So we have the claim of the lemma with $U = \{Q_H \mid H \in \mathfrak{p}_Q, \|H\|_Q < \epsilon\}$. \square

3.2 Perfection and eutaxy

We introduce now the two fundamental notions, and variants of them, which enters in characterizations of extremality.

17. DEFINITION. Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space. Let $Q \in \mathcal{Q}$ be a quadratic form, and X a nonempty finite subset of $V \setminus \{0\}$.

(a) (Q, X) is *eutactic* when, for every $H \in \mathfrak{g}$,

$$\exists x \in X, Q(x, Hx) > 0 \implies \exists y \in X, Q(y, Hy) < 0.$$

(b) (Q, X) is *strongly eutactic* when, for every $H \in \mathfrak{g}$,

$$\sum_{x \in X} \frac{Q(x, Hx)}{Q(x)} = 0.$$

(c) (Q, X) is *weakly perfect* when, for every $H \in \mathfrak{g}$,

$$\exists c \in \mathbb{R}, \forall x \in X, Q(x, Hx) = cQ(x) \implies \forall x \in X, Hx + H_Q^*x = 2cx.$$

(d) (Q, X) is *perfect* when, for every $H \in \mathfrak{g}$,

$$\exists c \in \mathbb{R}, \forall x \in X, Q(x, Hx) = cQ(x) \implies H + H_Q^* = 2c \operatorname{id}_V.$$

Moreover, Q is *eutactic*, *perfect*, etc., when (Q, Q_{\min}) is eutactic, perfect, etc.

In the definition above, we can replace $H \in \mathfrak{g}$ by $H \in \mathfrak{p}_Q$. This follows directly from the fact that any $H \in \mathfrak{g}$ can be decomposed in \mathfrak{g} as the sum of a Q -selfadjoint element and a Q -antiselfadjoint one, and that the conditions of the definition are trivially true for Q -antiselfadjoint linear transformations. Namely, we have:

(a) (Q, X) is eutactic iff, for every $H \in \mathfrak{p}_Q$,

$$\exists x \in X, Q(x, Hx) > 0 \implies \exists y \in X, Q(y, Hy) < 0.$$

(b) (Q, X) is strongly eutactic iff, for every $H \in \mathfrak{p}_Q$,

$$\sum_{x \in X} \frac{Q(x, Hx)}{Q(x)} = 0.$$

(c) (Q, X) is weakly perfect iff, for every $H \in \mathfrak{p}_Q$,

$$\exists c \in \mathbb{R}, \forall x \in X, Q(x, Hx) = cQ(x) \implies \forall x \in X, Hx = cx.$$

(d) (Q, X) is perfect iff, for every $H \in \mathfrak{p}_Q$,

$$\exists c \in \mathbb{R}, \forall x \in X, Q(x, Hx) = cQ(x) \implies H = cid_V.$$

We mention also a geometric interpretation of these notions, although we will not use it. Let us fix $Q \in \mathcal{Q}$ and consider, for $x \in V$, the linear form $\epsilon_x \in (\mathfrak{p}_Q \oplus \mathbb{R} \text{id}_V)^*$ defined by $\epsilon_x(H) = Q(x, Hx)/Q(x)$. Let τ be the linear form on $\mathfrak{g} \oplus \mathbb{R} \text{id}_V$ such that $\tau(H) = 0$ for any $H \in \mathfrak{g}$ and $\tau(\text{id}_V) = 1$; for most interesting examples of Voronoi spaces, the form τ is the normalized trace on V . Then

(a) (Q, X) is eutactic iff the form $\tau|_{\mathfrak{p}_Q \oplus \mathbb{R} \text{id}_V}$ is in the geometric interior of the convex hull of $\{\epsilon_x\}_{x \in X} \subseteq (\mathfrak{p}_Q \oplus \mathbb{R} \text{id}_V)^*$.

(b) (Q, X) is strongly eutactic iff

$$\frac{1}{|X|} \sum_{x \in X} \epsilon_x = \tau|_{\mathfrak{p}_Q \oplus \mathbb{R} \text{id}_V}.$$

(c) (Q, X) is weakly perfect iff the family $\{\tilde{\epsilon}_x\}_{x \in X}$ generates $((\mathfrak{p}_Q \oplus \mathbb{R} \text{id}_V)/\mathcal{A})^*$, where

$$\mathcal{A} = \{H \in \mathfrak{p}_Q \mid Hx = 0, \forall x \in X\},$$

$$\text{and } \tilde{\epsilon}_x(H + \mathcal{A}) = \epsilon_x(H).$$

(d) (Q, X) is perfect iff the family $\{\epsilon_x\}_{x \in X}$ generates $(\mathfrak{p}_Q \oplus \mathbb{R} \text{id}_V)^*$.

The geometric reformulation of Claim (a) can be deduced from the Hahn-Banach theorem; other claims follow from standard finite-dimensional linear space theory.

3.3 Characterization of extremality

The following result, that we call the *Voronoi characterization of extremality*, characterizes extreme and strictly extreme forms in terms of perfection and eutaxy. The characterization of strict extremality is known for many particular cases: it has been proved by Voronoi in the case of classic Voronoi space [Voro08], by Bergé and Martinet in the cases of dual-extremality in classic Voronoi space [BerMar89, Théorème 3.14], and family of quadratic forms in classic Voronoi space [BerMar91, Théorème 2.10], [BerMar95, Théorème 4.5], by Coulangéon in the case of exterior powers of classic Voronoi space [Coul96, Théorème 3.2.3], and Humbert forms [Coul01, Proposition 3]. Moreover, in many of the above cases, extremality is shown to be equivalent to strict extremality; but this question will be considered in the next paragraph. However, we do not know any appearance in the literature of a characterization of extremality, not necessarily strict, similar to the one we give here—although this absence is justified in the many cases where extremality implies strict extremality—, but a weaker result appears in [BerMar95, Théorème 4.5].

18. THEOREM (VORONOI CHARACTERIZATION). *Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space.*

- (i) *A quadratic form $Q \in \mathcal{Q}$ is strictly extreme if and only if it is perfect and eutactic.*
- (ii) *A quadratic form $Q \in \mathcal{Q}$ is extreme if and only if there exists a nonempty subset of Q_{\min} that is weakly perfect and eutactic.*

Before giving the proof, we reformulate the conditions of perfection and eutaxy:

19. LEMMA. *Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, and let X be a nonempty finite subset of V .*

- (i) *(Q, X) is perfect and eutactic if and only if, for each $H \in \mathfrak{p}_Q$, one of the two following conditions holds:*
 - (a) $H = 0$;
 - (b) *there exists $x \in X$ such that $Q(x, Hx) < 0$.*
- (ii) *(Q, X) is weakly perfect and eutactic if and only if, for each $H \in \mathfrak{p}_Q$, one of the two following conditions holds:*
 - (a) $Hx = 0$ for every $x \in X$;
 - (b) *there exists $x \in X$ such that $Q(x, Hx) < 0$.*

Proof. Let $H \in \mathfrak{p}_Q$. The eutaxy of (Q, X) implies that Condition (b) holds unless we have $Q(x, Hx) = 0$ for every $x \in X$. In the last case, the perfection, respectively the weak perfection, of (Q, X) implies Condition (a). \square

Proof of the Theorem. First, recall that a basis of neighborhoods of $Q \in \mathcal{Q}$ is given by $\{Q_H \mid H \in \mathfrak{p}_Q \text{ and } \|H\|_Q < \epsilon\}$, with $\epsilon > 0$ (Proposition 14).

(i, \Leftarrow) Suppose that Q is perfect and eutactic. Consider the following function on the sphere $\mathbb{S}(\mathfrak{p}_Q)$ of elements of \mathfrak{p}_Q whose Q -norm is equal to 1:

$$m(H) = \min_{x \in Q_{\min}} Q(x, Hx), \quad H \in \mathbb{S}(\mathfrak{p}_Q).$$

The perfection and eutaxy of Q imply that $m(H) < 0$ for each $H \in \mathbb{S}(\mathfrak{p}_Q)$ (Lemma 19). Then, the continuity of m and the compactness of $\mathbb{S}(\mathfrak{p}_Q)$ imply that there exists a $\delta > 0$ such that $m(H) < -\delta$ for every $H \in \mathbb{S}(\mathfrak{p}_Q)$. We infer that

$$\min_{x \in Q_{\min}} Q(x, Hx) < -\delta \|H\|_Q, \quad \forall H \in \mathfrak{p}_Q \setminus \{0\}.$$

Now, let $H \in \mathfrak{p}_Q \setminus \{0\}$ with $\|H\|_Q < 1$, and let $x \in Q_{\min}$ such that $Q(x, Hx) < -\delta \|H\|_Q$. By Lemma 15,

$$Q_H(x) \leq Q(x) + Q(x, Hx) + Q(Hx) < Q(x) - \delta \|H\|_Q + \|H\|_Q^2 Q(x).$$

Therefore, when $\|H\|_Q < \min(1, \delta/Q(x))$, we have $\gamma(Q_H) \leq Q_H(x) < Q(x) = \gamma(Q)$. So, Q is strictly extreme.

(ii, \Leftarrow) The pattern of the proof is similar as the one for (i, \Leftarrow) above. Suppose that there exists $X \subseteq Q_{\min}$ such that (Q, X) is weakly perfect and eutactic. Let $\mathcal{A} = \{H \in \mathfrak{p}_Q \mid Hx = 0, \forall x \in X\}$. We have a norm $\|\cdot\|_{\mathcal{A}}$ on $\mathfrak{p}_Q/\mathcal{A}$ defined by

$$\|H + \mathcal{A}\|_{\mathcal{A}} = \min_{J \in \mathcal{A}} \|H + J\|_Q, \quad H \in \mathfrak{p}_Q.$$

As $(H + J)x = Hx$ for every $J \in \mathcal{A}$ and every $x \in X$, we have

$$|Q(x, Hx)| \leq \|H + \mathcal{A}\|_Q Q(x) \quad \text{and} \quad Q(Hx) \leq \|H + \mathcal{A}\|_Q^2 Q(x), \quad \forall x \in X.$$

We define now a function m on the sphere $\mathbb{S}(\mathfrak{p}_Q/\mathcal{A})$ consisting of classes of norm 1 by

$$m(H + \mathcal{A}) = \min_{x \in X} Q(x, Hx), \quad H \in \mathfrak{p}_Q, \quad \|H + \mathcal{A}\|_Q = 1.$$

Now we use the weak perfection and the eutaxy of (Q, X) : Lemma 19 implies that $m(H + \mathcal{A})$ is negative on $\mathbb{S}(\mathfrak{p}_Q/\mathcal{A})$; therefore, by compactness, there exists a $\delta > 0$ such that $m(H + \mathcal{A}) < -\delta$ for every $H + \mathcal{A} \in \mathbb{S}(\mathfrak{p}_Q/\mathcal{A})$. We infer that

$$\min_{x \in Q_{\min}} Q(x, Hx) \leq -\delta \|H + \mathcal{A}\|_Q, \quad \forall H \in \mathfrak{p}_Q.$$

Now, let $H \in \mathfrak{p}_Q$ with $\|H\|_Q < 1$, and let $x \in X$ such that $Q(x, Hx) \leq -\delta \times \|H + \mathcal{A}\|_Q$. By Lemma 15, we have

$$Q_H(x) \leq Q(x) + Q(x, Hx) + Q(Hx) \leq Q(x) - \delta \|H + \mathcal{A}\|_Q + \|H + \mathcal{A}\|_Q^2 Q(x).$$

Therefore, when $\|H + \mathcal{A}\|_Q < \min(1, \delta/Q(x))$, we have $\gamma(Q_H) \leq Q_H(x) \leq Q(x) = \gamma(Q)$. So, Q is extreme.

(i, \Rightarrow) Suppose that Q is strictly extreme. Let $H \in \mathfrak{p}_Q \setminus \{0\}$ with $\|H\|_Q \leq 1$. By Lemma 15, we have, for each $x \in Q_{\min}$,

$$Q_{tH}(x) \geq Q(x) + tQ(x, Hx), \quad \forall t, \quad 0 < t < 1.$$

Since Q is strictly extreme, and $(Q_{tH})_{\min} \subseteq Q_{\min}$ for t sufficiently small (Lemma 16), we have $Q_{tH}(x) < Q(x)$ for some $x \in Q_{\min}$ and some $t > 0$. This implies $Q(x, Hx) < 0$ for some $x \in Q_{\min}$, which means, by Lemma 19, that (Q, Q_{\min}) is both perfect and eutactic.

(ii, \Rightarrow) We proceed by induction on the cardinal of Q_{\min} . Suppose that Q is extreme. Let $0 < \delta \leq 1$ such that, for every $H \in \mathfrak{p}_Q$ with $\|H\|_Q \leq \delta$, we have $(Q_H)_{\min} \subseteq Q_{\min}$ (by Lemma 16) and $\gamma(Q_H) \leq \gamma(Q)$ (by the assumption of extremality). Suppose that (Q, Q_{\min}) is *not* weakly perfect and eutactic (otherwise we are done). Then, by Lemma 19, there exists an $H \in \mathfrak{p}_Q$ and an $x_0 \in Q_{\min}$, such that $Q(x, Hx) \geq 0$ for every $x \in Q_{\min}$ and $Hx_0 \neq 0$. Moreover, we can choose H such that $\|H\|_Q < \delta$. By Lemma 15, we have,

$$Q_H(x) \geq Q(x) + Q(x, Hx), \quad \forall x \in Q_{\min}, \quad \text{with a strict inequality when } Hx \neq 0.$$

Then the extremality of Q and the choice of δ imply that $\gamma(Q_H) = \gamma(Q)$, with

$$(Q_H)_{\min} = \{x \in Q_{\min} \mid Hx = 0\},$$

and $|(Q_H)_{\min}| < |Q_{\min}|$ (since $Hx_0 \neq 0$), and Q_H is itself extreme. By induction, there exists a subset X of $(Q_H)_{\min}$ such that (Q_H, X) is weakly perfect and eutactic. It remains to show that (Q, X) is also weakly perfect and eutactic.

On the one hand, we have, for every $x \in X$ and every $J \in \mathfrak{g}$, using $Hx = 0$,

$$Q_H(x, Jx) = Q(\exp(H)x, Jx) = Q(x, Jx).$$

On the other hand, we have for every $x \in X$ and every $y \in V$,

$$\begin{aligned} Q(J_Q^*x, y) &= Q(x, Jy) = Q_H(\exp(-H)x, Jy) \\ &= Q_H(x, Jy) = Q_H(J_{Q_H}^*x, y) = Q(\exp(H)J_{Q_H}^*x, y), \end{aligned}$$

so $J_{Q_H}^* x = \exp(-H) J_Q^* x$ for every $x \in X$.

Let us now show that (Q, X) is weakly perfect and eutactic using Lemma 19. Let $J \in \mathfrak{p}_Q$. If there exists $x \in X$ such that $Q_H(x, Jx) < 0$, then, since $Q_H(x, Jx) = Q(x, Jx)$ for every $x \in X$, there exists $x \in X$ such that $Q(x, Jx) < 0$. Otherwise, since (Q_H, X) weakly perfect and eutactic, by Lemma 19, we have $Jx + J_{Q_H}^* x = 0$ for every $x \in X$. But since, $J_{Q_H}^* x = \exp(-H) J_Q^* x$, we have $(1 + \exp(-H)) Jx = 0$. Yet $(1 + \exp(-H))$ is an isomorphism, because it is positive Q -definite. Therefore $Jx = 0$ for every $x \in X$. \square

3.4 More about perfection

It is known that, in classic Voronoi spaces and many other interesting Voronoi spaces, extremality implies strict extremality. This is not true in general; but the following proposition gives a simple property of weakly perfect quadratic form, which turns to be an efficient tool to rule out the existence of extreme but not strictly extreme quadratic forms in many Voronoi spaces.

20. PROPOSITION. *Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, and let $Q \in \mathcal{Q}$ be a quadratic form and $X \subseteq V$ be a nonempty finite subset. If (Q, X) is weakly perfect but not perfect, then there exists a proper \mathbf{G}^0 -invariant subspace U of V such that $X \subseteq U$.*

Proof. Let $Q \in \mathcal{Q}$ and $X \subseteq V$ as in the hypothesis of the theorem. Consider the following subspace of $\mathfrak{p}_Q \oplus \mathbb{R} \text{id}_V$:

$$\mathcal{A} = \{H \in \mathfrak{p}_Q \oplus \mathbb{R} \text{id}_V \mid Hx = 0, \forall x \in X\}.$$

Since (Q, X) is weakly perfect but not perfect, \mathcal{A} is not reduced to $\{0\}$. We define U as

$$U = \bigcap_{H \in \mathcal{A}} \ker H,$$

which is a proper subspace of V containing X . It remains to show that it is \mathbf{G}^0 -invariant, or, equivalently, \mathfrak{g} -invariant.

(a) Let us first show that U is invariant under the action of \mathfrak{k}_Q . Let $H \in \mathfrak{k}_Q$. For any $J \in \mathcal{A}$ and any $x \in X$, we have

$$Q([J, H]x, x) = Q(JHx - HJx, x) \stackrel{Jx=0}{=} Q(JHx, x) \stackrel{J=J_Q^*}{=} Q(Hx, Jx) \stackrel{Jx=0}{=} 0.$$

Since $[J, H] \in \mathfrak{p}_Q$ and since (Q, X) is weakly perfect, we have $[J, H] \in \mathcal{A}$. Then, for any $u \in U$, we have

$$JHu = [J, H]u + HJu = 0,$$

so $Hu \in \ker J$ for any $J \in \mathcal{A}$, therefore $Hu \in U$ by definition of U .

(b) Let us now show that the elements of \mathfrak{p}_Q preserve U ; since $\mathfrak{g} = \mathfrak{k}_Q \oplus \mathfrak{p}_Q$, this will achieve the proof. Let $H \in \mathfrak{p}_Q$. For any $J \in \mathcal{A}$, and any $u \in U$, we have

$$\begin{aligned} Q([J, H]u, [J, H]u) &= Q(JHu - HJu, [J, H]u) \\ &\stackrel{Ju=0}{=} Q(JHu, [J, H]u) \stackrel{J=J_Q^*}{=} Q(Hu, J[J, H]u). \end{aligned}$$

Because we have $[J, H] \in \mathfrak{k}_Q$ and the elements of \mathfrak{k}_Q preserve U by the part (a) of the proof, we have $[J, H]u \subseteq U$, and therefore $J[J, H]u = 0$. We infer that $Q([J, H]u, [J, H]u) = 0$, that is $[J, H]u = 0$. Then

$$JHu = [J, H]u + HJu = 0,$$

so $Hu \in \ker J$ for any $J \in \mathcal{A}$, therefore $Hu \in U$ by definition of U . \square

The last proposition above implies that, in a Voronoi space $(V, \mathcal{Q}, Z, \mathbf{G})$ where an orbit of an element of Z under the action of \mathbf{G}^0 generates the space V , weak perfection implies perfection, and therefore, by Theorem 18, extremality implies strict extremality. This is the case in particular for

- the classic Voronoi space of rank n (Example 1 of Paragraph 1.5). Here, we have $\mathbf{G}^0 = \mathbf{SL}(n, \mathbb{R})$, which has two orbits on V , namely $\{0\}$ and $V \setminus \{0\}$;
- the m th exterior power $(V^{\wedge m}, \mathcal{Q}^{\wedge m}, Z^{\wedge m}, \mathbf{G}^{\wedge m})$ of the classic Voronoi space of rank n , with $m \leq n/2$ (Example 4 of Paragraph 1.5). Here, the orbit of an element of $Z^{\wedge m}$ under the action of \mathbf{G}^0 is equal to

$$\{x_1 \wedge \cdots \wedge x_m \mid x_1, \dots, x_m \text{ are linearly independent elements of } \mathbb{R}^n\},$$

and it spans $V^{\wedge m}$;

- the space $(V, \mathcal{Q}, Z, \mathbf{G})$ of Humbert forms of rank n over a number field (Example 5 of Paragraph 1.5). Here, the orbit of an element of Z under the action of \mathbf{G}^0 is equal to

$$\{x_1 \otimes \cdots \otimes x_{r+2s} \mid x_i \in \mathbb{R}^n \setminus \{0\} \text{ for } 1 \leq i \leq r, \\ x_{r+i} \in \mathbb{C}^n \setminus \{0\} \text{ and } x_{r+s+i} = \overline{x_{r+i}} \text{ for } 1 \leq i \leq s\},$$

and this set spans V .

- the “symplectic” Voronoi space of rank $2m$: Let $\sigma = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, and consider the Voronoi space of σ -isodual quadratic form of the classic Voronoi space of rank $2m$, as constructed in Example 3(b) of Paragraph 1.5. Here, we have $\mathbf{G}^0 = \mathbf{Sp}(2m)^0$, which has two orbits on V , namely $\{0\}$ and $V \setminus \{0\}$.

As another example, let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a classic Voronoi space, and let $(\tilde{V}, \tilde{\mathcal{Q}}, \tilde{Z}, \tilde{\mathbf{G}})$ be the Voronoi space constructed in Example 2 of Paragraph 1.5 (with $Q^\alpha = Q^{-1}$). For $Q \in \mathcal{Q}$ and $c \in \mathbb{R}^*$, let $\tilde{Q}_c \in \tilde{\mathcal{Q}}$ such that $\tilde{Q}_c((x_1, x_2)) = Q(cx_1) + Q^{-1}(c^{-1}x_2)$. If c is chosen such that $c^2\gamma(Q) < c^{-2}\gamma(Q^{-1})$, we have $(\tilde{Q}_c)_{\min} = Q_{\min} \times \{0\}$, and \tilde{Q}_c is weakly perfect if and only if Q is perfect; but \tilde{Q}_c is not perfect. However \tilde{Q}_c is not eutactic either, and it is easy to show that all examples of weakly perfect but not perfect quadratic forms in this space are of this type. Therefore, in this space, extremality implies strict extremality.

Another question is the finiteness (up to equivalence) of perfect forms as it is the case for the classic Voronoi space. We conjecture that the set of perfect forms in a Voronoi space is closed and discrete, but we have not been able to prove it, except under a very restrictive hypothesis, namely when the Lie algebra $\mathfrak{g} \oplus \mathbb{R} \text{id}_V$ is also an associative algebra. Since the set of Voronoi spaces given in Paragraph 1.5 for which that condition is satisfied is a strict subset of the set of Voronoi spaces for which finiteness of perfect forms is already known, we will not give the proof.

Also, an algorithm which would allow to enumerate theoretically all perfect form, as the one of Voronoi [Voro08], [Mart03, Chap. 7], which has been extended in various situations [Mart03, Chap. 13], [Bava01], [Bava07], would be a subject of high interest. However, a study of such an algorithm, is beyond the scope of this paper.

4 Alternative Voronoi spaces

The classic problems related to the Hermite function can be equivalently formulated in terms of quadratic forms or of Euclidean lattices. While the former formulation

is more convenient for the proof of the Voronoi characterization of extremality, the latter one is more appropriate for expressing the relation between spherical designs and extremality.

The same situation holds in our frame of Voronoi spaces. Hitherto, we have worked with the “quadratic forms” formulation, in which the subset Z of V is fixed and the quadratic forms Q varies. But in order to define designs, we need to work with the dual formulation, where the quadratic form Q is fixed and the corresponding subset Z of V varies; we name *alternative Voronoi space* this dual concept of Voronoi space. The Voronoi characterization of extremality is reformulated in terms of alternative Voronoi space in Theorem 26.

Regular alternative Voronoi spaces as defined in Paragraph 4.3 are a particular type of alternative Voronoi spaces for which designs may be defined in the most natural way. Finally, in the last paragraph of this section (Paragraph 4.4), we give the dual formulation of all the examples of Voronoi spaces given in Paragraph 1.5, and we add another example of alternative Voronoi space.

4.1 Alternative Voronoi space

21. DEFINITION. (Compare with Definition 1.) An *alternative Voronoi space* is the data of

- (a) a finite-dimensional real vector space V ,
- (b) a positive definite quadratic form Q of V ,
- (c) a closed (for the topology of simple convergence) set \mathcal{Z} of closed discrete subsets Z of V not containing 0 and spanning V ,
- (d) a closed linear Lie group $\mathbf{G} < \mathbf{GL}(V)$,

satisfying the following properties:

- (i) For $Z \in \mathcal{Z}$ and $c > 0$, we have $cZ \in \mathcal{Z}$ only if $c = 1$;
- (ii) for all $g \in \mathbf{G}$, we have $g_Q^* \in \mathbf{G}$ and $g_Q^*g \in \mathbf{G}^0$;
- (iii) for all $g \in \mathbf{G}$ and $Z \in \mathcal{Z}$, we have $gZ \in \mathcal{Z}$, and \mathbf{G} is transitive on \mathcal{Z} .

The supplementary requirement $g_Q^*g \in \mathbf{G}^0$ in condition (ii) is to be compared to Lemma 13; it replaces the condition of connectedness of \mathcal{Q} in the definition of Voronoi space.

22. DEFINITION. (Compare with Definition 2.) Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space. We define the following objects:

- $\mathbf{K} = \{g \in \mathbf{G} \mid g_Q^* = g^{-1}\}$, the group of Q -selfadjoint elements of \mathbf{G} , which is the *compact group* of $(V, Q, \mathcal{Z}, \mathbf{G})$;
- \mathfrak{g} , the Lie algebra of \mathbf{G} ;
- $\mathfrak{k} = \{H \in \mathfrak{g} \mid H + H_Q^* = 0\}$, the subalgebra of Q -antiselfadjoint elements of \mathfrak{g} , or the Lie algebra of \mathbf{K} ;
- $\mathfrak{p} = \{H \in \mathfrak{g} \mid H - H_Q^* = 0\}$, the set of Q -selfadjoint elements of \mathfrak{g}_Q ;
- \mathbf{G}^0 and \mathbf{K}^0 , the connected component of the identity of \mathbf{G} and \mathbf{K} respectively.

Note that we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

23. DEFINITION. (Compare with Definition 4.) Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space.

- (i) The *Hermite function* associated to $(V, Q, \mathcal{Z}, \mathbf{G})$ is the function γ defined on $\mathbf{K} \setminus \mathcal{Z}$ by

$$\gamma(\mathbf{K}Z) = \min_{x \in Z} Q(x), \quad Z \in \mathcal{Z}.$$

- (ii) For $r > 0$, the *layer* of square radius r of $Z \in \mathcal{Z}$ is the set

$$Z_r = \{x \in Z \mid Q(x) = r\}.$$

- (iii) The set of *minimal vectors* of a set $Z \in \mathcal{Z}$ is the layer of minimal radius, that is

$$Z_{\min} = Z_{\gamma(Z)} = \{x \in Z \mid Q(x) = \gamma(Q)\}.$$

- (iv) A set $Z \in \mathcal{Z}$ is *extreme*, respectively *strictly extreme*, when its class $\mathbf{K}Z$ is a local maximum, respectively a strict local maximum, of γ .

24. DEFINITION. (Compare with Definition 5.) The alternative Voronoi space $(V, Q, \mathcal{Z}, \mathbf{G})$ is *compact* when $\gamma : \mathbf{K} \setminus \mathcal{Z} \rightarrow]0, \infty[$ is bounded and proper.

4.2 Equivalence between Voronoi space and alternative Voronoi space

We construct standard transformations from Voronoi spaces to alternative Voronoi spaces and conversely as follows:

Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, and let $Q \in \mathcal{Q}$. We construct an alternative Voronoi space $(V, Q, \mathcal{Z}, \mathbf{G})$ by setting

$$\mathcal{Z} = \{gZ \mid g \in \mathbf{G}\}.$$

If γ is the Hermite function of (V, Q, Z, \mathbf{G}) and $\tilde{\gamma}$ the Hermite function of $(V, Q, \mathcal{Z}, \mathbf{G})$, we have $\tilde{\gamma}(gZ) = \gamma(Q \circ g)$.

Conversely, let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space, and let $Z \in \mathcal{Z}$. We construct a Voronoi space $(V, \mathcal{Q}, Z, \mathbf{G})$ by setting

$$\mathcal{Q} = \{Q \circ g \mid g \in \mathbf{G}\}.$$

The property $g_Q^* g \in \mathbf{G}^0$ for $g \in \mathbf{G}$ implies that \mathcal{Q} is connected. If γ is the Hermite function of (V, Q, Z, \mathbf{G}) and $\tilde{\gamma}$ the Hermite function of $(V, \mathcal{Q}, Z, \mathbf{G})$, we have $\tilde{\gamma}(Q \circ g) = \gamma(gZ)$.

The Voronoi characterization of extremality (Theorem 18) can be formulated in the language of alternative Voronoi space. First we have to complete the definitions of eutaxy and perfection (Definition 17):

25. DEFINITION. Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space, and let $Z \in \mathcal{Z}$. Then Z is *eutactic*, *perfect*, etc., when (Q, Z_{\min}) is eutactic, perfect, etc.

26. THEOREM (VORONOI CHARACTERIZATION). (Compare with Theorem 18.) Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space.

- (i) A set $Z \in \mathcal{Z}$ is *strictly extreme* if and only if it is *perfect and eutactic*.
- (ii) A set $Z \in \mathcal{Z}$ is *extreme* if and only if there exists a nonempty subset of Z_{\min} that is *weakly perfect and eutactic*.

Proof. Let $Z \in \mathcal{Z}$, and let $(V, \mathcal{Q}, Z, \mathbf{G})$ be the Voronoi space associated to $(V, Q, \mathcal{Z}, \mathbf{G})$ and Z . Then the extremality, perfection, eutaxy, etc., of Z in the alternative Voronoi space $(V, Q, \mathcal{Z}, \mathbf{G})$ is equivalent the extremality, perfection, eutaxy, etc., of Q in the Voronoi space $(V, \mathcal{Q}, Z, \mathbf{G})$. Therefore, the theorem is equivalent to the corresponding theorem for Voronoi spaces (Theorem 18). \square

Finally, we give a description of neighborhoods in alternative Voronoi space:

27. PROPOSITION. (Compare with Proposition 14.) *Let $(V, Q, \mathcal{Z}, \mathbf{G})$ an alternative Voronoi space, and let $Z \in \mathcal{Z}$. For $H \in \mathfrak{p}$, we define Z_H by*

$$Z_H = \exp(H/2)Z.$$

Then each $Z' \in \mathcal{Z}$ can be written as $Z = kZ_H$ for some $H \in \mathfrak{p}$ and some $k \in \mathbf{K}$. In particular, if, for $Z \in \mathcal{Z}$ and $\epsilon > 0$, we set

$$V_{Z,\epsilon} = \{\mathbf{K}Z_H \in \mathbf{K} \setminus \mathcal{Z} \mid H \in \mathfrak{p} \text{ and } \|H\|_Q < \epsilon\},$$

then the family $\{V_{Z,\epsilon}\}_{\epsilon>0}$ is a basis of neighborhoods of $\mathbf{K} \setminus \mathcal{Z}$.

Proof. Let $Z' \in \mathcal{Z}$, and let $g \in \mathbf{G}$ such that $Z' = gZ$. According to Theorem 9, we have $g_Q^*g \in \mathbf{P} = \exp(\mathfrak{p})$, therefore there exists an $H \in \mathfrak{p}$ such that $g_Q^*g = \exp(H)$. Let $k = g \exp(-H/2)$; we have $k \in \mathbf{K}$ and $g = k \exp(H)$, and we have $Z' = kZ_H$. The rest of the lemma is straightforward. \square

4.3 Regular alternative Voronoi space

There is not, in general, a natural notion of design for an arbitrary alternative Voronoi space; but we need to restrict to the following class of alternative Voronoi spaces.

28. DEFINITION. Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space, and let

$$\Omega = \overline{\bigcup_{Z \in \mathcal{Z}} [Z]} \in \mathbb{P}(V),$$

where $[]$ denotes the projection $V \setminus \{0\} \rightarrow \mathbb{P}(V)$ and \overline{E} denotes the adherence of E . The space $(V, Q, \mathcal{Z}, \mathbf{G})$ is *regular* when Ω is an orbit of \mathbf{G} acting on $\mathbb{P}(V)$.

The following fact is crucial for the definition of designs.

29. PROPOSITION. *Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be a regular alternative Voronoi space. Then its compact group \mathbf{K} is transitive on Ω .*

In order to show that statement, we need the following fact on triangular linear Lie groups:

30. LEMMA. *Let \mathbf{D} be a connected triangular subgroup of $\mathbf{GL}(V)$ and let Ω be a \mathbf{D} -invariant closed subset of the projective space $\mathbb{P}(V)$. Then \mathbf{D} has a fixed point in Ω .*

For a proof, see [OniVin94, Chap. 4, Prop. 4.6, p. 161].

Proof of Proposition 29. According to Theorem 9, there exists a connected triangular subgroup \mathbf{D} of \mathbf{G} such that $\mathbf{G} = \mathbf{KD}$. By Lemma 30, there exists $\xi \in \Omega$ such that $\mathbf{D}\xi = \xi$. As the alternative Voronoi space is regular, its group \mathbf{G} is transitive on Ω , and we have

$$\Omega = \mathbf{G}\xi = \mathbf{KD}\xi = \mathbf{K}\xi,$$

therefore \mathbf{K} is transitive on Ω . \square

4.4 Examples

We give here the alternative Voronoi spaces corresponding to the Voronoi spaces given in Paragraph 1.5, and we add another example of alternative Voronoi space.

1. *Classic spaces.* Let $V = \mathbb{R}^n$ with $n \geq 2$, let Q be the canonical quadratic form on \mathbb{R}^n , let \mathcal{Z} be the set of Euclidean lattices in \mathbb{R}^n of determinant 1 without their zero vector, and let $\mathbf{G} = \mathbf{SL}^\pm(n, \mathbb{R})$. Then $(V, Q, \mathcal{Z}, \mathbf{G})$ is a regular alternative Voronoi space with $\Omega = \mathbb{P}(\mathbb{R}^n)$.
2. *Duality.* Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space. Let $Z \mapsto Z^\alpha$ be an involution of \mathcal{Z} , and $g \mapsto g^\alpha$ be an involution and automorphism of \mathbf{G} such that $(g^*_Q)^\alpha = (g^\alpha)^*_Q$ and

$$(gZ)^\alpha = g^\alpha Z^\alpha, \quad \forall g \in \mathbf{G}, \quad \forall Z \in \mathcal{Z}.$$

We define an alternative Voronoi space $(\tilde{V}, \tilde{Q}, \tilde{\mathcal{Z}}, \tilde{\mathbf{G}})$ as follows:

$$\begin{aligned} \tilde{V} &= V \times V, \\ \tilde{Q}((x_1, x_2)) &= Q(x_1) + Q(x_2), \\ \tilde{\mathcal{Z}} &= \{(cZ \times \{0\}) \cup (\{0\} \times c^{-1}Z^\alpha) \mid Z \in \mathcal{Z}, c \in \mathbb{R}^*\}, \\ \tilde{\mathbf{G}} &= \left\{ \tilde{g} \in \mathbf{GL}(V \times V) \mid \tilde{g} = \begin{pmatrix} cg & 0 \\ 0 & c^{-1}g^\alpha \end{pmatrix} \text{ or } \begin{pmatrix} 0 & cg \\ c^{-1}g^\alpha & 0 \end{pmatrix} \right. \\ &\quad \left. \text{for some } g \in \mathbf{G} \text{ and some } c \in \mathbb{R}^* \right\}. \end{aligned}$$

We have

$$\tilde{\Omega} = (\Omega \times \{0\}) \cup (\{0\} \times \Omega),$$

and the space $(\tilde{V}, \tilde{Q}, \tilde{\mathcal{Z}}, \tilde{\mathbf{G}})$ is regular if $(V, Q, \mathcal{Z}, \mathbf{G})$ is regular. The local maxima of the Hermite function $\tilde{\gamma}$ of $(\tilde{V}, \tilde{Q}, \tilde{\mathcal{Z}}, \tilde{\mathbf{G}})$ correspond to the local maxima of the function $\gamma'(Z) = (\gamma(Z)\gamma(Z^\alpha))^{1/2}$ on \mathcal{Z} .

As a particular case, we take for $(V, Q, \mathcal{Z}, \mathbf{G})$ the classic alternative Voronoi space of rank n , with $g^\alpha = g^{*-1}$ and $Z^\alpha = Z^*$, where $Z^* \cup \{0\}$ is the dual lattice of $Z \cup \{0\}$.

3. *Family of lattices.* Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space, let Z be an element of \mathcal{Z} , and let \mathbf{H} be a connected closed subgroup of \mathbf{G} such that $g^*_Q \in \mathbf{H}$ when $g \in \mathbf{H}$. Let $\mathbf{H}Z$ be the orbit of Z in \mathcal{Z} . Then $(V, Q, \mathbf{H}Z, \mathbf{H})$ is an alternative Voronoi space, which is in general not regular, even when $(V, Q, \mathcal{Z}, \mathbf{G})$ is regular.

As special cases, we have:

- (a) Let Γ be a finite subgroup of \mathbf{K} , let $Z \in \mathcal{Z}$ be a Γ -invariant set, that is such that $\gamma Z = Z$ for every $\gamma \in \Gamma$, and let \mathbf{H} be the connected component of the identity of the centralizer of Γ in \mathbf{G} . Then \mathbf{H} is Q -selfadjoint and $\mathbf{H}Z$ is a connected component of the set of Γ -invariant elements of \mathcal{Z} . (Note that, contrary to what happens with the dual formulation in terms of Voronoi space, $\mathbf{H}Z$ is not necessary the entire set of Γ -invariant elements of \mathcal{Z} .) So, $(V, Q, \mathbf{H}Z, \mathbf{H})$ is an alternative Voronoi space.
- (b) Let $Z \mapsto Z^\alpha$ and $g \mapsto g^\alpha$ as in Example 2; let $\sigma \in \mathbf{G}$ such that $(\sigma^*)^{-1} = \sigma$. An element $Z \in \mathcal{Z}$ such that $Z^\alpha = \sigma Z$ is called σ -isodual.

Let Z be a σ -isodual element of \mathcal{Z} . Let \mathbf{H} be the connected component of the identity of the elements of $g \in \mathbf{G}$ satisfying $\sigma g = g^\alpha \sigma$. Then \mathbf{H} is Q -selfadjoint, and $\mathbf{H}Z$ is a connected component of the set of σ -isodual elements of \mathcal{Z} . So, $(V, Q, \mathbf{H}Z, \mathbf{H})$ is an alternative Voronoi space.

4. *Exterior powers or Grassmannian spaces.* Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space and let $1 \leq m \leq \dim(V)/2$. We define a space $(V^{\wedge m}, Q^{\wedge m}, \mathcal{Z}^{\wedge m}, \mathbf{G}^{\wedge m})$ as follows: $V^{\wedge m}$ is the m th exterior power of V , and

$$Q^{\wedge m}(x_1 \wedge \cdots \wedge x_m) = \det(Q(x_i, x_j)_{i,j=1}^m),$$

$$\mathcal{Z}^{\wedge m} = \{Z^{\wedge m} \mid Z \in \mathcal{Z}\}, \quad \mathbf{G}^{\wedge m} = \{\tilde{g} \mid g \in \mathbf{G}\},$$

where

$$Z^{\wedge m} = \{x_1 \wedge \cdots \wedge x_m \mid x_1, \dots, x_m \text{ are linearly independent elements of } Z\},$$

$$\tilde{g}(x_1 \wedge \cdots \wedge x_m) = gx_1 \wedge \cdots \wedge gx_m.$$

Note that the fact that $(V, Q, \mathcal{Z}, \mathbf{G})$ is regular does not imply that the space $(V^{\wedge m}, Q^{\wedge m}, \mathcal{Z}^{\wedge m}, \mathbf{G}^{\wedge m})$ is regular. If $(V, Q, \mathcal{Z}, \mathbf{G})$ is the classic Voronoi space of rank n , then $(V^{\wedge m}, Q^{\wedge m}, \mathcal{Z}^{\wedge m}, \mathbf{G}^{\wedge m})$ is regular, with

$$\Omega^{\wedge m} = \{[x_1 \wedge \cdots \wedge x_m] \mid x_1, \dots, x_m \text{ are linearly independent elements of } \mathbb{R}^n\},$$

which is the space of nonoriented Grassmannians of dimension m in \mathbb{R}^n .

5. *Humbert forms.* Let K be a number field of signature (r, s) , of integral ring \mathcal{O}_K , let $\sigma_1, \dots, \sigma_r$ the real embeddings of K , and $\sigma_{r+1}, \dots, \sigma_{r+2s}$ its complex embeddings, with $\sigma_{r+s+i} = \overline{\sigma_{r+i}}$, and let n be a positive integer. In what follows, ‘ \otimes ’ denotes always the tensor product as *real* vector space. Let V be the linear subspace of $(\mathbb{R}^n)^{\otimes r} \otimes (\mathbb{C}^n)^{\otimes 2s}$ generated by $x_1 \otimes \cdots \otimes x_{r+2s}$, where

$$x_i \in \mathbb{R}^n \quad \text{for } 1 \leq i \leq r,$$

$$x_{r+i} \in \mathbb{C}^n \text{ and } x_{r+s+i} = \overline{x_{r+i}} \quad \text{for } 1 \leq i \leq s.$$

Let Q be the quadratic form on V defined by

$$Q(x_1 \otimes \cdots \otimes x_{r+2s}) = \prod_{j=1}^{r+2s} \langle x_j \mid x_j \rangle.$$

Let \mathbf{G} be the linear group acting on V and whose elements g satisfy

$$g(x_1 \otimes \cdots \otimes x_{r+2s}) = g_1 x_1 \otimes \cdots \otimes g_{r+2s} x_{r+2s},$$

for some $g_i \in \mathbf{GL}(n, \mathbb{R})$ for $1 \leq i \leq r$, and some $g_{r+i} \in \mathbf{GL}(n, \mathbb{C})$ and $g_{r+s+i} = \overline{g_{r+i}}$ for $1 \leq i \leq s$, such that

$$\prod_{j=1}^{r+2s} \det(g_j) = 1.$$

Let $\mathcal{Z} = \{gZ \mid g \in \mathbf{G}\}$, where

$$Z = \{\sigma_1(x) \otimes \cdots \otimes \sigma_{r+s}(x) \mid x \in \mathcal{O}_n\}.$$

We obtain a regular alternative Voronoi space $(V, Q, \mathcal{Z}, \mathbf{G})$ with

$$\Omega = \{[x_1 \otimes \cdots \otimes x_{r+2s}] \mid x_i \in \mathbb{R}^n \setminus \{0\} \text{ for } 1 \leq i \leq r,$$

$$x_{r+i} \in \mathbb{C}^n \setminus \{0\} \text{ and } x_{r+s+i} = \overline{x_{r+i}} \text{ for } 1 \leq i \leq s\}.$$

6. *Isotropy in isodual lattices.* We add an example of alternative Voronoi space, which does not seem to have a natural formulation in terms of Voronoi space.

Let $V = \mathbb{R}^n$. An *isodual* lattice is the data of a Euclidean lattice $\Lambda \subseteq \mathbb{R}^n$ and an orthogonal transformation $\sigma \in \mathbf{O}(n)$ such that $\Lambda^* = \sigma(\Lambda)$. The lattice is *unimodular* when $\sigma = \pm \text{id}$, *orthogonal* when $\sigma^2 = \text{id}$ and $\sigma \neq \text{id}$, and *symplectic* when $\sigma^2 = -\text{id}$. For each isodual lattice (Λ, σ) , we set

$$Z_{(\Lambda, \sigma)} = \{z \in \Lambda \mid \langle z \mid \sigma z \rangle = 0\} \setminus \{0\}.$$

Let $\sigma \in \mathbf{O}(n)$; let \mathcal{Z} be a connected component of the set of $Z_{(\Lambda, \sigma)}$, where Λ is a lattice such that (Λ, σ) is isodual; let Q be the canonical quadratic form on \mathbb{R}^n , and let

$$\mathbf{G} = \{g \in \mathbf{GL}(n, \mathbb{R}) \mid \sigma g = g^{*-1} \sigma\}.$$

Then $(\mathbb{R}^n, Q, \mathcal{Z}, \mathbf{G})$ is an alternative Voronoi space of Lie algebra

$$\mathfrak{g} = \{H \in \mathfrak{gl}(n, \mathbb{R}) \mid \sigma H + H \sigma = 0\}.$$

The case where $\sigma^2 = \text{id}$ and σ is of signature $(n-1, 1)$ has been studied by Ch. Bavard in [Bava07].

5 Designs

The term of ‘‘spherical design’’ was introduced in a paper of Delsarte, Goethals and Seidel [DeGoSe77] as an equivalent on the Euclidean sphere of the notion of design in combinatorics, although it is also a particular case of the notion of cubature formula on sphere, as studied by Sobolev [Sobo62], [Sobo96].

Boris Venkov has shown that a lattice whose minimal vectors form a spherical 4-design is extreme [VenMar01], criterion which has been extended in few other situations.

In this section, we introduce an appropriate notion of designs in regular alternative Voronoi spaces, for which the criterion of Venkov holds (Corollary 36). The frame in which we define design is a particular case of the notion of *polynomial space* as defined and studied in [Pach06]. Other noteworthy, although more restrictive notions of polynomial space are found, e.g., in [Gods93, Chap. 14] and [HarPac05]. In the present paper, we do not develop the theory more than needed for our purpose.

In the last paragraph, we give a useful criterion (Corollary 38) which allows to check that a set is a design by considering the invariants of his automorphism group.

5.1 Definitions

We fix $(V, Q, \mathcal{Z}, \mathbf{G})$ a regular alternative Voronoi space (Definition 28). Recall that \mathbf{K} denotes the compact subgroup of Q -orthogonal elements of \mathbf{G} , and \mathfrak{p} denotes the subspace of Q -selfadjoint elements of the Lie algebra \mathfrak{g} of \mathbf{G} . As the group \mathbf{K} is compact and transitive on Ω (Proposition 29), there is a unique \mathbf{K} -invariant probability measure μ on Ω .

Consider the following linear subspaces of $\text{End}(V)$, the associative algebra of linear endomorphisms of V :

$$\begin{aligned} \mathfrak{p}_0 &= \mathbb{R} \text{id}_V, & \mathfrak{p}_1 &= \mathbb{R} \text{id}_V + \mathfrak{p}, \\ \mathfrak{p}_k &= \text{span}\{H_1 H_2 \cdots H_k + H_k H_{k-1} \cdots H_1 \mid H_i \in \mathfrak{p} \oplus \mathbb{R} \text{id}_V\}, & k &\geq 1. \end{aligned}$$

The elements of \mathfrak{p}_k are Q -selfadjoint, and we have, for $i, j \geq 0$,

$$H \in \mathfrak{p}_i, J \in \mathfrak{p}_j \Rightarrow HJ + JH \in \mathfrak{p}_{i+j}.$$

In other words, \mathfrak{p}_{i+j} is the subspace of Q -selfadjoint elements of the space generated by multiplications of elements of \mathfrak{p}_i with elements of \mathfrak{p}_j .

Now we define the following finite-dimensional subspaces of the space $\mathcal{C}(\Omega, \mathbb{R})$ of real continuous functions on Ω : Let $\{2k_1, \dots, 2k_d\}$ be a finite set with multiplicities whose elements are even positive integers. We set:

$$\mathcal{F}_\emptyset = \mathbb{R} \text{id}_\Omega \quad \mathcal{F}_{\{2k_1, \dots, 2k_d\}} = \text{span} \left\{ x \mapsto \prod_{i=1}^d \frac{Q(x, H_i x)}{Q(x)} \mid H_i \in \mathfrak{p}_{k_i} \right\}.$$

and, for τ an even nonnegative integer,

$$\mathcal{F}_\tau = \sum_{\substack{\{2k_1, \dots, 2k_d\} \\ 2k_1 + \dots + 2k_d \leq \tau}} \mathcal{F}_{\{2k_1, \dots, 2k_d\}}.$$

We have

$$\begin{aligned} \mathcal{F}_{\{2k_1, \dots, 2k_d\}} &\subseteq \mathcal{F}_{\{2k'_1, \dots, 2k'_d\}} \quad \text{if } d \leq d' \text{ and } 2k_i \leq 2k'_i \text{ for } i \leq d, \\ \mathcal{F}_{\{2k_1, \dots, 2k_d\}} \mathcal{F}_{\{2k_{d+1}, \dots, 2k_{d+d'}\}} &= \mathcal{F}_{\{2k_1, \dots, 2k_{d+d'}\}}, \end{aligned}$$

For $f \in \mathcal{C}(\Omega, \mathbb{R})$, we define his *average* on Ω by

$$\langle f \rangle = \int_{\Omega} f(\xi) d\mu(\xi),$$

where μ is the unique \mathbf{K} -invariant probability measure on Ω .

31. DEFINITION. Let A be either a finite set with multiplicities of even positive integers, or an even positive integer. A *weighted design* or *cubature formula of strength A* on Ω , or, in short a *weighted A -design*, is the data of a finite subset X of Ω and a function $W : X \rightarrow]0, 1]$, $\xi \mapsto W_\xi$, such that $\sum_{\xi \in X} W_\xi = 1$ and

$$\sum_{\xi \in X} W_\xi f(\xi) = \langle f \rangle, \quad \forall f \in \mathcal{F}_A.$$

A *design of strength A* on Ω , or an *A -design*, is a finite subset X of Ω such that (X, W) is a weighted A -design for W the constant function of value $|X|^{-1}$.

In particular, $\{2\}$ -designs and 2-designs are the same things, while 4-designs are sets which are simultaneously $\{4\}$ - and $\{2, 2\}$ -designs.

The notion of τ -design on $\Omega = \mathbb{P}(\mathbb{R}^n)$ for classic alternative Voronoi spaces coincides with the usual notion of antipodal spherical τ -designs on the Euclidean sphere of dimension $n - 1$, as defined in [DeGoSe77] (see also [GoeSei79], [VenMar01], etc.) When our alternative Voronoi space is an exterior power of the classic alternative Voronoi space, Ω is identified with a nonoriented Grassmannian manifold, and the notion of τ -design coincides with the one defined and studied in [BaCoNe02] and [BaBaCo04]. (Caution: the sets with multiplicities of even positive integers we use to define designs are not directly related to the partitions of even positive integers used to define designs on Grassmannian manifolds in the cited papers.)

When the Lie algebra $\mathfrak{g} \oplus \mathbb{R} \text{id}_V$ is also an associative algebra (as it is the case for classic alternative Voronoi spaces), then τ -designs are equivalent to $\{2, 2, \dots, 2\}$ -designs, where the multiplicity of 2 is equal to $\tau/2$.

5.2 Criterion for eutaxy and perfection

Although we have defined perfection and eutaxy for subsets of V , the definition is adaptable to subsets of $\mathbb{P}(V)$, as follows:

32. DEFINITION. Let Q be a quadratic form on V , and let $X \subseteq \mathbb{P}(V)$ be a nonempty finite subset.

(a) (Q, X) is *eutactic* when, for every $H \in \mathfrak{p}$,

$$\exists [x] \in X, \frac{Q(x, Hx)}{Q(x)} > 0 \implies \exists [y] \in X, \frac{Q(y, Hy)}{Q(y)} < 0.$$

(b) (Q, X) is *strongly eutactic* when, for every $H \in \mathfrak{p}$,

$$\sum_{[x] \in X} \frac{Q(x, Hx)}{Q(x)} = 0$$

(c) (Q, X) is *weakly perfect* when, for every $H \in \mathfrak{p}$,

$$\exists c \in \mathbb{R}, \forall [x] \in X, \frac{Q(x, Hx)}{Q(x)} = c \implies \forall [x] \in X, Hx = cx.$$

(d) (Q, X) is *perfect* when, for every $H \in \mathfrak{p}$,

$$\exists c \in \mathbb{R}, \forall [x] \in X, \frac{Q(x, Hx)}{Q(x)} = c \implies H = c \operatorname{id}_V.$$

The main goal of this section is to prove the following result, which has been proved by Venkov in the case of classic alternative Voronoi spaces [VenMar01, Proposition 6.2 and Théorème 6.4], and by Bachoc, Coulangéon and Nebe in the case of Grassmannians [BaCoNe02, Theorem 6.2].

33. THEOREM.

- (i) If (X, W) is a weighted design of strength $\{2\}$ then (Q, X) is eutactic.
- (ii) If X is a design of strength $\{2\}$ then (Q, X) is strongly eutactic.
- (iii) If (X, W) is a weighted design of strength $\{2, 2\}$ then (Q, X) is perfect and eutactic.

For this, we first need some lemmas:

34. LEMMA. Let $H \in \mathfrak{p}$ and let $f([x]) = Q(x, Hx)/Q(x)$. If f is constant on Ω , then $H = 0$.

Proof. First, we reformulate the assumption of the lemma as follows: There exists a constant $c \in \mathbb{R}^*$ such that

$$Q(x, (H - c)x) = 0 \quad \forall [x] \in \Omega.$$

Now, for $[y] \in \Omega$ and $t \in \mathbb{R}$, we have

$$[\exp(t(H - c))y] = [e^{-tc} \exp(tH)y] = [\exp(tH)y] \in \Omega,$$

because $\exp(tH)$ is an element of \mathbf{G} and Ω is \mathbf{G} -invariant. Therefore we have $Q(\exp(t(H-c))y, (H-c)\exp(t(H-c))y) = 0$, or, using $H_{\mathcal{Q}}^* = H$,

$$Q(\exp(2t(H-c))y, (H-c)y) = 0, \quad \forall y \in \Omega, t \in \mathbb{R}.$$

Differentiating this expression in t and setting $t = 0$, we obtain $2Q((H-c)y, (H-c)y) = 0$, therefore $Hy = cy$ for every $y \in \Omega$. As Ω spans V , we have $H = c \operatorname{id}_V$, and since $\mathbb{R} \operatorname{id}_V \cap \mathfrak{p} = \{0\}$, we have $c = 0$. \square

35. LEMMA. *Let $H \in \mathfrak{p}$ and let $f([x]) = Q(x, Hx)/Q(x)$. Then $\langle f \rangle = 0$.*

Proof. Using the transitivity of \mathbf{K} on Ω (Proposition 29), we get, for any $[y] \in \Omega$,

$$\begin{aligned} \langle f \rangle &= \int_{\Omega} \frac{Q(x, Hx)}{Q(x)} d\mu([x]) = \int_{\mathbf{K}} \frac{Q(gy, Hgy)}{Q(gy)} dg \\ &= \int_{\mathbf{K}} \frac{Q(y, g^{-1}Hgy)}{Q(y)} dg = \frac{Q(y, \int_{\mathbf{K}} g^{-1}Hg dg y)}{Q(y)} \\ &= \frac{Q(y, H^{\mathbf{K}}y)}{Q(y)}, \end{aligned}$$

where $H^{\mathbf{K}} = \int_{\mathbf{K}} g^{-1}Hg dg$ is an element of \mathfrak{p} . Applying Lemma 34 to $H^{\mathbf{K}}$, we get $H^{\mathbf{K}} = 0$; it follows that $\langle f \rangle = 0$. \square

Proof of Theorem 33. (i) Suppose that (X, W) is a weighted design of strength $\{2\}$. Let $H \in \mathfrak{p}$, and let $f([x]) = Q(x, Hx)/Q(x)$. As $\langle f \rangle = 0$ by Lemma 35, we get

$$\sum_{x \in X} W_x \frac{Q(x, Hx)}{Q(x)} = 0.$$

This implies that, if there exists an $x \in X$ such that $Q(x, Hx)/Q(x) > 0$, then there exists an $y \in X$ such that $Q(y, Hy)/Q(y) < 0$, that is (Q, X) is eutactic.

(ii) The argument is the same as in the proof of Claim (i).

(iii) Suppose that (X, W) is a weighted design of strength $\{2, 2\}$. As it is in particular a weighted design of strength $\{2\}$, (Q, X) is eutactic by Claim (i). It remains to show that (Q, X) is perfect.

Let $H \in \mathfrak{p}$, and suppose that there exists a $c \in \mathbb{R}$ such that $Q(x, Hx)/Q(x) = c$ for every $x \in X$. Consider the nonnegative function $f \in \mathcal{F}_{\{2, 2\}}$ defined by

$$f([x]) = \left(\frac{Q(x, Hx)}{Q(x)} - c \right)^2.$$

We have

$$\int_{\Omega} f(x) dx = \sum_{x \in X} W_x f(x) = 0,$$

and therefore $f(x) = 0$ for every $x \in \Omega$, that is

$$\frac{Q(x, Hx)}{Q(x)} = c, \quad \forall x \in \Omega.$$

By Lemma 34, $H = 0$ (and $c = 0$). Thus, (Q, X) is perfect. \square

As a direct consequence of Theorem 33, using Theorem 26, we get:

36. COROLLARY (VENKOV CRITERION). *Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space, and let $Z \in \mathcal{Z}$. If the minimum layer Z_{\min} of Z is a $\{2, 2\}$ -design, then Z is strictly extreme.*

In order to apply this criterion, we need some method for checking easily that a given finite set of V provide a design. In the next paragraph, we give a criterion exploiting the symmetry of the set to be checked.

5.3 Designs invariant under finite group

Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be a regular alternative Voronoi space, and Ω, \mathcal{F}_A , etc., as above. For A a set with multiplicities of even positive integers, or an even positive integer, and for \mathbf{F} a closed subgroup of \mathbf{K} we note $\mathcal{F}_A^{\mathbf{F}}$ the subspace of A -invariant elements of \mathcal{F}_A , that is

$$\mathcal{F}_A^{\mathbf{F}} = \{f \in \mathcal{F}_A \mid f \circ g = f, \forall g \in \mathbf{F}\}.$$

The technique of the following proposition has already been used by Sobolev in order to compute cubature formulas on spheres [Sobo62]; see also [Sobo96, Chap. 2, § 2, Theorem 2.3].

37. PROPOSITION. *Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be a regular alternative Voronoi space. Let \mathbf{F} be a finite subgroup of \mathbf{K} , let X be a finite subset of Ω , and $W : X \rightarrow \mathbb{R}$ a function with $W(\xi) = W_\xi > 0$ for all $\xi \in X$ and $\sum_{\xi \in X} W_\xi = 1$. Suppose that (X, W) is invariant under the action of \mathbf{F} , that is $g\xi \in X$ and $W_{g\xi} = W_\xi$ for $g \in \mathbf{F}$ and $\xi \in X$.*

Then (X, W) is a weighted A -design if and only if

$$\sum_{\xi \in X} W_\xi f(\xi) = \langle f \rangle, \quad \forall f \in \mathcal{F}_A^{\mathbf{F}}.$$

Proof. Let $f \in \mathcal{F}_A$. The invariance of (X, W) implies that

$$\sum_{\xi \in X} W_\xi f(\xi) = \sum_{\xi \in X} W_\xi f(g\xi), \quad \forall g \in \mathbf{F}.$$

Taking the sum on all elements $g \in \mathbf{F}$, we obtain

$$\sum_{\xi \in X} W_\xi f(\xi) = \sum_{\xi \in X} W_\xi f^{\mathbf{F}}(\xi), \quad \text{where } f^{\mathbf{F}} = \sum_{g \in \mathbf{F}} f \circ g.$$

In the same way, the invariance of the measure μ under the action of $\mathbf{F} \subseteq \mathbf{K}$ implies that

$$\langle f \rangle = \langle f^{\mathbf{F}} \rangle.$$

Therefore, we have

$$\sum_{\xi \in X} W_\xi f(\xi) = \langle f \rangle, \quad \text{iff} \quad \sum_{\xi \in X} W_\xi f^{\mathbf{F}}(\xi) = \langle f^{\mathbf{F}} \rangle.$$

The function $f^{\mathbf{F}}$ is an element of \mathcal{F}_A , which is invariant under the action of \mathbf{F} . It follows that, in order to have the equality $\sum_{\xi \in X} W_\xi f(\xi) = \langle f \rangle$ for every $f \in \mathcal{F}_A$, it suffices to check it for every $f \in \mathcal{F}_A^{\mathbf{F}}$. \square

As a particular case of the last proposition, we have:

38. COROLLARY. *Under the assumptions of Proposition 37, if the only \mathbf{F} -invariant functions in \mathcal{F}_A are the constant functions, then (X, W) is a weighted A -design.*

In particular, using the Venkov criterion of extremality (Corollary 36), we have:

39. COROLLARY. *Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be an alternative Voronoi space, and let $Z \in \mathcal{Z}$. Let $\mathbf{F} = \{g \in \mathbf{K} \mid gZ = Z\}$. If the only \mathbf{F} -invariant functions in $\mathcal{F}_{\{2,2\}}$ are the constant functions, then Z is strictly extreme.*

For example, let $((\mathbb{R}^n)^{\wedge m}, Q, \mathcal{Z}_{n,m}, \mathbf{SL}^{\pm}(n, \mathbb{R}))$ be the m th exterior power of the classic alternative Voronoi space of rank n (Example 4 of Paragraph 4.4). Recall that the elements of $\mathcal{Z}_{n,m}$ are of the form

$$Z(\Lambda, m) = \{x_1 \wedge \cdots \wedge x_m \mid x_1, \dots, x_m \text{ are linearly independent elements of } \Lambda\},$$

where Λ is a Euclidean lattice in \mathbb{R}^n of determinant 1; and the space $\Omega_{n,m} = \overline{[\bigcup_{Z \in \mathcal{Z}_{n,m}} Z]}$ is identified to the space of m -dimensional subspaces of \mathbb{R}^n . We denote by $\mathcal{F}_{\tau}(\Omega_{n,m}) = \mathcal{F}_{\tau}$ the corresponding spaces of function on $\Omega_{n,m}$. Consider the following list of Euclidean lattices in \mathbb{R}^n (the subscript indicates the dimension of the ambient space), which is taken from [Bach05] (see also [VenMar01]):

$$\begin{aligned} \mathbf{A}_2, \mathbf{D}_4 = BW_4, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8 = BW_8, \\ Q_{14}, \Lambda_{16} = BW_{16}, O_{16}, O_{22}, \Lambda_{22}, \Lambda_{22}[2], M_{22}, M_{22}[5], \\ O_{23}, \Lambda_{23}, M_{23}, M_{23}[2], \Lambda_{24}, \text{ and } BW_{2^k}, k \geq 2, \end{aligned}$$

where $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_n$ denote root lattices, Λ_n denotes laminated lattices, and BW_n denotes the Barnes-Wall lattices. It is known that the automorphism group of these lattices has no nonconstant invariant polynomial in $\mathcal{F}_4(\Omega_{n,m})$ for every $m \leq n/2$ (see [Bach05]); therefore, all these lattices and their duals provide strictly extreme sets for the alternative Voronoi spaces $((\mathbb{R}^n)^{\wedge m}, Q, \mathcal{Z}_{n,m}, \mathbf{SL}^{\pm}(n, \mathbb{R}))$.

6 Epstein zeta function

Hitherto, we have considered extremality relatively to the Hermite function; now, we turn towards extremality relatively to the Epstein zeta function as defined below. Delone and Ryshkov have formulated a characterization of the so-called *final ζ -extremality* [DelRyš67] in what we call classic Voronoi spaces. This characterization has some resemblance with Voronoi characterization of extremality, as it involves the notions of eutaxy and perfection. Likewise, a criterion of ζ -extremality using designs resembling to the criterion of Venkov for usual extremality has been found by Coulangeon [Coul06].

The goal of this section is to prove both results in our frame of Voronoi space. In this whole section, we suppose, for simplicity, that the Voronoi spaces (V, Q, Z, \mathbf{G}) we consider are antipodal, that is $Z = -Z$. All the results are also correct for non-antipodal spaces, and even for sets Z with positive weights attached to its elements and to the corresponding terms in the Epstein ζ -series, if correct weights are set in all conditions of eutaxy and designs.

6.1 Definitions

Let V be a vector space, Q a positive definite quadratic form, and Z a subset of $V \setminus \{0\}$, discrete and closed in V . We define the *Epstein zeta series* of (Q, Z) by

$$\zeta(Q, Z, s) = \sum_{x \in Z} Q(x)^{-s}, \quad s \in \mathbb{C}.$$

When Q , respectively Z , is implied, we write $\zeta(Z, s)$, respectively $\zeta(Q, s)$, instead of $\zeta(Q, Z, s)$.

40. LEMMA. *There exists a $s_0(Z) \in [-\infty, +\infty]$ (with $s_0 \geq 0$ when Z is infinite) depending only on Z such that $\zeta(Q, Z, s)$ converges absolutely for $s \in \mathbb{C}$ with $\Re s > s_0$ and diverges for $s \in \mathbb{R}$ with $s > s_0$.*

Proof. Suppose that $\zeta(Q, Z, s)$ converges for some positive definite quadratic form Q and some real number s , and let Q' be another positive definite quadratic form, and $s' \in \mathbb{C}$ such that $\Re s' < s$. There exists $\alpha > 0$ such that $Q'(x) \leq \alpha Q(x)$ for every $x \in V$. We have

$$\sum_{x \in Z} \left| Q'(x)^{-s'} \right| = \sum_{x \in Z} Q'(x)^{-\Re s'} \leq \alpha^{-s} \sum_{x \in Z} Q(x)^{-s} = \alpha^{-s} \zeta(Q, Z, s) < \infty.$$

So, we have the lemma with $s_0(Z)$ the supremum of the values of s for which $\zeta(Q, Z, s)$ converges. \square

41. LEMMA. *Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, let $s_0 = s_0(Z)$ as in the previous lemma. Then, for every $s > s_0$, the function $Q \mapsto \zeta(Q, s)$ converges and is indefinitely differentiable.*

Proof. It is a direct consequence of the proof of the previous lemma, using the fact that the successive derivatives of $Q \mapsto Q(x)^{-s}$ are bounded by a bound proportional to $Q(x)^{-s}$. \square

42. DEFINITION. Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space, and let $s_0 = s_0(Z)$ as in Lemma 40. Let $Q \in \mathcal{Q}$.

- (i) For $s > s_0$, Q is ζ -extreme at s when it is a local minimum of the function $Q \mapsto \zeta(Q, s)$.
- (ii) Q is finally ζ -extreme when there exists an $s > s_0$ such that Q is ζ -extreme at any $s' > s$.

6.2 Characterization of final ζ -extremality

In this section, $(V, \mathcal{Q}, Z, \mathbf{G})$ denotes a Voronoi space with $s_0 < \infty$, $Z = -Z$, and $|Z| = \infty$.

In order to study the ζ -extremality of a quadratic form Q , we need to approximate the function ζ in a neighborhood of Q :

43. LEMMA. *Let $s > s_0$, $Q \in \mathcal{Q}$, $H \in \mathfrak{p}_Q$, and $Q_H(x) = Q(x, \exp(H)x)$. We have*

$$\begin{aligned} \zeta(Q_H, s) = \sum_{x \in Z} Q(x)^{-s} & \left(1 - s \frac{Q(x, Hx)}{Q(x)} + \frac{s^2 Q(x, Hx)^2}{2 Q(x)^2} \right. \\ & \left. - \frac{s}{2} \left(\frac{Q(Hx)}{Q(x)} - \frac{Q(x, Hx)^2}{Q(x)^2} \right) + O(\|H\|_Q^3) \right). \end{aligned}$$

Proof. We have, using $H = H_Q^*$,

$$\begin{aligned} Q_H(x)^{-s} & = Q(x, \exp(H)x)^{-s} \\ & = \left(Q(x) + Q(x, Hx) + \frac{1}{2} Q(x, H^2x) + O(\|H\|_Q^3) \right)^{-s} \\ & = Q(x)^{-s} \left(1 + \frac{Q(x, Hx)}{Q(x)} + \frac{1}{2} \frac{Q(x, H^2x)}{Q(x)} + O(\|H\|_Q^3) \right)^{-s}. \end{aligned}$$

Using $(1 + \alpha)^{-s} = 1 - s\alpha + (s(s+1)/2)\alpha^2 + O(|\alpha|^3)$, we get:

$$\begin{aligned} Q_H(x)^{-s} &= Q(x)^{-s} \left(1 - s \frac{Q(x, Hx)}{Q(x)} - \frac{s}{2} \frac{Q(x, H^2x)}{Q(x)} \right. \\ &\quad \left. + \frac{s(s+1)}{2} \frac{Q(x, Hx)^2}{Q(x)^2} + O(\|H\|_Q^3) \right) \\ &= Q(x)^{-s} \left(1 - s \frac{Q(x, Hx)}{Q(x)} + \frac{s^2}{2} \frac{Q(x, Hx)^2}{Q(x)^2} \right. \\ &\quad \left. - \frac{s}{2} \left(\frac{Q(Hx, Hx)}{Q(x)} - \frac{Q(x, Hx)^2}{Q(x)^2} \right) \right) + O(\|H\|_Q^3). \end{aligned}$$

To get the claim of the lemma, it remains to sum on $x \in Z$ and to use the absolute convergence of the series. \square

Recall that, for $Q \in \mathcal{Q}$, the layer of square radius r is the set

$$Q_r = \{x \in Z \mid Q(x) = r\}.$$

The following result has been proved by Delone and Ryshkov in the case of classic Voronoi spaces [DelRyš67], and recently by R. Coulangeon in the case of Humbert forms [Coul08].

44. THEOREM. (Compare with Theorem 18.) *Let $(V, \mathcal{Q}, Z, \mathbf{G})$ be a Voronoi space such that $Z = -Z$ and $s_0 < \infty$, and let $Q \in \mathcal{Q}$.*

- (i) *If (Q, Q_r) is strongly eutactic for every $r > 0$ such that $Q_r \neq 0$, and if (Q, Q_{\min}) is perfect, then Q is finally ζ -extreme.*
- (ii) *If Q is finally ζ -extreme, then (Q, Q_r) is strongly eutactic for every $r > 0$ such that $Q_r \neq 0$, and (Q, Q_{\min}) is weakly perfect.*

Proof. Let $r_0 = \gamma(Q) < r_1 < r_2 < \dots$ be the square radii of the nonempty layers of Q . We have, according to Lemma 43,

$$\zeta(Q_H, s) = \zeta(Q, s) + A(H) + B(H) + O(\|H\|_Q^3),$$

where A is a linear form on \mathfrak{p}_Q and B is a quadratic form on \mathfrak{p}_Q . More precisely,

$$\begin{aligned} A(H) &= -s \sum_{i \geq 0} r_i^{-s} \sum_{x \in Q_{r_i}} \frac{Q(x, Hx)}{Q(x)}, \\ B(H) &= \sum_{i \geq 0} \left(\frac{s^2}{2} r_i^{-s} C_i(H) - \frac{s}{2} r_i^{-s} D_i(H) \right), \end{aligned}$$

where

$$\begin{aligned} C_i(H) &= \sum_{x \in Q_{r_i}} \frac{Q(x, Hx)^2}{Q(x)^2}, \\ D_i(H) &= \sum_{x \in Q_{r_i}} \left(\frac{Q(Hx)}{Q(x)} - \frac{Q(x, Hx)^2}{Q(x)^2} \right). \end{aligned}$$

Using the definition of the Q -norm and the triangular equality, we get

$$0 \leq C_i(H) \leq |Q_{r_i}| \|H\|_Q^2 \quad \text{and} \quad 0 \leq D_i(H) \leq |Q_{r_i}| \|H\|_Q^2.$$

To prove the theorem, we use the classic criterion for minima on twice continuously differentiable functions:

45. THEOREM. *Let W be a finite-dimensional real vector space, and let $f : W \rightarrow \mathbb{R}$ be a twice continuously differentiable function such that*

$$f(H) = f(0) + f'_0(H) + f''_0(H) + O(\|H\|^3),$$

where f'_0 and f''_0 are respectively linear and quadratic forms. Then, if $f'_0(H) = 0$ and $f''_0(H) > 0$ for all $H \in W \setminus \{0\}$, then 0 is a local minimum of f . Conversely, if 0 is a (strict) local minimum of f , then $f'_0(H) = 0$ and $f''_0(H) \geq 0$ for all $H \in W$.

Applying the criterion to the function $H \mapsto \zeta(Q_H, s)$, we get: *If $A(H) = 0$ and $B(H) > 0$ for all $H \in \mathfrak{p}_Q \setminus \{0\}$, then Q is (strictly) ζ -extreme at s . Conversely, if Q is ζ -extreme at s , then $A(H) = 0$ and $B(H) \geq 0$ for all $H \in \mathfrak{p}_Q$.* This criterion, together with Claims A and B below, achieves the proof.

Claim A. The following are equivalent:

- (a) $A(H) = 0$ for every $H \in \mathfrak{p}_Q$ and for s large enough;
- (b) (Q, Q_{r_i}) is strongly eutactic for every i .

It is clear that (b) implies (a). Conversely, suppose that there exists an index i such that (Q, Q_{r_i}) is not strongly eutactic. Let k be the smaller index for which (Q, Q_{r_k}) is not strongly eutactic, and let $H \in \mathfrak{p}_Q$ such that $p := \sum_{x \in Q_{r_i}} Q(x, Hx)/Q(x) \neq 0$. We have

$$A(H) = -sr_k^{-s}(p + S),$$

where

$$|S| \leq \left| \sum_{i=k+1}^{\infty} \left(\frac{r_i}{r_k}\right)^{-s} \sum_{x \in Q_{r_i}} \frac{Q(x, Hx)}{Q(x)} \right| \leq \|H\| \sum_{i=k+1}^{\infty} \left(\frac{r_i}{r_k}\right)^{-s}.$$

Now, for $s' = s(1 - \log(r_k)/\log(r_{k+1}))$, we have $(r_i/r_k)^{-s} \leq r_i^{-s'}$ for $i \geq k+1$. Therefore,

$$|S| \leq \|H\|_Q \sum_{i \geq 0} r_i^{-s'} = \|H\|_Q \zeta(Q, s').$$

As $\zeta(Q, s')$ tends to zero when s tends to the infinity, for s large enough, we have $|S| \leq p/2$, and therefore $A(H) \neq 0$. This achieves the proof of Claim A.

Claim B. If (Q, Q_{\min}) is perfect, then $B(H) > 0$ for every $H \in \mathfrak{p}_Q \setminus \{0\}$ and for s large enough. Conversely, if $B(H) \geq 0$ for every $H \in \mathfrak{p}_Q$ and for s large enough, and if (Q, Q_{\min}) is eutactic, then (Q, Q_{\min}) is weakly perfect.

If (Q, Q_{\min}) is perfect, then $C_0(H) > 0$ when $H \in \mathfrak{p}_Q \setminus \{0\}$. The function $f : [H] \mapsto C_0(H)/\|H\|_Q^2$ defined on $\mathbb{P}(\mathfrak{p}_Q)$ is continuous and positive, and $\mathbb{P}(\mathfrak{p}_Q)$ is compact; therefore there exists a $\delta > 0$ such that $f \geq \delta$, that is

$$C_0(H) \geq \delta \|H\|_Q^2, \quad \forall H \in \mathfrak{p}_Q.$$

Thus, we have

$$B(H) \geq \frac{s^2 r_0^{-s}}{2} (\delta \|H\|_Q^2 + S),$$

$$S = -s^{-1} D_0(H) + \sum_{i=1}^{\infty} \left(\frac{r_i}{r_k}\right)^{-s} (C_i(H) - s^{-1} D_i(H))$$

With an argument similar to the one of the proof of Claim A, it is shown that $|S| \leq \delta \|H\|_Q^2/2$ for s large enough, and therefore $B(H) > 0$ for every $H \in \mathfrak{p}_Q \setminus \{0\}$.

Conversely, suppose, that (Q, Q_{\min}) is eutactic but not weakly perfect. Then there exists an $H_0 \in \mathfrak{p}_Q$, an $x \in Q_{\min}$ and a $c \in \mathbb{R}$ such that $(H_0 - c)x \neq 0$ but $Q(y, (H_0 - c)y) = 0$ for every $y \in Q_{\min}$; the eutaxy of (Q, Q_{\min}) then implies that $c = 0$. So, we have $C_0(H_0) = 0$ and $D_0(H_0) = p > 0$. So,

$$B(H_0) = \frac{sr_0^{-s}}{2}(-p + S),$$

$$S = \sum_{i=1}^{\infty} \left(\frac{r_i}{r_k}\right)^{-s} (s C_i(H_0) - D_i(H_0)).$$

As before, $|S| \leq p/2$ for s sufficiently large, and therefore $B(H_0) < 0$. This achieves the proof of Claim B and of the theorem. \square

6.3 Criterion of ζ -extremality in terms of design

In this paragraph, we work with alternative Voronoi spaces, since they are more convenient when talking about designs. All the definitions and results from the two previous paragraphs can be formulated in the language of alternative Voronoi spaces.

The following result has been proved by Coulangeon in the case of the classic alternative Voronoi space [Coul06, Theorem 1]; note however that our result generalizes only partially the theorem of Coulangeon. Recall that we use the brackets $[\]$ to denote the image through the projection $V \setminus \{0\} \rightarrow \mathbb{P}(V)$.

46. THEOREM. (Compare with Corollary 36.) *Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be a regular alternative Voronoi space such that $Z = -Z$. Then there exists a $s_1 > 0$ with the following property: For $Z \in \mathcal{Z}$, if $[Z_r]$ is a 4-design for every $r \geq 0$ such that $Z_r \neq 0$, then Z is ζ -extreme for every $s > s_1$.*

The constant s_1 of the theorem can be computed explicitly by the formula given in the proof below. For example in the case of the classic alternative Voronoi space of rank n , we have $s_1 = n/2$ [Coul06].

Proof. Let $Z \in \mathcal{Z}$ and let $r_0 = \gamma(Q) < r_1 < r_2 < \dots$ be the square radii of the nonempty layers of Z ; and suppose that $[Z_{r_i}]$ is a 4-design for every i . Recall (Proposition 27) that a set in the neighborhood of Z is of the form kZ_H with $Z_H = \exp(H/2)Z$, $k \in \mathbf{K}$, and $H \in \mathfrak{p}$ with $\|H\|_Q$ small. From Lemma 43, we have

$$\zeta(Z_H, s) = \zeta(Z, s) + \sum_{i \geq 0} r_i^{-s} \left(-s A_i(H) + \frac{s^2}{2} B_i(H) - \frac{s}{2} C_i(H) + O(\|H\|_Q^3) \right),$$

where

$$A_i(H) = \sum_{x \in Z_{r_i}} \frac{Q(x, Hx)}{Q(x)}, \quad B_i(H) = \sum_{x \in Z_{r_i}} \frac{Q(x, Hx)^2}{Q(x)^2},$$

$$C_i(H) = \sum_{x \in Z_{r_i}} \left(\frac{Q(Hx)}{Q(x)} - \frac{Q(x, Hx)^2}{Q(x)^2} \right).$$

As $[Z_{r_i}]$ is in particular a $\{2\}$ -design, we have $A_i(H) = 0$ by Lemma 35. Let

$$\beta_H(x) = \frac{Q(x, Hx)^2}{Q(x)^2} \geq 0, \quad \gamma_H(x) = \frac{Q(Hx)}{Q(x)} - \frac{Q(x, Hx)^2}{Q(x)^2} \geq 0.$$

We have $\beta_H \in \mathcal{F}_{\{2,2\}} \subseteq \mathcal{F}_4$ and $\gamma_H \in \mathcal{F}_{\{2,2\}} + \mathcal{F}_{\{4\}} = \mathcal{F}_4$. Since $[Z_{r_i}]$ is a 4-design, we have $B_i(H) = |Z_{r_i}| \langle \beta_H \rangle$ and $C_i(H) = |Z_{r_i}| \langle \gamma_H \rangle$, so

$$\begin{aligned} \zeta(Z_H, s) &= \zeta(Z, s) + \sum_{i \geq 0} |Z_{r_i}| r_i^{-s} \left(\frac{s^2}{2} \langle \beta_H \rangle - \frac{s}{2} \langle \gamma_H \rangle + O(\|H\|_Q^3) \right), \\ &= \zeta(Z, s) \left(1 + \frac{s^2}{2} \langle \beta_H \rangle - \frac{s}{2} \langle \gamma_H \rangle + O(\|H\|_Q^3) \right). \end{aligned}$$

Now consider the function $f : \mathbb{P}(\mathfrak{p}) \rightarrow]0, \infty[$ defined by $f([H]) = \langle \gamma_H \rangle / \langle \beta_H \rangle$, and let s_1 be the maximal value of f . For $s > s_1$, we have $\langle \gamma_H \rangle \leq s_1 \langle \beta_H \rangle \leq s \|H\|_Q^2$, hence

$$\zeta(Z_H, s) \geq \zeta(Z, s) \left(1 + \frac{s(s-s_1)}{2} \|H\|_Q^2 + O(\|H\|_Q^3) \right).$$

We infer that $\zeta(kZ_H, s) = \zeta(Z_H, s) > \zeta(Z, s)$ for $\|H\|_Q$ sufficiently small. Note that the value of s_1 depends only on the structure of alternative Voronoi space of $(V, Q, \mathcal{Z}, \mathbf{G})$. \square

As for standard extremality, we can exploit the symmetry of the potentially extreme sets in order to check that there are indeed extreme. So, using Corollary 38, we draw the following consequence:

47. COROLLARY. (Compare with Corollary 39.) *Let $(V, Q, \mathcal{Z}, \mathbf{G})$ be a regular alternative Voronoi space, and let $Z \in \mathcal{Z}$. Let $\mathbf{F} = \{g \in \mathbf{K} \mid gZ = Z\}$. If the only \mathbf{F} -invariant functions in \mathcal{F}_4 are the constant functions, then Z is ζ -extreme at every $s > s_1$.*

For example, this criterion allows to show that all the lattices given at the end of Paragraph 5.3, are ζ -extreme at any $s > n/2$.

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