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A Metric Approach to nD Images Edge Detection With Clifford Algebras

Thomas Batard · Christophe Saint-Jean · Michel Berthier

Abstract The aim of this paper is to perform edge detection in color-infrared images from the point of view of Clifford algebras. The main idea is that such an image can be seen as a section of a Clifford bundle associated to the *RGBT*-space (Red, Green, Blue, Temperature) of acquisition. Dealing with geometric calculus and covariant derivatives of appropriate sections with respect to well-chosen connections allows to get various color and temperature information needed for the segmentation. We show in particular how to recover the first fundamental form of the image embedded in a *LSHT*-space (Luminance, Saturation, Hue, Temperature) equipped with a metric tensor. We propose applications to color edge detection with respect to a given hue interval and to edge detection in color-infrared images with constraints on temperature. Others applications related to different choices of connections, sections and embedding spaces for nD images may be considered from this general theoretical framework.

Keywords Edge detection · Clifford algebras bundle · Geometric calculus · Color · Infrared · Di Zenzo gradient

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1 Introduction

Clifford algebras appear to be a powerful tool in a wide range of applications to computer sciences, see [15] for examples. In particular, Sangwine & al. approach of color images segmentation with quaternions [12], [5] can be considered from this viewpoint since \mathbb{H} (the algebra of quaternions) is the Clifford algebra $\mathbb{R}_{0,2}$ and \mathbb{H}^1 (the group of unit quaternions) is the spinor group $Spin(3)$. Sangwine's idea is to associate a pure imaginary quaternion to each color of the RGB cube and then make geometric transformations on colors using the product of \mathbb{H} to compute some kind of gradient.

Working in the framework of Clifford algebras has several advantages:

- If we consider a color as a vector of the algebra $\mathbb{R}_{3,0}$ we can use the richness of the structure of this latter. In fact $\mathbb{R}_{3,0}$ is of dimension 8 over \mathbb{R} and contains elements of different degrees (scalars, vectors, bivectors and a pseudoscalar) that carry different information. We can also benefit from the efficiency of the calculus based on the geometric product.
- It is possible to generalize this approach to any dimension working with images with values in \mathbb{R}^n . We will consider in the sequel the example of color-infrared images where $n = 4$.
- A Clifford algebra is defined with respect to a chosen metric of a vector space. It's an asset when dealing with metric approach of edge detection. In particular, we can make the metric of the ambient space vary to detect edges of different kinds.

The metric approach we adopt in this paper was introduced by Di Zenzo [4], who considered implicitly a n-channels image as an embedded two-dimensionnal surface in the euclidean space \mathbb{R}^{n+2} . A measure of the "edge strength" and the direction wherein it is highest at each point are defined. This is done by computing

respectively the highest eigenvalue of the first fundamental form of the surface and the corresponding eigenvector. Using this method, Cumani [3] gives an explicit definition of an edge point, that is a point where the first directional derivative of the highest eigenvalue in the direction given by the highest eigenvector has a transversal zero-crossing. However, when $n > 1$, the smallest eigenvalue is no more constant; that's why Sapiro [13] suggests as a measure of the "edge strength" the square root of the difference of the two eigenvalues.

The idea of embedding the surface representing an image in a space endowed with a non-euclidean metric is proposed in [14], and is applied for color edge detection in [10]. In this latter, the author describes colors with hyperbolic coordinates and endows \mathbb{R}^5 with a corresponding metric.

It is our purpose to start from this general definition of a nD image and to develop a general approach of edge detection in color-infrared images based on metric information. Indeed, we define an image as a section of a fiber bundle. The fiber of this bundle is a Clifford algebra related to the acquisition space *RGBT* of the color-infrared image endowed with a metric that may vary with the base point. We show that it is possible, staying in the acquisition space, to recover the metric data of the image

$$\varphi : (x, y) \longmapsto (x, y, l, s, h, t)$$

embedded in a *LSHT*-space (l, s, h, t , denotes respectively the luminance, the saturation, the hue and the temperature) equipped with a metric tensor. This is done in both continuous and discrete cases by considering two sections of the bundle and their covariant derivatives with respect to a well chosen connection.

This paper is organized as follows. Sect. 2 is mainly devoted to basic notions and results on Clifford algebras and spinor groups. We show also how to derive the transition formulas from *RGBT* to *LSHT* with geometric calculus. In Sect. 3, we first compute the coefficients of the first fundamental form of a color-infrared image defined as a two-dimensionnal surface embedded in a *LSHT*-space equipped with a metric tensor. Then, we show how to interpret such an image as a section of a Clifford algebra bundle. The rest of the section is devoted to the continuous and discrete computations of the coefficients of the first fundamental form as explain above. In Sect. 4, we propose three applications: a comparison with the Di Zenzo method, an edge detection in a color image with respect to a given hue interval and an edge detection in a color-infrared image with constraints on temperature's values.

2 Clifford algebras and color-infrared spaces

2.1 Clifford algebras

Let V be a vector space of finite dimension n over \mathbb{R} equipped with a quadratic form Q . The Clifford algebra $Cl(V, Q)$ is the solution of the following universal problem. We search a couple $(Cl(V, Q), i_Q)$ where $Cl(V, Q)$ is an \mathbb{R} -algebra and $i_Q : V \longrightarrow Cl(V, Q)$ is \mathbb{R} -linear satisfying:

$$(i_Q(v))^2 = Q(v).1$$

for all v in V (1 denotes the unit of $Cl(V, Q)$) such that for all \mathbb{R} -algebra A and all \mathbb{R} -linear map $f : V \longrightarrow A$ with

$$(f(v))^2 = Q(v).1$$

for all v in V (1 denotes the unit of A), then there exists a unique morphism

$$g : Cl(V, Q) \longrightarrow A$$

of \mathbb{R} -algebras such that $f = g \circ i_Q$.

The solution is unique up to isomorphisms and is given as the (non commutative) quotient

$$T(V)/(v \otimes v - Q(v).1)$$

of the tensor algebra of V by the ideal generated by $v \otimes v - Q(v).1$, where v belongs to V (see [11] for a proof). It is well known that there exists a unique anti-automorphism t on $Cl(V, Q)$ such that

$$t(i_Q(v)) = i_Q(v)$$

for all v in V . It is called reversion and usually denoted by $x \longmapsto x^\dagger$, x in $Cl(V, Q)$. In the same way there exists a unique automorphism α on $Cl(V, Q)$ such that

$$\alpha(i_Q(v)) = -i_Q(v)$$

for all v in V . In the rest of this paper we write v for $i_Q(v)$ (according to the fact that i_Q embeds V in $Cl(V, Q)$).

As a vector space $Cl(V, Q)$ is of dimension 2^n on \mathbb{R} and a basis is given by the set

$$\{e_{i_1}e_{i_2}\cdots e_{i_k}, \quad i_1 < i_2 < \dots < i_k, \quad k \in \{1, \dots, n\}\}$$

and the unit 1. An element of degree k

$$\sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} e_{i_1} e_{i_2} \cdots e_{i_k}$$

is called a k -vector. A 0-vector is a scalar and $e_1 e_2 \cdots e_n$ is called the pseudoscalar. We will denote $\langle x \rangle_k$ the component of degree k of an element x of $Cl(V, Q)$.

The inner product of x_r of degree r and y_s of degree s is defined by

$$x_r \cdot y_s = \langle x_r y_s \rangle_{|r-s|}$$

if r and s are positive and by

$$x_r \cdot y_s = 0$$

otherwise.

The outer product of x_r of degree r and y_s of degree s is defined by

$$x_r \wedge y_s = \langle x_r y_s \rangle_{r+s}$$

These products extend by linearity on $Cl(V, Q)$. Clearly, if a and b are vectors of V , then the inner product of a and b coincides with the scalar product defined by Q . When it is defined (for example when x is a versor and Q is positive) we denote

$$\|x\| = \sqrt{xx^\dagger}$$

and say that x is a unit if $xx^\dagger = \pm 1$.

In the following, we deal in particular with the Clifford algebra of the euclidean \mathbb{R}^n denoted by $\mathbb{R}_{n,0}$. $\mathbb{R}_{n,0}^k$ is the subspace of elements of degree k and $\mathbb{R}_{n,0}^*$ is the group of elements that admit an inverse in $\mathbb{R}_{n,0}$.

Let a be a vector in $\mathbb{R}_{n,0}$ and B be the k -vector $a_1 \wedge a_2 \wedge \dots \wedge a_k$, then the orthogonal projection of a on the k -plane generated by the a_i 's is the vector

$$P_B(a) = (a \cdot B)B^{-1}$$

The vector

$$a - (a \cdot B)B^{-1} = (a \wedge B)B^{-1}$$

is called the rejection of a on B .

2.2 The spinor group $Spin(n)$

It is defined by

$$Spin(n) = \left\{ \prod_{i=1}^{2k} a_i, a_i \in \mathbb{R}_{n,0}^1, \|a_i\| = 1 \right\}$$

or equivalently

$$Spin(n) = \{x \in \mathbb{R}_{n,0}, \alpha(x) = x, xx^\dagger = 1, \\ xvx^{-1} \in \mathbb{R}_{n,0}^1 \forall v \in \mathbb{R}_{n,0}^1\}$$

It is well known that $Spin(n)$ is a connected compact Lie group that universally covers $SO(n)$ ($n \geq 3$). Its Lie algebra is $\mathbb{R}_{n,0}^2$ with Lie bracket

$$A \times B = AB - BA$$

As the exponential map from its Lie algebra to $Spin(n)$ is onto (see [8] for a proof), every spinor can be written as

$$S = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$$

for some bivector A .

From Hestenes and Sobczyk [9], we know that every A in $\mathbb{R}_{n,0}^2$ can be written as

$$A = A_1 + A_2 + \dots + A_m$$

where $m \leq n/2$ and

$$A_j = \|A_j\| a_j b_j, \quad j \in \{1, \dots, m\}$$

with

$$\{a_1, \dots, a_m, b_1, \dots, b_m\}$$

a set of orthonormal vectors. Thus

$$A_j A_k = A_k A_j = A_k \wedge A_j$$

whenever $j \neq k$ and

$$A_k^2 = -\|A_k\|^2 < 0$$

This means that the planes encoded by A_k and A_j are orthogonal and implies that

$$e^{A_1 + A_2 + \dots + A_m} = e^{A_{\sigma(1)}} e^{A_{\sigma(2)}} \dots e^{A_{\sigma(m)}}$$

for all σ in the permutation group $\mathfrak{S}(m)$. Actually, as A_k^2 is negative we have

$$e^{A_i} = \cos(\|A_i\|) + \sin(\|A_i\|) \frac{A_i}{\|A_i\|}$$

The corresponding rotation

$$R_i : x \mapsto e^{-A_i} x e^{A_i}$$

acts in the oriented plane defined by A_i as a plane rotation of angle $2\|A_i\|$. The vectors orthogonal to A_i are invariant under R_i .

It then appears that any element R of $SO(n)$ is a composition of commuting simple rotations, in the sense that they have only one invariant plane. The vectors left invariant by R are those of the orthogonal subspace to A . If $m = n/2$ this latter is trivial. The previous decomposition is not unique if $\|A_k\| = \|A_j\|$ for some j and k with $j \neq k$. In this case infinitely many planes are left invariant by R .

2.3 Color-infrared spaces

As mentioned before, Sangwine's approach of edge detection in color images relies on the fact that \mathbb{H}_0 (the set of pure imaginary quaternions) is isomorphic to \mathbb{R}^3 equipped with an action of \mathbb{H}^1 . In the same way, a color image can be treated as an application from \mathbb{R}^2 to \mathbb{R}^3 this latter being embedded in $\mathbb{R}_{3,0}$ and equipped with an action of $Spin(3)$. It is natural to extend this approach to nD images replacing \mathbb{R}^3 by \mathbb{R}^n and $\mathbb{R}_{3,0}$ by $\mathbb{R}_{n,0}$. We focus here on color-infrared images.

Beside RGB color space we consider HSL color space defined as follows. We set first

$$\begin{pmatrix} Y \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$

Then the luminance l , the saturation s and the hue h are respectively given by

$$\begin{aligned} l &= Y \\ s &= \sqrt{C_1^2 + C_2^2} \\ h &= \begin{cases} \arccos(C_2/s) & \text{if } C_2 > 0 \\ 2\pi - \arccos(C_2/s) & \text{otherwise} \end{cases} \end{aligned}$$

As it is well known, color spaces based on luminance (value), saturation and hue are more suitable to perception [6].

Let us denote \mathcal{CT} the Clifford algebra of (\mathbb{R}^4, Q) with Q the positive definite quadratic form given by

$$\begin{pmatrix} \beta/3 & 0 & 0 & 0 \\ 0 & \beta/3 & 0 & 0 \\ 0 & 0 & \beta/3 & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

Thus $e_1^2 = e_2^2 = e_3^2 = \beta/3$ and $e_4^2 = \delta$. Given a color-infrared vector $a = r(a)e_1 + g(a)e_2 + b(a)e_3 + t(a)e_4$, its color component is given by

$$\begin{aligned} c(a) &= r(a)e_1 + g(a)e_2 + b(a)e_3 \\ &= a \cdot (e_1e_2e_3)(e_1e_2e_3)^{-1} \end{aligned}$$

Let us denote

$$\mu = \frac{e_1 + e_2 + e_3}{\sqrt{\beta}}$$

the unit vector generating the achromatic axis, and $v(a)$ the rejection of $c(a)$ on μ , called the chrominance vector of a . Simple computations show that the luminance $l(a)$, the saturation $s(a)$ and the hue $h(a)$ of a can be written

$$\begin{aligned} l(a) &= \frac{1}{\sqrt{\beta}} \|(a \cdot \mu)\mu^{-1}\| \\ &= \frac{1}{\sqrt{\beta}} \sqrt{((a \cdot \mu)\mu^{-1})^2} \end{aligned}$$

$$\begin{aligned} s(a) &= \frac{3}{\sqrt{2\beta}} \|(c(a) \wedge \mu)\mu^{-1}\| \\ &= \frac{3}{\sqrt{2\beta}} \sqrt{((c(a) \wedge \mu)\mu^{-1})^2} \end{aligned}$$

$$h(a) = 2\pi + \text{sign}(g(a) - b(a)) \arccos\left(\frac{v(a)}{\|v(a)\|} \cdot \rho\right)$$

with ρ the unit chrominance vector of the red color and $h(a)$ defined modulo 2π . In other words, $h(a)$ is the oriented angle from ρ to $v(a)$.

Moreover, the dual of the achromatic axis in the vector space generated by (e_1, e_2, e_3) is a plane, called the chrominance plane, generated by the bivector

$$e_1e_2 - e_1e_3 + e_2e_3$$

and we have

$$v(a) = a \cdot (e_1e_2 - e_1e_3 + e_2e_3)(e_1e_2 - e_1e_3 + e_2e_3)^{-1}$$

The chrominance vector of a is therefore the orthogonal projection of a on the chrominance plane.

3 Edge detection in color-infrared images

3.1 First fundamental form of a surface and edge detectors

We recall in this subsection how to define an edge detector using metric information given by the first fundamental form. For this we consider a color-infrared image as a C^k map, $k \geq 1$,

$$\varphi(x, y) \longrightarrow (x, y, l(x, y), s(x, y), h(x, y), t(x, y))$$

from a rectangle D to \mathbb{R}^6 . In the following q denotes a point in D with image $p = \varphi(q)$ under the map φ . Note that we consider the hue h with values in the universal cover \mathbb{R} of $\mathbb{R}/2\pi\mathbb{Z}$. For the sake of a relevant definition of φ , we take $h = 0$ when $s = 0$. As we will see below, this has no consequences on the edge detection. Following Carron [2], let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f(t) = \frac{1}{\pi} \left(\frac{\pi}{2} + \arctan(0,07(t - 50)) \right)$$

We consider the domain

$$\Omega(p) = \{(x, y), \|(x, y) - \varphi^{-1}(p)\|_\infty \leq 1\}$$

and set

$$\xi(p) = \exp\left(\frac{1}{4} \int_{\Omega(p)} \ln(f \circ s(x, y)) dx dy\right)$$

if $s(u, v) \neq 0$ for all (u, v) in $\Omega(p)$ and $\xi(p) = 0$ otherwise. These functions are used to take into account the fact that the hue is irrelevant for small values of the saturation.

Next, we endow \mathbb{R}^6 with the following metric

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \lambda(p) & 0 & 0 & 0 \\ 0 & \lambda(p) & 0 & 0 \\ 0 & 0 & \kappa(p)\xi(p) & 0 \\ 0 & 0 & 0 & \eta(p) \end{pmatrix}$$

where λ , κ and η are positive functions. Strictly speaking, as ξ can vanish, this metric is not riemannian. However the metric induced on the surface $\varphi(D)$ is riemannian: it is the first fundamental form of $\varphi(D)$, usually denoted by

$$I(p) = \begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix}$$

The coefficients E , F and G are given by

$$E(p) = 1 + \lambda(p)t_x^2(p) + \lambda(p)s_x^2(p) + \kappa(p)\xi(p)h_x^2(p) + \eta(p)t_x^2(p)$$

$$F(p) = \lambda(p)l_x(p)l_y(p) + \lambda(p)s_x(p)s_y(p) + \kappa(p)\xi(p)h_x(p)h_y(p) + \eta(p)t_x(p)t_y(p)$$

$$G(p) = 1 + \lambda(p)t_y^2(p) + \lambda(p)s_y^2(p) + \kappa(p)\xi(p)h_y^2(p) + \eta(p)t_y^2(p)$$

We denote $\theta_+(p)$ and $\theta_-(p)$, $\theta_+(p) \geq \theta_-(p)$, the two eigenvalues of $I(p)$ and $\Theta_+(p)$, $\Theta_-(p)$ the corresponding eigenvectors. The edge detector is then given by

$$\varpi(p) = \sqrt{\theta_+(p) - \theta_-(p)}$$

More precisely, we say that q in D is an edge point if one of the following conditions holds:

1. the function ϖ has a local maximum at $\varphi(q)$ in the direction given by $\Theta_+(\varphi(q))$;
2. $\Theta_+(\varphi(q)) > 1$ and q is an endpoint of a curve of points satisfying 1.

3.2 Clifford bundle and color-infrared image

We explain how to consider a color/infrared image as a section of a Clifford bundle.

Let $\|\cdot\|_2$ be the euclidean norm on \mathbb{R}^n . We associate to each point q of D the Clifford algebra $\mathcal{CT}(q)$ of the four dimensional vector space containing the $RGBT$ -space endowed with the metric $Q(q)$

$$\begin{pmatrix} \lambda(p)/3 & 0 & 0 & 0 \\ 0 & \lambda(p)/3 & 0 & 0 \\ 0 & 0 & \lambda(p)/3 & 0 \\ 0 & 0 & 0 & \eta(p) \end{pmatrix}$$

(compare with Sect. 2.3 : β is replaced by $\lambda(p)$ and δ is replaced by $\eta(p)$). Let $\mathcal{CT}(D)$ be the disjoint union of $\mathcal{CT}(q)$ for q in D .

Proposition 1 $\mathcal{CT}(D)$ with the projection

$$\pi : \mathcal{CT}(D) \longrightarrow D$$

that maps $\zeta \in \mathcal{CT}(q)$ to q is a trivial fiber bundle $(\mathcal{CT}(D), D, \pi)$ with typical fiber $Cl(\mathbb{R}^4, \|\cdot\|_2)$.

Proof We have to show that there exists a diffeomorphism Φ from $\pi^{-1}(D)$ onto $D \times Cl(\mathbb{R}^4, \|\cdot\|_2)$ such that $\Phi \circ p_1 = \pi$ where p_1 denotes the projection on the first factor. As $\mathcal{CT}(q)$ is isomorphic to $Cl(\mathbb{R}^4, \|\cdot\|_2)$ by some Φ_q for all q in D , we can define Φ by

$$\Phi : (v \in \mathcal{CT}(q)) \longmapsto (q, \Phi_q(v))$$

It is clearly a diffeomorphism. \square

A color-infrared image I is now considered as a section $q \in D \longmapsto r(q)e_1(q) + g(q)e_2(q) + b(q)e_3(q) + t(q)e_4(q)$ of $\mathcal{CT}(D)$.

From the fact that $(\mathcal{CT}(D), D, \pi)$ is trivial we know that any connection on it can be written

$$\nabla = d + \omega$$

for some ω in $\Gamma(D, T^*D \otimes \text{End}(\mathcal{CT}(D)))$, d being the exterior differential [7]. If

$$X = (X_1, X_2)$$

belongs to TD and

$$Y = Y_0 1 + Y_1 e_1 + \dots + Y_{15} e_1 e_2 e_3 e_4$$

is a section of $\mathcal{CT}(D)$ then

$$(\omega(X)Y)_j = \sum_{k=0}^{15} (\Gamma_{1,k}^j X_1 + \Gamma_{2,k}^j X_2) Y_k$$

so that the connection is entirely determined by the symbols $\Gamma_{i,j}^k$, $i, j = 1, 2$ and $k = 0, \dots, 15$. In the next paragraph we deal with the following three objects:

i. the connection ∇_0 defined by

$$\Gamma_{ij}^k = \begin{cases} \frac{\partial_i \lambda}{\lambda} & \text{if } k = j \in \{6, 7, 9\} \\ 0 & \text{otherwise} \end{cases}$$

ii. the section ψ of $(\mathcal{CT}(D), D, \pi)$ given by

$$\psi = S^\dagger I S$$

with

$$S = \exp \left[-\frac{h}{2} \left(\frac{e_1 e_2 - e_1 e_3 + e_2 e_3}{\|e_1 e_2 - e_1 e_3 + e_2 e_3\|} \right) \right]$$

iii. the section γ of $(\mathcal{CT}(D), D, \pi)$ given by

$$\gamma = \frac{v}{\|v\|} \rho$$

where ρ is the unit chrominance vector of the red color, v is the chrominance vector, h is the hue (see Sect. 2.3) and $\|\cdot\|$ means that we take the norm of each fiber $\pi^{-1}(q)$.

3.3 Computing E, F, G with ∇_0

The main result of this part is that the preceding coefficients E, F and G can be computed using covariant derivatives with respect to ∇_0 .

Proposition 2 Let

i. P_1 (resp. P_2) be the section of $\text{End}(\mathcal{CT}(D))$ such that $P_1(q)$ (resp. $P_2(q)$) is the orthogonal projection on the plane generated by the luminance and the temperature (resp. on the chrominance plane) in the fiber $\pi^{-1}(q)$;

ii. dx (resp. dy) be the canonical $\mathcal{CT}(D)$ -valued 1-form $dx \otimes 1$ (resp. $dy \otimes 1$) and X (resp. Y) be the vector field on D of coordinates $(1, 0)$ (resp. $(0, 1)$);

iii. E, F, G be the coefficients of the first fundamental form of $\varphi(D)$ (see Sect. 3.1) and χ be the $\mathcal{CT}(D)$ -valued symmetric tensor of rank 2:

$$\chi = dx \otimes dx + dy \otimes dy + P_1(\nabla_0\psi)P_1(\nabla_0\psi) + \frac{9}{2}P_2(\nabla_0\psi) \otimes P_2(\nabla_0\psi) - \kappa \xi(\gamma^\dagger \nabla_0 \gamma) \otimes (\gamma^\dagger \nabla_0 \gamma)$$

then

$$E = \chi(X \otimes X) \quad F = \chi(X \otimes Y) \quad G = \chi(Y \otimes Y)$$

(see Sect. 3.1 for the definitions of κ and ξ). The symbol \otimes denotes the tensor product of $\mathcal{CT}(D)$ -valued 1-forms described in the appendix and $P_1(\nabla_0\psi)P_1(\nabla_0\psi)$ the symmetric product of $P_1(\nabla_0\psi)$ by itself (this tensor product is an analog of the wedge product for the so called “cliffforms”).

In other words, χ may be viewed as the metric on the surface $\varphi(D)$.

Proof From Sect. 2.3 we know that S is a $\text{Spin}(4)$ -valued section whose action on I for each $q \in D$, namely $S(q)^\dagger I(q)S(q)$, is a rotation. We explicit this rotation. The 4-dimensional vector subspace of $\mathcal{CT}(q)$ isomorphic to $\mathbb{R}_{4,0}^1$ by trivialisation can be decomposed into two orthogonal planes:

- i. the plane generated by the luminance and the temperature components, represented by the bivector $e_1(q)e_4(q) + e_2(q)e_4(q) + e_3(q)e_4(q)$;
- ii. the chrominance plane represented by the bivector $e_1(q)e_2(q) - e_1(q)e_3(q) + e_2(q)e_3(q)$.

From this we deduce that the rotation lets the luminance and temperature parts of $I(q)$ invariant and acts on the chrominance plane as a rotation of angle $-h(q)/2$. That is, it sends the chrominance vector $v(q)$ on the vector $\|v(q)\|\rho(q)$.

Since P_1 and P_2 are linear maps, we have

$$d(P_1(\psi)) = P_1 d(\psi)$$

From the definition of ∇_0 , it leads to

$$P_1(\nabla_0\psi) = \nabla_0 P_1(\psi)$$

Then

$$\begin{aligned} P_1(\nabla_0\psi) &= \nabla_0 P_1(\psi) \\ &= \nabla_0(l(e_1 + e_2 + e_3) + te_4) \\ &= dl \otimes (e_1 + e_2 + e_3) + dt \otimes e_4 \end{aligned}$$

hence

$$\begin{aligned} P_1(\nabla_0\psi)P_1(\nabla_0\psi) &= dl \otimes dl \otimes (e_1 + e_2 + e_3)^2 \\ &+ \frac{1}{2}(dl \otimes dt + dt \otimes dl) \otimes (e_1 + e_2 + e_3)e_4 \\ &+ \frac{1}{2}(dt \otimes dl + dl \otimes dt) \otimes e_4(e_1 + e_2 + e_3) \\ &+ dt \otimes dt \otimes (e_4)^2 \end{aligned}$$

So, we have

$$P_1(\nabla_0\psi)P_1(\nabla_0\psi) = dl \otimes dl \otimes \lambda + dt \otimes dt \otimes \eta$$

Simple computations show that $\rho = \sigma/\sqrt{\lambda}$ with

$$\sigma = \sqrt{2}e_1 - \frac{\sqrt{2}}{2}e_2 - \frac{\sqrt{2}}{2}e_3$$

then

$$\begin{aligned} P_2(\nabla_0\psi) &= \nabla_0 \left(\frac{\|v\|\sigma}{\sqrt{\lambda}} \right) \\ &= \left(d \frac{\|v\|}{\sqrt{\lambda}} \right) \otimes \sigma + \frac{\|v\|}{\sqrt{\lambda}} \nabla_0 \sigma \\ &= \left(d \frac{\|v\|}{\sqrt{\lambda}} \right) \otimes \sigma \end{aligned}$$

Recall that $\|v\| = \frac{\sqrt{2}}{3}\sqrt{\lambda}\sigma$ to obtain

$$P_2(\nabla_0\psi) = \frac{\sqrt{2}}{3}ds \otimes \sigma$$

which leads to

$$\begin{aligned} P_2(\nabla_0\psi) \otimes P_2(\nabla_0\psi) &= \frac{2}{9}ds \otimes ds \otimes \sigma^2 \\ &= \frac{2}{9}ds \otimes ds \otimes \lambda \end{aligned}$$

The section γ can be decomposed as

$$\gamma = \frac{v}{\|v\|} \cdot \rho + \frac{v}{\|v\|} \wedge \rho$$

Since

$$\frac{v}{\|v\|} \cdot \rho = \cos(h)$$

and

$$\frac{v}{\|v\|} \wedge \rho = \sin(h) \left(\frac{e_1 e_2 - e_1 e_3 + e_2 e_3}{\|e_1 e_2 - e_1 e_3 + e_2 e_3\|} \right)$$

we have then

$$\gamma = \cos(h) + \sin(h) \left(\frac{e_1 e_2 - e_1 e_3 + e_2 e_3}{\|e_1 e_2 - e_1 e_3 + e_2 e_3\|} \right)$$

The expression of $\nabla_0\gamma$ is therefore

$$\begin{aligned}\nabla_0\gamma &= -dh \otimes \sin(h) + \cos(h)\nabla_0 1 \\ &+ dh \otimes \cos(h) \left(\frac{e_1e_2 - e_1e_3 + e_2e_3}{\|e_1e_2 - e_1e_3 + e_2e_3\|} \right) \\ &+ \sqrt{3} \sin(h)\nabla_0 \left(\frac{e_1e_2}{\lambda} - \frac{e_1e_3}{\lambda} + \frac{e_2e_3}{\lambda} \right)\end{aligned}$$

However by definition of ∇_0

$$\nabla_0 1 = \nabla_0 \left(\frac{e_1e_2}{\lambda} \right) = \nabla_0 \left(\frac{e_1e_3}{\lambda} \right) = \nabla_0 \left(\frac{e_2e_3}{\lambda} \right) = 0$$

and so

$$\nabla_0\gamma = -dh \otimes \sin(h) + dh \otimes \cos(h) \left(\frac{e_1e_2 - e_1e_3 + e_2e_3}{\|e_1e_2 - e_1e_3 + e_2e_3\|} \right)$$

This implies that

$$\begin{aligned}\gamma^\dagger \nabla_0\gamma &= \left[\cos(h) - \sin(h) \left(\frac{e_1e_2 - e_1e_3 + e_2e_3}{\|e_1e_2 - e_1e_3 + e_2e_3\|} \right) \right] \times \\ &\left[(-dh \otimes \sin(h) + dh \otimes \cos(h) \left(\frac{e_1e_2 - e_1e_3 + e_2e_3}{\|e_1e_2 - e_1e_3 + e_2e_3\|} \right)) \right]\end{aligned}$$

and

$$\gamma^\dagger \nabla_0\gamma = dh \otimes \left(\frac{e_1e_2 - e_1e_3 + e_2e_3}{\|e_1e_2 - e_1e_3 + e_2e_3\|} \right)$$

Consequently

$$(\gamma^\dagger \nabla_0\gamma) \otimes (\gamma^\dagger \nabla_0\gamma) = -dh \otimes dh \otimes 1$$

Let $Z_1 = (Z_{11}, Z_{12})$ and $Z_2 = (Z_{21}, Z_{22})$ be two vector fields on D , then by definition of χ :

$$\begin{aligned}\chi(Z_1 \otimes Z_2) &= dx \otimes dx(Z_1 \otimes Z_2) + dy \otimes dy(Z_1 \otimes Z_2) \\ &+ P_1(\nabla_0\psi)P_1(\nabla_0\psi)(Z_1 \otimes Z_2) \\ &+ \frac{9}{2}P_2(\nabla_0\psi) \otimes P_2(\nabla_0\psi)(Z_1 \otimes Z_2) \\ &- \kappa\xi(\gamma^\dagger \nabla_0\gamma) \otimes (\gamma^\dagger \nabla_0\gamma)(Z_1 \otimes Z_2)\end{aligned}$$

From what we have shown above, we have

$$\begin{aligned}\chi(Z_1 \otimes Z_2) &= dx \otimes dx(Z_1 \otimes Z_2) + dy \otimes dy(Z_1 \otimes Z_2) \\ &+ dl \otimes dl \otimes \lambda(Z_1 \otimes Z_2) + dt \otimes dt \otimes \eta(Z_1 \otimes Z_2) \\ &+ ds \otimes ds \otimes \lambda(Z_1 \otimes Z_2) + \kappa\xi dh \otimes dh \otimes 1(Z_1 \otimes Z_2)\end{aligned}$$

Hence,

$$\begin{aligned}\chi(Z_1 \otimes Z_2) &= Z_{11}Z_{21} + Z_{12}Z_{22} + dl(Z_1)dl(Z_2)\lambda \\ &+ dt(Z_1)dt(Z_2)\eta + ds(Z_1)ds(Z_2)\lambda + \kappa\xi dh(Z_1)dh(Z_2)\end{aligned}$$

Taking $Z_1 = Z_2 = X$, we recover the expression of E . Similarly, taking $Z_1 = Z_2 = Y$, we recover the expression of G , and from $Z_1 = X, Z_2 = Y$ or $Z_1 = Y, Z_2 = X$, we recover the expression of F . \square

3.4 Discretisation and parallel transport

Let us now explain how to compute the coefficients E , F , and G when dealing with the integer coordinates points of D . In what follows, we use the matricial coordinates system.

We denote

$$\gamma_{(1,0)}^{(i,j)}, \quad \gamma_{(0,1)}^{(i,j)}, \quad \gamma_{(1,1)}^{(i,j)}, \quad \gamma_{(1,-1)}^{(i,j)}$$

the classes of curves on D form the point (i, j) that satisfy:

$$\dot{\gamma}_{(1,0)}^{(i,j)}(t) = (1, 0), \quad \dot{\gamma}_{(1,0)}^{(i,j)}(t) = (0, 1),$$

$$\dot{\gamma}_{(1,1)}^{(i,j)}(t) = (1, 1), \quad \dot{\gamma}_{(1,-1)}^{(i,j)}(t) = (1, -1)$$

for all t . The corresponding parallel transports (with respect to ∇_0)

$$\tau_{(1,0)}^{(i,j)}(t, \cdot), \quad \tau_{(0,1)}^{(i,j)}(t, \cdot), \quad \tau_{(1,1)}^{(i,j)}(t, \cdot), \quad \tau_{(1,-1)}^{(i,j)}(t, \cdot)$$

are linear maps from $\pi^{-1}(i, j)$ to $\pi^{-1}(\gamma_{(1,0)}^{(i,j)}(t))$ resp. $\pi^{-1}(\gamma_{(0,1)}^{(i,j)}(t))$, $\pi^{-1}(\gamma_{(1,1)}^{(i,j)}(t))$ and $\pi^{-1}(\gamma_{(1,-1)}^{(i,j)}(t))$.

By definition of ∇_0 , if γ is one of the preceding classes of curves and w is a vector of $\pi^{-1}(\gamma(0))$, i.e

$$\begin{aligned}w &= w_1e_1(\gamma(0)) + w_2e_2(\gamma(0)) \\ &+ w_3e_3(\gamma(0)) + w_4e_4(\gamma(0))\end{aligned}$$

then the parallel transport of w at $\pi^{-1}(\gamma(t))$ is the vector

$$\begin{aligned}w_1e_1(\gamma(t)) + w_2e_2(\gamma(t)) \\ + w_3e_3(\gamma(t)) + w_4e_4(\gamma(t))\end{aligned}$$

Let us consider the vector

$$\begin{aligned}\tau_1(i, j) &= \frac{1}{8} \left\{ \tau_{(1,-1)}^{(i+1, j-1)}(-1, \psi) + 2\tau_{(1,0)}^{(i+1, j)}(-1, \psi) \right. \\ &+ \tau_{(1,1)}^{(i+1, j+1)}(-1, \psi) - \tau_{(1,1)}^{(i-1, j-1)}(1, \psi) \\ &\left. - 2\tau_{(1,0)}^{(i-1, j)}(1, \psi) - \tau_{(1,-1)}^{(i-1, j+1)}(1, \psi) \right\}\end{aligned}$$

and the scalar

$$\tau_2(i, j) = \frac{1}{8} \{ \arccos(a \cdot b) + 2 \arccos(c \cdot d) + \arccos(e \cdot f) \}$$

where

$$a = \frac{\tau_{(1,1)}^{(i-1, j-1)}(1, v)}{\|\tau_{(1,1)}^{(i-1, j-1)}(1, v)\|} \quad b = \frac{\tau_{(1,-1)}^{(i+1, j-1)}(-1, v)}{\|\tau_{(1,-1)}^{(i+1, j-1)}(-1, v)\|}$$

$$c = \frac{\tau_{(1,0)}^{(i-1,j)}(1,v)}{\|\tau_{(1,0)}^{(i-1,j)}(1,v)\|} \quad d = \frac{\tau_{(1,0)}^{(i+1,j)}(-1,v)}{\|\tau_{(1,0)}^{(i+1,j)}(-1,v)\|}$$

$$e = \frac{\tau_{(1,-1)}^{(i-1,j+1)}(1,v)}{\|\tau_{(1,-1)}^{(i-1,j+1)}(1,v)\|} \quad f = \frac{\tau_{(1,1)}^{(i+1,j+1)}(-1,v)}{\|\tau_{(1,1)}^{(i+1,j+1)}(-1,v)\|}$$

These are elements of $\pi^{-1}(i,j)$.

The vector $\tau_1(i,j)$ (resp. the scalar $\tau_2(i,j)$) is a discrete approximation of $\nabla_0\psi$ (resp. $dh \otimes 1$) in the direction given by the tangent vector of coordinates $(1,0)$ at (i,j) . We obtain thus a discrete version of the coefficient G of the fundamental form I, namely

$$G_d = 1 + (P_1(\tau_1))^2 + \frac{9}{2}(P_2(\tau_1))^2 + \kappa\xi\tau_2^2$$

In the same way, we get a discrete version of the coefficients E and F .

4 Applications

We propose three applications: we first compare our approach with the method developed by Di Zenzo, then we focus on detecting edges in color images with respect to a given hue interval, and finally we deal with edge detection in color-infrared images with constraints on temperature.

4.1 Comparison with the Di Zenzo gradient

Let I be a color image seen as a section $I(q) = r(q)e_1(q) + g(q)e_2(q) + b(q)e_3(q)$ of the Clifford bundle $\mathcal{CT}(D)$ with λ the constant function equal to 1. Di Zenzo gradient relates to the highest eigenvalue of the first fundamental form whose coefficients are given by:

$$E = 1 + (r_x)^2 + (g_x)^2 + (b_x)^2$$

$$F = r_x r_y + g_x g_y + b_x b_y$$

$$G = 1 + (r_y)^2 + (g_y)^2 + (b_y)^2$$

The image is then considered as a surface in the euclidean ambient space. In this context we have:

$$\nabla_0 I = dr \otimes e_1 + dg \otimes e_2 + db \otimes e_3$$

and the symmetric product of ∇_0 by itself (see appendix) is therefore:

$$\nabla_0 I \nabla_0 I = dr \otimes dr \otimes \frac{1}{3} + dg \otimes dg \otimes \frac{1}{3} + db \otimes db \otimes \frac{1}{3}$$

So considering the symmetric tensor of rank 2

$$\underline{\chi} = dx \otimes dx + dy \otimes dy + 3\nabla_0 I \nabla_0 I$$

we get

$$E = \underline{\chi}(X \otimes X) \quad F = \underline{\chi}(X \otimes Y) \quad G = \underline{\chi}(Y \otimes Y)$$

(see Proposition 2 for the definition of X and Y). Furthermore we can split I into

$$(I \cdot \mu)\mu^{-1} + (I \wedge \mu)\mu^{-1} = l(e_1 + e_2 + e_3) + v$$

Since

$$\nabla_0 I = dl \otimes (e_1 + e_2 + e_3) + \nabla_0 v$$

we see that the method of Di Zenzo deals with the derivative $\nabla_0 v$ of the chrominance vector, which is by definition of ∇_0 the usual derivative of the vectorial function v . Moreover, since $(e_1 + e_2 + e_3)$ and v are orthogonal, then

$$\nabla_0 I \nabla_0 I = dl \otimes dl \otimes 1 + \nabla_0 v \nabla_0 v$$

If we apply now Proposition 2, the corresponding coefficients are given by

$$E = 1 + (l_x)^2 + (s_x)^2 + \kappa\xi(h_x)^2$$

$$F = l_x l_y + s_x s_y + \kappa\xi h_x h_y$$

$$G = 1 + (l_y)^2 + (s_y)^2 + \kappa\xi(h_y)^2$$

and the variation of the chrominance is measured by **both** the variations of the saturation and of the hue.

If for example we consider the case when the hue is locally constant, i.e. $dh = 0$, then

$$\nabla_0 v = d(\|v\|) \otimes \frac{v}{\|v\|}$$

and

$$\nabla_0 v \nabla_0 v = d(\|v\|) \otimes d(\|v\|) \otimes 1 = ds \otimes ds \otimes \frac{2}{9}$$

The coefficients for the Di Zenzo gradient are

$$E = 1 + (l_x)^2 + \frac{2}{9}s_x^2$$

$$F = l_x l_y + \frac{2}{9}s_x s_y$$

$$G = 1 + (l_y)^2 + \frac{2}{9}(s_y)^2$$

while those we defined are

$$E = 1 + (l_x)^2 + (s_x)^2$$

$$F = l_x l_y + s_x s_y$$

$$G = 1 + (l_y)^2 + (s_y)^2$$

We can see in this very particular case that the two methods differ from the way the saturation is taken into account.



Fig. 1 Original image



Fig. 2 Edge detection in the hue interval Red-Yellow

4.2 Color edge detection with respect to hue intervals

Let h_1 and h_2 be two angles representing two hues. The aim of this application is to detect color edges in the hue interval that they define.

Let $I(q) = r(q)e_1(q) + g(q)e_2(q) + b(q)e_3(q)$ be a color image. Here λ is taken small enough so that the luminance and the saturation have no influence in the edge detection and high enough so that the norm of a vector in any fiber is numerically computable. Consequently only κ will influence the detection.

Let θ in $[0, 2\pi]$. The unit chrominance vector v_θ of hue θ is

$$v_\theta = \cos(\theta)\rho + \sin(\theta)\rho B$$

where B is the unit bivector coding the chrominance plane. The angular distance δ between two hues h_3 and h_4 may be computed using the corresponding unit chrominance vectors

$$\delta(h_3, h_4) = \arccos(v_{h_3} \cdot v_{h_4})$$

Then we define κ as follows: $\kappa(p) = 1$ if

$$\max_{x \in \Omega(p)} (\max(\delta(h(x), h_1), \delta(h(x), h_2))) \leq \delta(h_1, h_2)$$

and extend it into a derivable function on $[0, \pi]$ that is strictly decreasing on $[\delta(h_1, h_2), \pi]$.

In the following illustration of this application, $h_1 = 0$ is the red hue, $h_2 = \pi/3$ the yellow hue, and κ equals 0 for any value greater than $\pi/3$.

As you can see Fig. 2, we principally detect edges of Fig. 1 inside petals. Let us explain this result. First, due to our definition of κ , as soon as there is a pixel in the neighborhood of a point q in D whose hue is not in the interval red-yellow, color variations at q are not detected. Then, one may find inside petals variations between yellow and red hues, which implies that κdh is maximal. Moreover, these colors have almost a full saturation, so ξ almost equals 1. In the same way,

variations where colors have few saturation are not detected, since ξ decreases strongly when the surrounding saturation declines.

4.3 Color edge detection with respect to temperature constraints

We consider a color-infrared image

$$I(q) = r(q)e_1(q) + g(q)e_2(q) + b(q)e_3(q) + t(q)e_4(q)$$

that is a section of the Clifford bundle $\mathcal{CT}(D)$. For the application we have in mind, we choose η a strictly positive function and λ defined by the gaussian

$$\lambda(p) = e^{-\frac{(t(p)-t_0)^2}{\sigma}} \quad \text{if} \quad \frac{1}{4} \int_{\Omega(p)} t(q) dx dy \leq t_0$$

and equals 1 otherwise.

We apply Proposition 2 with ψ the section given by

$$\psi = S^\dagger I' S$$

where $I' = I \cdot (e_1 e_2 e_3)(e_1 e_2 e_3)^{-1}$. The resulting coefficients are

$$E = 1 + \lambda(l_x)^2 + \lambda(s_x)^2 + \kappa \xi (h_x)^2$$

$$F = \lambda l_x l_y + \lambda s_x s_y + \kappa \xi h_x h_y$$

$$G = 1 + \lambda(l_y)^2 + \lambda(s_y)^2 + \kappa \xi (h_y)^2$$

We choose $\kappa = 255\lambda/\pi$ so that the hue component has the same weight in the detection as the two others components.

Let us consider the following situation: a man is standing in front of a wall and is handing a hot cup of coffee (see Fig. 3). We want to detect color variations with the additional constraint that on such points, the temperature is similar or higher than the human temperature. Fig. 4 represents the result of such an edge detection.



Fig. 3 Color and temperature information of the scene

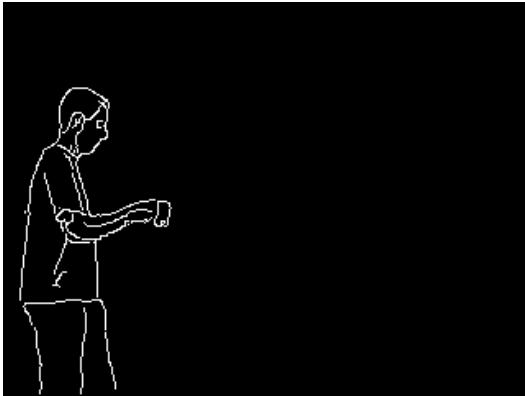


Fig. 4 Result of the edge detection

We see that the region delimited by the man and the cup of coffee is detected, as in the infrared image. Moreover, we detect color variations inside this domain, as wished.

5 Conclusion

In this paper, we have proposed a new framework for nD image processing, namely Clifford algebra bundles where the typical fiber is the Clifford algebra associated to the euclidean space of acquisition. A nD image is then considered as a section. The paper deals in particular with the reformulation, in this fiber bundle setting, of a well-known method for nD images edge detection, based on the computation of the first fundamental form of a surface embedded in a metric space. For this, we have introduced a connection on the fiber bundle, and use corresponding covariant derivative and parallel transport application on sections derived from the section defined by the image. We have treated the special case of color/infrared images where the space of acquisition is $RGBT$, and the metric space contain-

ing the surface is $LSHT$. We have recovered the first fundamental form by computations on $RGBT$, never computing explicitly derivatives of luminance, saturation, hue or temperature, whereas the approach based on surfaces does. It is therefore natural to envisage that this can be done whatever the metric space considered, whatever the dimension of the image. That's why we think, through the example developed by this paper, that (Clifford) fiber bundles framework is a more suitable global framework for nD image processing than surfaces, or manifolds, framework. This is motivated by the fact that fiber bundles theory generalises manifolds theory too. At last, let us remark that all the computations we have done were extremely simplified since we have considered a Clifford algebra bundle, and not only a vector bundle.

Appendix

We precise in this appendix how are defined the tensor product of $\mathcal{CT}(D)$ -valued 1-forms and the symmetric product used in particular in Proposition 2. As already said this tensor product is an analog of the wedge product of “clifforms”.

Let $(\mathcal{CT}(D), D, \pi)$ be the fiber bundle introduced in Sect. 3.2. We denote \mathcal{A} the ring of \mathbb{R} -valued functions of class C^k , $k \geq 1$, defined on D and \mathcal{B} the ring $\Gamma(\mathcal{CT}(D))$ of sections of $\mathcal{CT}(D)$.

Proposition 3 *The couple*

$$(\Gamma_1, \varphi) := (\Gamma(T^*D \otimes_{\mathcal{A}} T^*D \otimes_{\mathcal{A}} \mathcal{CT}(D)), \varphi)$$

is a solution of the universal problem defining the tensor product of the \mathcal{B} -bimodule

$$\Gamma_0 := \Gamma(T^*D \otimes_{\mathcal{A}} \mathcal{CT}(D))$$

with itself, where φ is the application from $\Gamma_0 \times \Gamma_0$ to Γ_1 defined by

$$((\omega_1 \otimes m_1), (\omega_2 \otimes m_2)) \mapsto (\omega_1 \otimes \omega_2) \otimes (m_1 m_2)$$

Proof Let ψ the ring homomorphism from \mathcal{A} to \mathcal{B} that makes \mathcal{B} a \mathcal{A} -algebra. Since $\psi(\mathcal{A})$ is in the center of \mathcal{B} there exists a \mathcal{B} -bimodule structure on Γ_0

$$\begin{aligned} \mathcal{B} \times \Gamma_0 &\longrightarrow \Gamma_0 & \Gamma_0 \times \mathcal{B} &\longrightarrow \Gamma_0 \\ (b, X \otimes c) &\longmapsto X \otimes bc & (X \otimes c, b) &\longmapsto X \otimes cb \end{aligned}$$

We consider the following universal problem:

we search a couple

$$(\Gamma_0 \otimes_{\mathcal{B}} \Gamma_0, \phi)$$

where $\Gamma_0 \otimes_{\mathcal{B}} \Gamma_0$ is a \mathcal{B} -bimodule and

$$\phi : \Gamma_0 \times \Gamma_0 \longrightarrow \Gamma_0 \otimes_{\mathcal{B}} \Gamma_0$$

is left \mathcal{B} -linear in the first variable and right \mathcal{B} -linear in the second variable with

$$\phi(xf, y) = \phi(x, fy)$$

for all x and y in Γ_0 and f in \mathcal{B} such that:

for all \mathcal{B} -bimodule N and all map

$$\eta : \Gamma_0 \times \Gamma_0 \longrightarrow N$$

which is left \mathcal{B} -linear in the first variable and right \mathcal{B} -linear in the second variable and satisfies

$$\eta(xf, y) = \eta(x, fy)$$

for all x and y in Γ_0 and f in \mathcal{B} , there exists a unique homomorphism

$$\gamma : \Gamma_0 \otimes_{\mathcal{B}} \Gamma_0 \longrightarrow N$$

of \mathcal{B} -bimodule such that:

$$\eta = \gamma \circ \phi$$

The solution is unique up to isomorphisms. A construction may be found in [1].

As above we can show that Γ_1 has a \mathcal{B} -bimodule structure. It is a fact that φ is bilinear with respect to the left-module structure in the first variable and the right-module structure in the second variable and satisfies

$$\varphi(xf, y) = \varphi(x, fy)$$

for all x and y in Γ_0 and f in \mathcal{B} . So there exists a unique \mathcal{B} -bimodule homomorphism γ from $\Gamma_0 \otimes_{\mathcal{B}} \Gamma_0$ to Γ_1 such that

$$\varphi = \gamma \circ \phi$$

Now γ is defined by

$$\gamma : (\omega_1 \otimes m_1) \otimes (\omega_2 \otimes m_2) \mapsto (\omega_1 \otimes \omega_2) \otimes m_1 m_2$$

The map δ from Γ_1 to $\Gamma_0 \otimes_{\mathcal{B}} \Gamma_0$ that sends $(\omega_1 \otimes \omega_2 \otimes m)$ to $(\omega_1 \otimes m) \otimes (\omega_2 \otimes 1)$ is a \mathcal{B} -bimodule homomorphism and is the inverse of γ .

Finally (Γ_1, φ) is a solution to our universal problem. \square

From the preceding proposition, $\Gamma_0 \otimes_{\mathcal{B}} \Gamma_0$ is isomorphic to the space of $\mathcal{CT}(D)$ -valued rank 2 tensors. If η_1 and η_2 belong to $\Gamma(T^*D)$ and s_1 and s_2 belong to \mathcal{B} then $(\eta_1 \otimes s_1) \otimes (\eta_2 \otimes s_2)$ may be identified with the $\mathcal{CT}(D)$ -valued rank 2 tensor that maps $(X \otimes Y)$ to $\eta_1(X)\eta_2(Y)s_1s_2$.

We denote

$$(\eta_1 \otimes s_1)(\eta_2 \otimes s_2) = (\eta_1\eta_2) \otimes (s_1s_2)$$

the symmetric product of $(\eta_1 \otimes s_1)$ and $(\eta_2 \otimes s_2)$. We extend it by linearity. This symmetric product can be identified with the $\mathcal{CT}(D)$ -valued symmetric tensor of rank 2 defined by

$$(X \otimes Y) \mapsto \frac{1}{2}(\eta_1(X)\eta_2(Y) + \eta_2(X)\eta_1(Y))s_1s_2$$

Finally we denote $(\eta_1 \otimes s_1) \otimes_{\mathcal{B}} (\eta_2 \otimes s_2)$ the element $(\eta_1 \otimes s_1) \otimes (\eta_2 \otimes s_2)$ of $\Gamma_0 \otimes_{\mathcal{B}} \Gamma_0$ to emphasize the fact that the tensor product is relative to \mathcal{B} .

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References

1. Bourbaki, N.: *Eléments de mathématiques. Algèbre. Chapitres 1 à 3.* Diffusion C.C.L.S Paris (1970)
2. Carron, T.: *Segmentation d'images couleur dans la base teinte-luminance-saturation: approche numérique et symbolique.* Thèse de doctorat, Université de Savoie (1995)
3. Cumani, A.: Edge detection in multispectral images. *Computer Vision, Graphics, and Image Processing: Graphical Models and Image Processing* **53**(1), 40–51 (1991)
4. DiZeno, S.: A note on the gradient of a multi-image. *Computer Vision, Graphics, and Image Processing* **33**(1), 116–125 (1986)
5. Evans, C., Ell, T., Sangwine, S.: Colour-sensitive edge detection using hypercomplex filters. In: *Proceedings EUSIPCO 2000*, pp. 107–110 (2000)
6. Fairchild, M.: *Color Appearance Models.* John Wiley and Sons, New York (2005)
7. Greub, W., Halperin, S., Vanstone, R.: *Connections, Curvature and Cohomology*, vol. I-III. Academic Press, New York (1972, 1973 and 1976)
8. Helgason, S.: *Differential geometry, Lie groups, and symmetric spaces.* Academic Press, London (1978)
9. Hestenes, D., Sobczyk, G.: *Clifford algebra to geometric calculus.* D. Reidel, Dordrecht (1984)
10. Lenz, R.: Color edge detectors for conical color spaces. In: *Proceedings of CGIP 2000. First International Conference on Color in Graphics and Image Processing*, pp. 284–289 (2000)
11. Postnikov, M.: *Leçons de Géométrie. Groupes et algèbres de Lie.* Editions Mir, Moscou (1982)
12. Sangwine, S., Ell, T.: Colour image filters based on hypercomplex convolution. *IEEE Proceedings-Vision, Image and Signal Processing* **147**(2), 89–93 (2000)

13. Sapiro, G.: Color snakes. *Computer Vision and Image understanding* **68**(2), 407–416 (1997)
14. Sochen, N., Kimmel, R., Malladi, R.: A general framework for low level vision. *IEEE Transactions on Image Processing* **7**, 310–318 (1998)
15. Sommer, G.: *Geometric computing with Clifford Algebras. Theoretical Foundations and Applications in Computer Vision and Robotics*. Springer-Verlag, Berlin (2001)