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Scalar Conservation Laws with Discontinuous Flux in Several Space Dimensions

Julien JIMENEZ

Université de Pau et des Pays de l'Adour
 Laboratoire de Mathématiques Appliquées - UMR 5142 CNRS
 BP 1155 - 64013 - PAU Cedex - FRANCE

Abstract

We deal with a scalar conservation law, set in a bounded multidimensional domain, and such that the convective term is discontinuous with respect to the space variable. We introduce a weak entropy formulation for the homogeneous Dirichlet problem associated with the first-order reaction-diffusion equation that we consider. We establish an existence and uniqueness property for the weak entropy solution. The method of doubling variables is used to state uniqueness while the vanishing viscosity method allows us to prove the existence result.

1 Introduction

We are interested in the existence and uniqueness properties for an hyperbolic first-order quasilinear equation set in a multidimensional bounded domain of \mathbb{R}^n , denoted by Ω . For any positive finite real T , the problem can be formally written as stated below:

Find a measurable and bounded function u on $Q =]0, T[\times \Omega$ such that:

$$\begin{cases} \partial_t u + \operatorname{div}_x(b(x)\mathbf{f}(u)) + g(t, x, u) &= 0 & \text{in } Q, \\ u &= 0 & \text{on (a part of) } \Sigma, \\ u(0, x) &= u_0(x) & \text{on } \Omega, \end{cases} \quad (1)$$

where b is a discontinuous function along an hypersurface.

Indeed we suppose that there exists two open disjoint domains Ω_L and Ω_R such that:

$$\overline{\Omega} = \overline{\Omega_R} \cup \overline{\Omega_L} \quad \text{and} \quad \overline{\Omega_L} \cap \overline{\Omega_R} = \Gamma_{L,R}.$$

We suppose that Ω_L and Ω_R admit a “regular deformable Lipschitz boundary” (see [25] or [10] Definition 2.1 for a rigorous definition). That will allow us to define a “strong” trace of a solution to (1).

Moreover, for $i = L, R$, $\mathcal{H}^{n-1}(\overline{\Gamma_{L,R}} \cap (\overline{\Gamma_i} \setminus \Gamma_{L,R})) = 0$, where \mathcal{H}^q is the q -dimensional Hausdorff measure in \mathbb{R}^n .

We set $\Sigma =]0, T[\times \Omega$, for $i = L, R$, $\Sigma_i =]0, T[\times \Omega_i$ and $\Sigma_{L,R} =]0, T[\times \Omega_{L,R}$.

The function b is such that:

$$b(x) = \begin{cases} b_L(x) & \text{if } x \in \Omega_L \\ b_R(x) & \text{if } x \in \Omega_R, \end{cases}$$

with $b_L \in W^{1,+\infty}(\Omega_L)$ and $b_R \in W^{1,+\infty}(\Omega_R)$.

The initial datum u_0 belongs to $L^\infty(\Omega)$ and takes values in $[m, M]$ where m and M are two fixed reals.

The vector function $\mathbf{f} = (f_1, \dots, f_n)$ belongs to $(C^1(\mathbb{R}))^n$. For $i = 1, \dots, n$, f_i is Lipschitzian on \mathbb{R} . We denote M_{f_i} its Lipschitz constant and we set $M_{\mathbf{f}} = \max_{i=1, \dots, n} M_{f_i}$.

The source term g is in $C^0([0, T] \times \bar{\Omega} \times \mathbb{R})$ such that

$$\exists M_g \in \mathbb{R}, \forall (t, x) \in Q, \forall (u, v) \in \mathbb{R}^2, |g(t, x, u) - g(t, x, v)| \leq M_g |u - v|.$$

We introduce a nondecreasing function M_1 and a nonincreasing function M_2 such that:

$$\begin{cases} M_1(0) \geq M, \\ \forall t \in [0, T], \\ M_1'(t) + g(t, x, M_1(t)) + \nabla_x b(x) \cdot \mathbf{f}(M_1(t)) \geq 0 \text{ a.e. on } \Omega_L \cup \Omega_R, \end{cases}$$

and

$$\begin{cases} M_2(0) \leq m, \\ \forall t \in [0, T], \\ M_2'(t) + g(t, x, M_2(t)) + \nabla_x b(x) \cdot \mathbf{f}(M_2(t)) \leq 0 \text{ a.e. on } \Omega_L \cup \Omega_R. \end{cases}$$

Generally we may choose:

$$M_1 : t \in [0, T] \longrightarrow M_1(t) = \operatorname{ess\,sup}_{\Omega} u_0^+ e^{N_1 t} + \frac{N_2}{N_1} (e^{N_1 t} - 1),$$

and

$$M_2 : t \in [0, T] \longrightarrow M_2(t) = \operatorname{ess\,inf}_{\Omega} (-u_0^-) e^{N_1 t} - \frac{N_2}{N_1} (e^{N_1 t} - 1),$$

with: $N_1 = \max(\|\nabla_x b\|_{L^\infty(\Omega_L)}, \|\nabla_x b(x)\|_{L^\infty(\Omega_R)}) \sum_{i=1}^n M_{f_i} + M_g$, and $N_2 =$

$$\sum_{i=L,R} \max_{[0, T] \times \bar{\Omega}} |g(t, x, 0) + \nabla_x b_i(x) \cdot \mathbf{f}(0)|.$$

Depending on the properties of the functions f and g , better choices of M_1 and M_2 can be done. For instance if $f(m) = f(M) = 0$ and $g(\cdot, M) \geq 0$, $g(\cdot, m) \leq 0$, we will choose $M_1 = M$ and $M_2 = m$.

We suppose also that the flux function \mathbf{f} is non-degenerate that is to say, for a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n, \xi \neq 0$, the function:

$$\begin{aligned} \lambda &\longmapsto \xi \cdot b(x) \mathbf{f}(\lambda) \text{ is not linear} \\ &\text{on any non degenerate interval included in } [M_2(T), M_1(T)], \end{aligned} \quad (2)$$

where \cdot denotes the scalar product in \mathbb{R}^n .

Scalar conservation laws with discontinuous flux have seen a great deal of interest (see for example [1], [2], [3] [5], [6], [8], [9], [12], [13], [15], [16], [23], [24] the list is far from being complete) but all these works treat the case of a one dimensional domain Ω . To our knowledge only two works consider the multidimensional case. In [14], the authors consider the particular case of a two dimensional domain and use the compensated compactness method to prove the existence of a weak solution. Note that this method cannot be generalized

to domains whose the dimension is greater than two. In [22], E. Yu. Panov, thanks the framework of the ‘‘H-measures’’ obtains an existence result for the Cauchy problem in $\mathbb{R}_+ \times \mathbb{R}^n$. In section 2, we give the definition of weak entropy solution u to (1) and we state the existence of ‘‘strong’’ traces for u . In section 3 we prove the uniqueness property thanks to the method of doubling variables and a pointwise reasoning along the interface $\Sigma_{L,R}$. In section 4, the existence result is established via the vanishing viscosity method. Finally in section 5, we prove, in some particular cases, that the method often used in one space dimension namely the regularization of the coefficient b , leads to the existence of a weak entropy solution to (1).

2 Notion of weak entropy solution

In this section we propose a definition extending that of J.D Towers in [24] - used in [13] for the homogeneous Dirichlet problem - to the multidimensional case. We say that:

Definition 1. *A function u of $L^\infty(Q)$ is a weak entropy solution to Problem (1) if:*

(i) $\forall \varphi \in \mathcal{C}_c^\infty(Q)$, $\varphi \geq 0$, $\forall k \in \mathbb{R}$,

$$\begin{aligned} & \int_Q \{ |u - k| \partial_t \varphi + b(x) \Phi(u, k) \cdot \nabla_x \varphi \} dx dt \\ & - \int_Q \text{sgn}(u - k) (g(t, x, u) + \nabla_x b(x) \cdot \mathbf{f}(k)) \varphi dx dt \\ & + \int_{\Sigma_{L,R}} |(b_R(\bar{\sigma}) - b_L(\bar{\sigma})) \mathbf{f}(k) \cdot \boldsymbol{\nu}_L(\bar{\sigma})| \varphi(\sigma) dt d\mathcal{H}^{n-1} \geq 0, \end{aligned} \quad (3)$$

where

$$\Phi(u, k) = (\Phi_1(u, k), \dots, \Phi_n(u, k)), \quad \Phi_i(u, k) = \text{sgn}(u - k) (f_i(u) - f_i(k))$$

(ii) u is a weak solution to (1):

$$\partial_t u + \text{div}_x (b(x) \mathbf{f}(u)) + g(t, x, u) = 0 \text{ in } \mathcal{D}'(Q). \quad (4)$$

(iii)

$$\text{ess lim}_{t \rightarrow 0^+} \int_\Omega |u(t, x) - u_0(x)| dx = 0, \quad (5)$$

(iv) for a.e. $t \in]0, T[$, \mathcal{H}^{n-1} -a.e., $\forall k \in \mathbb{R}$,

$$(\text{sgn}(u^\tau(\sigma) - k) + \text{sgn}(k)) b(\bar{\sigma}) (\mathbf{f}(u^\tau) - \mathbf{f}(k)) \cdot \boldsymbol{\nu} \geq 0 \quad (6)$$

where

$$u^\tau = \begin{cases} u_L^\tau & \text{on } \Sigma_L \cap \Sigma \\ u_R^\tau & \text{on } \Sigma_R \cap \Sigma. \end{cases}$$

In this definition u_L^τ and u_R^τ denote the traces of u respectively on Σ_L and Σ_R . Indeed it follows from [25] or [21]:

Lemma 1. Let u be a function in $L^\infty(Q)$ satisfying (3). Then under (2) there exists a function u_L^τ of $L^\infty(\Sigma_L)$ (resp. u_R^τ of $L^\infty(\Sigma_R)$), such that, for every compact K of Σ_L (resp. Σ_R) and every regular Lipschitz deformation Ψ of Ω_L (resp. Ω_R),

$$\operatorname{ess\,lim}_{s \rightarrow 0^+} \int_K |u(\Psi(s, \sigma)) - u_L^\tau(\sigma)| dt d\mathcal{H}^{n-1} = 0. \quad (7)$$

Lemma 2. Let $(\omega_\varepsilon)_{\varepsilon > 0}$ a sequence of functions such that, for every ε , $\omega_\varepsilon \in C_c^\infty(\bar{\Omega})$ and:

$$\begin{cases} 0 \leq \omega_\varepsilon \leq 1 \text{ on } \Omega, \\ \omega_\varepsilon(x) = 1 \text{ if } x \in \Gamma_{L,R}, \\ \omega_\varepsilon(x) = 0 \text{ if } d(x, \Gamma_{L,R}) > \varepsilon, \\ (\varepsilon \nabla_x \omega_\varepsilon)_\varepsilon \text{ is bounded on } \Omega. \end{cases} \quad (8)$$

Then, for $i = L, R$, for every φ in $C_c^\infty(Q)$,

$$\lim_{\varepsilon \rightarrow 0^+} \int_{Q_i} b(x) \Phi(u, k) \cdot \nabla_x \omega_\varepsilon \varphi dx dt = \int_{\Sigma_{L,R}} b_i \Phi(u_i^\tau, k) \cdot \nu_i \varphi dt d\mathcal{H}^{n-1}.$$

Proof. We prove the lemma when Q_i is the half-space i.e.:

$$\begin{aligned} \Omega_i &= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n < 0\}, \\ \nu_i &= (0, \dots, 0, 1) \in \mathbb{R}^n, \\ \Sigma_{L,R} &=]0, T[\times \mathbb{R}^{n-1} \times \{0\} \equiv]0, T[\times \mathbb{R}^{n-1}, \quad r = (t, x) \in \Sigma_i, \\ Q_i &= \{p = (r, x_n); r \in \Sigma_i, x_n < 0\}. \end{aligned}$$

To come back to the general case we can use a recovery argument.

In the case of the half-space, we can define a lipschitzian deformation ψ by:

$$\begin{aligned} \psi : [0, 1] \times \partial Q_i &\rightarrow \bar{Q}_i \\ (s, r) &\rightarrow r - s \cdot \nu_i, \end{aligned}$$

that implies:

$$\operatorname{ess\,lim}_{x_n \rightarrow 0^-} \int_{\Sigma_{L,R}} |u(r, x_n) - u_i^\tau(r)| dr = 0. \quad (9)$$

We also suppose (that is not restrictive) that:

$$\nabla \omega_\varepsilon(x) = (0, \dots, \partial_{x_n} \omega_\varepsilon(x)) \quad \text{and} \quad \varepsilon \|\partial_{x_n} \omega_\varepsilon\|_\infty \text{ is bounded.}$$

So we have to show that $\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon = 0$, where:

$$I_\varepsilon = \int_{\Sigma_{L,R}} \frac{1}{\varepsilon} \int_{-\varepsilon}^0 |b_i(x) \Phi_n(u, k) \varphi(r, x_n) - b_i(x') \Phi_n(u_i^\tau, k) \varphi(r, 0)| dx_n dr.$$

Since Φ_n is lipschitzian, we use the properties of b_i and φ , and equality (9) to conclude. \square

3 The uniqueness property

The proof relies on that proposed in [13] for the one dimensional case. First we focus on the transmission conditions along the interface $\Sigma_{L,R}$.

3.1 Interface conditions

We first look for a ‘‘Rankine-Hugoniot’’ condition along the interface of discontinuity. We consider a weak entropy solution u , in the sense of Definition 1. Since u is a weak solution ((4) is fulfilled) it follows:

Lemma 3. *Let u be an entropy solution to (1). Then, for a.e. σ in $\Sigma_{L,R}$,*

$$b_L \mathbf{f}(u_L^\tau) \cdot \boldsymbol{\nu}_L = b_R \mathbf{f}(u_R^\tau) \cdot \boldsymbol{\nu}_L. \quad (10)$$

Besides we can deduce from (3) an entropy inequality along $\Gamma_{L,R}$:

Lemma 4. *Let u be a weak entropy solution to (1). Then, for a.e. σ in $\Sigma_{L,R}$, for all real k ,*

$$\{b_L \Phi(u_L^\tau, k) - b_R \Phi(u_R^\tau, k)\} \cdot \boldsymbol{\nu}_L + |(b_L - b_R) \mathbf{f}(k) \cdot \boldsymbol{\nu}_L| \geq 0. \quad (11)$$

Proof. We consider a sequence of functions $(\omega_\varepsilon)_{\varepsilon>0}$ in $\mathcal{C}_c^\infty(\bar{\Omega})$ that fulfills the condition (8) of Lemma 2. For every positive ε , we choose in (3) the test function $\varphi\omega_\varepsilon$, $\varphi \in \mathcal{C}_c^\infty(Q)$, $\varphi \geq 0$. For any real k we obtain:

$$\begin{aligned} & \int_Q \{|u - k| \partial_t \varphi \omega_\varepsilon + b(x) \Phi(u, k) \cdot \nabla_x (\varphi \omega_\varepsilon)\} dx dt \\ & - \int_Q \text{sgn}(u - k) (g(t, x, u) + \nabla_x b(x) \cdot \mathbf{f}(k)) \varphi \omega_\varepsilon dx dt \\ & + \int_{\Sigma_{L,R}} |(b_R(\bar{\sigma}) - b_L(\bar{\sigma})) \mathbf{f}(k) \cdot \boldsymbol{\nu}_L(\bar{\sigma})| \varphi(\sigma) dt d\mathcal{H}^{n-1} \geq 0. \end{aligned}$$

We take now the ε -limit. Lemma 2 allows us to assert:

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \int_Q b(x) \Phi(u, k) \cdot \nabla_x (\varphi \omega_\varepsilon) dx dt \\ & = \int_{\Sigma_{L,R}} (b_L \Phi(u_L^\tau, k) \cdot \boldsymbol{\nu}_L + b_R \Phi(u_R^\tau, k) \cdot \boldsymbol{\nu}_R) \varphi(\sigma) dt d\mathcal{H}^{n-1}. \end{aligned}$$

Thanks to the dominated convergence Theorem the other terms go to 0 with ε (except the last one that do not depend on ε). The conclusion follows. \square

Remark 1. *The Rankine-Hugoniot condition (10) is included in (11) as soon as b_R , b_L and \mathbf{f} are such that:*

$$\left\{ \begin{array}{l} \exists k_1, \exists k_2 \in \mathbb{R}, k_2 \leq k_1 \text{ such that } u \in [k_2, k_1], \text{ and, for a.e. } \bar{\sigma} \text{ of } \Gamma_{L,R} \\ (b_R(\bar{\sigma}) - b_L(\bar{\sigma})) \mathbf{f}(k_1) \cdot \boldsymbol{\nu}_L(\bar{\sigma}) \geq 0, \\ (b_R(\bar{\sigma}) - b_L(\bar{\sigma})) \mathbf{f}(k_2) \cdot \boldsymbol{\nu}_L(\bar{\sigma}) \leq 0. \end{array} \right.$$

3.2 The uniqueness result

We are now able to give a uniqueness property through a Lipschitzian dependence in $L^1(\Omega)$ of a weak entropy solution with respect to corresponding initial data. To do so we suppose that for a.e. $\bar{\sigma}$ of $\Gamma_{L,R}$ the function:

$$\lambda \longmapsto \mathbf{f}(\lambda) \cdot \boldsymbol{\nu}_L(\bar{\sigma}) \quad (12)$$

changes no more than once its monotonicity on $[M_2(T), M_1(T)]$.

Theorem 1. *Let u and v be two weak entropy solutions to (1) that take values in $[M_2(T), M_1(T)]$, associated with initial conditions u_0 and v_0 in $L^\infty(\Omega)$ with values in $[m, M]$. Then, under (12), for a.e. t in $]0, T[$,*

$$\int_{\Omega} |u(t, x) - v(t, x)| dx \leq e^{M_g t} \int_{\Omega} |u_0(x) - v_0(x)| dx. \quad (13)$$

The proof of Theorem 1 is divided into several steps. First, we use the method of doubling variables, due to S. Kruzkov (see [17]) to show:

Lemma 5. *For any function φ in $C_c^\infty(Q)$ vanishing in a neighborhood of $\Sigma_{L,R}$, $\varphi \geq 0$,*

$$\int_Q \{|u - v| \partial_t \varphi + b(x) \Phi(u, v) \cdot \nabla_x \varphi - G(u, v) \varphi\} dx dt \geq 0, \quad (14)$$

where

$$G(u, v) = \text{sgn}(u - v)(g(t, x, u) - g(t, x, v)).$$

Proof. This result is proved in [13] for the one-dimensional case. The multidimensional one does not bring specific difficulties. \square

Our aim is now to obtain inequality (14) for any nonnegative function φ in $C_c^\infty(Q)$. So, for any positive real ε , we consider in (14) the test function $\varphi(1 - \omega_\varepsilon)$, such that ω_ε satisfies the assumptions of Lemma 2. We take the limit on ε . For $i = L, R$, Lemma 2 provides:

$$\lim_{\varepsilon \rightarrow 0^+} \int_{Q_i} b_i \Phi(u, v) \cdot \nabla_x (1 - \omega_\varepsilon) \varphi dx dt = - \int_{\Sigma_{L,R}} b_i \Phi(u_i^\tau, v_i^\tau) \cdot \nu_i \varphi dt d\mathcal{H}^{n-1},$$

and we can deduce that:

$$\begin{aligned} & \int_Q \{|u - v| \varphi_t + b(x) \Phi(u, v) \cdot \nabla_x \varphi - G(u, v) \varphi\} dx dt \\ & \geq \int_{\Sigma_{L,R}} \{b_L \Phi(u_L^\tau, v_L^\tau) - b_R \Phi(u_R^\tau, v_R^\tau)\} \cdot \nu_L \varphi dt d\mathcal{H}^{n-1}. \end{aligned}$$

Next we show the term in the right-hand side is nonnegative. Indeed we study, for a.e. σ , the sign of:

$$J = \{b_L(\bar{\sigma}) \Phi(u_L^\tau(\sigma), v_L^\tau(\sigma)) - b_R(\bar{\sigma}) \Phi(u_R^\tau(\sigma), v_R^\tau(\sigma))\} \cdot \nu_L.$$

Here we make a pointwise reasoning and we have to consider different cases.

If $\text{sgn}(u_L^\tau - v_L^\tau) = \text{sgn}(u_R^\tau - v_R^\tau)$,

$$J = \text{sgn}(u_L^\tau - v_L^\tau) \{b_L(\mathbf{f}(u_L^\tau) - \mathbf{f}(v_L^\tau)) - b_R(\mathbf{f}(u_R^\tau) - \mathbf{f}(v_R^\tau))\} \cdot \nu_L = 0,$$

by using (10).

If $\text{sgn}(u_L^\tau - v_L^\tau) = -\text{sgn}(u_R^\tau - v_R^\tau)$, we use (10) to have:

$$J = 2\text{sgn}(u_L^\tau - v_L^\tau) b_L(\mathbf{f}(u_L^\tau) - \mathbf{f}(v_L^\tau)) \cdot \nu_L = 2\text{sgn}(u_L^\tau - v_L^\tau) b_R(\mathbf{f}(u_R^\tau) - \mathbf{f}(v_R^\tau)) \cdot \nu_L.$$

We suppose that $\text{sgn}(u_L^\tau - v_L^\tau) = -1$ (i.e. $u_L^\tau < v_L^\tau$), $\text{sgn}(u_R^\tau - v_R^\tau) = 1$ (i.e. $u_R^\tau > v_R^\tau$), and $b_R < b_L$, the study of the other cases being similar.

- $u_L^\tau < v_L^\tau < v_R^\tau < u_R^\tau$

From Lemma 4 we deduce, for any k in $[u_L^\tau, u_R^\tau]$,

$$\{-b_L(\mathbf{f}(u_L^\tau) - \mathbf{f}(k)) - b_R(\mathbf{f}(u_R^\tau) - \mathbf{f}(k))\} \cdot \boldsymbol{\nu}_L + (b_L - b_R)|\mathbf{f}(k) \cdot \boldsymbol{\nu}_L(\bar{\sigma})| \geq 0. \quad (15)$$

If $\mathbf{f}(v_L^\tau) \cdot \boldsymbol{\nu}_L \geq 0$, we choose $k = v_L^\tau$ in (15) to obtain:

$$\{-b_L(\mathbf{f}(u_L^\tau) - \mathbf{f}(v_L^\tau)) - b_R\mathbf{f}(u_R^\tau) + b_L\mathbf{f}(v_L^\tau)\} \cdot \boldsymbol{\nu}_L \geq 0.$$

We refer to (10) to ensure that;

$$-2b_L(\mathbf{f}(u_L^\tau) - \mathbf{f}(v_L^\tau)) \cdot \boldsymbol{\nu}_L \geq 0, \text{ so } J \geq 0.$$

If $\mathbf{f}(v_R^\tau) \cdot \boldsymbol{\nu}_L \leq 0$, we choose $k = v_R^\tau$ in (15) and we have:

$$-2b_R(\mathbf{f}(u_R^\tau) - \mathbf{f}(v_R^\tau)) \cdot \boldsymbol{\nu}_L \geq 0 \text{ and thus } J \geq 0.$$

Finally, if $\mathbf{f}(v_L^\tau) \cdot \boldsymbol{\nu}_L < 0$ and $\mathbf{f}(v_R^\tau) \cdot \boldsymbol{\nu}_L > 0$, since $b_L > b_R$, by (10) we deduce that $b_L > 0$ and $b_R < 0$. If we suppose that $J < 0$, $J = -2b_L(\mathbf{f}(u_L^\tau) - \mathbf{f}(v_L^\tau)) \cdot \boldsymbol{\nu}_L$ implies $(\mathbf{f}(u_L^\tau) - \mathbf{f}(v_L^\tau)) \cdot \boldsymbol{\nu}_L > 0$. Similarly $J = -2b_R(\mathbf{f}(u_R^\tau) - \mathbf{f}(v_R^\tau)) \cdot \boldsymbol{\nu}_L$ implies $(\mathbf{f}(u_R^\tau) - \mathbf{f}(v_R^\tau)) \cdot \boldsymbol{\nu}_L < 0$. To sum up, we have:

$$\begin{aligned} \mathbf{f}(v_L^\tau) \cdot \boldsymbol{\nu}_L &< \mathbf{f}(v_R^\tau) \cdot \boldsymbol{\nu}_L, \mathbf{f}(u_L^\tau) \cdot \boldsymbol{\nu}_L > \mathbf{f}(v_L^\tau) \cdot \boldsymbol{\nu}_L \text{ et} \\ \mathbf{f}(v_R^\tau) \cdot \boldsymbol{\nu}_L &> \mathbf{f}(u_R^\tau) \cdot \boldsymbol{\nu}_L. \end{aligned}$$

Thus the function $\lambda \mapsto \mathbf{f}(\lambda) \cdot \boldsymbol{\nu}_L$ changes at least twice its monotonicity in $[M_2(T), M_1(T)]$ that contradicts the assumption (12).

- $u_L^\tau < v_R^\tau < v_L^\tau < u_R^\tau$

If $\mathbf{f}(v_L^\tau) \cdot \boldsymbol{\nu}_L \geq 0$ or $\mathbf{f}(v_R^\tau) \cdot \boldsymbol{\nu}_L \leq 0$ we can adapt the method used in the two first cases of the previous situation.

If $\mathbf{f}(v_L^\tau) \cdot \boldsymbol{\nu}_L < 0$ and $\mathbf{f}(v_R^\tau) \cdot \boldsymbol{\nu}_L > 0$, there exists α in $]v_R^\tau, v_L^\tau[$, such that $\mathbf{f}(\alpha) \cdot \boldsymbol{\nu}_L = 0$.

By choosing $k = \alpha$ in (15), we obtain:

$$-b_L\mathbf{f}(u_L^\tau) \cdot \boldsymbol{\nu}_L - b_R\mathbf{f}(u_R^\tau) \cdot \boldsymbol{\nu}_L \geq 0.$$

So, by (10),

$$-2b_L\mathbf{f}(u_L^\tau) \cdot \boldsymbol{\nu}_L \geq 0.$$

Likewise, we choose $k = \alpha$ in the inequality of Lemma 4 written for v et we use (10) to ensure:

$$2b_L\mathbf{f}(v_L^\tau) \cdot \boldsymbol{\nu}_L \geq 0.$$

We add up these two inequalities to have: $J \geq 0$.

All the others cases may be reduced to one of the previous situations. Then (14) still hold for any $\varphi \in \mathcal{C}_c^\infty(Q)$, $\varphi \geq 0$. Now we introduce the sequence of functions $(\beta_\varepsilon)_{\varepsilon>0}$ such that $\beta_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ and $\beta_\varepsilon = 1$ if $d(x, \partial\Omega) \geq \varepsilon$. We choose in (14) the sequence of test functions $(\alpha\beta_\varepsilon)_{\varepsilon>0}$, where $\alpha \in \mathcal{C}_c^\infty(]0, T[)$.

To take the limit on ε in the convective term, we adapt the proof of Lemma 2 (by taking $\omega_\varepsilon = 1 - \beta_\varepsilon$) to state:

$$\lim_{\varepsilon \rightarrow 0^+} \int_Q b(x) \Phi(u, v) \cdot \nabla_x \beta_\varepsilon \alpha(t) dx dt = - \int_\Sigma b(\bar{\sigma}) \Phi(u^\tau, v^\tau) \cdot \nu \alpha(t) dt d\mathcal{H}^{n-1}.$$

Then, passing to the limit with ε in (14) yields to:

$$\int_Q \{|u - v| \alpha'(t) - G(u, v) \alpha(t)\} dx dt - \int_\Sigma b(\bar{\sigma}) \Phi(u^\tau, v^\tau) \cdot \nu \alpha(t) dt d\mathcal{H}^{n-1} \geq 0.$$

Boundary condition (6) ensure, by reasoning for almost every point of Σ that:

$$\int_\Sigma b(\bar{\sigma}) \Phi(u^\tau, v^\tau) \cdot \nu \alpha(t) dt d\mathcal{H}^{n-1} \geq 0.$$

Lastly the Lipschitz condition on g provides:

$$- \int_Q \{|u - v| \alpha'(t)\} dx dt \leq M_g \int_Q |u - v| \alpha(t) dx dt.$$

For almost every t of $]0, T[$, we consider a sequence of functions $(\alpha_\varepsilon)_{\varepsilon > 0} \in \mathcal{C}_c^\infty([0, T])$ approximating the characteristic function $\mathbb{I}_{[0, t]}$. We use the initial condition (5) for u and v et we obtain (13) thanks to Gronwall's Lemma.

4 Existence

In order to state an existence result we use the vanishing viscosity method. To this purpose we consider the functional space $W(0, T)$ defined by:

$$W(0, T) = \{v \in L^2(0, T; H_0^1(\Omega)); \partial_t v \in L^2(0, T; H^{-1}(\Omega))\}.$$

Moreover we denote by $\langle \cdot, \cdot \rangle$ the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Then we introduce, for any positive real μ , the viscous problem related to (1),

Find a bounded and measurable function on Q , u_μ , such that:

$$\begin{cases} \partial_t u_\mu + \operatorname{div}_x(b(x) \mathbf{f}(u_\mu)) + g(t, x, u_\mu) &= \mu \Delta u_\mu & \text{dans } Q, \\ u_\mu(0, x) &= u_0(x) & \text{sur } \Omega, \\ u_\mu &= 0 & \text{sur } \Sigma. \end{cases} \quad (16)$$

First we show that Problem (16) admits a unique weak solution that is bounded independently of μ . Then the existence of a function u satisfying (3) will be provided by taking the limit on μ . Before this we give the definition of a weak solution to (16).

Definition 2. A function u in $W(0, T)$ is a weak solution to (16) if:

$$u_\mu(0, \cdot) = u_0 \quad \text{a.e. on } \Omega, \quad (17)$$

u_μ fulfills the variational equality, for a.e. $t \in]0, T[$, for any $v \in H_0^1(\Omega)$:

$$\langle \partial_t u_\mu, v \rangle + \int_\Omega ((\mu \nabla_x u_\mu - b(x) \mathbf{f}(u_\mu)) \cdot \nabla_x v + g(t, x, u_\mu) v) dx = 0. \quad (18)$$

In order to deal with bounded solutions, we use the following assumption on the flux function:

$$\forall t \in [0, T], \text{ for a.e. } \bar{\sigma} \in \Gamma_{L,R}, (b_R(\bar{\sigma}) - b_L(\bar{\sigma}))\mathbf{f}(M_1(t)) \cdot \boldsymbol{\nu}_L(\bar{\sigma}) \geq 0 \quad (19)$$

$$\forall t \in [0, T], \text{ for a.e. } \bar{\sigma} \in \Gamma_{L,R}, (b_R(\bar{\sigma}) - b_L(\bar{\sigma}))\mathbf{f}(M_2(t)) \cdot \boldsymbol{\nu}_L(\bar{\sigma}) \leq 0 \quad (20)$$

Remark 2. Conditions (19)-(20) are a little more general than the ones taken in the previous works. Indeed (19)-(20) are fulfilled as soon as b and \mathbf{f} are such that, for a.e. $\bar{\sigma}$ of $\Gamma_{L,R}$, the function $\lambda \mapsto (b_L - b_R)\mathbf{f}(\lambda) \cdot \boldsymbol{\nu}_L$ is nondecreasing and vanishes at a point. This kind of assumption is considered in [4]. Besides (19)-(20) are also fulfilled when:

$$\begin{cases} \mathbf{f}(m) = \mathbf{f}(M) = 0, \\ \text{for a.e. } (t, x) \in Q, g(t, x, M) \geq 0, \\ \text{for a.e. } (t, x) \in Q, g(t, x, m) \leq 0, \end{cases}$$

as we may choose in this case $M_1 \equiv M$ and $M_2 \equiv m$.

This previous hypothesis, with $g \equiv 0$, is in particular used in [1], [5], [6], [9], [16], [23], [24] for the one dimensional case, and in [14], [22] for the multidimensional case.

Now we prove:

Theorem 2. Under (19) and (20) there exists a unique weak solution u_μ to (16) such that:

$$\forall t \in [0, T], M_2(t) \leq u_\mu(t, \cdot) \leq M_1(t) \quad \text{a.e. on } \Omega, \quad (21)$$

Proof. (i) Existence

We use the Schauder-Tychonoff fixed point Theorem to obtain a function u_μ that satisfies (17), (18) and (21). First, for any real a, b, c , we define $\mathcal{B}(a, b, c) = \max\{a, \min\{b, c\}\}$ and we introduce, for $\mu > 0$, the problem:

$$\begin{cases} \text{Find } u_\mu \in W(0, T) \text{ such that a.e. on }]0, T[\text{ and for any } v \text{ of } H_0^1(\Omega), \\ \langle \partial_t u_\mu, v \rangle + \int_{\Omega} ((\mu \nabla_x u_\mu - b(x)\mathbf{f}(u_\mu^*)) \cdot \nabla_x v + g(t, x, u_\mu^*)v) dx = 0 \\ u_\mu(0, \cdot) = u_0 \text{ p.p. on } \Omega, \end{cases} \quad (22)$$

where $u_\mu^*(t, x) = \mathcal{B}(M_2(t), u_\mu(t, x), M_1(t))$. Let us remark that if u_μ is a solution to (22), then u_μ fulfills (21). Indeed we can consider in (22) the test function $v_\eta = \text{sgn}_\eta(u_\mu - M_1(t))^+$ (since $v_\eta \in L^2(0, T; H_0^1(\Omega))$). For any s of $]0, T[$, we integrate over $]0, s[$. Then,

$$\int_0^s \langle \partial_t u_\mu, v_\eta \rangle dt + \int_0^s \int_{\Omega} ((\mu \nabla_x u_\mu - b(x)\mathbf{f}(u_\mu^*)) \cdot \nabla_x v_\eta + g(t, x, u_\mu^*)v_\eta) dx dt = 0.$$

We write the evolution term under the form:

$$\int_0^s \langle \partial_t u_\mu, v_\eta \rangle dt = \int_0^s \langle \partial_t (u_\mu - M_1(t)), v_\eta \rangle dt + \int_{Q_s} M_1'(t) v_\eta dx dt,$$

and we use Mignot-Bamberger Lemma (see [11]) to obtain:

$$\int_0^s \langle \partial_t (u_\mu - M_1(t)), v_\eta \rangle dt = \int_{\Omega} \left(\int_0^{u_\mu(s, x) - M_1(s)} \text{sgn}_\eta(r - M_1(s))^+ dr \right) dx.$$

We use the definition of u_μ^* to write the convection term under the form:

$$- \int_{Q_s} b(x) \mathbf{f}(u_\mu^*) \cdot \nabla_x v_\eta dx dt = - \int_{Q_s} b(x) \mathbf{f}(M_1(t)) \cdot \nabla_x v_\eta dx dt.$$

We integrate by parts separately on Ω_L and Ω_R to have:

$$\begin{aligned} & - \int_0^s \int_{\Omega} b(x) \mathbf{f}(M_1(t)) \cdot \nabla v_\eta dx dt \\ &= \int_0^s \int_{\Gamma_{L,R}} (b_R - b_L) \mathbf{f}(M_1(t)) \cdot \boldsymbol{\nu}_L v_\eta d\mathcal{H}^{n-1} dt \\ &+ \sum_{i=L,R} \int_{Q_{i,s}} \nabla_x b(x) \cdot \mathbf{f}(M_1(t)) v_\eta dx dt. \end{aligned}$$

We notice that, thanks to (19) the interface integral is nonnegative. The diffusion term is equal to:

$$\int_0^s \int_{\Omega} \mu (\nabla_x u_\mu)^2 \operatorname{sgn}'_\eta(u_\mu - M_1(t))^+ dx dt,$$

and so, is nonnegative.

Moreover, by definition of u_μ^* ,

$$\int_0^s \int_{\Omega} g(t, x, u_\mu^*) v_\eta dx dt = \int_{Q_{i,s}} g(t, x, M_1(t)) v_\eta dx dt.$$

We take the η -limit. That yields to:

$$\begin{aligned} & \int_{\Omega} (u_\mu(s, x) - M_1(s))^+ dx \\ &+ \int_0^s \int_{\Omega} (M_1'(t) + g(t, x, M_1(t)) \operatorname{sgn}(u_\mu - M_1(t)))^+ dx dt \\ &+ \sum_{i=L,R} \int_0^s \int_{\Omega_i} \nabla_x b(x) \cdot \mathbf{f}(M_1(t)) \operatorname{sgn}(u_\mu - M_1(t))^+ dx dt \leq 0. \end{aligned}$$

By definition of M_1 , $M_1'(t) + g(t, x, M_1(t)) + \nabla_x b(x) \cdot \mathbf{f}(M_1(t)) \geq 0$, a.e. on Q_L and Q_R . Then we deduce the majorization of u_μ given in (21). To obtain the minorization of u_μ the reasoning is the same: we choose the test function $v_\eta = -\operatorname{sgn}_\eta(u_\mu - M_2(t))^-$ in (22). We use (20) to show the interface integral is nonnegative. Thus the existence of weak solution to (16) is proved as soon as (22) has a solution. To state this, for any w in $W(0, T)$, we consider the linearized problem:

$$\begin{cases} \text{Find } U \text{ in } W(0, T) \text{ such that a.e. in }]0, T[, \text{ for any } v \text{ of } H_0^1(\Omega), \\ \langle \partial_t U, v \rangle + \int_{\Omega} ((\mu \nabla_x U - b(x) \mathbf{f}(w^*)) \cdot \nabla v + g(t, x, w^*) v) dx = 0, \\ U(0, \cdot) = u_0. \end{cases} \quad (23)$$

Since Problem (23) admits a unique solution, we can define the operator:

$$\begin{aligned} \mathcal{T} : W(0, T) &\rightarrow W(0, T) \\ w &\rightarrow U \equiv \mathcal{T}(w) \end{aligned}$$

where U is the unique solution to (23). By taking $v = U$ in (23), as $w^*(t, x)$ takes values in $[M_2(t), M_1(t)]$, we have: $\|U\|_{L^2(0,T;H_0^1(\Omega))} \leq C_1$. This estimate implies, by using the definition of $L^2(0, T; H^{-1}(\Omega))$ -norm that: $\|\partial_t U\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2$, where C_1 and C_2 are two constants dependent on ε (but independent of w^*). So, with $C_3 = \sqrt{C_1^2 + C_2^2}$, the set:

$$\mathcal{C} = \{U \in W(0, T), \|U\|_{W(0,T)} \leq C_3, U(0, \cdot) = u_0 \text{ a.e. on } \Omega\},$$

is convex, bounded, weakly compact in $W(0, T)$ and such that $\mathcal{T}(\mathcal{C}) \subset \mathcal{C}$. It remains to prove the sequential continuity of \mathcal{T} for the weak topology

$\sigma(W(0, T), W'(0, T))$. Let $(w_n)_n$ a sequence converging towards w weakly in \mathcal{C} . Then the sequence $(U_n)_n = (\mathcal{T}(w_n))_n$ is bounded in $W(0, T)$ and so there exists U in $W(0, T)$ such that, up to a subsequence, $(\mathcal{T}(w_n))_n$ goes to U , weakly in $W(0, T)$, strongly in $L^2(Q)$. Moreover, $U_n(0, \cdot)$ goes to $U(0, \cdot) = u_0$ a.e. on Ω . We take the limit with respect to n in (23) to state that $U = \mathcal{T}(w)$. Since the solution to (23) is unique, we deduce that the whole sequence $(\mathcal{T}(w_n))_n$ goes to $\mathcal{T}(w)$ weakly in \mathcal{C} . Thus \mathcal{T} has (at least) one fixed point, denoted u_μ , that satisfies (22) and so (17), (18) and (21).

(ii) Uniqueness

We use an Holmgren-type duality method. Let u and \hat{u} be two weak solutions to (16). For the sake of simplicity we do not write the subscript μ . For any t in $[0, T[$, we introduce $z(t, \cdot)$ (resp. $\hat{z}(t, \cdot)$) the element of $H_0^1(\Omega)$ solution to the problem:

$$\begin{cases} \text{for any } v \in H_0^1(\Omega), \\ \int_{\Omega} \mu \nabla z(t, \cdot) \cdot \nabla v dx = \int_{\Omega} u(t, \cdot) v dx \\ \text{(resp. } \int_{\Omega} \mu \nabla \hat{z}(t, \cdot) \cdot \nabla v dx = \int_{\Omega} \hat{u}(t, \cdot) v dx). \end{cases} \quad (24)$$

Since $\partial_t u$ and $\partial_t \hat{u}$ belong to $L^2(0, T, H^{-1}(\Omega))$, we can assert that $\partial_t z$ (resp. $\partial_t \hat{z}$) is an element of $L^2(0, T; H^1(\Omega))$ such that, for a.e. $t \in]0, T[$, $\forall v \in H_0^1(\Omega)$,

$$\int_{\Omega} \mu \nabla \partial_t z \cdot \nabla v dx = \langle \partial_t u, v \rangle \quad (\text{resp. } \int_{\Omega} \mu \nabla \partial_t \hat{z} \cdot \nabla v dx = \langle \partial_t \hat{u}, v \rangle). \quad (25)$$

By choosing $v = z - \hat{z}$ in (18) written for u and \hat{u} , and by integrating from 0 to s , $s \in]0, T[$, we have:

$$\begin{aligned} & \int_0^s \langle \partial_t(u - \hat{u}), z - \hat{z} \rangle dt + \int_0^s \int_{\Omega} \mu \nabla(u - \hat{u}) \cdot \nabla(z - \hat{z}) dx dt \\ &= \int_0^s \int_{\Omega} b(x)(\mathbf{f}(u) - \mathbf{f}(\hat{u})) \cdot \nabla(z - \hat{z}) dx dt \\ & \quad - \int_0^s \int_{\Omega} (g(t, x, u) - g(t, x, \hat{u}))(z - \hat{z}) dx dt. \end{aligned}$$

For a.e. $t \in]0, T[$, we take $v = u(t, \cdot) - \hat{u}(t, \cdot)$ in (24) in order to have:

$$\mu \int_0^s \int_{\Omega} \nabla(u - \hat{u}) \cdot \nabla(z - \hat{z}) dx dt = \int_0^s \int_{\Omega} (u - \hat{u})^2 dx dt = \|u - \hat{u}\|_{L^2([0,s] \times \Omega)}^2.$$

In a same way, for a.e. $t \in]0, T[$ we choose $v = z(t, \cdot) - \hat{z}(t, \cdot)$ in (25) to obtain:

$$\begin{aligned}
\int_0^s \langle \partial_t(u - \hat{u}), z - \hat{z} \rangle dt &= \int_0^s \int_{\Omega} \mu \nabla \partial_t(z - \hat{z}) \cdot \nabla(z - \hat{z}) dx dt \\
&= \frac{1}{2} \int_{\Omega} \mu |\nabla(z - \hat{z})|^2(s, \cdot) dx dt.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega} \mu |\nabla(z - \hat{z})|^2(s, \cdot) dx + \|u - \hat{u}\|_{L^2(]0, s[\times \Omega)}^2 \leq \\
&\|u - \hat{u}\|_{L^2(]0, s[\times \Omega)} (2\|b\|_{\infty} M_f) \|\nabla(z - \hat{z})\|_{L^2(]0, s[\times \Omega)^n} + M_g \|z - \hat{z}\|_{L^2(]0, s[\times \Omega)}.
\end{aligned}$$

We use Poincaré Inequality to bound $\|z - \hat{z}\|_{L^2(]0, s[\times \Omega)}$ with $\|\nabla(z - \hat{z})\|_{L^2(]0, s[\times \Omega)^n}$, and then the Young inequality to state the existence of a positive real C such that, for a.e. $s \in]0, T[$:

$$\frac{1}{2} \int_{\Omega} \mu |\nabla(z - \hat{z})|^2(s, \cdot) dx \leq C \int_0^s \|\nabla(z - \hat{z})\|_{L^2(\Omega)^n} dt.$$

The conclusion follows thanks to Gronwall's Lemma. \square

Estimate (21) allows us to prove thanks to (18) the following Lemma, that will be useful to take the μ -limit.

Lemma 6. *There exists a positive real C such that:*

$$\mu \int_Q |\nabla_x u_{\mu}|^2 dx dt \leq C. \quad (26)$$

Now we take the limit on μ . The convergence of the sequence $(u_{\mu})_{\mu>0}$ is a consequence of E. Yu. Panov's work in [20]. Indeed it follows from Assumption (2):

Lemma 7. *The sequence of weak solutions $(u_{\mu})_{\mu>0}$ to problems (16) $_{\mu}$, contains a subsequence that converges in $L^1(Q)$.*

Proof. We first focus on Q_L . Since the convective term $b_L(x)f(u)$ is regular enough and the sequence $(u_{\mu})_{\mu>0}$ is bounded in $L^{\infty}(Q_L)$, we can apply E. Yu. Panov's result in [20] to state there exists a subsequence, still labelled $(u_{\mu})_{\mu>0}$ that converges in $L^1(Q_L)$. Then we use the same argument on the subdomain Q_R to extract from this last subsequence a new subsequence, still denoted by $(u_{\mu})_{\mu>0}$ that converges in $L^1(Q_R)$, and so in $L^1(Q)$. \square

We denote by u the limit of a subsequence $(u_{\mu})_{\mu}$ that converges in $L^1(Q)$. Let us show that u is a weak entropy solution to (1). First we prove that u fulfills the entropy inequality (3). To this aim we come back to the viscous problem (16). We choose in (18) the test function $sgn_{\eta}(u_{\mu} - k)\varphi_1\varphi_2$ where k is a real, φ_1 an element of $\mathcal{C}_c^{\infty}([0, T])$, φ_2 an element of $\mathcal{C}_c^{\infty}(\Omega)$, $\varphi_1, \varphi_2 \geq 0$. We integrate over $[0, T]$ to obtain:

$$\begin{aligned}
&\int_0^T \langle \partial_t u_{\mu}, sgn_{\eta}(u_{\mu} - k)\varphi_1\varphi_2 \rangle \\
&+ \int_Q (\mu \nabla_x u_{\mu} - b(x)\mathbf{f}(u_{\mu})) \cdot \nabla_x (sgn_{\eta}(u_{\mu} - k)\varphi_1\varphi_2) dx dt \quad (27) \\
&+ \int_Q g(t, x, u_{\mu}) sgn_{\eta}(u_{\mu} - k)\varphi_1\varphi_2 dx = 0.
\end{aligned}$$

Again we use the F. Mignot-A. Bamberger Lemma to transform the evolution term:

$$\int_0^T \langle \partial_t u_\mu, \operatorname{sgn}_\eta(u_\mu - k) \varphi_2 \rangle \varphi_1 dt = - \int_Q I_\eta(u_\mu) \varphi_2 \partial_t \varphi_1 dx dt - \int_\Omega I_\eta(u_0) \varphi_2 \varphi_1(0) dx$$

where

$$I_\eta(u_\mu) = \int_k^{u_\mu} \operatorname{sgn}_\eta(\tau - k) d\tau.$$

The diffusion term is transformed as below:

$$\begin{aligned} \int_Q \mu \nabla_x u_\mu \cdot \nabla_x (\operatorname{sgn}_\eta(u_\mu - k) \varphi_1 \varphi_2) dx dt &= \int_Q \mu (\nabla u_\mu)^2 \operatorname{sgn}'_\eta(u_\mu - k) \varphi_1 \varphi_2 dx dt \\ &+ \int_Q \mu \nabla u_\mu \cdot \nabla \varphi_2 \varphi_1 \operatorname{sgn}_\eta(u_\mu - k) dx dt, \end{aligned}$$

so that the first term on the right-hand side of the equality is nonnegative.

The convective term is studied separately on Q_L and Q_R . We write it under the form:

$$\begin{aligned} &- \sum_{i \in \{L, R\}} \int_{Q_i} b_i(x) \mathbf{f}(u_\mu) \cdot \nabla u_\mu \operatorname{sgn}'_\eta(u_\mu - k) \varphi_1 \varphi_2 dx dt \\ &- \sum_{i \in \{L, R\}} \int_{Q_i} b_i(x) \mathbf{f}(u_\mu) \cdot \nabla \varphi_2 \operatorname{sgn}_\eta(u_\mu - k) \varphi_1 dx dt \end{aligned}$$

We focus on the first integral for $i = L$ (the reasoning for $i = R$ being the same). We denote:

$$\begin{aligned} J_{\eta, \mu} &= - \int_{Q_L} b_L(x) \mathbf{f}(u_\mu) \cdot \nabla u_\mu \operatorname{sgn}'_\eta(u_\mu - k) \varphi_1 \varphi_2 dx dt \\ &= - \int_{Q_L} b_L(x) \operatorname{div}_x \mathbf{D}_\eta(u_\mu, k) \varphi_1 \varphi_2 dx dt, \end{aligned}$$

where

$$\mathbf{D}_\eta(u_\mu, k) = \int_k^{u_\mu} \mathbf{f}(\tau) \operatorname{sgn}'_\eta(\tau - k) d\tau.$$

Thanks to the Green formula, we have:

$$\begin{aligned} J_{\eta, \mu} &= \int_{Q_L} (b_L(x) \mathbf{D}_\eta(u_\mu, k) \cdot \nabla \varphi_2 + \nabla_x b_L(x) \cdot \mathbf{D}_\eta(u_\mu, k) \varphi_2 \varphi_1) dx dt \\ &- \int_{\Sigma_{L, R}} b_L(\bar{\sigma}) \mathbf{D}_\eta(u_\mu, k) \cdot \boldsymbol{\nu}_L \varphi_1 \varphi_2 dt d\mathcal{H}^{n-1}. \end{aligned}$$

Next we look at the interface integral. We have:

$$\begin{aligned} &\int_{\Sigma_{L, R}} b_L(\bar{\sigma}) \mathbf{D}_\eta(u_\mu, k) \cdot \boldsymbol{\nu}_L \varphi_1 \varphi_2 dt d\mathcal{H}^{n-1} \\ &= \int_{\Sigma_{L, R}} b_L(\bar{\sigma}) (\mathbf{D}_\eta(u_\mu, k) - \mathbf{f}(k) \operatorname{sgn}_\eta(u_\mu - k)) \cdot \boldsymbol{\nu}_L \varphi_1 \varphi_2 dt d\mathcal{H}^{n-1} \\ &+ \int_{\Sigma_{L, R}} b_L(\bar{\sigma}) \mathbf{f}(k) \cdot \boldsymbol{\nu}_L \operatorname{sgn}_\eta(u_\mu - k) \varphi_1 \varphi_2 dt d\mathcal{H}^{n-1}. \end{aligned}$$

We point out, by an integration by parts that, for any reals θ and k ,

$$\mathbf{D}_\eta(\theta, k) = - \int_k^\theta \mathbf{f}'(\tau) \operatorname{sgn}_\eta(\tau - k) d\tau + \mathbf{f}(\theta) \operatorname{sgn}_\eta(\theta - k).$$

where $\mathbf{f}' = (f'_1, \dots, f'_n)$.

Then we make sure that, for $i = 1, \dots, n$:

$$\left| - \int_k^{u_\mu} f'_i(\tau) \operatorname{sgn}_\eta(\tau - k) d\tau + f_i(u_\mu) - f_i(k) \operatorname{sgn}_\eta(u_\mu - k) \right| \leq 2\eta M_{f_i}.$$

So we deduce:

$$|(\mathbf{D}_\eta(u_\mu, k) - \mathbf{f}(k) \operatorname{sgn}_\eta(u_\mu - k))| \leq 2n\eta M_{\mathbf{f}}.$$

We use the same method to study the integral over Q_R and, for any positive μ and η , we obtain:

$$\begin{aligned} & \int_Q I_\eta(u_\mu) \partial_t \varphi_1 \varphi_2 dx dt + \int_\Omega I_\eta(u_0) \varphi_1(0) \varphi_2 dx \\ & + \sum_{i \in L, R} \int_{Q_i} b_i \operatorname{sgn}_\eta(u_\mu - k) \mathbf{f}(u_\mu) - \mathbf{D}_\eta(u_\mu, k) \cdot \nabla \varphi_2 \varphi_1 dx dt \\ & - \sum_{i \in L, R} \int_{Q_i} \nabla_x b_i(x) \cdot \mathbf{D}_\eta(u_\mu, k) \varphi_1 \varphi_2 dx dt \\ & + \int_{\Sigma_{L, R}} (b_L - b_R) \mathbf{f}(k) \cdot \boldsymbol{\nu}_L \operatorname{sgn}_\eta(u_\mu - k) \varphi_1 \varphi_2 dt d\mathcal{H}^{n-1} \\ & + \int_{\Sigma_{L, R}} 2n\eta M_{\mathbf{f}} (|b_L| + |b_R|) \varphi_1 \varphi_2 dt d\mathcal{H}^{n-1} \\ & - \int_Q g(t, x, u_\mu) \operatorname{sgn}_\eta(u_\mu - k) \varphi_1 \varphi_2 dx dt \\ & - \int_Q \mu \nabla u_\mu \cdot \nabla \varphi_2 \varphi_1 \operatorname{sgn}_\eta(u_\mu - k) dx dt \geq 0. \end{aligned} \tag{28}$$

However,

$$\begin{aligned} & \int_{\Sigma_{L, R}} (b_L - b_R) \mathbf{f}(k) \cdot \boldsymbol{\nu}_L \operatorname{sgn}_\eta(u_\mu - k) \varphi_1 \varphi_2 dt d\mathcal{H}^{n-1} \\ & \leq \int_{\Sigma_{L, R}} |(b_L - b_R) \mathbf{f}(k) \cdot \boldsymbol{\nu}_L| \varphi_1 \varphi_2 dt d\mathcal{H}^{n-1}. \end{aligned}$$

We take now the μ -limit in (28). By (26),

$$\lim_{\mu \rightarrow 0^+} \int_Q \mu \nabla u_\mu \cdot \nabla \varphi_2 \varphi_1 \operatorname{sgn}_\mu(u_\mu - k) dx dt = 0.$$

We use Lemma 7 and the Lebesgue dominated convergence Theorem to ensure there exists $u \in L^\infty(Q)$ such that,

$$\forall k \in \mathbb{R}, \forall \varphi_1 \in \mathcal{C}_c^\infty([0, T]), \forall \varphi_2 \in \mathcal{C}_c^\infty(\Omega), \forall \eta > 0,$$

$$\begin{aligned}
& \int_Q I_\eta(u) \partial_t \varphi_1 \varphi_2 dx dt + \int_\Omega I_\eta(u_0) \varphi_1(0) \varphi_2 dx \\
& - \int_Q g(t, x, u) \operatorname{sgn}_\eta(u - k) \varphi_1 \varphi_2 dx dt \\
& + \sum_{i \in L, R} \int_{Q_i} b_i (\operatorname{sgn}_\eta(u - k) \mathbf{f}(u) - \mathbf{D}_\eta(u, k)) \cdot \nabla \varphi_2 \varphi_1 dx dt \\
& - \sum_{i \in L, R} \int_{Q_i} \nabla_x b_i(x) \cdot \mathbf{D}_\eta(u, k) \varphi_1 \varphi_2 dx dt \\
& + \int_{\Sigma_{L, R}} (|(b_L - b_R) \mathbf{f}(k) \cdot \boldsymbol{\nu}_L| + 2n\eta M_{\mathbf{f}}(|b_L| + |b_R|) \varphi_1 \varphi_2) dt d\mathcal{H}^{n-1} \geq 0.
\end{aligned} \tag{29}$$

Since $\lim_{\eta \rightarrow 0} \mathbf{D}_\eta(u, k) = \operatorname{sgn}(u - k) \mathbf{f}(k)$, *a.e.* on Q , the η -limit gives (3).

Now let us establish that u fulfills (4), (5) and (6). We write (18) with a test function v in $\mathcal{D}(Q)$ and we take the μ -limit. Thanks to the L^1 -convergence of $(u_\mu)_\mu$ that yields to (4). To prove (5) we consider in (29) test functions such that φ_1 is in $\mathcal{C}_c^\infty([0, T[)$ and φ_2 belongs to $\mathcal{C}_c^\infty(\Omega_i)$, $i = \{L, R\}$. We take the η -limit and it comes:

$$\begin{aligned}
& - \int_{Q_i} (|u - k| \partial_t \varphi_1 \varphi_2 + b_i(x) \Phi(u, k) \cdot \nabla_x \varphi_2 \varphi_1) dx dt \\
& - \int_{Q_i} \operatorname{sgn}(u - k) ((g(t, x, u) + \nabla_x b(x) \cdot \mathbf{f}(k)) \varphi_1 \varphi_2) dx dt \\
& \leq \int_{\Omega_i} |u_0 - k| \varphi_1(0) \varphi_2(x) dx.
\end{aligned}$$

Then we use F. Otto's arguments (see [18] or [19]) to ensure that:

$$\operatorname{ess\,lim}_{t \rightarrow 0^+} \int_{\Omega_i} |u(t, x) - u_0(x)| dx = 0,$$

and (5) follows.

In order to show (6), we consider the functions H_δ and \mathbf{Q}_δ defined in [19], for any $\tau, k \in \mathbb{R}$ by:

$$H_\delta(\tau, k) = ((\operatorname{dist}(\tau, I[0, k]))^2 + \delta^2)^{\frac{1}{2}} - \delta,$$

and

$$\mathbf{Q}_\delta(\tau, k) = \int_k^\tau \partial_1 H_\delta(\lambda, k) \mathbf{f}'(\lambda) d\lambda.$$

where $I[0, k]$ denotes the closed interval bounded by 0 and k .

We choose in (18) the test function $\partial_1 H_\delta(u_\mu, k) \varphi$, $\varphi \in \mathcal{C}_c^\infty([0, T[\times \bar{\Omega})$, $\varphi \geq 0$, that vanishes on $\Sigma_{L, R}$. We integrate over $[0, T]$ and we use the same arguments as to obtain (28) from (27). In particular, for the convection term we use Green formulas in each subdomain Q_i , $i = L, R$. That yields to:

$$\begin{aligned}
& \int_Q \{H_\delta(u_\mu, k) \partial_t \varphi + b(x) \mathbf{Q}_\delta(u_\mu, k) \cdot \nabla_x \varphi - g(t, x, u_\mu) \partial_1 H_\delta(u_\mu, k) \varphi\} dx dt \\
& + \int_Q \nabla_x b(x) (\mathbf{Q}_\delta(u_\mu, k) - \partial_1 H_\delta(u_\mu, k) \mathbf{f}(u)) dx dt \\
& \geq \mu \int_Q \partial_1 H_\delta(u_\mu, k) \nabla u_\mu \cdot \nabla \varphi dx dt.
\end{aligned}$$

We take the μ -limit. By (26) and the boundness of the sequence $(u_\mu)_\mu$ in $L^\infty(Q)$, the term on the right hand side of the inequality goes to 0. We use Lemma 7 to obtain:

$$\int_Q \{H_\delta(u, k)\varphi_t + b(x)\mathbf{Q}_\delta(u, k) \cdot \nabla\varphi - g(t, x, u)\partial_1 H_\delta(u, k)\varphi\} dxdt \\ \int_Q \nabla_x b(x) \cdot (\mathbf{Q}_\delta(u, k) - \partial_1 H_\delta(u, k)\mathbf{f}(u)) dxdt \geq 0.$$

Then we consider in the previous inequality a sequence of test functions $(\varphi_n)_n$ such that $\varphi_n(t, x) = \beta(t)\alpha_n(x)$ where $\beta \in \mathcal{C}_c^\infty(]0, T[)$, and $\alpha_n \in \mathcal{C}^\infty(\bar{\Omega})$, $0 \leq \alpha_n \leq 1$,

$$\alpha_n(x) = \begin{cases} 1 & \text{si } x \in \partial\Omega \setminus \Gamma_{L,R} \text{ et } d(x, \Gamma_{L,R}) \geq \frac{1}{n}, \\ 0 & \text{sur } \Gamma_{L,R} \text{ ou si } d(x, \partial\Omega) \geq \frac{1}{n}, \end{cases}$$

and $(\frac{1}{n}\nabla_x \alpha_n)_n$ is bounded on Ω .

We refer to F. Otto's work in [18] to assert:

$$\lim_{n \rightarrow \infty} \int_Q b(x)\mathbf{Q}_\delta(u, k) \cdot \nabla(\alpha_n(x))\beta(t) dxdt \text{ exists and is nonnegative.}$$

Moreover we adapt the proof of Lemma 2 et we use the definition of u^τ to have:

$$\lim_{n \rightarrow \infty} \int_Q b(x)\mathbf{Q}_\delta(u, k) \cdot \nabla(\alpha_n(x))\beta(t) dxdt = \int_\Sigma b(\bar{\sigma})\mathbf{Q}_\delta(u^\tau, k) \cdot \boldsymbol{\nu}(\bar{\sigma})\beta(t) dt d\mathcal{H}^{n-1}.$$

Thus, when δ goes to 0:

$$\int_\Sigma b(\bar{\sigma})\mathcal{F}_0(u^\tau, k)\beta(t) dt d\mathcal{H}^{n-1} \geq 0,$$

that is equivalent to boundary condition (6).

5 A particular case

Another idea to prove an existence property in the one dimensional case, used for example in [5], [13], or [23] is to introduce a regularization of the coefficient b . Naturally we can wonder if we can apply it in the multidimensional case. In this section we show that, at least for some simple situations, an existence result can be obtained by regularization of the function b .

In this part we suppose $\Omega =]-1, 1[^n$ and $\Gamma_{L,R} = \{0\} \times]-1, 1[^{n-1}$.

We denote $\Omega_L =]-1, 0[\times]-1, 1[^{n-1}$ and $\Omega_R =]0, 1[\times]-1, 1[^{n-1}$. So on $\Sigma_{L,R}$, $\boldsymbol{\nu}_L = (1, 0, \dots, 0)$ and $\boldsymbol{\nu}_R = -\boldsymbol{\nu}_L$. Lastly, on $\Sigma_{L,R}$, $\bar{\sigma} = (x_2, \dots, x_n)$.

We suppose also:

$$b(x) = \begin{cases} b_L & \text{si } x \in \Omega_L \\ b_R & \text{si } x \in \Omega_R, \end{cases}$$

where b_L and b_R are two fixed reals.

Let us remark that when b_L and b_R depend on the space variable, we can apply the techniques we will use if, for all $\bar{\sigma}$ of $\Gamma_{L,R}$, $(b_L(\bar{\sigma}) - b_R(\bar{\sigma}))$ has the same sign.

We note that the entropy inequality (3) becomes:

$\forall \varphi \in C_c^\infty(Q), \varphi \geq 0, \forall k \in \mathbb{R},$

$$\begin{aligned} & \int_Q \{ |u - k| \partial_t \varphi + b(x) \Phi(u, k) \cdot \nabla_x \varphi - \text{sgn}(u - k) g(t, x, u) \varphi \} dx dt \\ & + |(b_R - b_L) f_1(k)| \int_0^T \int_{\Sigma_{L,R}} \varphi(t, \bar{\sigma}) d\bar{\sigma} dt \geq 0. \end{aligned} \quad (30)$$

We introduce a sequence of smooth functions $(b_\varepsilon)_{\varepsilon > 0}$ such that:

$$\forall x \in \Omega, b_\varepsilon(x) = \theta_\varepsilon(x_1),$$

with:

$$\theta_\varepsilon(x_1) = \begin{cases} b_L & \text{si } x_1 \leq -\varepsilon \\ b_R & \text{si } x_1 \geq \varepsilon, \end{cases}$$

and such that θ_ε is monotonic on $[-\varepsilon, \varepsilon]$ (depending on the sign of $(b_L - b_R)$). So,

$$\forall x \in \Omega, x_1 \neq 0, b_\varepsilon(x) \rightarrow b(x).$$

To state an existence result, we will use in this section assumptions (19)-(20) that is:

$$\forall t \in [0, T], (b_R - b_L) f_1(M_1(t)) \geq 0, \quad (31)$$

$$\forall t \in [0, T], (b_R - b_L) f_1(M_2(t)) \leq 0. \quad (32)$$

We consider also a sequence of functions $(u_0^j)_{j \in \mathbb{N}^*}$ such that:

$$\forall j \in \mathbb{N}^*, u_0^j \in C_c^\infty(\Omega), \text{ and } \lim_{j \rightarrow +\infty} u_0^j = u_0 \text{ in } L^1(\Omega).$$

For $j \in \mathbb{N}^*$ and $\varepsilon > 0$, we denote u_ε the unique entropy solution (see [7]) to the “regularized” problem:

Find a measurable and bounded function u in $BV(Q) \cap C([0, T]; L^1(\Omega))$ such that:

$$\begin{cases} \partial_t u_\varepsilon + \text{div}_x(b_\varepsilon(x) \mathbf{f}(u_\varepsilon)) + g(t, x, u_\varepsilon) = 0 & \text{in } Q, \\ u_\varepsilon(0, x) = u_0^j(x) & \text{on } \Omega, \\ u_\varepsilon = 0 & \text{on (a part of) } \Sigma. \end{cases} \quad (33)$$

We know that u_ε is bounded but the bounds depend on ε *a priori*. That is why we state:

Lemma 8. *Under (31) and (32), for a.e. $t \in [0, T]$,*

$$M_2(t) \leq u_\varepsilon(t, \cdot) \leq M_1(t) \text{ a.e. on } \Omega. \quad (34)$$

Proof. We come back to the viscous problem related to (33):

Find a measurable and bounded function $u_{\varepsilon, \mu}$ such that:

$$\begin{cases} \partial_t u_{\varepsilon, \mu} + \text{div}_x(b_\varepsilon(x) \mathbf{f}(u_{\varepsilon, \mu})) + g(t, x, u_{\varepsilon, \mu}) = \mu \Delta u_{\varepsilon, \mu} & \text{in } Q, \\ u_{\varepsilon, \mu}(0, x) = u_0^j(x) & \text{on } \Omega, \\ u_{\varepsilon, \mu} = 0 & \text{on } \Sigma. \end{cases} \quad (35)$$

For $\mu > 0$, (35) admits a unique solution $u_{\varepsilon, \mu}$ of $L^2(0, T; H^2(\Omega)) \cap C([0, T];$

$H^1(\Omega)$) and such that $\partial_t u_{\varepsilon,\mu} \in L^2(Q)$. Furthermore the sequence $(u_{\varepsilon,\mu})_\mu$ converges towards u_ε in $L^1(Q)$ when μ goes to 0^+ .

We multiply (35) by $(u_{\varepsilon,\mu} - M_1(t))^+$ and we integrate over $]0, s[\times \Omega$, $s \in]0, T[$. We have:

$$\mu \int_0^s \int_\Omega \Delta u_{\varepsilon,\mu} (u_{\varepsilon,\mu} - M_1(t))^+ dx dt = -\mu \int_0^s \int_\Omega [\nabla(u_{\varepsilon,\mu} - M_1(t))^+]^2 dx dt \leq 0.$$

For the evolution term, we write:

$$\begin{aligned} \int_0^s \int_\Omega \partial_t u_{\varepsilon,\mu} (u_{\varepsilon,\mu} - M_1(t))^+ dx dt &= \frac{1}{2} \| (u_{\varepsilon,\mu}(s, \cdot) - M_1(s))^+ \|_{L^2(\Omega)}^2 \\ &+ \int_0^s \int_\Omega M_1'(t) (u_{\varepsilon,\mu} - M_1(t))^+ dx dt. \end{aligned}$$

We introduce in the convective term, the term $\operatorname{div}(b_\varepsilon(x) \mathbf{f}(M_1(t)))$. By definition of $b_\varepsilon(x)$, that gives:

$$\begin{aligned} &\int_0^s \int_\Omega \operatorname{div}_x (b_\varepsilon(x) \mathbf{f}(u_{\varepsilon,\mu})) (u_{\varepsilon,\mu} - M_1(t))^+ dx dt \\ &= \int_0^s \int_\Omega \theta'_\varepsilon(x_1) f_1(M_1(t)) (u_{\varepsilon,\mu} - M_1(t))^+ dx dt \\ &+ \int_0^s \int_Q \operatorname{div}(b_\varepsilon(x) (\mathbf{f}(u_{\varepsilon,\mu}) - \mathbf{f}(M_1(t)))) (u_{\varepsilon,\mu} - M_1(t))^+ dx dt. \end{aligned}$$

For the reaction term,

$$\begin{aligned} &\int_0^s \int_\Omega g(t, x, u_{\varepsilon,\mu}) (u_{\varepsilon,\mu} - M_1(t))^+ dx dt \\ &= \int_0^s \int_\Omega g(t, x, M_1(t)) (u_{\varepsilon,\mu} - M_1(t))^+ dx dt \\ &+ \int_s \int_\Omega [g(t, x, u_{\varepsilon,\mu}) - g(t, x, M_1(t))] (u_{\varepsilon,\mu} - M_1(t))^+ dx dt. \end{aligned}$$

We gather all the terms to assert:

$$\begin{aligned} &\frac{1}{2} \| (u_{\varepsilon,\mu}(s, \cdot) - M_1(s))^+ \|_{L^2(\Omega)}^2 + \int_s \int_\Omega \Psi(t, x) (u_{\varepsilon,\mu} - M_1(t))^+ dx dt \\ &\leq (M_g + \frac{\|b\|_\infty M_F}{4\mu}) \int_s \int_\Omega ((u_{\varepsilon,\mu} - M_1(t))^+)^2 dx dt, \end{aligned}$$

where

$$\Psi(t, x) = \theta'_\varepsilon(x_1) f_1(M_1(t)) + M_1'(t) + g(t, x, M_1(t)).$$

Since $\operatorname{sgn}(\theta'_\varepsilon(x_1)) = \operatorname{sgn}(b_R - b_L)$, we use (31) to have:

$$\theta'_\varepsilon(x_1) f_1(M_1(t)) \geq 0.$$

As, by definition of M_1 , for every t in $]0, T[$ and every x in $\Omega_L \cup \Omega_R$, $M_1'(t) + g(t, x, M_1(t)) \geq 0$, we deduce that $\Psi(t, x) \geq 0$ a.e. on Q .

Then we use the Gronwall Lemma to ensure the sequence $(u_{\varepsilon,\mu}(t, \cdot))_{\varepsilon,\mu}$, so $u_\varepsilon(t, \cdot)$ is majorized by $M_1(t)$.

We multiply (35) by $-(u_{\varepsilon,\mu} - M_2(t))^-$ et we apply the same techniques as previously, especially condition (32) to prove that $u_\varepsilon(t, \cdot)$ is minorized by $M_1(t)$. \square

Nonlinear assumption (2) and Lemma 8 allow us to state, by referring to E. Yu. Panov's work in [20]:

Lemma 9. *There is a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ that converges in $L^1(Q)$.*

Now we have to prove that the limit highlighted, denoted u , fulfills the entropy inequality (30). It is known that u_ε satisfies the entropy inequality:

$$\begin{aligned} & \int_Q \{I_\eta(u_\varepsilon)\partial_t\varphi + b_\varepsilon(x)\Phi_\eta(u_\varepsilon, k) \cdot \nabla\varphi - \text{sgn}_\eta(u_\varepsilon - k)g(t, x, u_\varepsilon)\varphi\} dx dt \\ & + \int_Q \theta'_\varepsilon(x_1)(\Phi_{1,\eta}(u_\varepsilon, k) - I'_\eta(u_\varepsilon)f_1(u_\varepsilon))\varphi dx dt + \int_\Omega I_\eta(u_0^j)\varphi(0, x) dx \geq 0, \end{aligned}$$

where (I_η, Φ_η) is the regular entropy pair defined, for any real k by:

$$I_\eta(u_\varepsilon, k) = \int_k^{u_\varepsilon} \text{sgn}_\eta(\tau - k) d\tau \quad \text{and} \quad \Phi_\eta(u_\varepsilon, k) = \int_k^{u_\varepsilon} \text{sgn}_\eta(\tau - k) \mathbf{f}'(\tau) d\tau.$$

We take now the ε -limit. By definition of b_ε ,

$$\begin{aligned} & \int_Q \theta'_\varepsilon(x_1)(\Phi_{1,\eta}(u_\varepsilon, k) - I'_\eta(u_\varepsilon)f_1(u_\varepsilon))\varphi dx dt = \\ & - \int_{\Sigma_{L,R}} \left(\int_{-\varepsilon}^\varepsilon \theta'_\varepsilon(x_1)f_1(k)\text{sgn}_\eta(u_\varepsilon - k)\varphi dx_1 \right) d\mathcal{H}^{n-1} dt \\ & + \int_{\Sigma_{L,R}} \left(\int_{-\varepsilon}^\varepsilon \theta'_\varepsilon(x_1)\{\Phi_{1,\eta}(u_\varepsilon, k) - I'_\eta(u_\varepsilon)(f_1(u_\varepsilon) - f_1(k))\}\varphi dx_1 \right) d\mathcal{H}^{n-1} dt. \end{aligned}$$

We remark that $|\theta'_\varepsilon(x_1)| = \text{sgn}(b_R - b_L)\theta'_\varepsilon(x_1)$. So

$$\left| - \int_{-\varepsilon}^\varepsilon \theta'_\varepsilon(x_1)f_1(k)\text{sgn}_\eta(u_\varepsilon - k)\varphi dx_1 \right| \leq \text{sgn}(b_R - b_L)|f_1(k)| \int_{-\varepsilon}^\varepsilon \theta'_\varepsilon(x_1)\varphi dx_1.$$

Moreover,

$$|\Phi_{1,\eta}(u_\varepsilon) - I'_\eta(u_\varepsilon)(f_1(u_\varepsilon) - f_1(k))| \leq 2M_{f_1}\eta.$$

Then,

$$\begin{aligned} & \int_Q \theta'_\varepsilon(x_1)(\Phi_{1,\eta}(u_\varepsilon, k) - I'_\eta(u_\varepsilon)f_1(u_\varepsilon))\varphi dx dt \\ & \leq \text{sgn}(b_R - b_L)(|f_1(k)| + 2M_{f_1}\eta) \int_{\Sigma_{L,R}} \left(\int_{-\varepsilon}^\varepsilon \theta'_\varepsilon\varphi dx_1 \right) d\mathcal{H}^{n-1} dt. \end{aligned}$$

After an integration by parts with respect to x_1 , we pass to the limit with ε to obtain:

$$\begin{aligned} & \int_Q \{I_\eta(u)\partial_t\varphi + b(x)\Phi_\eta(u, k) \cdot \nabla\varphi - \text{sgn}_\eta(u - k)g(t, x, u)\varphi\} dx dt \\ & + (2M_{f_1}\eta + |f_1(k)|)(|b_R - b_L|) \int_{\Sigma_{L,R}} \varphi(t, 0, x_2) dt dx_2 \\ & + \int_\Omega I_\eta(u_0^j)\varphi(0, x) dx \geq 0. \end{aligned} \tag{36}$$

If we consider only functions $\varphi \in \mathcal{C}_c^\infty(Q)$, when η goes to 0^+ , we establish thanks to the Lebesgue dominated convergence Theorem, the entropy inequality (30).

To prove (4), we multiply (33) by φ , $\varphi \in \mathcal{C}_c^\infty(Q)$ and we integrate over Q . We obtain:

$$\int_Q (u_\varepsilon \partial_t \varphi + b_\varepsilon f(u_\varepsilon) \cdot \nabla_x \varphi - g(t, x, u_\varepsilon) \varphi) dx dt = 0.$$

We take the ε -limit and the conclusion follows.

By reasoning as in the previous section, we show that u satisfies the initial condition (5). To prove (6), we consider again, for any positive real δ , the sequence $(H_\delta, \mathbf{Q}_\delta)$ used in Section 4. We can assert that u_ε fulfills:

$$\begin{aligned} & \int_Q \{H_\delta(u_\varepsilon, k) \partial_t \varphi dx dt + b_\varepsilon(x) \mathbf{Q}_\delta(u_\varepsilon, k) \cdot \nabla \varphi - \partial_1 H_\delta(u_\varepsilon, k) g(t, x, u_\varepsilon) \varphi\} dx dt \\ & + \int_Q b'_\varepsilon(x) (\mathbf{Q}_\delta(u_\varepsilon, k) - \partial_1 H_\delta(u_\varepsilon, k) \mathbf{f}(u_\varepsilon)) dx dt \geq 0. \end{aligned}$$

If we consider functions φ vanishing in a neighborhood of $\Sigma_{L,R}$, we take the ε -limit without difficulties to have:

$$\begin{aligned} & \int_Q \{H_\delta(u, k) \varphi_t dx dt + b(x) \mathbf{Q}_\delta(u, k) \cdot \nabla \varphi - \partial_1 H_\delta(u, k) \varphi g(t, x, u)\} dx dt \\ & + \int_Q b'(x) (\mathbf{Q}_\delta(u, k) - \partial_1 H_\delta(u, k) \mathbf{f}(u)) \varphi dx dt \geq 0. \end{aligned} \tag{37}$$

We choose in the above inequality, a sequence of test functions, $\varphi_n(t, x) = \beta(t) \alpha_n(x)$ where $\beta \in \mathcal{C}_c^\infty(]0, T[)$, $\beta \geq 0$, and $\alpha_n \in \mathcal{C}^\infty(\overline{\Omega})$, $0 \leq \alpha_n \leq 1$,

$$\alpha_n(x) = \begin{cases} 1 & \text{if } x \in \partial\Omega \setminus \Gamma_{L,R} \text{ and } d(x, \Gamma_{L,R}) \geq \frac{2}{n}, \\ 0 & \text{if } d(x, \Gamma_{L,R}) \leq \frac{1}{n} \text{ where } d(x, \partial\Omega) \geq \frac{1}{n}. \end{cases}$$

We use F. Otto's arguments (see [18]) to state:

$$\lim_{n \rightarrow \infty} \int_Q b(x) \mathbf{Q}_\eta(u, k) \cdot \nabla (\alpha_n(x)) \beta(t) dx dt \text{ exists and is nonnegative.}$$

The conclusion follows.

Lastly we have to take the j -limit. We denote u_j the weak entropy solution to (1) associated with the initial condition u_0^j . For $j \neq j'$, the comparison result (13) ensures there exists a positive real C such that:

$$\int_Q |u_j(t, x) - u_{j'}(t, x)| \leq C \int_\Omega |u_0^j - u_0^{j'}| dx.$$

Then $(u_j)_j$ is a Cauchy sequence in $L^1(Q)$, and so converges towards a limit u . Moreover, for any j of \mathbb{N}^* , u_j fulfills (36) and (37). Thus by taking the j -limit in (36) and (37), we prove that u is a weak entropy solution to (1).

References

- [1] Adimurthi, J. Jaffre, G.D. Veerappa Gowda : *Godunov-type methods for conservation laws with a flux function discontinuous in space*, SIAM J. Numer. Anal., **42**, 179-208, 2004.

- [2] Adimurthi, S. Mishra, G.D. Veerappa Gowda : *Optimal entropy solutions for conservation laws with discontinuous flux function*, J. Hyperbolic Differ. Equ., **2**, 783-837, 2005.
- [3] Adimurthi, S. Mishra, G.D. Veerappa Gowda : *Existence and stability of entropy solutions for a conservation law with discontinuous non-convex fluxes*, Netw. Heterog. Media, **2**, 127-157, 2007.
- [4] E. Audusse, B. Perthame : *Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies*, Proc. Roy. Soc. Edinburgh Sect. A, **135**, 253-265, 2005.
- [5] F. Bachmann : *Analysis of a scalar conservation law with a flux function with discontinuous coefficients*, Advances in Differential equations, **9**:11-12, 1317-1338, 2004.
- [6] F. Bachmann, J. Vovelle : *Existence and uniqueness of entropy solution of scalar conservation laws with a flux function involving discontinuous coefficients*, Comm. Partial Differential Equations, **31**, 371-395, 2006.
- [7] C. Bardos, A.Y LeRoux, J.C Nédélec : *First order quasilinear equations with boundary conditions*, Comm. in partial differential equations, **4**, 1017-1034, 1979.
- [8] R. Bürger, K.H. Karlsen, N.H. Risebro, J.D. Towers : *Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units*, Numer. Math., **97**, 25-65, 2004.
- [9] R. Bürger, K.H. Karlsen, J.D. Towers : *A model of continuous sedimentation of flocculated suspensions in clarifier-thickener units*, SIAM J. Appl. Math., **65**, 882-840, 2005.
- [10] G.-Q. Chen, H. Frid : *Divergence-Measure fields and hyperbolic conservation laws*, Arch. Rational Mech. Anal., **147**, 89-118, 1999.
- [11] G. Gagneux, M. Madaune-Tort : *Analyse mathématique de modèles non-linéaires de l'ingénierie pétrolière*, Mathématiques et Applications, **22**, Springer-Verlag, Berlin, 1996.
- [12] T. Gimse, N.H. Risebro : *Solution of the Cauchy problem for a conservation law with a discontinuous flux function*, SIAM J. Math. Anal., **23**, 635-648, 1992.
- [13] J. Jimenez : *Some Scalar Conservation Laws with Discontinuous Flux*, International Journal of Evolution Equations, **2**, 3, 2007.
- [14] K.H. Karlsen, M. Rascle, E. Tadmor : *On the existence and compactness of a two-dimensional resonant system of conservation law*, Comm. Math. Sci. **5** (2), 253-265, 2007.
- [15] K.H. Karlsen, N.H. Risebro, J.D. Towers : *L^1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vid. Selsk., 49 pp, 2003.

- [16] K. H. Karlsen, J.D. Towers : *Convergence of the Lax-Friedrichs scheme and stability for conservation laws with a discontinuous space-time dependent flux*, Chin. Ann. Math., **25B**, 287-318, 2004.
- [17] S.N. Kruzkov : *First-order quasilinear equations with several independent variables*, Mat. Sb. **81**, 228-255, 1970.
- [18] J. Målex, J. Nečas, M. Rokyta, M. Růžička : *Weak and measure-valued solutions to evolutionary PDEs* Applied Mathematics and Mathematical Computation, **13**, Chapman & Hall, London, 1996.
- [19] F. Otto : *Initial-Boundary Value Problem for a Scalar Conservation Law*, C.R. Acad. Sci. Paris, **322**, serie I, pp. 729-734 (1996).
- [20] E. Yu. Panov : *Property of strong precompactness for bounded sets of measure valued solutions of a first-order quasilinear equation*, Sbornik: Mathematics, **190**:3, 427-446, 1999.
- [21] E. Yu. Panov : *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws*, Journal of Hyperbolic Differential Equations, **2**:4, 885-908, 2005.
- [22] E. Yu. Panov : *Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux*, Preprint.
- [23] N. Seguin, J. Vovelle : *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci. **13**:2, 221-257, 2003.
- [24] J.D. Towers : *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal. **38**, 681-698, 2000.
- [25] A. Vasseur : *Strong traces for solutions of multidimensional scalar conservation laws*, Arch. Rational Mech. Anal. **160**, 181-193, 2001.