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# GROTHENDIECK GROUP AND GENERALIZED MUTATION RULE FOR 2-CALABI-YAU TRIANGULATED CATEGORIES

YANN PALU

ABSTRACT. We compute the Grothendieck group of certain 2-Calabi–Yau triangulated categories appearing naturally in the study of the link between quiver representations and Fomin–Zelevinsky’s cluster algebras. In this setup, we also prove a generalization of Fomin–Zelevinsky’s mutation rule.

## INTRODUCTION

In their study [6] of the connections between cluster algebras (see [22]) and quiver representations, P. Caldero and B. Keller conjectured that a certain antisymmetric bilinear form is well–defined on the Grothendieck group of a cluster–tilted algebra associated with a finite–dimensional hereditary algebra. The conjecture was proved in [19] in the more general context of Hom-finite 2-Calabi–Yau triangulated categories. It was used in order to study the existence of a cluster character on such a category  $\mathcal{C}$ , by using a formula proposed by Caldero–Keller.

In the present paper, we restrict to the case where  $\mathcal{C}$  is algebraic (i.e. is the stable category of a Frobenius category). We first use this bilinear form to prove a generalized mutation rule for quivers of cluster–tilting subcategories in  $\mathcal{C}$ . When the cluster–tilting subcategories are related by a single mutation, this shows, via the method of [9], that their quivers are related by the Fomin–Zelevinsky mutation rule. This special case was already proved in [3], without assuming  $\mathcal{C}$  to be algebraic.

We also compute the Grothendieck group of the triangulated category  $\mathcal{C}$ . In particular, this allows us to improve on results by M. Barot, D. Kussin and H. Lenzing: We compare the Grothendieck group of a cluster category  $\mathcal{C}_A$  with the group  $\overline{K}_0(\mathcal{C}_A)$ . The latter group was defined in [1] by only considering the triangles in  $\mathcal{C}_A$  which are induced by those of the derived category. More precisely, we prove that those two groups are isomorphic for any cluster category associated with a finite dimensional hereditary algebra, with its triangulated structure defined by B. Keller in [16].

This paper is organized as follows: The first section is dedicated to notation and necessary background from [8], [9], [17], [19]. In section 2, we compute the Grothendieck group of the triangulated category  $\mathcal{C}$ . In section 3, we prove a generalized mutation rule for quivers of cluster–tilting subcategories in  $\mathcal{C}$ . In particular, this yields a new proof of the Fomin–Zelevinsky mutation rule, under the restriction that  $\mathcal{C}$  is algebraic. We finally show that  $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$  for any finite dimensional hereditary algebra  $A$ .

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## 1. NOTATIONS AND BACKGROUND

Let  $\mathcal{E}$  be a Frobenius category whose idempotents split and which is linear over a given algebraically closed field  $k$ . By a result of Happel [10], its stable category  $\mathcal{C} = \underline{\mathcal{E}}$  is triangulated. We assume moreover, that  $\mathcal{C}$  is Hom-finite, 2-Calabi–Yau and has a cluster–tilting subcategory (see section 1.2), and we denote by  $\Sigma$  its suspension functor. Note that we do not assume that  $\mathcal{E}$  is Hom-finite.

We write  $\mathcal{X}(\ , \ )$ , or  $\text{Hom}_{\mathcal{X}}(\ , \ )$ , for the morphisms in a category  $\mathcal{X}$  and  $\text{Hom}_{\mathcal{X}}(\ , \ )$  for the morphisms in the category of  $\mathcal{X}$ -modules. We also denote by  $X^\wedge$  the projective  $\mathcal{X}$ -module represented by  $X$ :  $X^\wedge = \mathcal{X}(\ ?, X)$ .

**1.1. Fomin–Zelevinsky mutation for matrices.** Let  $B = (b_{ij})_{i,j \in I}$  be a finite or infinite matrix, and let  $k$  be in  $I$ . The Fomin and Zelevinsky mutation of  $B$  (see [8]) in direction  $k$  is the matrix

$$\mu_k(B) = (b'_{ij})$$

defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{else.} \end{cases}$$

Note that  $\mu_k(\mu_k(B)) = B$  and that if  $B$  is skew-symmetric, then so is  $\mu_k(B)$ .

We recall two lemmas of [9], stated for infinite matrices, which will be useful in section 3. Note that lemma 7.2 is a restatement of [2, (3.2)]. Let  $S = (s_{ij})$  be the matrix defined by

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| - b_{ij}}{2} & \text{if } i = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

**Lemma 7.1** ([9, Geiss–Leclerc–Schröer]) : *Assume that  $B$  is skew-symmetric. Then,  $S^2 = 1$  and the  $(i, j)$ -entry of the transpose of the matrix  $S$  is given by*

$$s_{ij}^t = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| + b_{ij}}{2} & \text{if } j = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

The matrix  $S$  yields a convenient way to describe the mutation of  $B$  in the direction  $k$ :

**Lemma 7.2** ([9, Geiss–Leclerc–Schröer], [2, Berenstein–Fomin–Zelevinsky]) : *Assume that  $B$  is skew-symmetric. Then we have:*

$$\mu_k(B) = S^t BS.$$

Note that the product is well-defined since the matrix  $S$  has a finite number of non vanishing entries in each column.

**1.2. Cluster-tilting subcategories.** A cluster-tilting subcategory (see [17]) of  $\mathcal{C}$  is a full subcategory  $\mathcal{T}$  such that

- a)  $\mathcal{T}$  is a linear subcategory;
- b) for any object  $X$  in  $\mathcal{C}$ , the contravariant functor  $\mathcal{C}(?, X)|_{\mathcal{T}}$  is finitely generated;
- c) for any object  $X$  in  $\mathcal{C}$ , we have  $\mathcal{C}(X, \Sigma T) = 0$  for all  $T$  in  $\mathcal{T}$  if and only if  $X$  belongs to  $\mathcal{T}$ .

We now recall some results from [17], which we will use in the sequel. Let  $\mathcal{T}$  be a cluster-tilting subcategory of  $\mathcal{C}$ , and denote by  $\mathcal{M}$  its preimage in  $\mathcal{E}$ . In particular  $\mathcal{M}$  contains the full subcategory  $\mathcal{P}$  of  $\mathcal{E}$  formed by the projective-injective objects, and we have  $\underline{\mathcal{M}} = \mathcal{T}$ .

The following proposition will be used implicitly, extensively in this paper.

**Proposition** [17, Keller–Reiten] :

- a) *The category  $\text{mod } \underline{\mathcal{M}}$  of finitely presented  $\underline{\mathcal{M}}$ -modules is abelian.*
- b) *For each object  $X \in \mathcal{C}$ , there is a triangle*

$$\Sigma^{-1} X \longrightarrow T_1^X \longrightarrow T_0^X \longrightarrow X$$

*of  $\mathcal{C}$ , with  $T_0^X$  and  $T_1^X$  in  $\mathcal{T}$ .*

Recall that the perfect derived category  $\text{per } \mathcal{M}$  is the full triangulated subcategory of the derived category of  $\mathcal{D} \text{Mod } \mathcal{M}$  generated by the finitely generated projective  $\mathcal{M}$ -modules.

**Proposition** [17, Keller–Reiten] :

- a) *For each  $X \in \mathcal{E}$ , there are conflations*

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X \longrightarrow M^0 \longrightarrow M^1 \longrightarrow 0$$

*in  $\mathcal{E}$ , with  $M_0, M_1, M^0$  and  $M^1$  in  $\mathcal{M}$ .*

- b) *Let  $Z$  be in  $\text{mod } \underline{\mathcal{M}}$ . Then  $Z$  considered as an  $\mathcal{M}$ -module lies in the perfect derived category  $\text{per } \mathcal{M}$  and we have canonical isomorphisms*

$$D(\text{per } \mathcal{M})(Z, ?) \simeq (\text{per } \mathcal{M})(?, Z[3]).$$

**1.3. The antisymmetric bilinear form.** In section 3, we will use the existence of the antisymmetric bilinear form  $\langle \cdot, \cdot \rangle_a$  on  $K_0(\text{mod } \underline{\mathcal{M}})$ . We thus recall its definition from [6].

Let  $\langle \cdot, \cdot \rangle$  be a truncated Euler form on  $\text{mod } \underline{\mathcal{M}}$  defined by

$$\langle M, N \rangle = \dim \text{Hom}_{\underline{\mathcal{M}}}(M, N) - \dim \text{Ext}_{\underline{\mathcal{M}}}^1(M, N)$$

for any  $M, N \in \text{mod } \underline{\mathcal{M}}$ . Define  $\langle \cdot, \cdot \rangle_a$  to be the antisymmetrization of this form:

$$\langle M, N \rangle_a = \langle M, N \rangle - \langle N, M \rangle.$$

This bilinear form descends to the Grothendieck group  $K_0(\text{mod } \underline{\mathcal{M}})$ :

**Lemma** [19, section 3] : The antisymmetric bilinear form

$$\langle M, N \rangle_a : K_0(\text{mod } \underline{\mathcal{M}}) \times K_0(\text{mod } \underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$$

is well-defined.

## 2. GROTHENDIECK GROUPS OF ALGEBRAIC 2-CY CATEGORIES WITH A CLUSTER-TILTING SUBCATEGORY

We fix a cluster-tilting subcategory  $\mathcal{T}$  of  $\mathcal{C}$ , and we denote by  $\mathcal{M}$  its preimage in  $\mathcal{E}$ . In particular  $\mathcal{M}$  contains the full subcategory  $\mathcal{P}$  of  $\mathcal{E}$  formed by the projective-injective objects, and we have  $\underline{\mathcal{M}} = \mathcal{T}$ .

We denote by  $\mathcal{H}^b(\mathcal{E})$  and  $\mathcal{D}^b(\mathcal{E})$  respectively the bounded homotopy category and the bounded derived category of  $\mathcal{E}$ . We also denote by  $\mathcal{H}_{\mathcal{E}\text{-}ac}^b(\mathcal{E})$ ,  $\mathcal{H}^b(\mathcal{P})$ ,  $\mathcal{H}^b(\mathcal{M})$  and  $\mathcal{H}_{\mathcal{E}\text{-}ac}^b(\mathcal{M})$  the full subcategories of  $\mathcal{H}^b(\mathcal{E})$  whose objects are the  $\mathcal{E}$ -acyclic complexes, the complexes of projective objects in  $\mathcal{E}$ , the complexes of objects of  $\mathcal{M}$  and the  $\mathcal{E}$ -acyclic complexes of objects of  $\mathcal{M}$ , respectively.

### 2.1. A short exact sequence of triangulated categories.

**Lemma 1.** *Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be thick, full triangulated subcategories of a triangulated category  $\mathcal{A}$  and let  $\mathcal{B}$  be  $\mathcal{A}_1 \cap \mathcal{A}_2$ . Assume that for any object  $X$  in  $\mathcal{A}$  there is a triangle  $X_1 \rightarrow X \rightarrow X_2 \rightarrow \Sigma X_1$  in  $\mathcal{A}$ , with  $X_1$  in  $\mathcal{A}_1$  and  $X_2$  in  $\mathcal{A}_2$ . Then the induced functor  $\mathcal{A}_1/\mathcal{B} \rightarrow \mathcal{A}/\mathcal{A}_2$  is a triangle equivalence.*

*Proof.* Under these assumptions, denote by  $F$  the induced triangle functor from  $\mathcal{A}_1/\mathcal{B}$  to  $\mathcal{A}/\mathcal{A}_2$ . We are going to show that the functor  $F$  is a full, conservative, dense functor. Since any full conservative triangle functor is fully faithful,  $F$  will then be an equivalence of categories.

We first show that  $F$  is full. Let  $X_1$  and  $X'_1$  be two objects in  $\mathcal{A}_1$ . Let  $f$  be a morphism from  $X_1$  to  $X'_1$  in  $\mathcal{A}/\mathcal{A}_2$  and let

$$\begin{array}{ccc} & Y & \\ & \swarrow & \searrow^w \\ X_1 & & X'_1 \end{array}$$

be a left fraction which represents  $f$ . The morphism  $w$  is in the multiplicative system associated with  $\mathcal{A}_2$  and thus yields a triangle  $\Sigma^{-1}A_2 \rightarrow Y \xrightarrow{w} X'_1 \rightarrow A_2$  where  $A_2$  lies in the subcategory  $\mathcal{A}_2$ . Moreover, by assumption, there exists a triangle  $Y_1 \rightarrow Y \rightarrow Y_2 \rightarrow \Sigma Y_1$  with  $Y_i$  in  $\mathcal{A}_i$ . Applying the octahedral axiom to the composition  $Y_1 \rightarrow Y \rightarrow X'_1$  yields a commutative diagram whose two middle rows and columns are triangles in  $\mathcal{A}$

$$\begin{array}{ccccccc} & & \Sigma^{-1}A_2 & \xlongequal{\quad} & \Sigma^{-1}A_2 & & \\ & & \downarrow & & \downarrow & & \\ Y_1 & \longrightarrow & Y & \longrightarrow & Y_2 & \longrightarrow & \Sigma Y_1 \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ Y_1 & \longrightarrow & X'_1 & \longrightarrow & Z & \longrightarrow & \Sigma Y_1 \\ & & \downarrow & & \downarrow & & \\ & & A_2 & \xlongequal{\quad} & A_2 & & . \end{array}$$

Since  $Y_2$  and  $A_2$  belong to  $\mathcal{A}_2$ , so does  $Z$ . And since  $X'_1$  and  $Y_1$  belong to  $\mathcal{A}_1$ , so does  $Z$ . This implies, that  $Z$  belongs to  $\mathcal{B}$ . The morphism  $Y_1 \rightarrow X'_1$  is in the multiplicative system of  $\mathcal{A}_1$  associated with  $\mathcal{B}$  and the diagram

$$\begin{array}{ccc} & Y_1 & \\ & \swarrow & \searrow \\ X_1 & & X'_1 \end{array}$$

is a left fraction which represents  $f$ . This implies that  $f$  is the image of a morphism in  $\mathcal{A}_1/\mathcal{B}$ . Therefore the functor  $F$  is full.

We now show that  $F$  is conservative. Let  $X_1 \xrightarrow{f} Y_1 \rightarrow Z_1 \rightarrow \Sigma X_1$  be a triangle in  $\mathcal{A}_1$ . Assume that  $Ff$  is an isomorphism in  $\mathcal{A}/\mathcal{A}_2$ , which implies that  $Z_1$  is an object of  $\mathcal{A}_2$ . Therefore,  $Z_1$  is an object of  $\mathcal{B}$  and  $f$  is an isomorphism in  $\mathcal{A}_1/\mathcal{B}$ .

We finally show that  $F$  is dense. Let  $X$  be an object of the category  $\mathcal{A}/\mathcal{A}_2$ , and let  $X_1 \rightarrow X \rightarrow X_2 \rightarrow \Sigma X_1$  be a triangle in  $\mathcal{A}$  with  $X_i$  in  $\mathcal{A}_i$ . Since  $X_2$  belongs to  $\mathcal{A}_2$ , the image of the morphism  $X_1 \rightarrow X$  in  $\mathcal{A}/\mathcal{A}_2$  is an isomorphism. Thus  $X$  is isomorphic to the image by  $F$  of an object in  $\mathcal{A}_1/\mathcal{B}$ .  $\square$

As a corollary, we have the following:

**Lemma 2.** *The following sequence of triangulated categories is short exact:*

$$0 \longrightarrow \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M}) \longrightarrow \mathcal{D}^b(\mathcal{E}) \longrightarrow 0.$$

Remark: This lemma remains true if  $\mathcal{C}$  is  $d$ -Calabi–Yau and  $\underline{\mathcal{M}}$  is  $(d-1)$ -cluster-tilting, using section 5.4 of [17].

*Proof.* For any object  $X$  in  $\mathcal{H}^b(\mathcal{E})$ , the existence of an object  $M$  in  $\mathcal{H}^b(\mathcal{M})$  and of a quasi-isomorphism  $w$  from  $M$  to  $X$  is obtained using the approximation conflations of Keller–Reiten (see section 1.2). Since the cone of the morphism  $w$  belongs to  $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{E})$ , lemma 1 applies to the subcategories  $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$ ,  $\mathcal{H}^b(\mathcal{M})$  and  $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{E})$  of  $\mathcal{H}^b(\mathcal{E})$ .  $\square$

**Proposition 3.** *The following diagram is commutative with exact rows and columns:*

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \uparrow & & \uparrow & \\
0 & \longrightarrow & \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) & \xrightarrow{i_{\mathcal{M}}} & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) & \longrightarrow & \mathcal{E} \longrightarrow 0 \\
& & \parallel & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) & \longrightarrow & \mathcal{H}^b(\mathcal{M}) & \longrightarrow & \mathcal{D}^b(\mathcal{E}) \longrightarrow 0 & (D) \\
& & & & \uparrow & & \uparrow i_{\mathcal{P}} \\
& & & & \mathcal{H}^b(\mathcal{P}) & \xlongequal{\quad} & \mathcal{H}^b(\mathcal{P}) \longrightarrow 0 \\
& & & & \uparrow & & \uparrow \\
& & & & 0 & & 0
\end{array}$$

*Proof.* The column on the right side has been shown to be exact in [18] and [20]. The second row is exact by lemma 2. The subcategories  $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$  and  $\mathcal{H}^b(\mathcal{P})$  of  $\mathcal{H}^b(\mathcal{M})$  are left and right orthogonal to each other. This implies that the induced functors  $i_{\mathcal{M}}$  and  $i_{\mathcal{P}}$  are fully faithful and that taking the quotient of  $\mathcal{H}^b(\mathcal{M})$  by those two subcategories either in one order or in the other gives the same category. Therefore the first row is exact.  $\square$

**2.2. Invariance under mutation.** A natural question is then to which extent the diagram (D) depends on the choice of a particular cluster-tilting subcategory. Let thus  $\mathcal{T}'$  be another cluster-tilting subcategory of  $\mathcal{C}$ , and let  $\mathcal{M}'$  be its preimage in  $\mathcal{E}$ . Let  $\text{Mod } \mathcal{M}$  (resp.  $\text{Mod } \mathcal{M}'$ ) be the category of  $\mathcal{M}$ -modules (resp.  $\mathcal{M}'$ -modules), i.e. of  $k$ -linear contravariant functors from  $\mathcal{M}$  (resp.  $\mathcal{M}'$ ) to the category of  $k$ -vector spaces.

Let  $X$  be the  $\mathcal{M}$ - $\mathcal{M}'$ -bimodule which sends the pair of objects  $(M, M')$  to the  $k$ -vector space  $\mathcal{E}(M, M')$ . The bimodule  $X$  induces a functor  $F =? \otimes_{\mathcal{M}'} X : \text{Mod } \mathcal{M}' \longrightarrow \text{Mod } \mathcal{M}$  denoted by  $T_X$  in [15, section 6.1].

Recall that the perfect derived category per  $\mathcal{M}$  is the full triangulated subcategory of the derived category  $\mathcal{D}\text{Mod } \mathcal{M}$  generated by the finitely generated projective  $\mathcal{M}$ -modules.

**Proposition 4.** *The left derived functor*

$$\mathbb{L}F : \mathcal{D}\text{Mod } \mathcal{M}' \longrightarrow \mathcal{D}\text{Mod } \mathcal{M}$$

*is an equivalence of categories.*

*Proof.* Recall that if  $X$  is an object in a category  $\mathcal{X}$ , we denote by  $X^\wedge$  the functor  $\mathcal{X}(?, X)$  represented by  $X$ . By [15, 6.1], it is enough to check the following three properties:

1. For all objects  $M', M''$  of  $\mathcal{M}$ , the group  $\text{Hom}_{\mathcal{D}\text{Mod } \mathcal{M}}(\mathbb{L}FM', \mathbb{L}FM''[n])$  vanishes for  $n \neq 0$  and identifies with  $\text{Hom}_{\mathcal{M}'}(M', M'')$  for  $n = 0$ ;
2. for any object  $M'$  of  $\mathcal{M}'$ , the complex  $\mathbb{L}FM'$  belongs to per  $\mathcal{M}$ ;
3. the set  $\{\mathbb{L}FM', M' \in \mathcal{M}'\}$  generates  $\mathcal{D}\text{Mod } \mathcal{M}$  as a triangulated category with infinite sums.

Let  $M'$  be an object of  $\mathcal{M}'$ , and let  $M_1 \twoheadrightarrow M_0 \twoheadrightarrow M'$  be a conflation in  $\mathcal{E}$ , with  $M_0$  and  $M_1$  in  $\mathcal{M}$ , and whose deflation is a right  $\mathcal{M}$ -approximation (c.f. section 4 of [17]). The surjectivity of the map  $M_0^\wedge \rightarrow \mathcal{E}(?, M')|_{\mathcal{M}}$  implies that the complex  $P = (\cdots \rightarrow 0 \rightarrow M_1^\wedge \rightarrow M_0^\wedge \rightarrow 0 \rightarrow \cdots)$  is quasi-isomorphic to  $\mathbb{L}FM' = \mathcal{E}(?, M')|_{\mathcal{M}}$ . Therefore  $\mathbb{L}FM'$  belongs to the subcategory per  $\mathcal{M}$  of  $\mathcal{D}\text{Mod } \mathcal{M}$ . Moreover, we have, for any  $n \in \mathbb{Z}$  and any  $M'' \in \mathcal{M}'$ , the equality

$$\text{Hom}_{\mathcal{D}\text{Mod } \mathcal{M}}(\mathbb{L}FM', \mathbb{L}FM''[n]) = \text{Hom}_{\mathcal{H}^b \text{Mod } \mathcal{M}}(P, \mathcal{E}(?, M'')|_{\mathcal{M}}[n])$$

where the right-hand side vanishes for  $n \neq 0, 1$ . In case  $n = 1$  it also vanishes, since  $\text{Ext}_{\mathcal{E}}^1(M', M'')$  vanishes. Now,

$$\begin{aligned} \text{Hom}_{\mathcal{H}^b \text{Mod } \mathcal{M}}(P, \mathcal{E}(?, M'')|_{\mathcal{M}}) &\simeq \text{Ker}(\mathcal{E}(M_0, M'') \rightarrow \mathcal{E}(M_1, M'')) \\ &\simeq \mathcal{E}(M', M''). \end{aligned}$$

It only remains to be shown that the set  $R = \{\mathbb{L}FM', M' \in \mathcal{M}'\}$  generates  $\mathcal{D}\text{Mod } \mathcal{M}$ . Denote by  $\mathcal{R}$  the full triangulated subcategory with infinite sums of  $\mathcal{D}\text{Mod } \mathcal{M}$  generated by the set  $R$ . The set  $\{M, M \in \mathcal{M}\}$  generates  $\mathcal{D}\text{Mod } \mathcal{M}$  as a triangulated category with infinite sums. Thus it is enough to show that, for any object  $M$  of  $\mathcal{M}$ , the complex  $M^\wedge$  concentrated in degree 0 belongs to the subcategory  $\mathcal{R}$ . Let  $M$  be an object of  $\mathcal{M}$ , and let  $M \xrightarrow{i} M'_0 \xrightarrow{p} M'_1$  be a conflation of  $\mathcal{E}$  with  $M'_0$  and  $M'_1$  in  $\mathcal{M}'$ . Since  $\text{Ext}_{\mathcal{E}}^1(?, M)|_{\mathcal{M}}$  vanishes, we have a short exact sequence of  $\mathcal{M}$ -modules

$$0 \longrightarrow \mathcal{E}(?, M)|_{\mathcal{M}} \longrightarrow \mathcal{E}(?, M'_0)|_{\mathcal{M}} \longrightarrow \mathcal{E}(?, M'_1)|_{\mathcal{M}} \longrightarrow 0,$$

which yields the triangle

$$M^\wedge \longrightarrow \mathbb{L}FM'_0 \longrightarrow \mathbb{L}FM'_1 \longrightarrow \Sigma M^\wedge.$$

□

As a corollary of proposition 4, up to equivalence the diagram (D) does not depend on the choice of a cluster-tilting subcategory. To be more precise: Let  $G$  be the functor which sends an object  $X$  in the category  $\mathcal{H}^b(\mathcal{M}')$  to a representative of  $(\mathbb{L}F)X^\wedge$  in  $\mathcal{H}^b(\mathcal{M})$ , and a morphism in  $\mathcal{H}^b(\mathcal{M}')$  to the induced one in  $\mathcal{H}^b(\mathcal{M})$ .

**Corollary 5.** *The following diagram is commutative*

$$\begin{array}{ccccc}
 & & \mathcal{D}\text{Mod } \mathcal{M}' & \xrightarrow{\text{LF}} & \mathcal{D}\text{Mod } \mathcal{M} \\
 & \nearrow & \uparrow & & \nearrow \\
 \mathcal{H}^b(\mathcal{M}') & \xrightarrow{G} & \mathcal{H}^b(\mathcal{M}) & & \mathcal{H}^b(\mathcal{M}) \\
 \downarrow & \searrow & \downarrow & & \downarrow \\
 & & \mathcal{H}^b(\mathcal{P}) & \xlongequal{\quad} & \mathcal{H}^b(\mathcal{P}) \\
 \downarrow & \swarrow & \downarrow & & \downarrow \\
 \mathcal{D}^b(\mathcal{E}) & \xlongequal{\quad} & \mathcal{D}^b(\mathcal{E}) & & \mathcal{D}^b(\mathcal{E})
 \end{array}$$

and the functor  $G$  is an equivalence of categories.

We denote by  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$  the full subcategory of  $\text{per } \mathcal{M}$  whose objects are the complexes with homologies in  $\text{mod } \underline{\mathcal{M}}$ . The following lemma will allow us to compute the Grothendieck group of  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$  in section 2.3:

**Lemma 6.** *The canonical t-structure on  $\mathcal{D}\text{Mod } \mathcal{M}$  restricts to a t-structure on  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$ , whose heart is  $\text{mod } \underline{\mathcal{M}}$ .*

*Proof.* By [13], it is enough to show that for any object  $M^\bullet$  of  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$ , its truncation  $\tau_{\leq 0}M^\bullet$  in  $\mathcal{D}\text{Mod } \mathcal{M}$  belongs to  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$ . Since  $M^\bullet$  is in  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$ ,  $\tau_{\leq 0}M^\bullet$  is bounded, and is thus formed from the complexes  $H^i(M^\bullet)$  concentrated in one degree by taking iterated extensions. But, for any  $i$ , the  $\mathcal{M}$ -module  $H^i(M^\bullet)$  actually is an  $\underline{\mathcal{M}}$ -module. Therefore, by [17] (see section 1.2), it is perfect as an  $\mathcal{M}$ -module and it lies in  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$ .  $\square$

The next lemma already appears in [21]. For the convenience of the reader, we include a proof.

**Lemma 7.** *The Yoneda equivalence of triangulated categories  $\mathcal{H}^b(\mathcal{M}) \rightarrow \text{per } \mathcal{M}$  induces a triangle equivalence  $\mathcal{H}_{\mathcal{E}\text{-ac}}^b(\mathcal{M}) \rightarrow \text{per}_{\underline{\mathcal{M}}}\mathcal{M}$ .*

*Proof.* We first show that the cohomology groups of an  $\mathcal{E}$ -acyclic bounded complex  $M$  vanish on  $\mathcal{P}$ . Let  $P$  be a projective object in  $\mathcal{E}$  and let  $E$  be a kernel in  $\mathcal{E}$  of the map  $M^n \rightarrow M^{n+1}$ . Since  $M$  is  $\mathcal{E}$ -acyclic, such an object exists, and moreover, it is an image of the map  $M^{n-1} \rightarrow M^n$ . Any map from  $P$  to  $M^n$  whose composition with  $M^n \rightarrow M^{n+1}$  vanishes factors through the kernel  $E \rightarrow M^n$ . Since  $P$  is projective, this factorization factors through the deflation  $M^{n-1} \rightarrow E$ .

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow & \downarrow & \searrow & \\
 & & M^n & \xrightarrow{0} & M^{n+1} \\
 & \swarrow & \downarrow & \searrow & \\
 M^{n-1} & \xrightarrow{\quad} & M^n & \xrightarrow{\quad} & M^{n+1} \\
 & \swarrow & \downarrow & \searrow & \\
 & & E & & 
 \end{array}$$

Therefore, we have  $H^n(M)(P) = 0$  for all projective objects  $P$ , and  $H^n(M)$  belongs to  $\text{mod } \underline{\mathcal{M}}$ . Thus the Yoneda functor induces a fully faithful functor from  $\mathcal{H}_{\mathcal{E}\text{-ac}}^b(\mathcal{M})$  to  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$ . To prove that it is dense, it is enough to prove that any object of the heart  $\text{mod } \underline{\mathcal{M}}$  of the t-structure on  $\text{per}_{\underline{\mathcal{M}}}\mathcal{M}$  is in its essential image.

But this was proved in [17, section 4] (see section 1.2).  $\square$



**Proposition 8.** *There is a triangle equivalence of categories*

$$\text{per}_{\underline{\mathcal{M}}} \mathcal{M} \xrightarrow{\simeq} \text{per}_{\underline{\mathcal{M}'}} \mathcal{M}'$$

*Proof.* Since the categories  $\mathcal{H}^b(\mathcal{P})$  and  $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$  are left-right orthogonal in  $\mathcal{H}^b(\mathcal{M})$ , this is immediate from corollary 5 and lemma 7.  $\square$

**2.3. Grothendieck groups.** For a triangulated (resp. additive, resp. abelian) category  $\mathcal{A}$ , we denote by  $K_0^{\text{tri}}(\mathcal{A})$  or simply  $K_0(\mathcal{A})$  (resp.  $K_0^{\text{add}}(\mathcal{A})$ , resp.  $K_0^{\text{ab}}(\mathcal{A})$ ) its Grothendieck group (with respect to the mentioned structure of the category). For an object  $A$  in  $\mathcal{A}$ , we also denote by  $[A]$  its class in the Grothendieck group of  $\mathcal{A}$ .

The short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \underline{\mathcal{E}} \longrightarrow 0$$

given by proposition 3 induces an exact sequence in the Grothendieck groups

$$(*) \quad K_0(\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})) \longrightarrow K_0(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})) \longrightarrow K_0(\underline{\mathcal{E}}) \longrightarrow 0.$$

**Lemma 9.** *The exact sequence (\*) is isomorphic to an exact sequence*

$$(**) \quad K_0^{\text{ab}}(\text{mod } \underline{\mathcal{M}}) \xrightarrow{\varphi} K_0^{\text{add}}(\underline{\mathcal{M}}) \longrightarrow K_0^{\text{tri}}(\underline{\mathcal{E}}) \longrightarrow 0.$$

*Proof.* First, note that, by [21], see also lemma 7, we have an isomorphism between the Grothendieck groups  $K_0(\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}))$  and  $K_0(\text{per}_{\underline{\mathcal{M}}} \mathcal{M})$ . The t-structure on  $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$  whose heart is  $\text{mod } \underline{\mathcal{M}}$ , see lemma 6, in turn yields an isomorphism between the Grothendieck groups  $K_0^{\text{tri}}(\text{per}_{\underline{\mathcal{M}}} \mathcal{M})$  and  $K_0^{\text{ab}}(\text{mod } \underline{\mathcal{M}})$ . Next, we show that the canonical additive functor  $\underline{\mathcal{M}} \xrightarrow{\alpha} \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$  induces an isomorphism between the Grothendieck groups  $K_0^{\text{add}}(\underline{\mathcal{M}})$  and  $K_0^{\text{tri}}(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}))$ . For this, let us consider the canonical additive functor  $\underline{\mathcal{M}} \xrightarrow{\beta} \mathcal{H}^b(\underline{\mathcal{M}})$  and the triangle functor  $\mathcal{H}^b(\mathcal{M}) \xrightarrow{\gamma} \mathcal{H}^b(\underline{\mathcal{M}})$ . The following diagram describes the situation:

$$\begin{array}{ccc} \mathcal{H}^b(\underline{\mathcal{M}}) & \xleftarrow{\gamma} & \mathcal{H}^b(\mathcal{M}) \\ \beta \uparrow & \swarrow \gamma & \downarrow \\ \underline{\mathcal{M}} & \xrightarrow{\alpha} & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \end{array}$$

The functor  $\gamma$  vanishes on the full subcategory  $\mathcal{H}^b(\mathcal{P})$ , thus inducing a triangle functor, still denoted by  $\gamma$ , from  $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$  to  $\mathcal{H}^b(\underline{\mathcal{M}})$ . Furthermore, the functor  $\beta$  induces an isomorphism at the level of Grothendieck groups, whose inverse  $K_0(\beta)^{-1}$  is given by

$$\begin{aligned} K_0^{\text{tri}}(\mathcal{H}^b(\underline{\mathcal{M}})) &\longrightarrow K_0^{\text{add}}(\underline{\mathcal{M}}) \\ [M] &\longmapsto \sum_{i \in \mathbb{Z}} (-1)^i [M^i]. \end{aligned}$$

As the group  $K_0^{\text{tri}}(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}))$  is generated by objects concentrated in degree 0, it is straightforward to check that the morphisms  $K_0(\alpha)$  and  $K_0(\beta)^{-1} K_0(\gamma)$  are inverse to each other.  $\square$

As a consequence of the exact sequence (\*\*), we have an isomorphism between  $K_0^{\text{tri}}(\underline{\mathcal{E}})$  and  $K_0^{\text{add}}(\underline{\mathcal{M}})/\text{Im } \varphi$ . In order to compute  $K_0^{\text{tri}}(\underline{\mathcal{E}})$ , the map  $\varphi$  has to be made explicit. We first recall some results from Iyama–Yoshino [12] which generalize results from [4]: For any indecomposable  $M$  of  $\underline{\mathcal{M}}$  not in  $\mathcal{P}$ , there exists  $M^*$  unique up to isomorphism such that  $(M, M^*)$  is an exchange pair. This means that  $M$  and  $M^*$  are not isomorphic and that the full additive subcategory of  $\mathcal{C}$  generated

by all the indecomposable objects of  $\underline{\mathcal{M}}$  but those isomorphic to  $M$ , and by the indecomposable objects isomorphic to  $M^*$  is again a cluster-tilting subcategory. Moreover,  $\dim \underline{\mathcal{E}}(M, \Sigma M^*) = 1$ . We can thus fix two (non-split) exchange triangles

$$M^* \rightarrow B_M \rightarrow M \rightarrow \Sigma M^* \text{ and } M \rightarrow B_{M^*} \rightarrow M^* \rightarrow \Sigma M.$$

We may now state the following:

**Theorem 10.** *The Grothendieck group of the triangulated category  $\underline{\mathcal{E}}$  is the quotient of that of the additive subcategory  $\underline{\mathcal{M}}$  by all relations  $[B_{M^*}] - [B_M]$ :*

$$K_0^{tri}(\underline{\mathcal{E}}) \simeq K_0^{add}(\underline{\mathcal{M}})/([B_{M^*}] - [B_M])_M.$$

*Proof.* We denote by  $S_M$  the simple  $\underline{\mathcal{M}}$ -module associated to the indecomposable object  $M$ . This means that  $S_M(M')$  vanishes for all indecomposable objects  $M'$  in  $\underline{\mathcal{M}}$  not isomorphic to  $M$  and that  $S_M(M)$  is isomorphic to  $k$ . The abelian group  $K_0^{ab}(\text{mod } \underline{\mathcal{M}})$  is generated by all classes  $[S_M]$ . In view of lemma 9, it is sufficient to prove that the image of the class  $[S_M]$  under  $\varphi$  is  $[B_{M^*}] - [B_M]$ . First note that the  $\mathcal{M}$ -module  $\text{Ext}_{\mathcal{E}}^1(?, M^*)|_{\mathcal{M}}$  vanishes on the projectives; it can thus be viewed as an  $\underline{\mathcal{M}}$ -module, and as such, is isomorphic to  $S_M$ . After replacing  $B_M$  and  $B_{M^*}$  by isomorphic objects of  $\underline{\mathcal{E}}$ , we can assume that the exchange triangles  $M^* \rightarrow B_M \rightarrow M \rightarrow \Sigma M^*$  and  $M \rightarrow B_{M^*} \rightarrow M^* \rightarrow \Sigma M$  come from conflations  $M^* \twoheadrightarrow B_M \twoheadrightarrow M$  and  $M \twoheadrightarrow B_{M^*} \twoheadrightarrow M^*$ . The spliced complex

$$(\cdots \rightarrow 0 \rightarrow M \rightarrow B_{M^*} \rightarrow B_M \rightarrow M \rightarrow 0 \rightarrow \cdots)$$

denoted by  $C^\bullet$ , is then an  $\mathcal{E}$ -acyclic complex, and it is the image of  $S_M$  under the functor  $\text{mod } \underline{\mathcal{M}} \subset \text{per } \underline{\mathcal{M}} \mathcal{M} \simeq \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$ . Indeed, we have two long exact sequences induced by the conflations above:

$$0 \rightarrow \mathcal{M}(?, M) \rightarrow \mathcal{M}(?, B_{M^*}) \rightarrow \mathcal{E}(?, M^*)|_{\mathcal{M}} \rightarrow \text{Ext}_{\mathcal{E}}^1(?, M)|_{\mathcal{M}} = 0 \text{ and}$$

$$0 \rightarrow \mathcal{E}(?, M^*)|_{\mathcal{M}} \rightarrow \mathcal{M}(?, B_M) \rightarrow \mathcal{M}(?, M) \rightarrow \text{Ext}_{\mathcal{E}}^1(?, M^*)|_{\mathcal{M}} \rightarrow \text{Ext}_{\mathcal{E}}^1(?, B_M)|_{\mathcal{M}}.$$

Since  $B_M$  belongs to  $\mathcal{M}$ , the functor  $\text{Ext}_{\mathcal{E}}^1(?, B_M)$  vanishes on  $\mathcal{M}$ , and the complex:

$$(C^\bullet) : (\cdots \rightarrow 0 \rightarrow M^\wedge \rightarrow (B_{M^*})^\wedge \rightarrow (B_M)^\wedge \rightarrow M^\wedge \rightarrow 0 \rightarrow \cdots)$$

is quasi-isomorphic to  $S_M$ .

Now, in the notations of the proof of lemma 9,  $\varphi[S_M]$  is the image of the class of the  $\mathcal{E}$ -acyclic complex  $C^\bullet$  under the morphism  $K_0(\beta)^{-1} K_0(\gamma)$ . This is  $[M] - [B_M] + [B_{M^*}] - [M]$  which equals  $[B_{M^*}] - [B_M]$  as claimed.  $\square$

### 3. THE GENERALIZED MUTATION RULE

Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two cluster-tilting subcategories of  $\mathcal{C}$ . Let  $Q$  and  $Q'$  be the quivers obtained from their Auslander-Reiten quivers by removing all loops and oriented 2-cycles.

Our aim, in this section, is to give a rule relating  $Q'$  to  $Q$ , and to prove that it generalizes the Fomin-Zelevinsky mutation rule.

*Remark:*

- . Assume that  $\mathcal{C}$  has cluster-tilting objects. Then it is proved in [3, Theorem I.1.6], without assuming that  $\mathcal{C}$  is algebraic, that the Auslander-Reiten quivers of two cluster-tilting objects having all but one indecomposable direct summands in common (up to isomorphism) are related by the Fomin-Zelevinsky mutation rule.
- . To prove that the generalized mutation rule actually generalizes the Fomin-Zelevinsky mutation rule, we use the ideas of section 7 of [9].

**3.1. The rule.** As in section 2, we fix a cluster-tilting subcategory  $\mathcal{T}$  of  $\mathcal{C}$ , and write  $\mathcal{M}$  for its preimage in  $\mathcal{E}$ , so that  $\mathcal{T} = \underline{\mathcal{M}}$ . Define  $Q$  to be the quiver obtained from the Auslander–Reiten quiver of  $\underline{\mathcal{M}}$  by deleting its loops and its oriented 2-cycles. Its vertex corresponding to an indecomposable object  $L$  will also be labeled by  $L$ . We denote by  $a_{LN}$  the number of arrows from vertex  $L$  to vertex  $N$  in the quiver  $Q$ . Let  $B_{\mathcal{M}}$  be the matrix whose entries are given by  $b_{LN} = a_{LN} - a_{NL}$ .

Let  $R_{\mathcal{M}}$  be the matrix of  $\langle \ , \ \rangle_a : K_0(\text{mod } \underline{\mathcal{M}}) \times K_0(\text{mod } \underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$  in the basis given by the classes of the simple modules.

**Lemma 11.** *The matrices  $R_{\mathcal{M}}$  and  $B_{\mathcal{M}}$  are equal:  $R_{\mathcal{M}} = B_{\mathcal{M}}$ .*

*Proof.* Let  $L$  and  $N$  be two non-projective indecomposable objects in  $\mathcal{M}$ . Then  $\dim \text{Hom}(S_L, S_N) - \dim \text{Hom}(S_N, S_L) = 0$  and we have:

$$\langle [S_L], [S_N] \rangle_a = \dim \text{Ext}^1(S_N, S_L) - \dim \text{Ext}^1(S_L, S_N) = b_{L,N}.$$

□

Let  $\mathcal{T}'$  be another cluster-tilting subcategory of  $\mathcal{C}$ , and let  $\mathcal{M}'$  be its preimage in the Frobenius category  $\mathcal{E}$ . Let  $(M'_i)_{i \in I}$  (resp.  $(M_j)_{j \in J}$ ) be representatives for the isoclasses of non-projective indecomposable objects in  $\mathcal{M}'$  (resp.  $\mathcal{M}$ ). The equivalence of categories  $\text{per}_{\underline{\mathcal{M}}} \mathcal{M} \xrightarrow{\sim} \text{per}_{\underline{\mathcal{M}'}} \mathcal{M}'$  of proposition 8 induces an isomorphism between the Grothendieck groups  $K_0(\text{mod } \underline{\mathcal{M}})$  and  $K_0(\text{mod } \underline{\mathcal{M}'})$  whose matrix, in the bases given by the classes of the simple modules, is denoted by  $S$ . The equivalence of categories  $\mathcal{D} \text{Mod } \mathcal{M} \xrightarrow{\sim} \mathcal{D} \text{Mod } \mathcal{M}'$  restricts to the identity on  $\mathcal{H}^b(\mathcal{P})$ , so that it induces an equivalence  $\text{per } \mathcal{M} / \text{per } \mathcal{P} \xrightarrow{\sim} \text{per } \mathcal{M}' / \text{per } \mathcal{P}$ . Let  $T$  be the matrix of the induced isomorphism from  $K_0(\text{proj } \mathcal{M}) / K_0(\text{proj } \mathcal{P})$  to  $K_0(\text{proj } \mathcal{M}') / K_0(\text{proj } \mathcal{P})$ , in the bases given by the classes  $[\mathcal{M}(?, M_j)]$ ,  $j \in J$ , and  $[\mathcal{M}'(? , M'_i)]$ ,  $i \in I$ . The matrix  $T$  is much easier to compute than the matrix  $S$ . Its entries  $t_{ij}$  are given by the approximation triangles of Keller and Reiten in the following way: For all  $j$ , there exists a triangle of the form

$$\Sigma^{-1} M_j \longrightarrow \bigoplus_i \beta_{ij} M'_i \longrightarrow \bigoplus_i \alpha_{ij} M'_i \longrightarrow M_j.$$

Then, we have:

**Theorem 12.** a) (Generalized mutation rule) *The following equalities hold:*

$$t_{ij} = \alpha_{ij} - \beta_{ij}$$

and

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^t.$$

- b) *The category  $\mathcal{C}$  has a cluster-tilting object if and only if all its cluster-tilting subcategories have a finite number of pairwise non-isomorphic indecomposable objects.*  
c) *All cluster-tilting objects of  $\mathcal{C}$  have the same number of indecomposable direct summands (up to isomorphism).*

Note that point c) was shown in [11, 5.3.3(1)] (see also [3, I.1.8]) and, in a more general context, in [7]. Note also that, for the generalized mutation rule to hold, the cluster-tilting subcategories do not need to be related by a sequence of mutation.

*Proof.* Assertions b) and c) are consequences of the existence of an isomorphism between the Grothendieck groups  $K_0(\text{mod } \underline{\mathcal{M}})$  and  $K_0(\text{mod } \underline{\mathcal{M}'})$ . Let us prove the equalities a). Recall from [19, section 3.3], that the antisymmetric bilinear form

$\langle \cdot, \cdot \rangle_a$  on  $\text{mod } \underline{\mathcal{M}}$  is induced by the usual Euler form  $\langle \cdot, \cdot \rangle_E$  on  $\text{per } \underline{\mathcal{M}} \mathcal{M}$ . The following commutative diagram

$$\begin{array}{ccc} \text{per } \underline{\mathcal{M}} \mathcal{M} \times \text{per } \underline{\mathcal{M}} \mathcal{M} & \xrightarrow{\cong} & \text{per } \underline{\mathcal{M}'} \mathcal{M}' \times \text{per } \underline{\mathcal{M}'} \mathcal{M}' \\ & \searrow \langle \cdot, \cdot \rangle_E & \swarrow \langle \cdot, \cdot \rangle_E \\ & \mathbb{Z} & \end{array},$$

thus induces a commutative diagram

$$\begin{array}{ccc} K_0(\text{mod } \underline{\mathcal{M}}) \times K_0(\text{mod } \underline{\mathcal{M}}) & \xrightarrow{S \times S} & K_0(\text{mod } \underline{\mathcal{M}'}) \times K_0(\text{mod } \underline{\mathcal{M}'}) \\ & \searrow \langle \cdot, \cdot \rangle_a & \swarrow \langle \cdot, \cdot \rangle_a \\ & \mathbb{Z} & \end{array}.$$

This proves the equality  $R_{\mathcal{M}} = S^t R_{\mathcal{M}'} S$ , or, by lemma 11,

$$(1) \quad B_{\mathcal{M}} = S^t B_{\mathcal{M}'} S.$$

Any object of  $\text{per } \underline{\mathcal{M}} \mathcal{M}$  becomes an object of  $\text{per } \mathcal{M} / \text{per } \mathcal{P}$  through the composition  $\text{per } \underline{\mathcal{M}} \mathcal{M} \hookrightarrow \text{per } \mathcal{M} \twoheadrightarrow \text{per } \mathcal{M} / \text{per } \mathcal{P}$ . Let  $M$  and  $N$  be two non-projective indecomposable objects in  $\mathcal{M}$ . Since  $S_N$  vanishes on  $\mathcal{P}$ , we have

$$\begin{aligned} \text{Hom}_{\text{per } \mathcal{M} / \text{per } \mathcal{P}}(\mathcal{M}(?, M), S_N) &= \text{Hom}_{\text{per } \mathcal{M}}(\mathcal{M}(?, M), S_N) \\ &= \text{Hom}_{\text{Mod } \mathcal{M}}(\mathcal{M}(?, M), S_N) \\ &= S_N(M). \end{aligned}$$

Thus  $\dim \text{Hom}_{\text{per } \mathcal{M} / \text{per } \mathcal{P}}(\mathcal{M}(?, M), S_N) = \delta_{MN}$ , and the commutative diagram

$$\begin{array}{ccc} \text{per } \mathcal{M} / \text{per } \mathcal{P} \times \text{per } \mathcal{M} / \text{per } \mathcal{P} & \xrightarrow{\cong} & \text{per } \mathcal{M}' / \text{per } \mathcal{P} \times \text{per } \mathcal{M}' / \text{per } \mathcal{P} \\ & \searrow R\mathcal{H}om & \swarrow R\mathcal{H}om \\ & \text{per } k & \end{array},$$

induces a commutative diagram

$$\begin{array}{ccc} K_0(\text{proj } \mathcal{M}) / K_0(\text{proj } \mathcal{P}) \times K_0(\text{mod } \underline{\mathcal{M}}) & \xrightarrow{T \times S} & K_0(\text{proj } \mathcal{M}') / K_0(\text{proj } \mathcal{P}) \times K_0(\text{mod } \underline{\mathcal{M}'}) \\ & \searrow \text{Id} & \swarrow \text{Id} \\ & \mathbb{Z} & \end{array}.$$

In other words, the matrix  $S$  is the inverse of the transpose of  $T$ :

$$(2) \quad S = T^{-t}$$

Equalities (1) and (2) imply what was claimed, that is

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^t.$$

Let us compute the matrix  $T$ : Let  $M$  be indecomposable non-projective in  $\mathcal{M}$ , and let

$$\Sigma^{-1}M \longrightarrow M'_1 \longrightarrow M'_0 \longrightarrow M$$

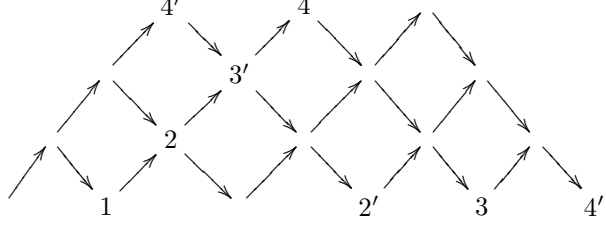
be a Keller–Reiten approximation triangle of  $M$  with respect to  $\mathcal{M}'$ , which we may assume to come from a conflation in  $\mathcal{E}$ . This conflation yields a projective resolution

$$0 \longrightarrow (M'_1)^\wedge \longrightarrow (M'_0)^\wedge \longrightarrow \mathcal{E}(?, M)|_{\mathcal{M}'} \longrightarrow \text{Ext}_{\mathcal{E}}^1(?, M'_1)|_{\mathcal{M}'} = 0.$$

so that  $T$  sends the class of  $M^\wedge$  to  $[(M'_0)^\wedge] - [(M'_1)^\wedge]$ . Therefore,  $t_{ij}$  equals  $\alpha_{ij} - \beta_{ij}$ .  $\square$

### 3.2. Examples.

3.2.1. As a first example, let  $\mathcal{C}$  be the cluster category associated with the quiver of type  $A_4$ :  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ . Its Auslander–Reiten quiver is the Moebius strip:



Let  $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$ , where the indecomposable  $M_i$  corresponds to the vertex labelled by  $i$  in the picture. Let also  $M' = M'_1 \oplus M'_2 \oplus M'_3 \oplus M'_4$ , where  $M'_1 = M_1$ , and where the indecomposable  $M'_i$  corresponds to the vertex labelled by  $i'$  if  $i \neq 1$ . One easily computes the following Keller–Reiten approximation triangles:

$$\begin{aligned} \Sigma^{-1}M_1 &\longrightarrow 0 \longrightarrow M'_1 \longrightarrow M_1, \\ \Sigma^{-1}M_2 &\longrightarrow M'_2 \longrightarrow M'_1 \longrightarrow M_2, \\ \Sigma^{-1}M_3 &\longrightarrow M'_4 \longrightarrow 0 \longrightarrow M_4 \text{ and} \\ \Sigma^{-1}M_4 &\longrightarrow M'_4 \longrightarrow M'_3 \longrightarrow M_4; \end{aligned}$$

so that the matrix  $T$  is given by:

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

We also have

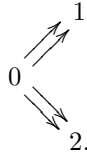
$$B_{M'} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let maple compute

$$T^{-1}B_{M'}T^{-t} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

which is  $B_M$ .

3.2.2. Let us look at a more interesting example, where one cannot easily read the quiver of  $M'$  from the Auslander–Reiten quiver of  $\mathcal{C}$ . Let  $\mathcal{C}$  be the cluster category associated with the quiver  $Q$ :

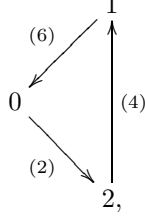


For  $i = 0, 1, 2$ , let  $M_i$  be (the image in  $\mathcal{C}$  of) the projective indecomposable (right)  $kQ$ -module associated with vertex  $i$ . Their dimension vectors are respectively  $[1, 0, 0]$ ,  $[2, 1, 0]$  and  $[2, 0, 1]$ . Let  $M$  be the direct sum  $M_0 \oplus M_1 \oplus M_2$ . Let  $M'$  be the direct sum  $M'_0 \oplus M'_1 \oplus M'_2$ , where  $M'_0, M'_1$  and  $M'_2$  are (the images in  $\mathcal{C}$  of) the indecomposable regular  $kQ$ -modules with dimension vectors  $[1, 2, 0]$ ,  $[0, 1, 0]$

and  $[2, 4, 1]$  respectively. As one can check, using [14],  $M$  and  $M'$  are two cluster-tilting objects of  $\mathcal{C}$ . To compute Keller–Reiten’s approximation triangles, amounts to computing projective resolutions in  $\text{mod } kQ$ , viewed as  $\text{mod } \text{End}_{\mathcal{C}}(M)$ . One easily computes these projective resolutions, by considering dimension vectors:

$$\begin{aligned} 0 &\longrightarrow 8M_0 \longrightarrow M_2 \oplus 4M_1 \longrightarrow M'_2 \longrightarrow 0, \\ 0 &\longrightarrow 2M_0 \longrightarrow M_1 \longrightarrow M'_1 \longrightarrow 0 \text{ and} \\ 0 &\longrightarrow 3M_0 \longrightarrow 2M_1 \longrightarrow M'_0 \longrightarrow 0. \end{aligned}$$

By applying the generalized mutation rule, one gets the following quiver



which is therefore the quiver of  $\text{End}_{\mathcal{C}}(M')$  since by [5], there are no loops or 2-cycles in the quiver of the endomorphism algebra of a cluster-tilting object in a cluster category.

**3.3. Back to the mutation rule.** We assume in this section that the Auslander–Reiten quiver of  $\mathcal{T}$  has no loops nor 2-cycles. Under the notations of section 3.1, let  $k$  be in  $I$  and let  $(M_k, M'_k)$  be an exchange pair (see section 2.3). We choose  $\underline{\mathcal{M}}'$  to be the cluster-tilting subcategory of  $\mathcal{C}$  obtained from  $\underline{\mathcal{M}}$  by replacing  $M_k$  by  $M'_k$ , so that  $M'_i = M_i$  for all  $i \neq k$ . Recall that  $T$  is the matrix of the isomorphism  $\text{K}_0(\text{proj } \mathcal{M}) / \text{K}_0(\text{proj } \mathcal{P}) \longrightarrow \text{K}_0(\text{proj } \mathcal{M}') / \text{K}_0(\text{proj } \mathcal{P})$ .

**Lemma 13.** *Then, the  $(i, j)$ -entry of the matrix  $T$  is given by*

$$t_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| + b_{ij}}{2} & \text{if } j = k \\ \delta_{ij} & \text{else.} \end{cases}$$

*Proof.* Let us apply theorem 12 to compute the matrix  $T$ . For all  $j \neq k$ , the triangle  $\Sigma^{-1}M_j \rightarrow 0 \rightarrow M'_j = M_j$  is a Keller–Reiten approximation triangle of  $M_j$  with respect to  $\mathcal{M}'$ . We thus have  $t_{ij} = \delta_{ij}$  for all  $j \neq k$ . There is a triangle unique up to isomorphism

$$M'_k \longrightarrow B_{M_k} \longrightarrow M_k \longrightarrow \Sigma M'_k$$

where  $B_{M_k} \rightarrow M_k$  is a right  $\mathcal{T} \cap \mathcal{T}'$ -approximation. Since the Auslander–Reiten quiver of  $\mathcal{T}$  has no loops and no 2-cycles,  $B_{M_k}$  is isomorphic to the direct sum:  $\bigoplus_{i \in I} (M'_i)^{a_{ik}}$ . We thus have  $t_{ik} = -\delta_{ik} + a_{ik}$ , which equals  $\frac{|b_{ik}| + b_{ik}}{2}$ . Remark that, by lemma 7.1 of [9], as stated in section 1.1, we have  $T^2 = Id$ , so that  $S = T^t$  and

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| - b_{ij}}{2} & \text{if } i = k \\ \delta_{ij} & \text{else.} \end{cases}$$

□

**Theorem 14.** *The matrix  $B_{\mathcal{M}'}$  is obtained from the matrix  $B_{\mathcal{M}}$  by the Fomin–Zelevinski mutation rule in the direction  $M$ .*

*Proof.* By [2] (see section 1.1), and by lemma 13, we know that the mutation of the matrix  $B_{\mathcal{M}}$  in direction  $M$  is given by  $TB_{\mathcal{M}}T^t$ , which is  $B_{\mathcal{M}'}$ , by the generalized mutation rule (theorem 12). □

**3.4. Cluster categories.** In [1], the authors study the Grothendieck group of the cluster category  $\mathcal{C}_A$  associated to an algebra  $A$  which is either hereditary or canonical, endowed with any admissible triangulated structure. A triangulated structure on the category  $\mathcal{C}_A$  is called admissible in [1] if the projection functor from the bounded derived category  $\mathcal{D}^b(\text{mod } A)$  to  $\mathcal{C}_A$  is exact (triangulated). They define a Grothendieck group  $\overline{K}_0(\mathcal{C}_A)$  with respect to the triangles induced by those of  $\mathcal{D}^b(\text{mod } A)$ , and show that it coincides with the usual Grothendieck group of the cluster category in many cases:

**Theorem 15.** [Barot–Kussin–Lenzing] *We have  $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$  in each of the following three cases:*

- (i)  *$A$  is canonical with weight sequence  $(p_1, \dots, p_t)$  having at least one even weight.*
- (ii)  *$A$  is tubular,*
- (iii)  *$A$  is hereditary of finite representation type.*

Under some restriction on the triangulated structure of  $\mathcal{C}_A$ , we have the following generalization of case (iii) of theorem 15:

**Theorem 16.** *Let  $A$  be a finite-dimensional hereditary algebra, and let  $\mathcal{C}_A$  be the associated cluster category with its triangulated structure defined in [16]. Then we have  $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$ .*

*Proof.* By lemma 3.2 in [1], this theorem is a corollary of the following lemma.  $\square$

**Lemma 17.** *Under the assumptions of section 3.1, and if moreover  $\underline{M}$  has a finite number  $n$  of non-isomorphic indecomposable objects, then we have an isomorphism  $K_0(\mathcal{C}) \simeq \mathbb{Z}^n / \text{Im } B_{\mathcal{M}}$ .*

*Proof.* This is a restatement of theorem 10.  $\square$

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