



Grothendieck Group and Generalized Mutation Rule for 2-Calabi–Yau Triangulated Categories

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GROTHENDIECK GROUP AND GENERALIZED MUTATION RULE FOR 2-CALABI-YAU TRIANGULATED CATEGORIES

YANN PALU

ABSTRACT. We compute the Grothendieck group of certain 2-Calabi-Yau triangulated categories appearing naturally in the study of the link between quiver representations and Fomin-Zelevinsky's cluster algebras. In this setup, we also prove a generalization of Fomin-Zelevinsky's mutation rule.

INTRODUCTION

In their study [5] of the connections between cluster algebras (see [21]) and quiver representations, P. Caldero and B. Keller conjectured that a certain antisymmetric bilinear form is well-defined on the Grothendieck group of a cluster-tilted algebra associated with a finite-dimensional hereditary algebra. The conjecture was proved in [18] in the more general context of Hom-finite 2-Calabi-Yau triangulated categories. It was used in order to study the existence of a cluster character on such a category $\mathcal{C}$, by using a formula proposed by Caldero-Keller.

In the present paper, we restrict to the case where $\mathcal{C}$ is algebraic (i.e. is the stable category of a Frobenius category). We first use this bilinear form to prove a generalized mutation rule for quivers of cluster-tilting subcategories in $\mathcal{C}$. When the cluster-tilting subcategories are related by a single mutation, this shows, via the method of [8], that their quivers are related by the Fomin-Zelevinsky mutation rule. This special case was already proved in [2], without assuming $\mathcal{C}$ to be algebraic.

We also compute the Grothendieck group of the triangulated category $\mathcal{C}$. In particular, this allows us to improve on results by M. Barot, D. Kussin and H. Lenzing: We compare the Grothendieck group of a cluster category $\mathcal{C}_A$ with the group $\overline{K}_0(\mathcal{C}_A)$. The latter group was defined in [1] by only considering the triangles in $\mathcal{C}_A$ which are induced by those of the derived category. More precisely, we prove that those two groups are isomorphic for any cluster category associated with a finite dimensional hereditary algebra, with its triangulated structure defined by B. Keller in [15].

This paper is organized as follows: The first section is dedicated to notation and necessary background from [7], [8], [16], [18]. In section 2, we compute the Grothendieck group of the triangulated category $\mathcal{C}$. In section 3, we prove a generalized mutation rule for quivers of cluster-tilting subcategories in $\mathcal{C}$. In particular, this yields a new proof of the Fomin-Zelevinsky mutation rule, under the restriction that $\mathcal{C}$ is algebraic. We finally show that $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$ for any finite dimensional hereditary algebra A .

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1. NOTATIONS AND BACKGROUND

Let $\mathcal{E}$ be a Frobenius category whose idempotents split and which is linear over a given algebraically closed field k . By a result of Happel [9], its stable category $\mathcal{C} = \underline{\mathcal{E}}$ is triangulated. We assume moreover, that $\mathcal{C}$ is Hom-finite, 2-Calabi–Yau and has a cluster-tilting subcategory (see section 1.2), and we denote by Σ its suspension functor. Note that we do not assume that $\mathcal{E}$ is Hom-finite.

We write $\mathcal{X}(\ , \)$, or $\text{Hom}_{\mathcal{X}}(\ , \)$, for the morphisms in a category $\mathcal{X}$ and $\text{Hom}_{\mathcal{X}}(\ , \)$ for the morphisms in the category of $\mathcal{X}$ -modules. We also denote by $X^\wedge$ the projective $\mathcal{X}$ -module represented by X : $X^\wedge = \mathcal{X}(\ ?, X)$.

1.1. Fomin–Zelevinsky mutation for matrices. Let $B = (b_{ij})_{i,j \in I}$ be a finite or infinite matrix, and let k be in I . The Fomin and Zelevinsky mutation of B (see [7]) in direction k is the matrix

$$\mu_k(B) = (b'_{ij})$$

defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{else.} \end{cases}$$

Note that $\mu_k(\mu_k(B)) = B$ and that if B is skew-symmetric, then so is $\mu_k(B)$.

We recall some results from [8, section 7], stated for infinite matrices, which will be useful in section 3. Let $S = (s_{ij})$ be the matrix defined by

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| - b_{ij}}{2} & \text{if } i = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

Lemma 7.1 [8, Geiss–Leclerc–Schröer]: *Assume that B is skew-symmetric. Then, $S^2 = 1$ and the (i, j) -entry of the transpose of the matrix S is given by*

$$s_{ij}^t = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| + b_{ij}}{2} & \text{if } j = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

The matrix S yields a convenient way to describe the mutation of B in the direction k :

Lemma 7.2 [8, Geiss–Leclerc–Schröer] : *Assume that B is skew-symmetric. Then we have:*

$$\mu_k(B) = S^t B S.$$

Note that the product is well-defined since the matrix S has a finite number of non vanishing entries in each column.

1.2. Cluster-tilting subcategories. A cluster-tilting subcategory (see [16]) of $\mathcal{C}$ is a full subcategory $\mathcal{T}$ such that

- a) $\mathcal{T}$ is a linear subcategory;
- b) for any object X in $\mathcal{C}$, the contravariant functor $\mathcal{C}(?, X)|_{\mathcal{T}}$ is finitely generated;
- c) for any object X in $\mathcal{C}$, we have $\mathcal{C}(X, \Sigma T) = 0$ for all T in $\mathcal{T}$ if and only if X belongs to $\mathcal{T}$.

We now recall some results from [16], which we will use in the sequel. Let $\mathcal{T}$ be a cluster-tilting subcategory of $\mathcal{C}$, and denote by $\mathcal{M}$ its preimage in $\mathcal{E}$. In particular $\mathcal{M}$ contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective-injective objects, and we have $\underline{\mathcal{M}} = \mathcal{T}$.

The following proposition will be used implicitly, extensively in this paper.

Proposition [16, Keller–Reiten] :

- a) *The category $\text{mod } \underline{\mathcal{M}}$ of finitely presented $\underline{\mathcal{M}}$ -modules is abelian.*
- b) *For each object $X \in \mathcal{C}$, there is a triangle*

$$\Sigma^{-1} X \longrightarrow T_1^X \longrightarrow T_0^X \longrightarrow X$$

of $\mathcal{C}$, with T_0^X and T_1^X in $\mathcal{T}$.

Recall that the perfect derived category $\text{per } \mathcal{M}$ is the full triangulated subcategory of the derived category of $\mathcal{D} \text{Mod } \mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$ -modules.

Proposition [16, Keller–Reiten] :

- a) *For each $X \in \mathcal{E}$, there are conflations*

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X \longrightarrow M^0 \longrightarrow M^1 \longrightarrow 0$$

in $\mathcal{E}$, with M_0, M_1, M^0 and M^1 in $\mathcal{M}$.

- b) *Let Z be in $\text{mod } \underline{\mathcal{M}}$. Then Z considered as an $\mathcal{M}$ -module lies in the perfect derived category $\text{per } \mathcal{M}$ and we have canonical isomorphisms*

$$D(\text{per } \mathcal{M})(Z, ?) \simeq (\text{per } \mathcal{M})(?, Z[3]).$$

1.3. The antisymmetric bilinear form. In section 3, we will use the existence of the antisymmetric bilinear form $\langle \ , \ \rangle_a$ on $K_0(\text{mod } \underline{\mathcal{M}})$. We thus recall its definition from [5].

Let $\langle \ , \ \rangle$ be a truncated Euler form on $\text{mod } \underline{\mathcal{M}}$ defined by

$$\langle M, N \rangle = \dim \text{Hom}_{\underline{\mathcal{M}}}(M, N) - \dim \text{Ext}_{\underline{\mathcal{M}}}^1(M, N)$$

for any $M, N \in \text{mod } \underline{\mathcal{M}}$. Define $\langle \ , \ \rangle_a$ to be the antisymmetrization of this form:

$$\langle M, N \rangle_a = \langle M, N \rangle - \langle N, M \rangle.$$

This bilinear form descends to the Grothendieck group $K_0(\text{mod } \underline{\mathcal{M}})$:

Lemma [18, section 3] : The antisymmetric bilinear form

$$\langle M, N \rangle_a : K_0(\text{mod } \underline{\mathcal{M}}) \times K_0(\text{mod } \underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$$

is well-defined.

2. GROTHENDIECK GROUPS OF ALGEBRAIC 2-CY CATEGORIES WITH A CLUSTER-TILTING SUBCATEGORY

We fix a cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{C}$, and we denote by $\mathcal{M}$ its preimage in $\mathcal{E}$. In particular $\mathcal{M}$ contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective-injective objects, and we have $\underline{\mathcal{M}} = \mathcal{T}$.

We denote by $\mathcal{H}^b(\mathcal{E})$ and $\mathcal{D}^b(\mathcal{E})$ respectively the bounded homotopy category and the bounded derived category of $\mathcal{E}$. We also denote by $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{E})$, $\mathcal{H}^b(\mathcal{P})$, $\mathcal{H}^b(\mathcal{M})$ and $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$ the full subcategories of $\mathcal{H}^b(\mathcal{E})$ whose objects are the $\mathcal{E}$ -acyclic complexes, the complexes of projective objects in $\mathcal{E}$, the complexes of objects of $\mathcal{M}$ and the $\mathcal{E}$ -acyclic complexes of objects of $\mathcal{M}$, respectively.

2.1. A short exact sequence of triangulated categories.

Lemma 1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be thick, full triangulated subcategories of a triangulated category $\mathcal{A}$ and let $\mathcal{B}$ be $\mathcal{A}_1 \cap \mathcal{A}_2$. Assume that for any object X in $\mathcal{A}$ there is a triangle $X_1 \rightarrow X \rightarrow X_2 \rightarrow \Sigma X_1$ in $\mathcal{A}$, with X_1 in $\mathcal{A}_1$ and X_2 in $\mathcal{A}_2$. Then the induced functor $\mathcal{A}_1/\mathcal{B} \rightarrow \mathcal{A}/\mathcal{A}_2$ is a triangle equivalence.*

Proof. Under these assumptions, denote by F the induced triangle functor from $\mathcal{A}_1/\mathcal{B}$ to $\mathcal{A}/\mathcal{A}_2$. We are going to show that the functor F is a full, conservative, dense functor. Since any full conservative triangle functor is fully faithful, F will then be an equivalence of categories.

We first show that F is full. Let X_1 and X'_1 be two objects in $\mathcal{A}_1$. Let f be a morphism from X_1 to X'_1 in $\mathcal{A}/\mathcal{A}_2$ and let

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow w \\ X_1 & & X'_1 \end{array}$$

be a left fraction which represents f . The morphism w is in the multiplicative system associated with $\mathcal{A}_2$ and thus yields a triangle $\Sigma^{-1}A_2 \rightarrow Y \xrightarrow{w} X'_1 \rightarrow A_2$ where A_2 lies in the subcategory $\mathcal{A}_2$. Moreover, by assumption, there exists a triangle $Y_1 \rightarrow Y \rightarrow Y_2 \rightarrow \Sigma Y_1$ with Y_i in $\mathcal{A}_i$. Applying the octahedral axiom to the composition $Y_1 \rightarrow Y \rightarrow X'_1$ yields a commutative diagram whose two middle rows and columns are triangles in $\mathcal{A}$

$$\begin{array}{ccccccc} & & \Sigma^{-1}A_2 & \xlongequal{\quad} & \Sigma^{-1}A_2 & & \\ & & \downarrow & & \downarrow & & \\ Y_1 & \longrightarrow & Y & \longrightarrow & Y_2 & \longrightarrow & \Sigma Y_1 \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ Y_1 & \longrightarrow & X'_1 & \longrightarrow & Z & \longrightarrow & \Sigma Y_1 \\ & & \downarrow & & \downarrow & & \\ & & A_2 & \xlongequal{\quad} & A_2 & & . \end{array}$$

Since Y_2 and A_2 belong to $\mathcal{A}_2$, so does Z . And since X'_1 and Y_1 belong to $\mathcal{A}_1$, so does Z . This implies, that Z belongs to $\mathcal{B}$. The morphism $Y_1 \rightarrow X'_1$ is in the multiplicative system of $\mathcal{A}_1$ associated with $\mathcal{B}$ and the diagram

$$\begin{array}{ccc} & Y_1 & \\ \swarrow & & \searrow \\ X_1 & & X'_1 \end{array}$$

is a left fraction which represents f . This implies that f is the image of a morphism in $\mathcal{A}_1/\mathcal{B}$. Therefore the functor F is full.

We now show that F is conservative. Let $X_1 \xrightarrow{f} Y_1 \rightarrow Z_1 \rightarrow \Sigma X_1$ be a triangle in $\mathcal{A}_1$. Assume that Ff is an isomorphism in $\mathcal{A}/\mathcal{A}_2$, which implies that Z_1 is an object of $\mathcal{A}_2$. Therefore, Z_1 is an object of $\mathcal{B}$ and f is an isomorphism in $\mathcal{A}_1/\mathcal{B}$.

We finally show that F is dense. Let X be an object of the category $\mathcal{A}/\mathcal{A}_2$, and let $X_1 \rightarrow X \rightarrow X_2 \rightarrow \Sigma X_1$ be a triangle in $\mathcal{A}$ with X_i in $\mathcal{A}_i$. Since X_2 belongs to $\mathcal{A}_2$, the image of the morphism $X_1 \rightarrow X$ in $\mathcal{A}/\mathcal{A}_2$ is an isomorphism. Thus X is isomorphic to the image by F of an object in $\mathcal{A}_1/\mathcal{B}$. $\square$

As a corollary, we have the following:

Lemma 2. *The following sequence of triangulated categories is short exact:*

$$0 \longrightarrow \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M}) \longrightarrow \mathcal{D}^b(\mathcal{E}) \longrightarrow 0.$$

Remark: This lemma remains true if $\mathcal{C}$ is d -Calabi–Yau and $\underline{\mathcal{M}}$ is $(d-1)$ -cluster-tilting, using section 5.4 of [16].

Proof. For any object X in $\mathcal{H}^b(\mathcal{E})$, the existence of an object M in $\mathcal{H}^b(\mathcal{M})$ and of a quasi-isomorphism w from M to X is obtained using the approximation conflations of Keller–Reiten (see section 1.2). Since the cone of the morphism w belongs to $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{E})$, lemma 1 applies to the subcategories $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$, $\mathcal{H}^b(\mathcal{M})$ and $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{E})$ of $\mathcal{H}^b(\mathcal{E})$. $\square$

Proposition 3. *The following diagram is commutative with exact rows and columns:*

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) & \xrightarrow{i_{\mathcal{M}}} & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) & \longrightarrow & \mathcal{E} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) & \longrightarrow & \mathcal{H}^b(\mathcal{M}) & \longrightarrow & \mathcal{D}^b(\mathcal{E}) \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow i_{\mathcal{P}} \\
 & & & & \mathcal{H}^b(\mathcal{P}) & \xlongequal{\quad} & \mathcal{H}^b(\mathcal{P}) \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array} \quad (D)$$

Proof. The column on the right side has been shown to be exact in [17] and [19]. The second row is exact by lemma 2. The subcategories $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$ and $\mathcal{H}^b(\mathcal{P})$ of $\mathcal{H}^b(\mathcal{M})$ are left and right orthogonal to each other. This implies that the induced functors $i_{\mathcal{M}}$ and $i_{\mathcal{P}}$ are fully faithful and that taking the quotient of $\mathcal{H}^b(\mathcal{M})$ by those two subcategories either in one order or in the other gives the same category. Therefore the first row is exact. $\square$

2.2. Invariance under mutation. A natural question is then to which extent the diagram (D) depends on the choice of a particular cluster-tilting subcategory. Let thus $\mathcal{T}'$ be another cluster-tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be its preimage in $\mathcal{E}$. Let $\text{Mod } \mathcal{M}$ (resp. $\text{Mod } \mathcal{M}'$) be the category of $\mathcal{M}$ -modules (resp. $\mathcal{M}'$ -modules), i.e. of k -linear contravariant functors from $\mathcal{M}$ (resp. $\mathcal{M}'$) to the category of k -vector spaces.

Let X be the $\mathcal{M}$ - $\mathcal{M}'$ -bimodule which sends the pair of objects (M, M') to the k -vector space $\mathcal{E}(M, M')$. The bimodule X induces a functor $F = ? \otimes_{\mathcal{M}'} X : \text{Mod } \mathcal{M}' \longrightarrow \text{Mod } \mathcal{M}$ denoted by T_X in [14, section 6.1].

Recall that the perfect derived category $\text{per } \mathcal{M}$ is the full triangulated subcategory of the derived category $\mathcal{D} \text{Mod } \mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$ -modules.

Proposition 4. *The left derived functor*

$$\mathbb{L}F : \mathcal{D} \text{Mod } \mathcal{M}' \longrightarrow \mathcal{D} \text{Mod } \mathcal{M}$$

is an equivalence of categories.

Proof. Recall that if X is an object in a category $\mathcal{X}$, we denote by $X^\wedge$ the functor $\mathcal{X}(?, X)$ represented by X . By [14, 6.1], it is enough to check the following three properties:

1. For all objects M', M'' of $\mathcal{M}$, the group $\text{Hom}_{\mathcal{D} \text{Mod } \mathcal{M}}(\mathbb{L}FM', \mathbb{L}FM''[n])$ vanishes for $n \neq 0$ and identifies with $\text{Hom}_{\mathcal{M}'}(M', M'')$ for $n = 0$;
2. for any object M' of $\mathcal{M}'$, the complex $\mathbb{L}FM'$ belongs to $\text{per } \mathcal{M}$;
3. the set $\{\mathbb{L}FM', M' \in \mathcal{M}'\}$ generates $\mathcal{D} \text{Mod } \mathcal{M}$ as a triangulated category with infinite sums.

Let M' be an object of $\mathcal{M}'$, and let $M_1 \rightrightarrows M_0 \twoheadrightarrow M'$ be a conflation in $\mathcal{E}$, with M_0 and M_1 in $\mathcal{M}$, and whose deflation is a right $\mathcal{M}$ -approximation (c.f. section 4 of [16]). The surjectivity of the map $M_0^\wedge \rightarrow \mathcal{E}(?, M')|_{\mathcal{M}}$ implies that the complex $P = (\cdots \rightarrow 0 \rightarrow M_1^\wedge \rightarrow M_0^\wedge \rightarrow 0 \rightarrow \cdots)$ is quasi-isomorphic to $\mathbb{L}FM' = \mathcal{E}(?, M')|_{\mathcal{M}}$. Therefore $\mathbb{L}FM'$ belongs to the subcategory $\text{per } \mathcal{M}$ of $\mathcal{D} \text{Mod } \mathcal{M}$. Moreover, we have, for any $n \in \mathbb{Z}$ and any $M'' \in \mathcal{M}'$, the equality

$$\text{Hom}_{\mathcal{D} \text{Mod } \mathcal{M}}(\mathbb{L}FM', \mathbb{L}FM''[n]) = \text{Hom}_{\mathcal{H}^b \text{Mod } \mathcal{M}}(P, \mathcal{E}(?, M'')|_{\mathcal{M}}[n])$$

where the right-hand side vanishes for $n \neq 0, 1$. In case $n = 1$ it also vanishes, since $\text{Ext}_{\mathcal{E}}^1(M', M'')$ vanishes. Now,

$$\begin{aligned} \text{Hom}_{\mathcal{H}^b \text{Mod } \mathcal{M}}(P, \mathcal{E}(?, M'')|_{\mathcal{M}}) &\simeq \text{Ker}(\mathcal{E}(M_0, M'') \rightarrow \mathcal{E}(M_1, M'')) \\ &\simeq \mathcal{E}(M', M''). \end{aligned}$$

It only remains to be shown that the set $R = \{\mathbb{L}FM', M' \in \mathcal{M}'\}$ generates $\mathcal{D} \text{Mod } \mathcal{M}$. Denote by $\mathcal{R}$ the full triangulated subcategory with infinite sums of $\mathcal{D} \text{Mod } \mathcal{M}$ generated by the set R . The set $\{M, M \in \mathcal{M}\}$ generates $\mathcal{D} \text{Mod } \mathcal{M}$ as a triangulated category with infinite sums. Thus it is enough to show that, for any object M of $\mathcal{M}$, the complex $M^\wedge$ concentrated in degree 0 belongs to the subcategory $\mathcal{R}$. Let M be an object of $\mathcal{M}$, and let $M \rightrightarrows M'_0 \xrightarrow{p} M'_1$ be a conflation of $\mathcal{E}$ with M'_0 and M'_1 in $\mathcal{M}'$. Since $\text{Ext}_{\mathcal{E}}^1(?, M)|_{\mathcal{M}}$ vanishes, we have a short exact sequence of $\mathcal{M}$ -modules

$$0 \longrightarrow \mathcal{E}(?, M)|_{\mathcal{M}} \longrightarrow \mathcal{E}(?, M'_0)|_{\mathcal{M}} \longrightarrow \mathcal{E}(?, M'_1)|_{\mathcal{M}} \longrightarrow 0,$$

which yields the triangle

$$M^\wedge \longrightarrow \mathbb{L}FM'_0 \longrightarrow \mathbb{L}FM'_1 \longrightarrow \Sigma M^\wedge.$$

□

As a corollary of proposition 4, up to equivalence the diagram (D) does not depend on the choice of a cluster-tilting subcategory. To be more precise: Let G be the functor which sends an object X in the category $\mathcal{H}^b(\mathcal{M}')$ to a representative of $(\mathbb{L}F)X^\wedge$ in $\mathcal{H}^b(\mathcal{M})$, and a morphism in $\mathcal{H}^b(\mathcal{M}')$ to the induced one in $\mathcal{H}^b(\mathcal{M})$.

Corollary 5. *The following diagram is commutative*

$$\begin{array}{ccccc}
 & \mathcal{D}\text{Mod } \mathcal{M}' & \xrightarrow{\text{LF}} & \mathcal{D}\text{Mod } \mathcal{M} & \\
 & \nearrow & & \nearrow & \\
 \mathcal{H}^b(\mathcal{M}') & \xrightarrow{G} & \mathcal{H}^b(\mathcal{M}) & & \\
 & \searrow & & \searrow & \\
 & \mathcal{H}^b(\mathcal{P}) & \xlongequal{\quad} & \mathcal{H}^b(\mathcal{P}) & \\
 & \nwarrow & & \nwarrow & \\
 \mathcal{D}^b(\mathcal{E}) & \xlongequal{\quad} & \mathcal{D}^b(\mathcal{E}) & &
 \end{array}$$

and the functor G is an equivalence of categories.

We denote by $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$ the full subcategory of $\text{per } \mathcal{M}$ whose objects are the complexes with homologies in $\text{mod } \underline{\mathcal{M}}$. The following lemma will allow us to compute the Grothendieck group of $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$ in section 2.3:

Lemma 6. *The canonical t -structure on $\mathcal{D}\text{Mod } \mathcal{M}$ restricts to a t -structure on $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$, whose heart is $\text{mod } \underline{\mathcal{M}}$.*

Proof. By [12], it is enough to show that for any object $M^\bullet$ of $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$, its truncation $\tau_{\leq 0} M^\bullet$ in $\mathcal{D}\text{Mod } \mathcal{M}$ belongs to $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$. Since $M^\bullet$ is in $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$, $\tau_{\leq 0} M^\bullet$ is bounded, and is thus formed from the complexes $H^i(M^\bullet)$ concentrated in one degree by taking iterated extensions. But, for any i , the $\mathcal{M}$ -module $H^i(M^\bullet)$ actually is an $\underline{\mathcal{M}}$ -module. Therefore, by [16] (see section 1.2), it is perfect as an $\mathcal{M}$ -module and it lies in $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$. $\square$

The next lemma already appears in [20]. For the convenience of the reader, we include a proof.

Lemma 7. *The Yoneda equivalence of triangulated categories $\mathcal{H}^b(\mathcal{M}) \rightarrow \text{per } \mathcal{M}$ induces a triangle equivalence $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) \rightarrow \text{per}_{\underline{\mathcal{M}}} \mathcal{M}$.*

Proof. We first show that the cohomology groups of an $\mathcal{E}$ -acyclic bounded complex M vanish on $\mathcal{P}$. Let P be a projective object in $\mathcal{E}$ and let E be a kernel in $\mathcal{E}$ of the map $M^n \rightarrow M^{n+1}$. Since M is $\mathcal{E}$ -acyclic, such an object exists, and moreover, it is an image of the map $M^{n-1} \rightarrow M^n$. Any map from P to M^n whose composition with $M^n \rightarrow M^{n+1}$ vanishes factors through the kernel $E \hookrightarrow M^n$. Since P is projective, this factorization factors through the deflation $M^{n-1} \twoheadrightarrow E$.

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow & \downarrow & \searrow & \\
 M^{n-1} & \xrightarrow{\quad} & M^n & \xrightarrow{\quad} & M^{n+1} \\
 & \searrow & \downarrow & \swarrow & \\
 & & E & &
 \end{array}$$

Therefore, we have $H^n(M)(P) = 0$ for all projective objects P , and $H^n(M)$ belongs to $\text{mod } \underline{\mathcal{M}}$. Thus the Yoneda functor induces a fully faithful functor from $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$ to $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$. To prove that it is dense, it is enough to prove that any object of the heart $\text{mod } \underline{\mathcal{M}}$ of the t -structure on $\text{per}_{\underline{\mathcal{M}}} \mathcal{M}$ is in its essential image. But this was proved in [16, section 4] (see section 1.2). $\square$

Proposition 8. *There is a triangle equivalence of categories*

$$\mathrm{per}_{\underline{\mathcal{M}}} \mathcal{M} \xrightarrow{\simeq} \mathrm{per}_{\underline{\mathcal{M}'}} \mathcal{M}'$$

Proof. Since the categories $\mathcal{H}^b(\mathcal{P})$ and $\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$ are left-right orthogonal in $\mathcal{H}^b(\mathcal{M})$, this is immediate from corollary 5 and lemma 7. $\square$

2.3. Grothendieck groups. For a triangulated (resp. additive, resp. abelian) category $\mathcal{A}$, we denote by $K_0^{\mathrm{tri}}(\mathcal{A})$ or simply $K_0(\mathcal{A})$ (resp. $K_0^{\mathrm{add}}(\mathcal{A})$, resp. $K_0^{\mathrm{ab}}(\mathcal{A})$) its Grothendieck group (with respect to the mentioned structure of the category). For an object A in $\mathcal{A}$, we also denote by $[A]$ its class in the Grothendieck group of $\mathcal{A}$.

The short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P}) \longrightarrow \underline{\mathcal{E}} \longrightarrow 0$$

given by proposition 3 induces an exact sequence in the Grothendieck groups

$$(*) \quad K_0(\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})) \longrightarrow K_0(\mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P})) \longrightarrow K_0(\underline{\mathcal{E}}) \longrightarrow 0.$$

Lemma 9. *The exact sequence (*) is isomorphic to an exact sequence*

$$(**) \quad K_0^{\mathrm{ab}}(\mathrm{mod} \underline{\mathcal{M}}) \xrightarrow{\varphi} K_0^{\mathrm{add}}(\underline{\mathcal{M}}) \longrightarrow K_0^{\mathrm{tri}}(\underline{\mathcal{E}}) \longrightarrow 0.$$

Proof. First, note that, by [20], see also lemma 7, we have an isomorphism between the Grothendieck groups $K_0(\mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M}))$ and $K_0(\mathrm{per}_{\underline{\mathcal{M}}} \mathcal{M})$. The t-structure on $\mathrm{per}_{\underline{\mathcal{M}}} \mathcal{M}$ whose heart is $\mathrm{mod} \underline{\mathcal{M}}$, see lemma 6, in turn yields an isomorphism between the Grothendieck groups $K_0^{\mathrm{tri}}(\mathrm{per}_{\underline{\mathcal{M}}} \mathcal{M})$ and $K_0^{\mathrm{ab}}(\mathrm{mod} \underline{\mathcal{M}})$. Next, we show that the canonical additive functor $\underline{\mathcal{M}} \xrightarrow{\alpha} \mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P})$ induces an isomorphism between the Grothendieck groups $K_0^{\mathrm{add}}(\underline{\mathcal{M}})$ and $K_0^{\mathrm{tri}}(\mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P}))$. For this, let us consider the canonical additive functor $\underline{\mathcal{M}} \xrightarrow{\beta} \mathcal{H}^b(\underline{\mathcal{M}})$ and the triangle functor $\mathcal{H}^b(\mathcal{M}) \xrightarrow{\gamma} \mathcal{H}^b(\underline{\mathcal{M}})$. The following diagram describes the situation:

$$\begin{array}{ccc} \mathcal{H}^b(\underline{\mathcal{M}}) & \xleftarrow{\gamma} & \mathcal{H}^b(\mathcal{M}) \\ \beta \uparrow & \nwarrow \gamma & \downarrow \\ \underline{\mathcal{M}} & \xrightarrow{\alpha} & \mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P}) \end{array}$$

The functor γ vanishes on the full subcategory $\mathcal{H}^b(\mathcal{P})$, thus inducing a triangle functor, still denoted by γ , from $\mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P})$ to $\mathcal{H}^b(\underline{\mathcal{M}})$. Furthermore, the functor β induces an isomorphism at the level of Grothendieck groups, whose inverse $K_0(\beta)^{-1}$ is given by

$$\begin{aligned} K_0^{\mathrm{tri}}(\mathcal{H}^b(\underline{\mathcal{M}})) &\longrightarrow K_0^{\mathrm{add}}(\underline{\mathcal{M}}) \\ [M] &\longmapsto \sum_{i \in \mathbb{Z}} (-1)^i [M^i]. \end{aligned}$$

As the group $K_0^{\mathrm{tri}}(\mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P}))$ is generated by objects concentrated in degree 0, it is straightforward to check that the morphisms $K_0(\alpha)$ and $K_0(\beta)^{-1} K_0(\gamma)$ are inverse to each other. $\square$

As a consequence of the exact sequence (**), we have an isomorphism between $K_0^{\mathrm{tri}}(\underline{\mathcal{E}})$ and $K_0^{\mathrm{add}}(\underline{\mathcal{M}}) / \mathrm{Im} \varphi$. In order to compute $K_0^{\mathrm{tri}}(\underline{\mathcal{E}})$, the map φ has to be made explicit. We first recall some results from Iyama–Yoshino [11] which generalize results from [3]: For any indecomposable M of $\underline{\mathcal{M}}$ not in $\mathcal{P}$, there exists M^* unique up to isomorphism such that (M, M^*) is an exchange pair. This means that M and M^* are not isomorphic and that the full additive subcategory of $\mathcal{C}$ generated

by all the indecomposable objects of $\underline{\mathcal{M}}$ but those isomorphic to M , and by the indecomposable objects isomorphic to M^* is again a cluster-tilting subcategory. Moreover, $\dim \underline{\mathcal{E}}(M, \Sigma M^*) = 1$. We can thus fix two (non-split) exchange triangles

$$M^* \rightarrow B_M \rightarrow M \rightarrow \Sigma M^* \text{ and } M \rightarrow B_{M^*} \rightarrow M^* \rightarrow \Sigma M.$$

We may now state the following:

Theorem 10. *The Grothendieck group of the triangulated category $\underline{\mathcal{E}}$ is the quotient of that of the additive subcategory $\underline{\mathcal{M}}$ by all relations $[B_{M^*}] - [B_M]$:*

$$K_0^{tri}(\underline{\mathcal{E}}) \simeq K_0^{add}(\underline{\mathcal{M}})/([B_{M^*}] - [B_M])_M.$$

Proof. We denote by S_M the simple $\underline{\mathcal{M}}$ -module associated to the indecomposable object M . This means that $S_M(M')$ vanishes for all indecomposable objects M' in $\underline{\mathcal{M}}$ not isomorphic to M and that $S_M(M)$ is isomorphic to k . The abelian group $K_0^{ab}(\text{mod } \underline{\mathcal{M}})$ is generated by all classes $[S_M]$. In view of lemma 9, it is sufficient to prove that the image of the class $[S_M]$ under φ is $[B_{M^*}] - [B_M]$. First note that the $\mathcal{M}$ -module $\text{Ext}_{\mathcal{E}}^1(?, M^*)|_{\mathcal{M}}$ vanishes on the projectives; it can thus be viewed as an $\underline{\mathcal{M}}$ -module, and as such, is isomorphic to S_M . After replacing B_M and B_{M^*} by isomorphic objects of $\underline{\mathcal{E}}$, we can assume that the exchange triangles $M^* \rightarrow B_M \rightarrow M \rightarrow \Sigma M^*$ and $M \rightarrow B_{M^*} \rightarrow M^* \rightarrow \Sigma M$ come from conflations $M^* \twoheadrightarrow B_M \twoheadrightarrow M$ and $M \twoheadrightarrow B_{M^*} \twoheadrightarrow M^*$. The spliced complex

$$(\cdots \rightarrow 0 \rightarrow M \rightarrow B_{M^*} \rightarrow B_M \rightarrow M \rightarrow 0 \rightarrow \cdots)$$

denoted by $C^\bullet$, is then an $\mathcal{E}$ -acyclic complex, and it is the image of S_M under the functor $\text{mod } \underline{\mathcal{M}} \subset \text{per } \underline{\mathcal{M}} \mathcal{M} \simeq \mathcal{H}_{\mathcal{E}-ac}^b(\mathcal{M})$. Indeed, we have two long exact sequences induced by the conflations above:

$$0 \rightarrow \mathcal{M}(?, M) \rightarrow \mathcal{M}(?, B_{M^*}) \rightarrow \mathcal{E}(?, M^*)|_{\mathcal{M}} \rightarrow \text{Ext}_{\mathcal{E}}^1(?, M)|_{\mathcal{M}} = 0 \text{ and}$$

$$0 \rightarrow \mathcal{E}(?, M^*)|_{\mathcal{M}} \rightarrow \mathcal{M}(?, B_M) \rightarrow \mathcal{M}(?, M) \rightarrow \text{Ext}_{\mathcal{E}}^1(?, M^*)|_{\mathcal{M}} \rightarrow \text{Ext}_{\mathcal{E}}^1(?, B_M)|_{\mathcal{M}}.$$

Since B_M belongs to $\mathcal{M}$, the functor $\text{Ext}_{\mathcal{E}}^1(?, B_M)$ vanishes on $\mathcal{M}$, and the complex:

$$(C^\bullet) : (\cdots \rightarrow 0 \rightarrow M^\wedge \rightarrow (B_{M^*})^\wedge \rightarrow (B_M)^\wedge \rightarrow M^\wedge \rightarrow 0 \rightarrow \cdots)$$

is quasi-isomorphic to S_M .

Now, in the notations of the proof of lemma 9, $\varphi[S_M]$ is the image of the class of the $\mathcal{E}$ -acyclic complex $C^\bullet$ under the morphism $K_0(\beta)^{-1} K_0(\gamma)$. This is $[M] - [B_M] + [B_{M^*}] - [M]$ which equals $[B_{M^*}] - [B_M]$ as claimed. $\square$

3. THE GENERALIZED MUTATION RULE

Let $\mathcal{T}$ and $\mathcal{T}'$ be two cluster-tilting subcategories of $\mathcal{C}$. Let Q and Q' be the quivers obtained from their Auslander-Reiten quivers by removing all loops and oriented 2-cycles.

Our aim, in this section, is to give a rule relating Q' to Q , and to prove that it generalizes the Fomin-Zelevinsky mutation rule.

Remark:

- . Assume that $\mathcal{C}$ has cluster-tilting objects. Then it is proved in [2, Theorem I.1.6], without assuming that $\mathcal{C}$ is algebraic, that the Auslander-Reiten quivers of two cluster-tilting objects having all but one indecomposable direct summands in common (up to isomorphism) are related by the Fomin-Zelevinsky mutation rule.
- . To prove that the generalized mutation rule actually generalizes the Fomin-Zelevinsky mutation rule, we use the ideas of section 7 of [8].

3.1. The rule. As in section 2, we fix a cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{C}$, and write $\mathcal{M}$ for its preimage in $\mathcal{E}$, so that $\mathcal{T} = \underline{\mathcal{M}}$. Define Q to be the quiver obtained from the Auslander–Reiten quiver of $\underline{\mathcal{M}}$ by deleting its loops and its oriented 2-cycles. Its vertex corresponding to an indecomposable object L will also be labeled by L . We denote by a_{LN} the number of arrows from vertex L to vertex N in the quiver Q . Let $B_{\mathcal{M}}$ be the matrix whose entries are given by $b_{LN} = a_{LN} - a_{NL}$.

Let $R_{\mathcal{M}}$ be the matrix of $\langle \cdot, \cdot \rangle_a : K_0(\text{mod } \underline{\mathcal{M}}) \times K_0(\text{mod } \underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$ in the basis given by the classes of the simple modules.

Lemma 11. *The matrices $R_{\mathcal{M}}$ and $B_{\mathcal{M}}$ are equal: $R_{\mathcal{M}} = B_{\mathcal{M}}$.*

Proof. Let L and N be two non-projective indecomposable objects in $\mathcal{M}$. Then $\dim \text{Hom}(S_L, S_N) - \dim \text{Hom}(S_N, S_L) = 0$ and we have:

$$\langle [S_L], [S_N] \rangle_a = \dim \text{Ext}^1(S_N, S_L) - \dim \text{Ext}^1(S_L, S_N) = b_{L,N}.$$

□

Let $\mathcal{T}'$ be another cluster-tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be its preimage in the Frobenius category $\mathcal{E}$. Let $(M'_i)_{i \in I}$ (resp. $(M_j)_{j \in J}$) be representatives for the isoclasses of non-projective indecomposable objects in $\mathcal{M}'$ (resp. $\mathcal{M}$). The equivalence of categories $\text{per } \underline{\mathcal{M}} \xrightarrow{\sim} \text{per } \underline{\mathcal{M}'}$ of proposition 8 induces an isomorphism between the Grothendieck groups $K_0(\text{mod } \underline{\mathcal{M}})$ and $K_0(\text{mod } \underline{\mathcal{M}'})$ whose matrix, in the bases given by the classes of the simple modules, is denoted by S . The equivalence of categories $\mathcal{D} \text{Mod } \mathcal{M} \xrightarrow{\sim} \mathcal{D} \text{Mod } \mathcal{M}'$ restricts to the identity on $\mathcal{H}^b(\mathcal{P})$, so that it induces an equivalence $\text{per } \mathcal{M} / \text{per } \mathcal{P} \xrightarrow{\sim} \text{per } \mathcal{M}' / \text{per } \mathcal{P}$. Let T be the matrix of the induced isomorphism from $K_0(\text{proj } \mathcal{M}) / K_0(\text{proj } \mathcal{P})$ to $K_0(\text{proj } \mathcal{M}') / K_0(\text{proj } \mathcal{P})$, in the bases given by the classes $[\mathcal{M}(?, M_j)]$, $j \in J$, and $[\mathcal{M}'(? , M'_i)]$, $i \in I$. The matrix T is much easier to compute than the matrix S . Its entries t_{ij} are given by the approximation triangles of Keller and Reiten in the following way: For all j , there exists a triangle of the form

$$\Sigma^{-1} M_j \longrightarrow \bigoplus_i \beta_{ij} M'_i \longrightarrow \bigoplus_i \alpha_{ij} M'_i \longrightarrow M_j.$$

Then, we have:

Theorem 12. a) (Generalized mutation rule) *The following equalities hold:*

$$t_{ij} = \alpha_{ij} - \beta_{ij}$$

and

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^t.$$

- b) *The category $\mathcal{C}$ has a cluster-tilting object if and only if all its cluster-tilting subcategories have a finite number of pairwise non-isomorphic indecomposable objects.*
- c) *All cluster-tilting objects of $\mathcal{C}$ have the same number of indecomposable direct summands (up to isomorphism).*

Note that point c) was shown in [10, 5.3.3(1)] (see also [2, I.1.8]) and, in a more general context, in [6]. Note also that, for the generalized mutation rule to hold, the cluster-tilting subcategories do not need to be related by a sequence of mutation.

Proof. Assertions b) and c) are consequences of the existence of an isomorphism between the Grothendieck groups $K_0(\text{mod } \underline{\mathcal{M}})$ and $K_0(\text{mod } \underline{\mathcal{M}'})$. Let us prove the equalities a). Recall from [18, section 3.3], that the antisymmetric bilinear form

$\langle \cdot, \cdot \rangle_a$ on $\text{mod } \underline{\mathcal{M}}$ is induced by the usual Euler form $\langle \cdot, \cdot \rangle_E$ on $\text{per } \underline{\mathcal{M}} \mathcal{M}$. The following commutative diagram

$$\begin{array}{ccc} \text{per } \underline{\mathcal{M}} \mathcal{M} \times \text{per } \underline{\mathcal{M}} \mathcal{M} & \xrightarrow{\simeq} & \text{per } \underline{\mathcal{M}'} \mathcal{M}' \times \text{per } \underline{\mathcal{M}'} \mathcal{M}' \\ & \searrow \langle \cdot, \cdot \rangle_E & \swarrow \langle \cdot, \cdot \rangle_E \\ & \mathbb{Z} & \end{array},$$

thus induces a commutative diagram

$$\begin{array}{ccc} K_0(\text{mod } \underline{\mathcal{M}}) \times K_0(\text{mod } \underline{\mathcal{M}}) & \xrightarrow{S \times S} & K_0(\text{mod } \underline{\mathcal{M}'}) \times K_0(\text{mod } \underline{\mathcal{M}'}) \\ & \searrow \langle \cdot, \cdot \rangle_a & \swarrow \langle \cdot, \cdot \rangle_a \\ & \mathbb{Z} & \end{array}.$$

This proves the equality $R_{\mathcal{M}} = S^t R_{\mathcal{M}'} S$, or, by lemma 11,

$$(1) \quad B_{\mathcal{M}} = S^t B_{\mathcal{M}'} S.$$

Any object of $\text{per } \underline{\mathcal{M}} \mathcal{M}$ becomes an object of $\text{per } \mathcal{M} / \text{per } \mathcal{P}$ through the composition $\text{per } \underline{\mathcal{M}} \mathcal{M} \hookrightarrow \text{per } \mathcal{M} \twoheadrightarrow \text{per } \mathcal{M} / \text{per } \mathcal{P}$. Let M and N be two non-projective indecomposable objects in $\mathcal{M}$. Since S_N vanishes on $\mathcal{P}$, we have

$$\begin{aligned} \text{Hom}_{\text{per } \mathcal{M} / \text{per } \mathcal{P}}(\mathcal{M}(?, M), S_N) &= \text{Hom}_{\text{per } \mathcal{M}}(\mathcal{M}(?, M), S_N) \\ &= \text{Hom}_{\text{Mod } \mathcal{M}}(\mathcal{M}(?, M), S_N) \\ &= S_N(M). \end{aligned}$$

Thus $\dim \text{Hom}_{\text{per } \mathcal{M} / \text{per } \mathcal{P}}(\mathcal{M}(?, M), S_N) = \delta_{MN}$, and the commutative diagram

$$\begin{array}{ccc} \text{per } \mathcal{M} / \text{per } \mathcal{P} \times \text{per } \mathcal{M} / \text{per } \mathcal{P} & \xrightarrow{\simeq} & \text{per } \mathcal{M}' / \text{per } \mathcal{P} \times \text{per } \mathcal{M}' / \text{per } \mathcal{P} \\ & \searrow R\mathcal{H}om & \swarrow R\mathcal{H}om \\ & \text{per } k & \end{array},$$

induces a commutative diagram

$$\begin{array}{ccc} K_0(\text{proj } \mathcal{M}) / K_0(\text{proj } \mathcal{P}) \times K_0(\text{mod } \underline{\mathcal{M}}) & \xrightarrow{T \times S} & K_0(\text{proj } \mathcal{M}') / K_0(\text{proj } \mathcal{P}) \times K_0(\text{mod } \underline{\mathcal{M}'}) \\ & \searrow \text{Id} & \swarrow \text{Id} \\ & \mathbb{Z} & \end{array}.$$

In other words, the matrix S is the inverse of the transpose of T :

$$(2) \quad S = T^{-t}$$

Equalities (1) and (2) imply what was claimed, that is

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^t.$$

Let us compute the matrix T : Let M be indecomposable non-projective in $\mathcal{M}$, and let

$$\Sigma^{-1}M \longrightarrow M'_1 \longrightarrow M'_0 \longrightarrow M$$

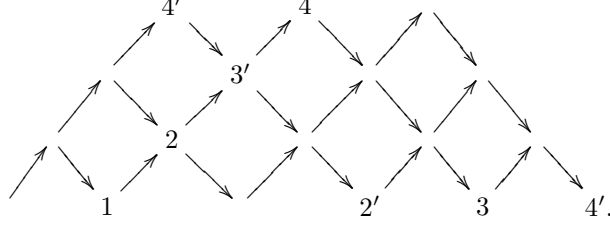
be a Keller–Reiten approximation triangle of M with respect to $\mathcal{M}'$, which we may assume to come from a conflation in $\mathcal{E}$. This conflation yields a projective resolution

$$0 \longrightarrow (M'_1)^\wedge \longrightarrow (M'_0)^\wedge \longrightarrow \mathcal{E}(?, M)|_{\mathcal{M}'} \longrightarrow \text{Ext}_{\mathcal{E}}^1(?, M'_1)|_{\mathcal{M}'} = 0.$$

so that T sends the class of M to $[(M'_0)^\wedge] - [(M'_1)^\wedge]$. Therefore, t_{ij} equals $\alpha_{ij} - \beta_{ij}$. $\square$

3.2. Examples.

3.2.1. As a first example, let $\mathcal{C}$ be the cluster category associated with the quiver of type A_4 : $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. Its Auslander–Reiten quiver is the Moebius strip:



Let $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where the indecomposable M_i corresponds to the vertex labelled by i in the picture. Let also $M' = M'_1 \oplus M'_2 \oplus M'_3 \oplus M'_4$, where $M'_1 = M_1$, and where the indecomposable M'_i corresponds to the vertex labelled by i' if $i \neq 1$. One easily computes the following Keller–Reiten approximation triangles:

$$\begin{aligned} \Sigma^{-1}M_1 &\longrightarrow 0 \longrightarrow M'_1 \longrightarrow M_1, \\ \Sigma^{-1}M_2 &\longrightarrow M'_2 \longrightarrow M'_1 \longrightarrow M_2, \\ \Sigma^{-1}M_3 &\longrightarrow M'_4 \longrightarrow 0 \longrightarrow M_4 \text{ and} \\ \Sigma^{-1}M_4 &\longrightarrow M'_4 \longrightarrow M'_3 \longrightarrow M_4; \end{aligned}$$

so that the matrix T is given by:

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

We also have

$$B_{M'} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let maple compute

$$T^{-1}B_{M'}T^{-t} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

which is B_M .

3.2.2. Let us look at a more interesting example, where one cannot easily read the quiver of M' from the Auslander–Reiten quiver of $\mathcal{C}$. Let $\mathcal{C}$ be the cluster category associated with the quiver Q :

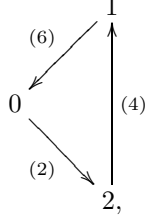


For $i = 0, 1, 2$, let M_i be (the image in $\mathcal{C}$ of) the projective indecomposable (right) kQ -module associated with vertex i . Their dimension vectors are respectively $[1, 0, 0]$, $[2, 1, 0]$ and $[2, 0, 1]$. Let M be the direct sum $M_0 \oplus M_1 \oplus M_2$. Let M' be the direct sum $M'_0 \oplus M'_1 \oplus M'_2$, where M'_0, M'_1 and M'_2 are (the images in $\mathcal{C}$ of) the indecomposable regular kQ -modules with dimension vectors $[1, 2, 0]$, $[0, 1, 0]$

and $[2, 4, 1]$ respectively. As one can check, using [13], M and M' are two cluster-tilting objects of $\mathcal{C}$. To compute Keller–Reiten’s approximation triangles, amounts to computing projective resolutions in $\text{mod } kQ$, viewed as $\text{mod } \text{End}_{\mathcal{C}}(M)$. One easily computes these projective resolutions, by considering dimension vectors:

$$\begin{aligned} 0 &\longrightarrow 8M_0 \longrightarrow M_2 \oplus 4M_1 \longrightarrow M'_2 \longrightarrow 0, \\ 0 &\longrightarrow 2M_0 \longrightarrow M_1 \longrightarrow M'_1 \longrightarrow 0 \text{ and} \\ 0 &\longrightarrow 3M_0 \longrightarrow 2M_1 \longrightarrow M'_0 \longrightarrow 0. \end{aligned}$$

By applying the generalized mutation rule, one gets the following quiver



which is therefore the quiver of $\text{End}_{\mathcal{C}}(M')$ since by [4], there are no loops or 2-cycles in the quiver of the endomorphism algebra of a cluster-tilting object in a cluster category.

3.3. Back to the mutation rule. We assume in this section that the Auslander–Reiten quiver of $\mathcal{T}$ has no loops nor 2-cycles. Under the notations of section 3.1, let k be in I and let (M_k, M'_k) be an exchange pair (see section 2.3). We choose $\underline{\mathcal{M}}'$ to be the cluster-tilting subcategory of $\mathcal{C}$ obtained from $\underline{\mathcal{M}}$ by replacing M_k by M'_k , so that $M'_i = M_i$ for all $i \neq k$. Recall that T is the matrix of the isomorphism $K_0(\text{proj } \mathcal{M})/K_0(\text{proj } \mathcal{P}) \longrightarrow K_0(\text{proj } \mathcal{M}')/K_0(\text{proj } \mathcal{P})$.

Lemma 13. *Then, the (i, j) -entry of the matrix T is given by*

$$t_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| + b_{ij}}{2} & \text{if } j = k \\ \delta_{ij} & \text{else.} \end{cases}$$

Proof. Let us apply theorem 12 to compute the matrix T . For all $j \neq k$, the triangle $\Sigma^{-1}M_j \rightarrow 0 \rightarrow M'_j = M_j$ is a Keller–Reiten approximation triangle of M_j with respect to $\mathcal{M}'$. We thus have $t_{ij} = \delta_{ij}$ for all $j \neq k$. There is a triangle unique up to isomorphism

$$M'_k \longrightarrow B_{M_k} \longrightarrow M_k \longrightarrow \Sigma M'_k$$

where $B_{M_k} \longrightarrow M_k$ is a right $\mathcal{T} \cap \mathcal{T}'$ -approximation. Since the Auslander–Reiten quiver of $\mathcal{T}$ has no loops and no 2-cycles, B_{M_k} is isomorphic to the direct sum: $\bigoplus_{i \in I} (M'_i)^{a_{ik}}$. We thus have $t_{ik} = -\delta_{ik} + a_{ik}$, which equals $\frac{|b_{ik}| + b_{ik}}{2}$. Remark that, by lemma 7.1 of [8], as stated in section 1.1, we have $T^2 = Id$, so that $S = T^t$ and

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| - b_{ij}}{2} & \text{if } i = k \\ \delta_{ij} & \text{else.} \end{cases}$$

□

Theorem 14. *The matrix $B_{\mathcal{M}'}$ is obtained from the matrix $B_{\mathcal{M}}$ by the Fomin–Zelevinski mutation rule in the direction M .*

Proof. By lemma 7.2 of [8], as it is stated in section 1.1, and by lemma 13, we know that the mutation of the matrix $B_{\mathcal{M}}$ in direction M is given by $T B_{\mathcal{M}} T^t$, which is $B_{\mathcal{M}'}$, by the generalized mutation rule (theorem 12). □

3.4. Cluster categories. In [1], the authors study the Grothendieck group of the cluster category $\mathcal{C}_A$ associated to an algebra A which is either hereditary or canonical, endowed with any admissible triangulated structure. A triangulated structure on the category $\mathcal{C}_A$ is called admissible in [1] if the projection functor from the bounded derived category $\mathcal{D}^b(\text{mod } A)$ to $\mathcal{C}_A$ is exact (triangulated). They define a Grothendieck group $\overline{K}_0(\mathcal{C}_A)$ with respect to the triangles induced by those of $\mathcal{D}^b(\text{mod } A)$, and show that it coincides with the usual Grothendieck group of the cluster category in many cases:

Theorem 15. [Barot–Kussin–Lenzing] *We have $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$ in each of the following three cases:*

- (i) *A is canonical with weight sequence $(p_1, \dots, p_t)$ having at least one even weight.*
- (ii) *A is tubular,*
- (iii) *A is hereditary of finite representation type.*

Under some restriction on the triangulated structure of $\mathcal{C}_A$, we have the following generalization of case (iii) of theorem 15:

Theorem 16. *Let A be a finite-dimensional hereditary algebra, and let $\mathcal{C}_A$ be the associated cluster category with its triangulated structure defined in [15]. Then we have $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$.*

Proof. By lemma 3.2 in [1], this theorem is a corollary of the following lemma. $\square$

Lemma 17. *Under the assumptions of section 3.1, and if moreover $\underline{M}$ has a finite number n of non-isomorphic indecomposable objects, then we have an isomorphism $K_0(\mathcal{C}) \simeq \mathbb{Z}^n / \text{Im } B_{\mathcal{M}}$.*

Proof. This is a restatement of theorem 10. $\square$

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