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Homogenization of semi-linear PDEs with discontinuous coefficients

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Abstract

We study the asymptotic behavior of the solution of semi-linear PDEs. Neither periodicity nor ergodicity assumptions are assumed. The coefficients admit only a limit in a Česaro sense. In such a case, the limit coefficients may have discontinuity. We use probabilistic approach based on weak convergence techniques for the associated backward stochastic differential equation in the S -topology. We establish weak continuity for the flow of the limit diffusion process and related the PDE limit to the backward stochastic differential equation via the representation of L^p -viscosity solution.

Keys words: *Backward stochastic differential equations (BSDEs), L^p -viscosity solution for PDEs, homogenization, S -topology, limit in Česaro sense.*

MSC 2000 subject classifications, 60H20, 60H30, 35K60.

1 Introduction

In this paper, we study the limit of the solution of the semi-linear PDEs of the form

$$\begin{cases} \frac{\partial v^\varepsilon}{\partial s}(s, x_1, x_2) = \mathcal{L}^\varepsilon(x_1, x_2)v^\varepsilon(s, x_1, x_2) + f(\frac{x_1}{\varepsilon}, x_2, v^\varepsilon(s, x_1, x_2)) & s \in (0, t) \\ v^\varepsilon(0, x_1, x_2) = H(x_1, x_2) \end{cases} \quad (1.1)$$

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The infinitesimal generator \mathcal{L}^ε is associated to the $\mathbb{R} \times \mathbb{R}^d$ -diffusion process $(x_t^{1,\varepsilon}, x_t^{2,\varepsilon})$

$$\begin{cases} x_t^{1,\varepsilon} = x_1 + \frac{1}{\varepsilon} \int_0^t \varphi(x_s^{1,\varepsilon}, x_s^{2,\varepsilon}) dW_s \\ x_t^{2,\varepsilon} = x_2 + \int_0^t b^{(1)}(x_s^{1,\varepsilon}, x_s^{2,\varepsilon}) ds + \int_0^t \sigma^{(1)}(x_s^{1,\varepsilon}, x_s^{2,\varepsilon}) d\widetilde{W}_s \end{cases} \quad (1.2)$$

where $x_t^{1,\varepsilon}$ is a null-recurrent fast component and $x_t^{2,\varepsilon}$ is a slow component. The function φ (resp. σ) is \mathbb{R} -valued (resp. $\mathbb{R}^{d \times (k-1)}$ -valued). W is a one dimensional standard Brownian motion and \widetilde{W} a \mathbb{R}^{k-1} -standard Brownian motion with independent components. W and \widetilde{W} are independent. The system (1.2) have been considered by Krylov and Khasminskii [4] studying weak convergence without ergodicity and periodicity assumptions. They defined averaged coefficients as a limit in $\dot{\text{C}}\text{esaro}$ sense. With the additional assumption that the presumed SDE limit is weakly unique, they proved that the process $(\varepsilon x_t^{1,\varepsilon}, x_t^{2,\varepsilon})$ converges in distribution towards a Markov diffusion (X_t^1, X_t^2) . As a byproduct, if the the limit PDE admits a unique weak solution, they derived the limit behavior of the PDE (1.1) in the linear case.

In this work, we consider the averaged coefficients as a limits in $\dot{\text{C}}\text{esaro}$ sense too. In such a case, the limit coefficients may have discontinuity. In our framework, in light of Krylov [5] weak assumption on the SDE limit is dropped out. The BSDE limit admits a unique strong solution. But, the classical probabilistic representation of viscosity solution for PDE fails due, to the discontinuity of the coefficients. Then, we use a probabilistic representation of L^p -viscosity solution of nonlinear PDE to make sense the connection to BSDE. Even if the notion of L^p -viscosity solution is available for PDEs with non-smooth coefficients, one require continuity property for such solutions. In our case, the lack of strong continuity property for the flow $(X^{1,x}, X^{2,x})$ transfer the difficulty to the backward one. To overcome, we establish weak continuity for the flow $x \mapsto (X^{1,x}, X^{2,x})$ and using the fact that Y_0^x is deterministic, we derive a strong continuity property for Y_0^x . The method used is a probabilistic arguments based on weak convergence techniques of the corresponding BSDE in the \mathbf{S} -topology. Let also note that, in a periodic media, some authors have studied the asymptotic behavior of the PDE (1.1) by probabilistic approach. We refer to Pardoux [7], Pardoux [8], Pardoux and Verotennikov [9].

The paper is organized as follows: in section 2, we make some notations, precise the problem and state the assumptions. In section 3, we state some facts on the FBSDE limit and L^p -viscosity solution for the corresponding PDE. In section 4, we deal with the convergence of the BSDEs while in section 5, we deduce the asymptotic behavior of the solutions of PDEs.

2 Statement of the problem and assumptions

2.1 Notations and problem's formulation

Recall here the PDE,

$$\begin{cases} \frac{\partial v^\varepsilon}{\partial s}(s, x_1, x_2) = \mathcal{L}^\varepsilon(x_1, x_2)v^\varepsilon(s, x_1, x_2) + f\left(\frac{x_1}{\varepsilon}, x_2, v^\varepsilon(s, x_1, x_2)\right) & s \in (0, t) \\ v^\varepsilon(0, x_1, x_2) = H(x_1, x_2) \end{cases} \quad (2.1)$$

where

$$\mathcal{L}^\varepsilon(x_1, x_2) = a_{00}\left(\frac{x_1}{\varepsilon}, x_2\right)\frac{\partial^2}{\partial^2 x_1} + \sum_{i,j} a_{ij}\left(\frac{x_1}{\varepsilon}, x_2\right)\frac{\partial^2}{\partial x_{2i}\partial x_{2j}} + \sum_i b_i^{(1)}\left(\frac{x_1}{\varepsilon}, x_2\right)\frac{\partial}{\partial x_{2i}},$$

is the infinitesimal generator associated to the $\mathbb{R} \times \mathbb{R}^d$ -diffusion process $(\varepsilon x_t^{1,\varepsilon}, x_t^{2,\varepsilon})$ defined by (1.2) and $a_{00} = \frac{1}{2}\varphi^2$, $a_{ij} = \frac{1}{2}(\sigma^{(1)}\sigma^{(1)*})_{ij}$, $i, j = 1, \dots, d$.

The measurable \mathbb{R} -valued functions f and H are defined on $\mathbb{R}^{d+1} \times \mathbb{R}$ and \mathbb{R}^{d+1} respectively. We denote $X_t^{1,\varepsilon} := \varepsilon x_t^{1,\varepsilon}$, $X_t^{2,\varepsilon} := x_t^{2,\varepsilon}$, $X^\varepsilon := (X^{1,\varepsilon}, X^{2,\varepsilon})$, $b = (0, b^{(1)})^*$, $B = (W, \widetilde{W})$ and $\sigma = \begin{pmatrix} \varphi & 0 \\ 0 & \sigma^{(1)} \end{pmatrix}$. One has $\sigma \in \mathbb{R}^{(d+1) \times k}$ with

$$\begin{cases} \sigma_{00} = \varphi, \\ \sigma_{0j} = 0, j = 1, \dots, k-1 \\ \sigma_{i0} = 0, i = 1, \dots, d \\ \sigma_{ij} = \sigma_{ij}^{(1)}, i = 1, \dots, d, j = 1, \dots, k-1 \end{cases}$$

The PDEs (2.1) is connected to the sequence of decoupled FBSDEs,

$$\begin{cases} X_s^\varepsilon = X_0^\varepsilon + \int_0^s b\left(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}\right)du + \int_0^s \sigma\left(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}\right)dB_u, \\ Y_s^\varepsilon = H(X_t^\varepsilon) + \int_s^t f\left(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon\right)du - \int_s^t Z_u^\varepsilon dM_u^{X^\varepsilon}, \forall s \in [0, t] \end{cases} \quad (2.2)$$

thanks to the probabilistic representation $v^\varepsilon(t, x) = Y_0^\varepsilon$. (M^{X^ε} is a martingale part of the process $X^\varepsilon := (X^{1,\varepsilon}, X^{2,\varepsilon})$).

For a given function $g(x_1, x_2)$, we define $g^+(x_2) := \lim_{x_1 \rightarrow +\infty} \frac{1}{x_1} \int_0^{x_1} g(t, x_2)dt$, $g^-(x_2) := \lim_{x_1 \rightarrow -\infty} \frac{1}{x_1} \int_0^{x_1} g(t, x_2)dt$ and $g_\pm(x_1, x_2) := g^+(x_2)1_{\{x_1 > 0\}} + g^-(x_2)1_{\{x_1 \leq 0\}}$. We denote by $\bar{b}(x_1, x_2)$, $\bar{a}(x_1, x_2)$ and $\bar{f}(x_1, x_2, y)$, the averaged coefficients defined as follows:

$$\begin{aligned} \bar{b}_i(x_1, x_2) &= \frac{(\rho b_i)_\pm(x_1, x_2)}{\rho_\pm(x_1, x_2)}, i = 1, \dots, d \\ \bar{a}_{ij}(x_1, x_2) &= \frac{(\rho a_{ij})_\pm(x_1, x_2)}{\rho_\pm(x_1, x_2)}, i, j = 0, 1, \dots, d \\ \bar{f}(x_1, x_2, y) &= \frac{(\rho f)_\pm(x_1, x_2, y)}{\rho_\pm(x_1, x_2)}, \end{aligned}$$

where the weight ρ is defined by $\rho(x_1, x_2) = a_{00}(x_1, x_2)^{-1} = [\frac{1}{2}\varphi^2(x_1, x_2)]^{-1}$. It's worth noting that \bar{b} , \bar{a} and \bar{f} are discontinuous at $x_1 = 0$. Using the asymptotic behavior of the system (2.2), we shall show that v^ε tends towards v , which is a L^p -viscosity solution of the following averaged system,

$$\begin{cases} \frac{\partial v}{\partial s}(s, x_1, x_2) = \bar{L}(x_1, x_2)v(s, x_1, x_2) + \bar{f}(x_1, x_2, v(s, x_1, x_2)) & 0 < s \leq t \\ v(0, x_1, x_2) = H(x_1, x_2) \end{cases} \quad (2.3)$$

2.2 Assumptions

We assume that the following conditions hold for the coefficients.

(A1) The function $b^{(1)}$, σ^1 , φ are Lipschitz continuous in (x_1, x_2) and, for each x_1 , their derivative in x_2 up to and including second order derivatives are bounded continuous functions of x_2 .

(A2) $a = (\sigma^{(1)}\sigma^{(1)*}) \geq \Lambda I$ for some $\Lambda > 0$. Moreover, there exists positive constants C_1, C_2, C_3 such that

$$\begin{cases} (i) C_1 \leq a_{00}(x_1, x_2) \leq C_2 \\ (ii) \sum_{i=1}^d [a_{ii}(x_1, x_2) + b_i^2(x_1, x_2)] \leq C_3(1 + |x_2|^2) \end{cases}$$

(B1) We denote $\rho(x_1, x_2) := a_{00}(x_1, x_2)^{-1}$. We assume that, as x_1 tends to $\pm\infty$, $\frac{1}{x_1} \int_0^{x_1} \rho(t, x_2) dt$ (resp. $\frac{1}{x_1} \int_0^{x_1} D_{x_2} \rho(t, x_2) dt$, resp. $\frac{1}{x_1} \int_0^{x_1} D_{x_2}^2 \rho(t, x_2) dt$) converges to $\rho^\pm(x_2)$ (resp. $D_{x_2} \rho^\pm(x_2)$, resp. $D_{x_2}^2 \rho^\pm(x_2)$) uniformly in x_2 . Here $D_{x_2} u$, $D_{x_2}^2 u$ denote respectively the gradient vector and the matrix of second derivatives in x_2 of u . We refer to $\rho^\pm(x_2)$ as a limit in *Çesaro sense*.

(B2) For $i = 1, \dots, d, j = 0, \dots, d$, the coefficients $\rho b_i, D_{x_2}(\rho b_i), D_{x_2}^2(\rho b_i), \rho a_{ij}, D_{x_2}(\rho a_{ij}), D_{x_2}^2(\rho a_{ij})$ have averages in a *Çesaro sense*.

(B3) For any function $k \in \{\rho, \rho b_i, D_{x_2}(\rho b_i), D_{x_2}^2(\rho b_i), \rho a_{ij}, D_{x_2}(\rho a_{ij}), D_{x_2}^2(\rho a_{ij})\}$, there exists a bounded function α such that

$$\begin{cases} \frac{1}{x_1} \int_0^{x_1} k(t, x_2) dt - k_\pm(x_1, x_2) = (1 + |x_2|^2) \alpha(x_1, x_2), \\ \lim_{|x_1| \rightarrow \infty} \sup_{x_2 \in \mathbb{R}^d} |\alpha(x_1, x_2)| = 0. \end{cases} \quad (2.4)$$

(C1) There are positive constants C_4, C_5 such that the \mathbb{R} -valued functions H and f satisfy:

$$\begin{cases} (i) \langle y - y', f(x_1, x_2, y) - f(x_1, x_2, y') \rangle \leq C_4 |y - y'|, \forall (x_1, x_2, y, y') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^2 \\ (ii) H \text{ is a continuous bounded function and } |f(x_1, x_2, y)| \leq C_5(1 + |x_2| + |y|) \end{cases}$$

(C2) ρf has a limit in *Çesaro sense* and there exists a measurable and bounded function β such that

$$\begin{cases} \frac{1}{x_1} \int_0^{x_2} \rho(t, x_2) f(t, x_2, y) dt - (\rho f)_\pm(x_1, x_2, y) = (1 + |x_2|^2 + |y|^2) \beta(x_1, x_2, y) \\ \lim_{|x_1| \rightarrow \infty} \sup_{(x_2, y) \in \mathbb{R}^d \times \mathbb{R}} |\beta(x_1, x_2, y)| = 0, \end{cases} \quad (2.5)$$

where $(\rho f)_\pm(x_1, x_2, y) := (\rho f)^+(x_2, y)1_{\{x_1 > 0\}} + (\rho f)^-(x_2, y)1_{\{x_1 \leq 0\}}$.

(C3) For each x_1 , ρf has a derivatives up to a second order in x_2 uniformly in y and these derivatives are bounded and satisfy (C2).

Throughout the paper, **(A)** stands for conditions (A1), (A2); **(B)** for conditions (B1), (B2), (B3) and **(C)** for (C1), (C2), (C3).

We have the following:

Lemma 2.1. *Assume **(A)**, **(B)**, (C2), (C3). For each $y \in \mathbb{R}$, let $V^{\varepsilon, y}(x_1, x_2)$ denote the solution of the following equation:*

$$\begin{cases} a_{00}(\frac{x_1}{\varepsilon}, x_2) D_{x_1}^2 u(x_1, x_2) = f(\frac{x_1}{\varepsilon}, x_2, y) - \bar{f}(x_1, x_2, y) \\ u(0, x_2) = D_{x_1} u(0, x_2) = 0. \end{cases} \quad (2.6)$$

Then,

$$(i) \quad D_{x_1} V^{\varepsilon, y}(x_1, x_2) = x_1(1 + |x_2|^2 + |y|^2) \beta(\frac{x_1}{\varepsilon}, x_2, y) - x_1(1 + |x_2|^2) m(x_1, x_2, y),$$

(ii) for any $K^{\varepsilon, y}(x_1, x_2) \in \{V^{\varepsilon, y}, D_{x_2} V^{\varepsilon, y}, D_{x_2}^2 V^{\varepsilon, y}, D_{x_1} D_{x_2} V^{\varepsilon, y}\}$ it holds,

$$K^{\varepsilon, y}(x_1, x_2) = x_1^2(1 + |x_2|^2 + |y|^2) \beta(\frac{x_1}{\varepsilon}, x_2, y) + x_1^2(1 + |x_2|^2) m(x_1, x_2, y)$$

where $m(x_1, x_2, y) := \frac{(\rho f)^\pm(x_1, x_2, y)}{\rho^\pm(x_1, x_2)} \alpha(\frac{x_1}{\varepsilon}, x_2)$ and $\alpha(x_1, x_2)$, $\beta(x_1, x_2, y)$ are various bounded functions which satisfy property (2.4) and (2.5) respectively.

Proof. For any fixed y , we set

$$F(\frac{x_1}{\varepsilon}, x_2, y) := \frac{1}{x_1} \int_0^{x_1} \rho(\frac{t}{\varepsilon}, x_2) g(\frac{t}{\varepsilon}, x_2, y) dt$$

where $g(\frac{x_1}{\varepsilon}, x_2, y) := f(\frac{x_1}{\varepsilon}, x_2, y) - \bar{f}(x_1, x_2, y)$.

For $x_1 > 0$, we have

$$\begin{aligned} F(\frac{x_1}{\varepsilon}, x_2, y) &= \frac{1}{x_1} \int_0^{x_1} \rho(\frac{t}{\varepsilon}, x_2) f(\frac{t}{\varepsilon}, x_2, y) dt - (\rho f)^+(x_2, y) \\ &+ (\rho f)^+(x_2, y) - \frac{(\rho f)^+(x_2, y)}{\rho^+(x_2)} \frac{1}{x_1} \int_0^{x_1} \rho(\frac{t}{\varepsilon}, x_2) dt \\ &= (1 + |x_2|^2 + |y|^2) \beta_1(\frac{x_1}{\varepsilon}, x_2, y) \\ &+ (\rho f)^+(x_2, y) \left[1 - \frac{1}{\rho^+(x_2) x_1} \int_0^{x_1} \rho(\frac{t}{\varepsilon}, x_2) dt \right] \\ &= (1 + |x_2|^2 + |y|^2) \beta_1(\frac{x_1}{\varepsilon}, x_2, y) - (1 + |x_2|^2) \frac{(\rho f)^+(x_2, y)}{\rho^+(x_2)} \alpha_1(\frac{x_1}{\varepsilon}, x_2) \end{aligned}$$

Since, $D_{x_1} V^{\varepsilon, y}(x_1, x_2) = x_1 F(\frac{x_1}{\varepsilon}, x_2, y)$, we derive the result for $D_{x_1} V^{\varepsilon, y}(x_1, x_2)$. Further, by integrating, we get

$$\begin{aligned} V^{\varepsilon, y}(x_1, x_2) &= x_1^2(1 + |x_2|^2 + |y|^2) \left(\left(\frac{\varepsilon}{x_1}\right)^2 \int_0^{\frac{x_1}{\varepsilon}} t \beta_1(t, x_2, y) dt \right) \\ &- (1 + |x_2|^2) \frac{(\rho f)^+(x_2, y)}{\rho^+(x_2)} \left(\left(\frac{\varepsilon}{x_1}\right)^2 \int_0^{\frac{x_1}{\varepsilon}} t \alpha_1(t, x_2) dt \right) \end{aligned}$$

Clearly, $\beta(\frac{x_1}{\varepsilon}, x_2, y) = \left(\frac{\varepsilon}{x_1}\right)^2 \int_0^{\frac{x_1}{\varepsilon}} t \beta_1(t, x_2, y) dt$, $\alpha(\frac{x_1}{\varepsilon}, x_2) = \left(\frac{\varepsilon}{x_1}\right)^2 \int_0^{\frac{x_1}{\varepsilon}} t \alpha_1(t, x_2) dt$ satisfy (2.4) and (2.5) respectively. For $x_1 < 0$, the proof is the same as previous. The result for $D_{x_2} V^{\varepsilon, y}(x_1, x_2)$, $D_{x_2}^2 V^{\varepsilon, y}(x_1, x_2)$ and $D_{x_1} D_{x_2} V^{\varepsilon, y}(x_1, x_2)$ is obtained by using similar arguments. ■

3 FBSDE limit and L^p -viscosity solution of PDE

Consider the equation

$$X_s^{t,x} = x + \int_t^s \bar{b}(X_u^{t,x}) du + \int_t^s \bar{\sigma}(X_u^{t,x}) dB_u, \quad t \leq s \leq T \quad (3.1)$$

From assumption **(A)**, **(B)**, we deduce, thanks to Krylov [5] that the forward SDE (3.1) admits a unique weak solution. Hence, the result of Khasminskii and Krylov [4] remain valid. We state it for the sake of completeness. Namely, we have:

Theorem 3.1. (*Khasminskii and Krylov [4]*)

Assume **(A)**, **(B)**. Then the process $X^\varepsilon = (X^{1,\varepsilon}, X^{2,\varepsilon})$ converges in distribution to the process $X = (X^1, X^2)$ which is solution to forward SDE (3.1).

Now, we define the notion of L^p -viscosity solution of nonlinear PDE with non-smooth coefficients. We refer the reader to Crandall et al. [2], Caffarelli et al. [1] for a wide presentation on this topic.

Let $g : [0, T] \times \mathbb{R}^{d+1} \times \mathbb{R} \rightarrow \mathbb{R}$ be a \mathcal{F}_t -progressively measurable which satisfies (C1), and $\bar{L}(x_1, x_2) = \sum_{i,j} \bar{a}_{ij}(x_1, x_2) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \bar{b}_i(x_1, x_2) \frac{\partial}{\partial x_i}$, the associated generator of the process $X^{t,x}$. Consider the parabolic equation,

$$\begin{cases} \frac{\partial v}{\partial s}(s, x_1, x_2) + \bar{L}(x_1, x_2)v(s, x_1, x_2) + g(s, x_1, x_2, v(s, x_1, x_2)) = 0, & t \leq s < T \\ v(T, x_1, x_2) = H(x_1, x_2). \end{cases} \quad (3.2)$$

Definition 3.2. Let p be an integer such that $p > N = d + 2$.

(a) A function $v \in \mathcal{C}([0, T] \times \mathbb{R}^{d+1}, \mathbb{R})$ is a L^p -viscosity sub-solution of the system (3.2), if for every $x \in \mathbb{R}^{d+1}$, $v(T, x) \leq H(x)$ and for every $\varphi \in W_{p,loc}^{1,2}([0, T] \times \mathbb{R}^{d+1}, \mathbb{R})$ and $(\hat{t}, \hat{x}) \in [0, T] \times \mathbb{R}^{d+1}$ at which $v - \varphi$ has a local maximum, one has

$$\text{ess} \lim_{(s,x) \rightarrow (\hat{t}, \hat{x})} \inf \left\{ -\frac{\partial \varphi}{\partial s}(s, x_1, x_2) - G(s, x, \varphi(s, x)) \right\} \leq 0.$$

(b) A function $v \in \mathcal{C}([0, T] \times \mathbb{R}^{d+1}, \mathbb{R})$ is a L^p -viscosity super-solution of the system (3.2), if for every $x \in \mathbb{R}^{d+1}$, $v(T, x) \geq H(x)$ and for every $\varphi \in W_{p,loc}^{1,2}([0, T] \times \mathbb{R}^{d+1}, \mathbb{R})$ and $(\hat{t}, \hat{x}) \in [0, T] \times \mathbb{R}^{d+1}$ at which $v - \varphi$ has a local minimum, one has

$$\text{ess} \lim_{(s,x) \rightarrow (\hat{t}, \hat{x})} \sup \left\{ -\frac{\partial \varphi}{\partial s}(s, x_1, x_2) - G(s, x, \varphi(s, x)) \right\} \geq 0.$$

where

$$G(s, x, \varphi(s, x)) = \bar{L}(x_1, x_2)\varphi(s, x_1, x_2) + g(s, x_1, x_2, \varphi(s, x_1, x_2))$$

is assumed to be merely measurable on the variable $x = (x_1, x_2)$.

(c) A function $v \in \mathcal{C}([0, T] \times \mathbb{R}^{d+1}, \mathbb{R})$ is a L^p -viscosity solution of the system (3.2) if it is both a L^p -viscosity sub-solution and super-solution.

Since the SDE (3.1) is weakly unique, the martingale problem associated to $X = (X^1, X^2)$ is well posed. We have the following:

Proposition 3.3. *Assume that g satisfies (C1). Then,*

(i) *For a fixed positive number T , the BSDE*

$$Y_s^{t,x} = H(X_T^{t,x}) + \int_s^T g(u, X_u^{t,x}, Y_u^{t,x}) du - \int_s^T Z_u^{t,x} dM_u^{X^{t,x}}, t \leq s \leq T.$$

admit a unique strong solution $(Y_s^{t,x}, Z_s^{t,x})_{t \leq s \leq T}$. Moreover, $(Y_s^{t,x})_{t \leq s \leq T}$ is bounded.

(ii) *If the function $(t, x) \in [0, T] \times \mathbb{R}^{d+1} \mapsto v(t, x) = Y_t^{t,x}$ is continuous, then $v(t, x) = Y_t^{t,x}$ is a L^p -viscosity solution of the system of PDE (3.2).*

Proof. (i) Thanks to Remark 3.5 in Pardoux [7], it is enough to prove existence and uniqueness for the BSDE, $Y_s^{t,x} = H(X_T^{t,x}) + \int_s^T g(u, X_u^{t,x}, Y_u^{t,x}) du - \int_s^T Z_u^{t,x} dB_u, t \leq s \leq T$; which can be proved by usual argument for BSDE. For instance, it's obvious that uniqueness holds under (C1) and, we can prove the existence of the solution by using a Picard type approximation.

(ii) We assume that $v(t, x)$ is continuous. We only prove that v is L^p -viscosity sub-solution since one can similarly verify that it is also a L^p -viscosity super-solution. Note that the definition (a) is equivalent to the following: for every $\varepsilon > 0, r > 0$, there exists a set $A \subset B_r(\hat{t}, \hat{x})$ of positive measure such that, $\forall (s, x) \in A$,

$$-\frac{\partial \varphi}{\partial s}(s, x_1, x_2) - \bar{L}(x_1, x_2)\varphi(s, x_1, x_2) - g(s, x_1, x_2, v(s, x_1, x_2)) \leq \varepsilon.$$

For any $\varphi \in W_{p,loc}^{1,2}([0, T] \times \mathbb{R}^{d+1}, \mathbb{R})$, take $(\hat{t}, \hat{x}) \in [0, T] \times \mathbb{R}^{d+1}$ a point which is a local maximum of $v - \varphi$. By the choice of $p > d + 2$, φ has a continuous version which we consider below. We assume without loss of generality that $v(\hat{t}, \hat{x}) = \varphi(\hat{t}, \hat{x})$. Assume that there exists $\varepsilon_0, r_0 > 0$ such that

$$\begin{cases} \frac{\partial \varphi}{\partial s}(s, x_1, x_2) + \bar{L}(x_1, x_2)\varphi(s, x_1, x_2) + g(s, x_1, x_2, v(s, x_1, x_2)) < -\varepsilon_0, \lambda\text{-a.s in } B_{r_0}(\hat{t}, \hat{x}) \\ v(s, x) \leq \varphi(s, x) + \varepsilon_0(s - \hat{t}), \lambda\text{-a.s in } B_{r_0}(\hat{t}, \hat{x}) \end{cases}$$

Let $A_0 \subset B_{r_0}(\hat{t}, \hat{x})$ a set of positive measure such that $(\hat{t}, \hat{x}) \in A_0$. Define

$$\tau = \inf \{s \geq \hat{t}; (s, X_s^{t,x}) \notin A_0\} \wedge (\hat{t} + r_0)$$

The process $(\bar{Y}_s, \bar{Z}_s) = ((Y_{s \wedge \tau}^{t,x}), 1_{[0, \tau]}(s)(Z_s^{t,x}))_{s \in [\hat{t}, \hat{t} + r_0]}$ solves the BSDE

$$\bar{Y}_s = v_l(\tau, X_\tau^{t,x}) + \int_s^{\hat{t} + r_0} 1_{[0, \tau]}(u)g(u, X_u^{t,x}, v(u, X_u^{t,x}))du - \int_s^{\hat{t} + r_0} \bar{Z}_u dM_u^{X^{t,x}}, s \in [\hat{t}, \hat{t} + r_0].$$

On other hand, setting $\psi(s, x) = \varphi(s, x) + \varepsilon_0(s - \hat{t})$, we have by Itô-Krylov's formula that the process $(\hat{Y}_s, \hat{Z}_s) = (\psi(s, X_{s \wedge \tau}^{t,x}), 1_{[0, \tau]}(s)\nabla \varphi(s, X_s^{t,x}))_{s \in [\hat{t}, \hat{t} + r_0]}$ solves the BSDE

$$\hat{Y}_s = \psi(\tau, X_\tau^{t,x}) - \int_s^{\hat{t} + r_0} 1_{[0, \tau]}(u)[\varepsilon_0 + (\frac{\partial \varphi}{\partial u} + \bar{L}\varphi)(u, X_u^{t,x})]du - \int_s^{\hat{t} + r_0} \hat{Z}_u dM_u^{X^{t,x}}.$$

From the choice of τ , $(\tau, X_\tau^{t,x}) \in A_0$. Therefore $v(\tau, X_\tau^{t,x}) \leq \psi(\tau, X_\tau^{t,x})$ and thanks to the comparison theorem [7], we deduce that $\bar{Y}_{\hat{t}} < \hat{Y}_{\hat{t}}$, i.e $v_l(\hat{t}, \hat{x}) < \varphi(\hat{t}, \hat{x})$, which contradicts our assumptions. \blacksquare

Remark 3.4. (i) Whenever g does not depends on t ; $v(t, x) = \tilde{Y}_0^x$ is a L^p -viscosity solution of the PDE

$$\begin{cases} \frac{\partial v}{\partial s}(s, x_1, x_2) = \bar{L}(x_1, x_2)v(s, x_1, x_2) + g(x_1, x_2, v(s, x_1, x_2)) \\ v(0, x_1, x_2) = H(x_1, x_2), \quad s > 0, x = (x_1, x_2) \in \mathbb{R}^{d+1} \end{cases}$$

where $(X^x, \tilde{Y}_s^x, \tilde{Z}_s^x; \quad 0 \leq s \leq t)$, solves the decoupled FBSDE

$$\begin{cases} X_s^x = x + \int_0^s \bar{b}(X_u^x)du + \int_0^s \bar{\sigma}(X_u^x)dB_u, \quad 0 \leq s \leq t. \\ \tilde{Y}_s^x = H(X_t^x) + \int_s^t g(X_u^x, \tilde{Y}_u^x)du - \int_s^t \tilde{Z}_u^x dM_u^{X^x}, \quad 0 \leq s \leq t \end{cases}$$

(ii) Since f satisfies **(C)** and p is bounded, one can easily verify that \bar{f} satisfies **(C1)**. Therefore, for a fixed positive number t , the BSDE with data $(H(X_t^x), \bar{f})$ admit a unique strong solution $(Y_s^x, Z_s^x)_{0 \leq s \leq t}$. Moreover, if the function $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d+1} \mapsto v(t, x) = Y_0^x$ is continuous, it is a L^p -viscosity solution of the system of PDE (2.3).

4 Convergence results for BSDE

4.1 Tightness and convergence results

Proposition 4.1. *There exists a positive constant C which does not depend on ε such that*

$$\sup_{\varepsilon} \left\{ \mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_s^\varepsilon|^2 + \int_0^t |Z_s^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_s \right) \right\} \leq C.$$

Proof. Throughout this proof, K, C are positive constants which depends only on (s, t) and may change from line to line. It is easy to check that for all $k \geq 1$,

$$\sup_{\varepsilon} \mathbb{E} \left(\sup_{0 \leq s \leq t} [|X_s^{1,\varepsilon}|^{2k} + |X_s^{2,\varepsilon}|^{2k}] \right) < +\infty. \quad (4.1)$$

Using Itô's formula, we get:

$$\begin{aligned} |Y_s^\varepsilon|^2 + \int_s^t |Z_u^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_u &\leq |H(X_t^\varepsilon)|^2 + K \int_s^t |Y_u^\varepsilon|^2 du + \int_s^t |f(\bar{X}_u^{1,\varepsilon}, X_u^{2,\varepsilon}, 0)|^2 du \\ &\quad - 2 \int_s^t \langle Y_u^\varepsilon, Z_u^\varepsilon dM_s^{X^\varepsilon} \rangle. \end{aligned}$$

Passing to expectation, it is then follows by using Gronwall's lemma that, there exists a constant which does not depend on ε such that,

$$\mathbb{E} (|Y_s^\varepsilon|^2) \leq C \mathbb{E} \left(|H(X_t^\varepsilon)|^2 + \int_0^t |f(\bar{X}_u^{1,\varepsilon}, X_u^{2,\varepsilon}, 0)|^2 du \right), \quad \forall s \in [0, t]$$

We deduce that

$$\mathbb{E} \left(\int_s^t |Z_u^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_u \right) \leq C \mathbb{E} \left(|H(X_t^\varepsilon)|^2 + \int_0^t |f(\bar{X}_u^{1,\varepsilon}, X_u^{2,\varepsilon}, 0)|^2 du \right) \quad (4.2)$$

Combining (4.2) and the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_t^\varepsilon|^2 + \frac{1}{2} \int_0^t |Z_u^\varepsilon|^2 d\langle M^{X^\varepsilon} \rangle_u \right) \leq C \mathbb{E} \left(|H(X_t^\varepsilon)|^2 + \int_0^t |f(\bar{X}_u^{1,\varepsilon}, X_u^{2,\varepsilon}, 0)|^2 du \right)$$

In view of condition **(C1 - iii)** and (4.1), the proof is complete. \blacksquare

Proposition 4.2. *The sequence $(Y^\varepsilon, M^\varepsilon)_{\varepsilon>0}$ is tight on the space $\mathcal{D}([0, t], \mathbb{R}) \times \mathcal{D}([0, t], \mathbb{R})$ endowed with the \mathbf{S} -topology.*

Proof. Since M^ε is a martingale, the Meyer-Zheng tightness criteria is fulfilled whenever

$$\sup_{\varepsilon} \left(CV(Y^\varepsilon) + \mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_s^\varepsilon| + |M_t^\varepsilon| \right) \right) < +\infty. \quad (4.3)$$

see [6] or [3]. The conditional variation CV is defined in appendix A. From [9], $CV(Y^\varepsilon)$ satisfies

$$CV(Y^\varepsilon) \leq K \mathbb{E} \left(\int_0^t |f(\bar{X}_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)|^2 ds \right),$$

where K is a constant which only depends on t .

Combining condition (C1) and Proposition 4.1, we deduce (4.3). \blacksquare

Theorem 4.3. *There exists (\bar{Y}, \bar{M}) and a countable subset \mathbf{D} of $[0, t]$ such that along a subsequence of ε ,*

- (i) $(Y^\varepsilon, M^\varepsilon) \xrightarrow{\text{law}} (\bar{Y}, \bar{M})$ on $\mathcal{D}([0, t], \mathbb{R}) \times \mathcal{D}([0, t], \mathbb{R})$ endowed with the \mathbf{S} -topology.
- (ii) $(Y^\varepsilon, M^\varepsilon) \rightarrow (\bar{Y}, \bar{M})$ in finite-distribution on \mathbf{D}^c .

Proof. From Proposition 4.2, the family $(Y^\varepsilon, M^\varepsilon)_\varepsilon$ is tight on $\mathcal{D}([0, t], \mathbb{R}) \times \mathcal{D}([0, t], \mathbb{R})$ endowed with the \mathbf{S} -topology. Hence, along a subsequence (still denoted by ε), $(Y^\varepsilon, M^\varepsilon)_\varepsilon$ converges in law on $\mathcal{D}([0, t], \mathbb{R}) \times \mathcal{D}([0, t], \mathbb{R})$ towards a càd-làg process (\bar{Y}, \bar{M}) . Moreover, thanks to Theorem 3.1 in Jakubowski [3], there exists a countable set \mathbf{D} such that along a subsequence the convergence in law as well as in finite-distribution hold on \mathbf{D}^c . \blacksquare

Theorem 4.4. *As $\varepsilon \rightarrow 0$, $(X^{1,\varepsilon}, X^{2,\varepsilon}, Y^\varepsilon) \Rightarrow (X^1, X^2, \bar{Y})$, in the sense of weak convergence in $C(\mathbb{R}_+ \times \mathbb{R}^{d+1}) \times D(\mathbb{R}_+ \times \mathbb{R})$, equipped with the product of the locally uniform convergence and the \mathbf{S} topologies.*

The following two lemmas are fundamentals for the convergence result.

Lemma 4.5. $\sup_{0 \leq s \leq t} \left| \int_0^s \left(f\left(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon\right) - \bar{f}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon) \right) du \right|$ tends to zero in probability as $\varepsilon \rightarrow 0$.

Proof. We set $h(\bar{X}_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon) = f\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon\right) - \bar{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)$. We shall show that for any $0 \leq s \leq t$, $\left| \int_0^s h(\bar{X}_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon) du \right|$ tends to zero in probability as ε tends to zero.

Let $V^\varepsilon := V^{Y^\varepsilon, \varepsilon}$ denote the solution of equation (2.6). Note that the first and second derivatives of V^ε in (x_1, x_2) are locally integrable. Then, as in [4], since the matrix a is non degenerate, we can use Itô-Krylov's formula to get

$$\begin{aligned} V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}) &= V^\varepsilon(\varepsilon x_1, x_2) + \int_0^s h(\bar{X}_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon) ds \\ &+ \int_0^s a_{ij}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \frac{\partial^2 V^\varepsilon}{\partial x_{2i} \partial x_{2j}}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}) du + \int_0^s b_j(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \frac{\partial V^\varepsilon}{\partial x_{2j}}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) du \\ &+ \int_0^s \left[\frac{\partial V^\varepsilon}{\partial x_1}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \varphi(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) + \frac{\partial V^\varepsilon}{\partial x_2}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \sigma(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \right] dW_u. \end{aligned} \quad (4.4)$$

In view of Lemma 2.1, it is obvious to see that $V^\varepsilon(\varepsilon x_1, x_2)$ tends to zero. Once again, from Lemma 2.1, we have

$$\begin{aligned} |V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon})| &\leq \varepsilon \left[(1 + |X_s^{2,\varepsilon}|^2 + |Y_s^\varepsilon|^2) \left| \beta\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon\right) \right| \right] \\ &+ \varepsilon \left[(1 + |X_s^{2,\varepsilon}|^2) \frac{(pf)^\pm(X_s^{2,\varepsilon}, Y_s^\varepsilon)}{p^\pm(X_s^{2,\varepsilon})} \left| \alpha\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}\right) \right| \right] \\ &+ 1_{\{|X_s^{1,\varepsilon}| \geq \sqrt{\varepsilon}\}} |X_s^{1,\varepsilon}|^2 \left[(1 + |X_s^{2,\varepsilon}|^2 + |Y_s^\varepsilon|^2) \left| \beta\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n}\right) \right| \right] \\ &+ 1_{\{|X_s^{1,\varepsilon}| \geq \sqrt{\varepsilon}\}} |X_s^{1,\varepsilon}|^2 \left[(1 + |X_s^{2,\varepsilon}|^2) \frac{(pf)^\pm(X_s^{2,\varepsilon}, Y_s^\varepsilon)}{p^\pm(X_s^{2,\varepsilon})} \left| \alpha\left(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}\right) \right| \right] \end{aligned}$$

From the uniform estimates of the processes $X_s^{1,\varepsilon}$, $X_s^{2,\varepsilon}$, Y_s^ε and the fact that $(pf)^\pm$ satisfies **(C1 – iii)**, we deduce that

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |V^{\varepsilon,n}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon})| \right) \leq K \left(\varepsilon + \sup_{|x_1| \geq \sqrt{\varepsilon}} \sup_{(x_2, y)} \left| \beta\left(\frac{x^1}{\varepsilon}, x^2, y\right) \right| + \sup_{|x_1| \geq \sqrt{\varepsilon}} \sup_{x^2} \left| \alpha\left(\frac{x^1}{\varepsilon}, x^2\right) \right| \right)$$

Then, since α and β satisfy (2.4) and (2.5) respectively, the right hand side of the previous inequality tends to zero as $\varepsilon \rightarrow 0$. Similarly, one can show that

$$\begin{aligned} &\int_0^s a_{ij}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \frac{\partial^2 V^\varepsilon}{\partial x_{2i} \partial x_{2j}}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) du + \int_0^s b_j(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \frac{\partial V^\varepsilon}{\partial x_{2j}}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) du \\ &+ \int_0^s \left[\frac{\partial V^\varepsilon}{\partial x_1}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \varphi(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) + \frac{\partial V^{\varepsilon,n}}{\partial x_2}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \sigma(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}) \right] dW_u \end{aligned}$$

converge to zero in probability. ■

Lemma 4.6. $\int_0^\cdot \bar{f}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon) du \xrightarrow{law} \int_0^\cdot \bar{f}(X_u^1, X_u^2, \bar{Y}_u) du$ on $(\mathcal{C}([0, t], \mathbb{R}), \|\cdot\|_\infty)$ as $\varepsilon \rightarrow 0$.

For the proof of this Lemma, we need first the two following results. Let

$$X_s^1 = x_1 + \int_0^s \bar{\varphi}(X_u^1, X_u^2) dW_u, \quad 0 \leq s \leq t.$$

Lemma 4.7. Assume (A2-i), (B1).

Let for each $\varepsilon > 0$, let $D_n^\varepsilon = \left\{ s : s \in [0, t] / X_s^{1,\varepsilon} \in B(0, \frac{1}{n}) \right\}$. Define also

$D_n = \left\{ s : s \in [0, t] / X_s^1 \in B(0, \frac{1}{n}) \right\}$. Then there exists a constant $c > 0$ such that for each $n \geq 1$, $\varepsilon > 0$,

$$\mathbb{E}|D_n^\varepsilon| \leq \frac{c}{n}, \quad \mathbb{E}|D_n| \leq \frac{c}{n},$$

where $|\cdot|$ stands for the Lebesgue measure on $[0, t]$.

Proof. Consider the function define as follows: for every $n \in \mathbb{N}^*$,

$$\Psi_n(x) = \begin{cases} -x/n - 1/2n^2 & \text{if } x \leq -1/n \\ x^2/2 & \text{if } -1/n \leq x \leq 1/n \\ x/n - 1/2n^2 & \text{if } x \geq 1/n \end{cases}$$

By using the Itô formula, we get

$$\Psi_n(X_s^1) = \Psi_n(X_0^1) + \int_0^s \Psi_n'(X_s^1) \bar{\varphi}(X_s^1, X_s^2) dW_s + \frac{1}{2} \int_0^s \Psi_n''(X_s^1) \bar{\varphi}^2(X_s^1, X_s^2) ds, \quad s \in [0, t]$$

Since $\bar{\varphi} = \bar{a}_{00} = \rho(x_1, x_2)^{-1}$ is lower bounded by C_1 , taking the expectation, we get

$$\begin{aligned} C_1 \mathbb{E} \int_0^t 1_{[-\frac{1}{n}, \frac{1}{n}]}(X_s^1) ds &\leq \mathbb{E} \int_0^t \Psi_n''(X_s^1) \bar{\varphi}^2(X_s^1, X_s^2) ds \\ &= 2\mathbb{E} [\Psi_n(X_t^1) - \Psi_n(x_1)] \end{aligned}$$

It follows that $\mathbb{E}(|D_n|) \leq 2C_1^{-1} \mathbb{E} [\Psi_n(X_t^1) - \Psi_n(x_1)] \leq c/n$. The same argument applies to D_n^ε . \blacksquare

Lemma 4.8. *Consider a collection $\{Z^\varepsilon, \varepsilon > 0\}$ of real valued random variables, and a real valued random variable Z be such that for each $n \geq 1$, we have the decompositions*

$$\begin{aligned} Z^\varepsilon &= Z_n^{1,\varepsilon} + Z_n^{2,\varepsilon} \\ Z &= Z_n^1 + Z_n^2, \end{aligned}$$

such that for each fixed $n \geq 1$,

$$\begin{aligned} Z_n^{1,\varepsilon} &\Rightarrow Z_n^1 \\ \mathbb{E}|Z_n^{2,\varepsilon}| &\leq \frac{c}{\sqrt{n}} \\ \mathbb{E}|Z_n^2| &\leq \frac{c}{\sqrt{n}}. \end{aligned}$$

Then $Z^\varepsilon \Rightarrow Z$, as $\varepsilon \rightarrow 0$.

Proof. The above assumptions imply that the collection of random variables $\{Z^\varepsilon, \varepsilon > 0\}$ is tight. Hence the result will follow from the fact that

$$\mathbb{E}\Phi(Z^\varepsilon) \rightarrow \mathbb{E}\Phi(Z), \quad \text{as } \varepsilon \rightarrow 0$$

for all $\Phi \in C_b(\mathbb{R})$ which is uniformly Lipschitz. Let Φ be such a function, and denote by K its Lipschitz constant. Then

$$\begin{aligned} |\mathbb{E}\Phi(Z^\varepsilon) - \mathbb{E}\Phi(Z)| &\leq \mathbb{E}|\Phi(Z^\varepsilon) - \Phi(Z_n^{1,\varepsilon})| + |\mathbb{E}\Phi(Z_n^{1,\varepsilon}) - \mathbb{E}\Phi(Z_n^1)| + \mathbb{E}|\Phi(Z_n^1) - \Phi(Z)| \\ &\leq |\mathbb{E}\Phi(Z_n^{1,\varepsilon}) - \mathbb{E}\Phi(Z_n^1)| + 2K \frac{c}{\sqrt{n}}. \end{aligned}$$

Hence

$$\limsup_{\varepsilon \rightarrow 0} |\mathbb{E}\Phi(Z^\varepsilon) - \mathbb{E}\Phi(Z)| \leq 2K \frac{c}{\sqrt{n}},$$

for all $n \geq 1$. The result follows. \blacksquare

Proof of Lemma 4.6 For each $n \geq 1$, define a function $\theta_n \in C(\mathbb{R}, [0, 1])$ such that $\theta_n(x) = 0$ for $|x| \leq 1/(2n)$, and $\theta_n(x) = 1$ for $|x| \geq 1/n$. Now

$$\begin{aligned} \int_0^t \bar{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon) ds &= \int_0^t \bar{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon) \theta_n(X_s^{1,\varepsilon}) ds + \int_0^t \bar{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon) [1 - \theta_n(X_s^{1,\varepsilon})] ds \\ &= Z_n^{1,\varepsilon} + Z_n^{2,\varepsilon} \\ \int_0^t \bar{f}(X_s^1, X_s^2, Y_s) ds &= \int_0^t \bar{f}(X_s^1, X_s^2, Y_s) \theta_n(X_s^1) ds + \int_0^t \bar{f}(X_s^1, X_s^2, Y_s) [1 - \theta_n(X_s^1)] ds \\ &= Z_n^1 + Z_n^2 \end{aligned}$$

Now the mapping

$$(x^1, x^2, y) \rightarrow \int_0^t \bar{f}(x_s^1, x_s^2, y_s) \theta_n(x_s^1) ds$$

is continuous from $C([0, t]) \times D([0, t])$ equipped with the product of the sup-norm and the \mathbf{S} topologies into \mathbb{R} . Hence from Theorem 4.4, $Z_n^{1,\varepsilon} \Rightarrow Z_n^1$ as $\varepsilon \rightarrow 0$, for each fixed $n \geq 1$. Moreover, from Lemma 4.7, the linear growth property of \bar{f} , Proposition 4.1 and (4.1), we deduce that

$$E|Z_n^{2,\varepsilon}| \leq \frac{c}{\sqrt{n}}, \quad E|Z_n^2| \leq \frac{c}{\sqrt{n}}.$$

The Lemma now follows from Lemma 4.8. \blacksquare

The following proposition is obtained by passing to the limit on the backward component of the equation (2.2).

Proposition 4.9. *Let (\bar{Y}, \bar{M}) , the limit process defined in Theorem 4.3. Then,*
(i) *For every $s \in [0, t] - \mathbf{D}$,*

$$\begin{cases} \bar{Y}_s = H(X_t) + \int_s^t \bar{f}(X_u^1, X_u^2, \bar{Y}) du - (\bar{M}_t - \bar{M}_s), \\ \mathbb{E}(\sup_{0 \leq s \leq t} |\bar{Y}_s|^2 + |X_s^1|^2 + |X_s^2|^2) \leq C. \end{cases} \quad (4.5)$$

(ii) *Moreover, \bar{M} is \mathcal{F}_s -martingale, where $\mathcal{F}_s = \sigma\{X_u, \bar{Y}_u, \bar{M}_u, \quad 0 \leq u \leq s\}$.*

Proof. (i) Passing to the limit in the backward component of the equation (2.2) and using Lemmas 4.5 and 4.6, we derive (i).

(ii) Let $\mathcal{F}_s = \sigma\{X_u, \bar{Y}_u, \bar{M}_u, \quad 0 \leq u \leq s\}$ be the filtration generated by X, \bar{Y}, \bar{M} and completed by the \mathbb{IP} -null sets. Combining the uniform estimates in Proposition 4.1, the inequality (4.1), Lemma (A.4) and Lemma (A.5) in Appendix A, we show that \bar{M} is \mathcal{F}_s -martingale. \blacksquare

4.2 Identification of the limits

Proposition 4.10. *Let $(Y_s, Z_s, 0 \leq s \leq t)$ be the unique solution of the BSDE $(H(X_t), \bar{f})$. Then, for every $s \in [0, t]$,*

$$\mathbb{E}|Y_s - Y_s|^2 + \mathbb{E}\left(\left[\bar{M} - \int_0^\cdot Z_u dM_u^X\right]_t - \left[\bar{M} - \int_0^\cdot Z_u dM_u^X\right]_s\right) = 0.$$

Proof. For every $s \in [0, t] - \mathbf{D}$, we have

$$\begin{cases} Y_s = H(X_t) + \int_s^t \bar{f}(X_u, Y_u) du - \int_s^t Z_u dM_u^X \\ \bar{Y}_s = H(X_t) + \int_s^t \bar{f}(X_u, \bar{Y}_u) du - (\bar{M}_t - \bar{M}_s) \end{cases}$$

Since \bar{f} satisfies condition (C1), we get by Itô's formula, that

$$\mathbb{E}|\bar{Y}_s - Y_s|^2 + \mathbb{E} \left([\bar{M} - \int_0^\cdot Z_u dM_u^X]_t - [\bar{M} - \int_0^\cdot Z_u dM_u^X]_s \right) \leq C \mathbb{E} \int_s^t |\bar{Y}_u - Y_u|^2 du.$$

Therefore, Gronwall's lemma yields that $\mathbb{E}|\bar{Y}_s - Y_s|^2 = 0, \forall s \in [0, t] - \mathbf{D}$. But, Y is continuous, \bar{Y} is càd-lag and \mathbf{D} is countable. Hence, $\bar{Y}_s = Y_s, \mathbb{P}$ -a.s, $\forall s \in [0, t]$. Moreover, we deduce that, $\mathbb{E} \left([\bar{M} - \int_0^\cdot Z_u dM_u^X]_t - [\bar{M} - \int_0^\cdot Z_u dM_u^X]_s \right) = 0. \quad \blacksquare$

As a consequence of Proposition 4.10, we have

$$\text{Corollary 4.11.} \quad \left(Y^\varepsilon, \int_0^\cdot Z_u^\varepsilon dM_u^{X^\varepsilon} \right) \xrightarrow{\text{law}} \left(Y, \int_0^\cdot Z_u dM_u^X \right).$$

5 Application to the convergence of PDE.

Under assumptions (A), (B), the SDE (3.1) has a unique weak solution [5]. And we have the following:

Proposition 5.1. *(Continuity in law of the flow $x \mapsto X^x$)*

Assume (A), (B). Let X_s^x be the unique weak solution of the SDE (3.1), and

$$X_s^n := x_n + \int_0^s \bar{b}(X_u^n) du + \int_0^s \bar{\sigma}(X_u^n) dB_u, \quad 0 \leq s \leq t$$

Assume that x_n converges towards $x = (x^1, x^2) \in \mathbb{R}^{1+d}$. Then, $X^n \xrightarrow{\text{law}} X^x$.

Proof. Since \bar{b} and $\bar{\sigma}$ satisfy (A), (B), one can easily check that the sequence X^n is tight in $\mathcal{C}([0, t] \times \mathbb{R}^{d+1})$. By Prokhorov's theorem, there exists a subsequence (denoted also by X^n) which converges weakly to a process \hat{X} . In the sequel, we show that \hat{X} is a weak solution of SDE (3.1).

• *Step 1:* For every $\varphi \in C_c^\infty(\mathbb{R}^{1+d})$,

$$\forall u \in [0, t], \quad \varphi(\hat{X}_u) - \int_0^u \bar{L}\varphi(\hat{X}_v) dv \quad \text{is a } \mathcal{F}^{\hat{X}}\text{-martingale.}$$

We need to show that for every $0 \leq s \leq u$ and any function Φ_s of $X_r^{x_n}, 0 \leq r < s$ which is bounded and continuous in the topology of the uniform convergence,

$$\begin{aligned} 0 &= \mathbb{E} \left\{ [\varphi(X_u^{x_n}) - \varphi(X_s^{x_n}) - \int_s^u \bar{L}\varphi(X_v^{x_n}) dv] \Phi_s(X^{x_n}) \right\} \\ &\xrightarrow{n} \mathbb{E} \left\{ [\varphi(\hat{X}_u) - \varphi(\hat{X}_s) - \int_s^u \bar{L}\varphi(\hat{X}_v) dv] \Phi_s(\hat{X}) \right\} \end{aligned}$$

Indeed, since φ, Φ are continuous functions and \bar{L} is continuous out of the set $\{x_1 = 0\}$, similar argument as in the proof of Lemma 4.6 yields that

$$[\varphi(X_u^{x_n}) - \varphi(X_s^{x_n}) - \int_s^u \bar{L}\varphi(X_v^{x_n})dv]\Phi_s(X^{x_n}) \xrightarrow{law} [\varphi(\hat{X}_u) - \varphi(\hat{X}_s) - \int_s^u \bar{L}\varphi(\hat{X}_v)dv]\Phi_s(\hat{X})$$

Moreover φ, Φ are bounded functions and $\sup_n \mathbb{E}(\sup_{s \in [0, t]} |X^{x_n}|^2) < \infty$, the result follows by the the uniform integrability criterium. Hence

$$\mathbb{E} \left\{ [\varphi(\hat{X}_u) - \varphi(\hat{X}_s) - \int_s^u \bar{L}\varphi(\hat{X}_v)dv]\Phi_s(\hat{X}) \right\} = 0 \text{ and } \varphi(\hat{X}_u) - \varphi(\hat{X}_s) - \int_s^u \bar{L}\varphi(\hat{X}_v)dv \text{ is a } \mathcal{F}^{\hat{X}}\text{-martingale for any } \varphi \in C_c^\infty(\mathbb{R}^{1+d}).$$

•*Step 2:* From *step 1*, there exists a $\mathcal{F}^{\hat{X}}$ -Brownian motion \hat{B} such that \hat{X} ,

$$\hat{X}_s = x + \int_0^s \bar{b}(\hat{X}_u)du + \int_0^s \bar{\sigma}(\hat{X}_u)d\hat{B}_u, \quad 0 \leq s \leq t.$$

Weak uniqueness tell us that (X^x, B) and (\hat{X}, \hat{B}) have the same law on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Hence $X^{x_n} \xrightarrow{law} X^x$. \blacksquare

Theorem 5.2. *Assume (A), (B), (C). Let $p > d + 2$. Then,*

$$(i) \lim_{\varepsilon \rightarrow 0} \mathbb{E}|Y_0^\varepsilon - v(t, x)|^2 = 0.$$

$$(ii) Y_0^{t, x} = v(t, x) \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^{d+1}), \text{ and it is a } L^p\text{-viscosity solution of the PDE (2.3).}$$

Proof. (i) We shall prove that $\lim_{\varepsilon \rightarrow 0} \mathbb{E}|Y_0^\varepsilon - \bar{Y}_0|^2 = 0$. We have,

$$\begin{cases} Y_0^\varepsilon = H(X_t^\varepsilon) + \int_0^t f(\bar{X}_u^\varepsilon, X_u^{2, \varepsilon}, Y_u^\varepsilon)du - M_t^\varepsilon \\ \bar{Y}_0 = H(X_t) + \int_0^t \bar{f}(X_u, \bar{Y}_u)du - \bar{M}_t \end{cases}$$

From Jakubowski [3], the projection: $y \mapsto y_t$ is continuous in the \mathbf{S} -topology. We then deduce that Y_0^ε converges towards \bar{Y}_0 in distribution. Moreover, since Y_0^ε and Y_0 are deterministic and bounded, we have $\lim_{\varepsilon \rightarrow 0} \mathbb{E}|Y_0^\varepsilon - \bar{Y}_0|^2 = 0$. That is $\lim_{\varepsilon \rightarrow 0} \mathbb{E}|v^\varepsilon(t, x) - v(t, x)|^2 = 0$.

(ii) For every fixed $t \in \mathbb{R}_+$, we check that the map $(t, x) \mapsto Y_0^{t, x}$ is continuous. Let $(t_n, x_n) \rightarrow (t, x)$. We assume that $t > t_n > 0$. We have,

$$Y_s^{t_n, x_n} = H(X_{t_n}^{x_n}) + \int_s^{t_n} \bar{f}(X_u^{x_n}, Y_u^{t_n, x_n})du - \int_s^{t_n} Z_u^{t_n, x_n} dM_u^{X^{x_n}}, \quad 0 \leq s \leq t_n, \quad (5.1)$$

where $X^{x_n} \xrightarrow{law} X^x$.

Since H is a continuous and bounded function and \bar{f} satisfies (C1), one can easily show that the sequence $\{(Y^{t_n, x_n}, \int_0^{\cdot} 1_{[s, t_n]}(u)Z_u^{t_n, x_n} dM_u^{X^{x_n}})\}_{n \in \mathbb{N}^*}$ is tight in $\mathcal{D}([0, t] \times \mathbb{R} \times \mathbb{R})$.

Let us rewrite the equation (5.1) as follows,

$$\begin{aligned} Y_s^{t_n, x_n} &= H(X_{t_n}^{x_n}) + \int_s^t \bar{f}(X_u^{x_n}, Y_u^{t_n, x_n})du - \int_s^t 1_{[s, t_n]}(u)Z_u^{t_n, x_n} dM_u^{X^{x_n}} \\ &- \int_{t_n}^t \bar{f}(X_u^{x_n}, Y_u^{t_n, x_n})du, \quad 0 \leq s \leq t. \\ &= A_n^1 + A_n^2 \end{aligned} \quad (5.2)$$

- *Convergence of A_n^2*

One has $\mathbb{E} \left| \int_{t_n}^t \bar{f}(X_u^{x_n}, Y_u^{t_n, x_n}) du \right| \leq K(|x|)|t - t_n|$. Hence A_n^2 tends to zero in probability.

- *Convergence of A_n^1*

Denote by (Y', M') the weak limit of $\{(Y^{t_n, x_n}, \int_0^{\cdot} 1_{[s, t_n]}(u) Z_u^{x_n} dM_u^{X^{x_n}})\}_{n \in \mathbb{N}^*}$. In view of Lemma 4.6, one has $\int_s^t \bar{f}(X_u^{x_n}, Y_u^{t_n, x_n}) du \xrightarrow{law} \int_s^t \bar{f}(X_u^x, Y_u') du$.

We now pass into the limit in (5.2) and obtain that

$$Y'_s = H(X_s^x) + \int_s^t \bar{f}(X_u^x, Y'_u) du - (M'_t - M'_s), \quad s \in [0, t] \cap D^c.$$

The uniqueness of the considered BSDE ensures that $\forall s \in [0, t]$, $Y'_s = Y_s^{t, x}$ \mathbb{P} -ps. Hence $Y^{t_n, x_n} \xrightarrow{law} Y^{t, x}$. As in (i), one derive that $Y_0^{t_n, x_n} \xrightarrow{law} Y_0^{t, x}$ which yields to the continuity of $Y_0^{t, x}$. Therefore, in view of Remark 3.4, we conclude that $v(t, x) = Y_0^{x, t}$ is a L^p -viscosity solution of the system of PDEs (2.3). \blacksquare

Remark 5.3. The result is extended to the multi-dimensional case under our standing assumptions. In this case Y is \mathbb{R}^L -values process and the strategy is line by line the same.

A Appendix: S-topology

The \mathbf{S} -topology has been introduced by Jakubowski ([3], 1997) as a topology defined on the Skorohod space of càdlàg functions: $\mathcal{D}([0, T]; \mathbb{R})$. This topology is weaker than the Skorohod topology but tightness criteria are easier to establish. These criteria are the same as the one used in Meyer-Zheng topology, ([6], 1984).

Let $N^{a, b}(z)$ denotes the number of up-crossing of the function $z \in \mathcal{D}([0, T]; \mathbb{R})$ in a given level $a < b$. We recall some facts about the \mathbf{S} -topology.

Proposition A.1. *(A criteria for S-tight). A sequence $(Y^\varepsilon)_{\varepsilon > 0}$ is S-tight if and only if it is relatively compact on the S-topology.*

Let $(Y^\varepsilon)_{\varepsilon > 0}$ be a family of stochastic processes in $\mathcal{D}([0, T]; \mathbb{R})$. Then this family is tight for the S-topology if and only if $(\|Y^\varepsilon\|_\infty)_{\varepsilon > 0}$ and $(N^{a, b}(Y^\varepsilon))_{\varepsilon > 0}$ are tight for each $a < b$.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ be a stochastic basis. If $(Y)_{0 \leq t \leq T}$ is a process in $\mathcal{D}([0, T]; \mathbb{R})$ such that Y_t is integrable for any t , the conditional variation of Y is defined by

$$CV(Y) = \sup_{0 \leq t_1 < \dots < t_n = T, \text{ partition of } [0, T]} \sum_{i=1}^{n-1} \mathbb{E}[|\mathbb{E}[Y_{t_{i+1}} - Y_{t_i} | \mathcal{F}_{t_i}]|].$$

The process is call *quasimartingale* if $CV(Y) < +\infty$. When Y is a \mathcal{F}_t -martingale, $CV(Y) = 0$. A variation of Doob inequality (cf. lemma 3, p.359 in Meyer and Zheng, 1984, where it is assumed that $Y_T = 0$) implies that

$$\mathbb{P} \left[\sup_{t \in [0, T]} |Y_t| \geq k \right] \leq \frac{2}{k} \left(CV(Y) + \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t| \right] \right),$$

$$\mathbb{E} [N^{a,b}(Y)] \leq \frac{1}{b-a} \left(|a| + CV(Y) + \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t| \right] \right).$$

It follows that a sequence $(Y^\varepsilon)_{\varepsilon>0}$ is S-tight if

$$\sup_{\varepsilon>0} \left(CV(Y^\varepsilon) + \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^\varepsilon| \right] \right) < +\infty.$$

Theorem A.2. *Let $(Y^\varepsilon)_{\varepsilon>0}$ be a S-tight family of stochastic process in $\mathcal{D}([0, T]; \mathbb{R})$. Then there exists a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ decreasing to zero, some process $Y \in \mathcal{D}([0, T]; \mathbb{R})$ and a countable subset $D \in [0, T]$ such that for any n and any $(t_1, \dots, t_n) \in [0, T] \setminus D$,*

$$(Y_{t_1}^{\varepsilon_k}, \dots, Y_{t_n}^{\varepsilon_k}) \xrightarrow{Dist} (Y_{t_1}, \dots, Y_{t_n})$$

Remark A.3. The projection $:\pi_T y \in (\mathcal{D}([0, T]; \mathbb{R}), S) \mapsto y(T)$ is continuous (see Remark 2.4, p.8 in Jakubowski,1997), but $y \mapsto y(t)$ is not continuous for each $0 \leq t \leq T$.

Lemma A.4. *Let $(X^\varepsilon, M^\varepsilon)$ be a multidimensional process in $\mathcal{D}([0, T]; \mathbb{R}^p)$ ($p \in \mathbb{N}^*$) converging to (Y, M) in the S-topology. Let $(\mathcal{F}_t^{X^\varepsilon})_{t \geq 0}$ (resp. $(\mathcal{F}_t^X)_{t \geq 0}$) be the minimal complete admissible filtration for X^ε (resp. X). We assume that $\sup_{\varepsilon>0} \mathbb{E} [\sup_{0 \leq t \leq T} |M_t^\varepsilon|^2] < C_T \forall T > 0$, M^ε is a $\mathcal{F}^{X^\varepsilon}$ -martingale and M is a \mathcal{F}^X -adapted. Then M is a \mathcal{F}^X -martingale.*

Lemma A.5. *Let $(Y^\varepsilon)_{\varepsilon>0}$ be a sequence of process converging weakly in $\mathcal{D}([0, T]; \mathbb{R}^p)$ to Y . We assume that $\sup_{\varepsilon>0} \mathbb{E} [\sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2] < +\infty$. Hence, for any $t \geq 0$, $E [\sup_{0 \leq t \leq T} |Y_t|^2] < +\infty$.*

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