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## EDGE CONNECTIVITY IN GRAPHS: AN EXPANSION THEOREM

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ABSTRACT. We show that if a graph is  $k$ -edge-connected, and we adjoin to it another graph satisfying a “contracted diameter  $\leq 2$ ” condition, with minimal degree  $\geq k$ , and some natural hypothesis on the edges connecting one graph to the other, the resulting graph is also  $k$ -edge-connected.

### 1. INTRODUCTION

Let  $G$  be a simple graph (*i.e.* a graph with no loops, no multiple edges) with vertex set  $V(G)$  and edge set  $E(G)$  (we follow in notation the book [Wes01]). Given  $A, B \subset V(G)$ ,  $[A, B]$  is the set of edges of the form  $ab$ , joining a vertex  $a \in A$  to a vertex  $b \in B$ . As we consider edges without orientation,  $[A, B] = [B, A]$ . Abusing of notation, for  $v \in V(G)$ ,  $A \subset V(G)$ , we write  $[v, A]$  instead of  $[\{v\}, A]$ . The *degree* of a vertex  $v \in V(G)$  is  $\deg_G(v) \doteq |[v, V(G)]|$ . The *neighbourhood* of a vertex  $v$ ,  $N(v)$ , is the set of vertexes  $w$  such that  $vw \in E(G)$ . Given  $A \subset V(G)$ ,  $G(A)$  is the graph  $G'$  such that  $V(G') = A$  and  $E(G')$  is the set of edges in  $E(G)$  having both endpoints in  $A$ . Given  $v, w \in V(G)$ ,  $d_G(v, w)$  is the distance in  $G$  from  $v$  to  $w$ , that is the minimum length of a path from  $v$  to  $w$ . If  $v \in V(G)$ ,  $A \subset V(G)$  we set  $d_G(v, A) \doteq \min_{w \in A} d_G(v, w)$ .

An edge cut in  $G$  is a set of edges  $[S, \bar{S}]$ , where  $S \subset V(G)$  is non void, and  $\bar{S} \doteq V(G) \setminus S$  is also assumed to be non void.

The edge-connectivity of  $G$ ,  $k'(G)$ , is the minimum cardinal of the cuts in  $G$ . We say that  $G$  is  $k$ -edge-connected if  $k'(G) \geq k$ . Menger’s theorem has as a consequence that given two vertices  $v, w$  in  $V(G)$ , if  $G$  is  $k$ -edge-connected there are at least  $k$ -edge-disjoint paths joining  $v$  to  $w$  (see [Wes01], pp.153-169).

In this paper we address the following expansion problem: given a  $k$ -edge-connected graph  $G_2$ , give conditions under which the result of adjoining to  $G_2$  a graph  $G_1$  will be also  $k$  edge-connected (see Corollary 1 below).

### 2. AN EXPANSION THEOREM

Let  $G$  be a simple graph. Let  $V_1 \subset V(G)$ ,  $V_2 \doteq V(G) \setminus V_1$ , and set  $G_1 = G(V_1)$ ,  $G_2 = G(V_2)$ . We assume in the sequel that  $V_1$  and  $V_2$  are non void. We define, for  $x, y \in V_1$ , the *contracted distance*

$$\delta(x, y) \doteq \min\{d_{G_1}(x, y), d_G(x, V_2) + d_G(y, V_2)\}$$

and for  $x \in V_1, y \in V_2$

$$\delta(x, y) = \delta(y, x) \doteq d_G(x, V_2)$$

If  $x \in V$  and  $A \subset V$ , we set  $\delta(x, A) \doteq \min_{a \in A} \delta(x, a)$ .

Notice that with these definitions, if  $\delta(x, y) = 2$  for some  $x, y \in V$ , then there exists  $z \in V$  such that  $\delta(x, z) = \delta(z, y) = 1$ .

We shall also use the notations

$$\begin{aligned}\partial^j V_1 &\doteq \{x \in V_1 : |[x, V_2]| \geq j\} \\ i^j V_1 &\doteq \{x \in V_1 : |[x, V_2]| < j\}\end{aligned}$$

Under these settings, we consider also

$$\Phi \doteq \sum_{x \in V_1} \min\{\max\{1, |[x, i^2 V_1]|\}, |[x, V_2]|\}$$

In this general framework, we have

**Theorem 1.** *If  $\max_{x,y \in V} \delta(x,y) \leq 2$  (i.e. the contracted diameter of  $V$  is  $\leq 2$ ),  $[S, \bar{S}]$  is an edge cut in  $G$  such that  $V_2 \subset S$ , and  $k \doteq \min_{x \in V_1} \deg_G(v) > |[S, \bar{S}]|$ , then*

- (1)  $\exists \bar{s} \in \bar{S} : \delta(\bar{s}, S) = 2$ .
- (2)  $\forall s \in S : \delta(s, \bar{S}) = 1$ .
- (3)  $|S \cap V_1| < |[S, \bar{S}]| < k < |\bar{S}|$ .
- (4)  $S \cap V_1 \subset \partial^2 V_1, \bar{S} \supset i^2 V_1$ .
- (5)  $\Phi \leq |[S, \bar{S}]|$ .

(See the examples in Figure 1.)

*Proof.*

- (1) Arguing by contradiction, suppose that for any  $\bar{s} \in \bar{S}$ :  $\delta(\bar{s}, S) = 1$ . Let  $\bar{s} \in \bar{S}$ . Then we have  $k_1$  edges  $\bar{s}s_i, 1 \leq i \leq k_1$  with  $s_i \in S$  and (eventually)  $k_2$  edges  $\bar{s}\bar{s}_j, \bar{s}_j \in \bar{S}$ . But each  $\bar{s}_j$  satisfies  $\delta(\bar{s}_j, S) = 1$ , thus we have  $k_2$  new edges (here we used that  $G$  is simple, because we assumed that the vertices  $\bar{s}_j$  are different)  $\bar{s}_j s'_j$ , with  $s'_j \in S$ , whence

$$|[S, \bar{S}]| \geq k_1 + k_2 = \deg_G(\bar{s}) \geq k$$

which contradicts our hypothesis.

- (2) Let  $\bar{s}_0 \in \bar{S}$  be such that  $\delta(\bar{s}_0, S) = 2$ . Then for each  $s \in S$ , as  $\delta(\bar{s}_0, s) = 2$ , there exists  $\bar{s}'$  such that  $\delta(\bar{s}_0, \bar{s}') = \delta(\bar{s}', s) = 1$ . But, again, as  $\delta(\bar{s}_0, S) = 2$ , it follows that  $\bar{s}' \in \bar{S}$ , hence  $\delta(s, \bar{S}) = 1$ .
- (3) By the previous point, we have for each  $s \in S \cap V_1$  some edge in  $[S, \bar{S}]$  incident in  $s$ , and for some  $v \in V_2$  we have also some edge in  $[S, \bar{S}]$  incident in  $v$ , thus

$$|S \cap V_1| + 1 \leq |[S, \bar{S}]|$$

On the other hand, if  $\bar{s} \in \bar{S}$  satisfies  $\delta(\bar{s}, S) = 2$  (such  $\bar{s}$  exists by our first point), then  $N(\bar{s}) \subset \bar{S}$  (recall that  $N(\bar{s})$  is the neighbourhood of  $\bar{s}$ ), whence

$$|\bar{S}| \geq 1 + |N(\bar{s})| \geq 1 + k$$

and our statement follows.

- (4) Let  $s \in S \cap V_1$ . By our second point, and using again that there is at least one edge in  $[S, V_2]$ , we have

$$\begin{aligned}|N(s) \cap V_1| + |[s, \bar{S}]| &\leq |[S, \bar{S}]| - 1 \\ &< \deg_G(s) - 1\end{aligned}$$

and the first of our statements follows if we notice that

$$\deg_G(s) = |N(s) \cap V_1| + |[s, \bar{S}]| + |[s, V_2]|$$

Now,  $\bar{S} = V_1 \setminus S \cap V_1$ , and our second statement follows immediately.

(5) By our previous points, if  $s \in S \cap V_1$  then

$$|[s, \bar{S}]| \geq \max\{1, |[s, i^2 V_1]|\}$$

and of course for  $\bar{s} \in \bar{S}$ ,  $|\bar{s}, S| \geq |\bar{s}, V_2|$ , thus

$$\begin{aligned} |[S, \bar{S}]| &= |[S \cap V_1, \bar{S}]| + |\bar{S}, V_2| \\ &\geq \sum_{s \in S \cap V_1} \max\{1, |[s, i^2 V_1]|\} + \sum_{\bar{s} \in \bar{S}} |\bar{s}, V_2| \\ &\geq \Phi \end{aligned}$$

**Corollary 1.** *Assume that*

- (1)  $\deg_G(x) \geq k, x \in V(G)$
- (2)  $G_2$  is  $k$ -edge connected
- (3)  $\max_{x, y \in V} \delta(x, y) \leq 2$

*Then any of the following*

- (1)  $\Phi \geq k$
- (2)  $|\partial^1 V_1| \geq k$
- (3)  $V_1 = \partial^1 V_1$

*implies that  $G$  is  $k$ -edge-connected.*

(See the examples in Figure 2.)

**Proof.** Let  $[S, \bar{S}]$  be any cut in  $G$ . We shall show that, under the listed hypotheses and any of the alternatives,  $|[S, \bar{S}]| \geq k$ .

If  $S \cap V_2 \neq \emptyset$  and  $\bar{S} \cap V_2 \neq \emptyset$ , then, as

$$[S \cap V_2, \bar{S} \cap V_2] \subset [S, \bar{S}]$$

is a cut in  $G_2$ , which we assumed to be  $k$ -edge connected, we obtain  $|[S, \bar{S}]| \geq k$ .

Without loss of generality, we assume in the sequel that  $V_2 \subset S$ . We argue by contradiction assuming that there exists some  $S$  such that  $|[S, \bar{S}]| < k$ , so that we are under the hypothesis of Theorem 1.

The first of our alternative hypothesis contradicts the last of the conclusions of Theorem 1.

When  $x \in \partial^1 V_1$ ,

$$\min\{\max\{1, |[x, i^2 V_1]|\}, |[x, V_2]|\} \geq 1$$

so that we have  $|\partial^1 V_1| \leq \Phi$  *i.e.* the second of our alternative hypothesis implies the first one.

To finish our proof, notice that if  $V_1 = \partial^1 V_1$ , as  $\bar{S} \subset V_1$ , we have  $\delta(\bar{s}, S) = 1$  for any  $\bar{s} \in \bar{S}$ , contradicting the first of the conclusions in Theorem 1.

### 3. FINAL REMARKS

Corollary 1 is related to a well known theorem of Plesník (see [Ple75], Theorem 6), which states that in a simple graph of diameter 2 the edge connectivity is equal to the minimum degree.

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- [Wes01] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2001.

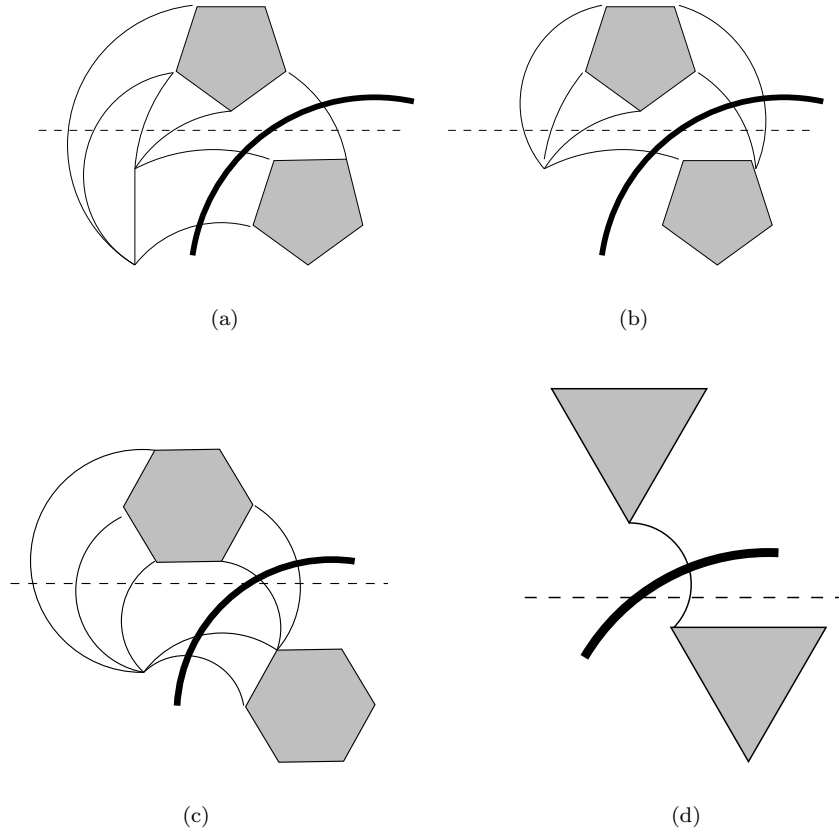


FIGURE 1. **Conventions:** 1.Filled polygons represent cliques, and curved arcs represent edges. 2.The dotted line separates  $G_2$  (the upper graph) from  $G_1$  (the lower graph). 3. The widest arc shows the cut  $[S, \bar{S}]$ . **Descriptions:** (a) Here  $|[S, \bar{S}]| = 3 < k = 4$ ,  $|S \cap V_1| = 2$ ,  $|\bar{S}| = 5$ ,  $\Phi = 3$ ,  $S \cap V_1 = \partial^2 V_1$ . (b) Here  $|[S, \bar{S}]| = 3 < k = 4$ ,  $|S \cap V_1| = 1$ ,  $|\bar{S}| = 5$ ,  $\Phi = 3$ ,  $S \cap V_1 \neq \partial^2 V_1$ . (c) Here  $|[S, \bar{S}]| = 4 < k = 5$ ,  $|S \cap V_1| = 1$ ,  $|\bar{S}| = 6$ ,  $\Phi = 3$ ,  $S \cap V_1 \neq \partial^2 V_1$ . (d) Here  $|[S, \bar{S}]| = 1 < k = 2$ ,  $|S \cap V_1| = 0$ ,  $|\bar{S}| = 3$ ,  $\Phi = 1$ ,  $S \cap V_1 = \partial^2 V_1 = \emptyset$ .

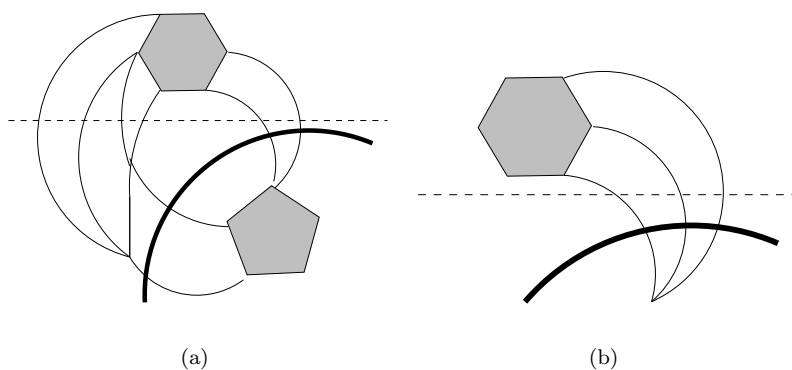


FIGURE 2. **Conventions:** 1. Filled polygons represent cliques, and curved arcs represent edges. 2. The dotted line separates  $G_2$  (the upper graph) from  $G_1$  (the lower graph). 3. The widest arc shows a minimal cut  $[S, \bar{S}]$ . **Descriptions:** (a) Here  $|[S, \bar{S}]| = k = 4$ ,  $\Phi = 4$ ,  $|\partial^1 V_1| = 3$ . (b) Here  $|[S, \bar{S}]| = k = 3$ ,  $\Phi = 1$ ,  $|\partial^1 V_1| = 1$ ,  $V_1 = \partial^1 V_1$ . This example shows that Corollary 1 includes an edge-connectivity version of the Expansion Lemma in [Wes01], Lemma 4.2.3.