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Stochastic Volterra Equations with Singular Kernels

L. Coutin and L. Decreusefond

1 Introduction

Motivated by the potential applications to the fractional Brownian motion (cf. [3]), we study Volterra stochastic differential of the form :

$$X_t = x + \int_0^t K(t, s)b(s, X_s)ds + \int_0^t K(t, s)\sigma(s, X_s) dB_s, \quad (\text{E})$$

where $(B_s, s \in [0, 1])$ is a one-dimensional standard Brownian motion and $(K(t, s), t, s \in [0, 1])$ is a deterministic kernel whose properties will be pre-cised below but for which we don't assume any boundedness property.

Actually, when σ is a constant and K is given by (2.2), we obtain :

$$X_t = x + \int_0^t K(t, s)b(s, X_s)ds + \sigma W_t^H,$$

where W^H is the fractional Brownian motion of Hurst parameter H – see the example below. In this particular case, the main feature is that K is highly singular as a kernel but the integral map canonically associated to it, i.e.,

$$Kf(t) = \int_0^t K(t, s)f(s) ds,$$

is a regularizing operator. That explains why we work as much as possible with the properties of the map K and not with those of the kernel $K(t, s)$. The problem is then in the treatment of the stochastic integral. Actually, one of the main difficulties is to control the (Hölder) regularity, with respect to t , of the stochastic integral in the right-hand-side of (E). This has been the object of a previous paper [2], the hypothesis of which we simplify here. With the result obtained in that paper, the proof of existence and uniqueness of the solution of (E) is achieved as usual by a fixed point technique.

However, some other problems arise when we study the Gross-Sobolev regularity of the solution of (E) and the expression of its derivative. They are of two sorts : on one hand the singularity of the kernel and on the other hand the fact that the Gross-Sobolev derivative of the solution is the solution of a linear but time dependent stochastic differential equation. We eventually give a somewhat explicit expression for the Gross-Sobolev derivative of the solution – in this part, the approach owes much to the ideas developed in [5].

Note that the specific form of the drift ensures both a symmetric role to b and σ and the existence of weak solutions to (E) – cf. [1] for the application of this notion to the non-linear filtering theory with fractional Gaussian noise. For other stochastic differential equations related to the fractional Brownian motion, we refer to [7, 8, 16].

The equations we have to deal with are of Volterra type but our work does not seem to be subsumed by previous articles on this subject (see for instance [6, 11, 12]) because our kernel is weakly regular and we are looking for classical solutions and not distribution-valued ones. Our work is only done in one dimension but it is straightforward to extend it to higher dimensions.

This paper is organized as follows: in the next section, we show how hypothesis (A) is sufficient to entail those of the main theorem of [2]. As an example, we address the case of the fractional Brownian motion. In Section 3, we prove the main theorem of existence and uniqueness of the solution of (E). In Section 4, we prove that under an extra boundedness assumption on σ , this solution is Gross-Sobolev differentiable and we give an integral representation of it.

2 Preliminaries

Consider a measurable kernel $(K(t, s), s, t \in [0, 1])$ and denote also by K the (formal) linear map :

$$Kf(t) = \int_0^t K(t, s)f(s) ds.$$

Hypothesis (A)– We assume once for all that there exist $\gamma > 0$ such that K is a continuous both from $\mathcal{L}^1([0, t])$ into $\mathcal{I}_{\gamma,2}([0, t])$ and from $\mathcal{L}^2([0, t])$ into $\mathcal{I}_{\gamma+1/2,2}([0, t])$ for any $t \in [0, 1]$.

Theorem 2.1 *Let u be an adapted process belonging to $L^r(\Omega \times [0, 1], P \otimes dt)$. If $r \geq 2$, the family of random variables $(M_t(u) = \int_0^t K(t, s)u_s dB_s, t \in [0, 1])$ has a version whose trajectories belong almost surely to $\mathcal{I}_{\gamma-\epsilon, r}$ for any $\epsilon \in (0, \gamma)$. Consequently, if $\gamma > 1/r$, the sample-paths are almost surely $(\gamma - 1/r - \epsilon)$ -Hölder continuous for any $\epsilon \in (0, \gamma)$. Moreover, the following maximal inequality holds :*

$$\|M(u)\|_{L^r(\Omega; \mathcal{I}_{\gamma-\epsilon, r}([0, t]))} \leq c\|u\|_{L^r(\Omega \times [0, t])}. \quad (2.1)$$

Proof. It is sufficient to prove that hypothesis I to III of [2] are satisfied. Let $\delta = \gamma - \epsilon$, we have to show that

1. K is continuous from \mathcal{L}^2 into \mathcal{B} , the space of bounded functions on $[0, 1]$.
2. K and $K_\delta = I_{0+}^{-\delta} \circ K$ are Hilbert-Schmidt from \mathcal{L}^2 into itself,

Since $\gamma > 0$, it is well known (see section A of the appendix) that $\mathcal{I}_{\gamma+1/2, 1/2} \subset \mathcal{H}_\gamma \subset \mathcal{B}$ and the first point follows.

Since $\gamma > 0$, the embedding of $\mathcal{I}_{\gamma+1/2, 2}$ in \mathcal{L}^2 is Hilbert-Schmidt (see [14]) and so is K from \mathcal{L}^2 into itself. Moreover, K_δ is continuous from \mathcal{L}^2 into $\mathcal{I}_{1/2+\epsilon, 2}$ and the embedding of $\mathcal{I}_{1/2+\epsilon, 2}$ in \mathcal{L}^2 is Hilbert-Schmidt, hence K_δ is Hilbert-Schmidt from \mathcal{L}^2 into itself. \square

As an example (in fact the motivating one), consider $K = K_H$ the kernel which is related to the fractional Brownian motion. For any H in $(0, 1)$, the fractional Brownian motion of index (Hurst parameter) H , $\{W_t^H; t \in [0, 1]\}$ is the unique centered Gaussian process whose covariance kernel is given by

$$R_H(s, t) = \mathbb{E}[W_s^H W_t^H] = \frac{V_H}{2} (s^{2H} + t^{2H} - |t - s|^{2H})$$

where

$$V_H = \frac{\Gamma(2 - 2H) \cos(\pi H)}{\pi H(1 - 2H)}.$$

It has been proved (see [3]) that there exists a standard Brownian motion such that almost surely :

$$W_t^H = \int_0^t K_H(t, s) dB_s$$

where

$$K_H(t, r) = \frac{(t-r)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{r}\right) 1_{[0,t)}(r). \quad (2.2)$$

The Gauss hyper-geometric function $F(\alpha, \beta, \gamma, z)$ (see [10]) is the analytic continuation on $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \setminus \{-1, -2, \dots\} \times \{z \in \mathbb{C}, \text{Arg}|1-z| < \pi\}$ of the power series

$$\sum_{k=0}^{+\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k.$$

Here $(\alpha)_k$ denotes the Pochhammer symbol defined by

$$(a)_0 = 1 \text{ and } (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1).$$

It is well known from [13] that K_H maps continuously \mathcal{L}^p into $\mathcal{I}_{H+1/2,p}$ for any $p \geq 1$, so that, in this case, hypothesis (A) is fulfilled for any $\gamma < H$, since $\mathcal{I}_{H+1/2,1} \subset \mathcal{I}_{H-\epsilon,2}$ for any ϵ sufficiently small.

3 Existence and uniqueness of the solution of (E)

In the following, we denote by c any irrelevant constant appearing in the computations.

Definition 3.1 *By a solution of the equation (E), we mean a real-valued, progressively measurable stochastic process $X = \{X_t, t \in I\}$ such that X belongs to $L^2(\Omega \times [0, 1], P \otimes dt)$ and for any t , X_t is a.s. a solution of (E).*

IN THE SEQUEL, r WILL DENOTE A FIXED REAL STRICTLY GREATER THAN $\max(2, \gamma^{-1})$.

Theorem 3.2 *Let b and σ be L -Lipschitz continuous with respect to their second variable, uniformly with respect to their first variable : For all t in $[0, 1]$, for all x, y in \mathbf{R} ,*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|.$$

Assume also that there exist x_0 and y_0 in \mathbb{R} , such that $b(\cdot, x_0)$ and $\sigma(\cdot, y_0)$ belong to \mathcal{L}^r . The differential equation (E) has then a unique continuous solution which belongs to $\mathcal{H}_{\gamma-1/r-\epsilon}$, for any $\epsilon \in (0, \gamma)$.

Proof. The proof proceeds as usual by a fixed point technique. It is sufficient to note that for any u and v two progressively measurable processes belonging to $L^r(\Omega \times [0, 1], \mathbb{P} \otimes dt)$, we have :

1. The process $\int_0^t K(t, s)u_s ds + \int_0^t K(t, s)v_s dB_s$ is continuous (according to (A) and theorem 2.1) and adapted, hence progressively measurable;
2. since $\mathcal{I}_{\gamma+1/2, 2}$ is continuously embedded in \mathcal{B} , according to hypothesis (A) , for any $1/r < \delta < \gamma$,

$$\mathbb{E}[\sup_{s \leq t} |\int_0^t K(t, s)u_s ds|^r] \leq \mathbb{E}[\|\int_0^t K(\cdot, s)u_s ds\|_{\mathcal{I}_{\delta, r}}^r] \quad (3.3)$$

$$\leq c \mathbb{E}[\int_0^t |u_s|^r ds]; \quad (3.4)$$

3. according to Theorem 2.1, for any $1/r < \delta < \gamma$,

$$\mathbb{E}[\sup_{s \leq t} |\int_0^t K(t, s)v_s dB_s|^r] \leq c \mathbb{E}[\|\int_0^t K(\cdot, s)v_s dB_s\|_{\mathcal{I}_{\delta, r}}^r] \quad (3.5)$$

$$\leq c \|v\|_{L^r(\Omega \times [0, 1])}^r. \quad (3.6)$$

The uniqueness is then a consequence of (3.4), (3.6) and the Gronwall lemma. According to (3.4), (3.6), the Picard sequence defined by :

$$X_t^0 = x, \quad X_t^n = x + \int_0^t K(t, s)b(s, X_s^{n-1}) ds + \int_0^t K(t, s)\sigma(s, X_s^{n-1}) dB_s$$

is a Cauchy sequence in $L^r(\Omega \times [0, 1], \mathbb{P} \otimes dt)$. We denote by X its limit. It is clearly a continuous and adapted solution of (E). Furthermore, inequalities (3.3) and (3.5) entail that the convergence also holds in $L^r(\Omega; \mathcal{I}_{\delta, r})$, so that the solution has a.s. $(\delta - 1/r)$ -Hölder continuous sample-paths. \square

Remark : Note that in the case of the fBm of Hurst index $H < 1/2$, we cannot work as usual in L^2 but only in $L^{1/H}$, hence we have to ensure stronger regularity on the coefficients, i.e., $b(\cdot, x_0)$ and $\sigma(\cdot, x_0)$ must belong to $\mathcal{L}^{1/H}$.

By the same techniques, we can prove that :

Theorem 3.3 *Under the hypothesis of the previous theorem, the map which sends x to the solution of (E) with initial condition x is continuous from \mathbb{R} in $\mathcal{L}^r(\Omega \times [0, 1], \mathbb{P} \otimes dt)$.*

4 Gross-Sobolev regularity of X

We are now interested in the Gross-Sobolev differentiability of X_t .

Lemma 4.1 *For u adapted belonging to $\mathcal{L}^2(\Omega \times [0, 1]; \mathbb{D}_{2,1})$, for any $t \in [0, 1]$, the distribution $\nabla(\int_0^t K(t, s)u_s dB_s)$ exists as a $L^2(\Omega \times [0, t])$ random variable and satisfies :*

$$\|\nabla(\int_0^t K(t, s)u_s ds)\|_{L^2(\Omega \times [0, t])} \leq c \|u\|_{L^2(\Omega \times [0, t])} \quad (4.7)$$

Lemma 4.2 *For u bounded, adapted belonging to $\mathcal{L}^2(\Omega \times [0, 1]; \mathbb{D}_{2,1})$, for any $t \in [0, 1]$, the distribution $\nabla(\int_0^t K(t, s)u_s dB_s)$ exists as a $L^2(\Omega \times [0, t])$ random variable and satisfies :*

$$\|\nabla(\int_0^t K(t, s)u_s dB_s)\|_{L^2(\Omega \times [0, t])} \leq c \|u\|_{L^2(\Omega \times [0, t]; \mathbb{D}_{2,1})}. \quad (4.8)$$

Proof. We only prove (4.8) since inequality (4.7) is simpler to show and its proof proceeds along the same lines. According to (A), for any t , the random variable $\int_0^t K(t, s)u_s dB_s$ belongs to $L^r(\Omega, \mathbb{P})$ and thus has a derivative in the distributional sense. For any $\xi \in \mathbb{D}_\infty(\mathcal{L}^2)$,

$$\begin{aligned} \langle \nabla(\int_0^t K(t, s)u_s dB_s), \xi \rangle_{\mathbb{D}_{-\infty}, \mathbb{D}_\infty} &= \mathbb{E}[\int_0^t K(t, s)u_s dB_s \delta(\xi)] \\ &= \mathbb{E}[\int_0^t K(t, s)u_s \xi_s ds] + \mathbb{E}[\int_0^t K(t, s) \left(\int_0^t \nabla_r u_s \xi_r dr \right) dB_s]. \end{aligned}$$

Hypothesis (A) induces that for any $\xi \in \mathbb{D}_\infty(\mathcal{L}^2)$, we have :

$$\begin{aligned} \mathbb{E}[\int_0^t K(t, s)u_s \xi_s ds] &\leq c \|K(t, \cdot)u\|_{L^2(\Omega \times [0, 1])} \|\xi\|_{L^2(\Omega \times [0, 1])} \\ &\leq c \|u\|_\infty \|K^*(\epsilon_t)\|_{\mathcal{L}^2} \|\xi\|_{L^2(\Omega \times [0, 1])}. \end{aligned}$$

Since ϵ_t belongs to the dual of $\mathcal{I}_{\gamma+1/2, 2}$, according to (A), we have :

$$\|K^*(\epsilon_t)\|_{\mathcal{L}^2} \leq c \|\epsilon_t\|_{\mathcal{I}_{\gamma+1/2, 2}^*} = ct^\gamma. \quad (4.9)$$

According to theorem 2.1 for $r = 2$, since $\mathcal{I}_{\gamma-\epsilon,2} \subset \mathcal{L}^2$, for any $\xi \in \mathbb{D}_\infty(\mathcal{L}^2)$,

$$\begin{aligned} \mathbb{E}[|\int_0^t \xi_\tau \int_0^t K(t,s) \nabla_\tau u_s dB_s d\tau|] \\ \leq c \|\xi\|_{L^2(\Omega \times [0,1])} \mathbb{E}[\int_0^t \int_0^s |\nabla_\tau u_s|^2 d\tau ds]^{1/2}. \end{aligned} \quad (4.10)$$

The result is then a consequence of (4.9) and (4.10) and Proposition 3 of [15, page 37]. \square

Remark : If $\gamma > 1/2$, inequalities (4.7) and (4.8) are true uniformly with respect to t , i.e., for instance,

$$\|\sup_{t \leq T} \nabla(\int_0^t K(t,s) u_s dB_s)\|_{L^2(\Omega \times [0,T])} \leq c \|u\|_{L^2(\Omega \times [0,T]; \mathbb{D}_{2,1})}.$$

Theorem 4.3 *The hypothesis of Theorem 3.2 are assumed to hold. Moreover, b and σ are supposed to be once continuously differentiable with respect to their space variable, with bounded derivative; assume furthermore that σ is bounded. For any $t \in I$, the value at t of the solution of Eqn. (E), denoted by X_t , belongs to $\mathbb{D}_{2,1}$. For any $\xi \in \mathbb{H}$,*

$$\begin{aligned} \langle X_t, \xi \rangle_{\mathbb{H}} &= \int_0^t K(t,s) \sigma(X_s) \xi_s ds \\ &+ \int_0^t K(t,u) \frac{\partial b}{\partial x}(u, X_u) \langle \nabla X_u, \xi \rangle_{\mathbb{H}} du \\ &+ \int_0^t K(t,u) \frac{\partial \sigma}{\partial x}(u, X_u) \langle \nabla X_u, \xi \rangle_{\mathbb{H}} dB_u. \end{aligned} \quad (4.11)$$

Moreover, for $\xi \in L^r(\Omega \times [0,1])$, $(\langle \nabla X_t, \xi \rangle_{\mathbb{H}}, t \in [0,1])$ belongs to $L^r(\Omega \times [0,1])$.

Proof. Let X^n be the Picard sequence already defined in the proof of theorem 3.2. Using lemmas 4.1 and 4.2, we prove by induction on n that X^n belongs to $\mathbb{D}_{2,1}$ and that :

$$\|X_t^n\|_{\mathbb{D}_{2,1}}^2 \leq c \int_0^t \|X_s^{n-1}\|_{\mathbb{D}_{2,1}}^2 ds \leq \frac{(ct)^n}{n!} x^2.$$

It follows that $\sup_n \|X_t^n\|_{\mathbb{D}_{2,1}}$ is finite and thus that there exists a weakly convergent subsequence in $\mathbb{D}_{2,1}$. Since X_t^n converges to X_t in $L^2(\Omega)$, the closability of ∇ entails that X_t belongs to $\mathbb{D}_{2,1}$. Since for any n ,

$$\begin{aligned} \langle X_t^n, \xi \rangle_{\mathbb{H}} &= \int_0^t K(t, s) \sigma(X_s^{n-1}) \xi_s ds \\ &\quad + \int_0^t K(t, u) \frac{\partial b}{\partial x}(u, X_u^{n-1}) \langle \nabla X_u^{n-1}, \xi \rangle_{\mathbb{H}} du \\ &\quad + \int_0^t K(t, u) \frac{\partial \sigma}{\partial x}(u, X_u^{n-1}) \langle \nabla X_u^{n-1}, \xi \rangle_{\mathbb{H}} dB_u, \end{aligned}$$

a straightforward application of the dominated convergence theorem yields to (4.11). Since \mathcal{L}^r is continuously imbedded in \mathcal{L}^2 and $\mathcal{I}_{\gamma+1/2,2}$ is continuously embedded in $\mathcal{C}_0([0, 1]; \mathbb{R})$, we have :

$$\begin{aligned} \mathbb{E}[\sup_{s \leq t} |\langle \nabla X_s, \xi \rangle_{\mathbb{H}}|^r] &\leq c \left(\mathbb{E}[\int_0^1 |\xi_s|^r ds] + \mathbb{E}[\int_0^t |\langle \nabla X_s, \xi \rangle_{\mathbb{H}}|^r ds] \right) \\ &\leq c \left(\mathbb{E}[\int_0^1 |\xi_s|^r ds] + \mathbb{E}[\int_0^t \sup_{u \leq s} |\langle \nabla X_u, \xi \rangle_{\mathbb{H}}|^r ds] \right). \end{aligned}$$

By Gronwall lemma, it follows that $\langle \nabla X, \xi \rangle_{\mathbb{H}}$ belongs to $L^r(\Omega \times [0, 1])$. \square

Theorem 4.4 *Assume that the hypothesis of Theorem 4.3 hold. For any $\xi \in L^r(\Omega \times [0, 1])$, the equation*

$$\begin{aligned} Y_t = &\langle K(t, \cdot) \sigma \circ X, \xi \rangle_{\mathbb{H}} + \int_0^t K(t, u) \frac{\partial b}{\partial x}(u, X_u) Y_u du \\ &+ \int_0^t K(t, u) \frac{\partial \sigma}{\partial x}(u, X_u) Y_u dB_u \quad (4.12) \end{aligned}$$

has one and only one solution belonging to $L^r(\Omega \times [0, 1])$.

Proof. Theorem 4.3 stands that the process $(\langle \nabla X_t, \xi \rangle_{\mathbb{H}}, t \in [0, 1])$ is a solution of (4.12) with the desired integrability property.

If Y and Z are two such solutions, according to hypothesis (A) and to theorem 2.1, we have :

$$\mathbb{E}[|Y_t - Z_t|^r] \leq c \int_0^t \mathbb{E}[|Y_s - Z_s|^r] ds.$$

By iteration, this induces that $Y = Z$, $\mathbb{P} \otimes dt$ almost everywhere. \square

Theorem 4.5 *Assume that the hypothesis of theorem 4.3 hold. Let $V_0(t, s)$ be a measurable deterministic kernel such that :*

$$\int_0^1 \int_0^1 |V_0(u, s)|^r du ds < \infty. \quad (4.13)$$

For $n \geq 1$, consider

$$\begin{aligned} V_{n+1}(t, s) = \int_s^t K(t, u) \frac{\partial b}{\partial x}(u, X_u) V_n(u, s) du \\ + \int_s^t K(t, u) \frac{\partial \sigma}{\partial x}(u, X_u) V_n(u, s) dB_u. \end{aligned}$$

The two following properties hold :

V1 $L(t, s) = \sum_{n=0}^{+\infty} V_n(t, s)$ is a convergent series in $L^r(\Omega \times [0, 1]^2)$.

V2 $L(t, s)$ is a solution of

$$\begin{aligned} L(t, s) - V_0(t, s) = \int_s^t K(t, u) \frac{\partial b}{\partial x}(u, X_u) L(u, s) du \\ + \int_s^t K(t, u) \frac{\partial \sigma}{\partial x}(u, X_u) L(u, s) dB_u. \quad (4.14) \end{aligned}$$

Proof. By the techniques used above, we can show that :

$$\int_0^\zeta \int_0^1 |V_{n+1}(t, s)|^r dt ds \leq c \int_0^\zeta \int_0^1 \int_s^t |V_n(u, s)|^r du ds dt,$$

hence if we set

$$\psi_n(t) = \int_0^t \int_0^1 |V_n(u, s)|^r ds du,$$

the previous equation reads as

$$\psi_{n+1}(\zeta) \leq c \int_0^\zeta \psi_n(t) dt.$$

Applying Gronwall lemma and (4.13), it follows that the series $\sum_n V_n$ is convergent in $L^r(\Omega \times [0, 1]^2)$. Moreover, it is straightforward that :

$$\begin{aligned} & \int_s^t K(t, u) \frac{\partial b}{\partial x}(u, X_u) \sum_{j=0}^n V_j(u, s) du \\ & + \int_s^t K(t, u) \frac{\partial \sigma}{\partial x}(u, X_u) \sum_{j=0}^n V_j(u, s) dB_u = \sum_{j=0}^{n+1} V_j(t, s) - V_0(t, s). \end{aligned}$$

According to hypothesis (A) and theorem 2.1, the left-hand-side of the last equation converges to the right-hand-side of (4.14). \square

Hypothesis (B)– We assume that there exist g an almost surely positive function such that $V_0(t, s) = K(t, s)g(s)$ satisfies (4.13).

Theorem 4.6 (Parameter variation formula) *Assume that all the hypothesis made so far hold. Consider the space \mathbb{H}_0 of elements of \mathbb{H} satisfying $\xi g^{-1} \in \mathcal{L}^r$. Let L be defined as in the previous theorem with the value of V_0 taken in hypothesis (B). For any $\xi \in \mathbb{H}_0$, let $Y_t = \int_0^t L(t, s) \sigma(X_s) g^{-1}(s) \xi(s) ds$. For any $t \in [0, 1]$, we have $\langle \nabla X_t, \xi \rangle_{\mathbb{H}} = Y_t$, P almost surely.*

Proof. For $\xi \in \mathbb{H}_0$, since σ is bounded, Y belongs to $L^r(\Omega \times [0, 1])$. It is clear that Y is a formal solution of (4.12) hence by theorem 4.4, the equality $\langle \nabla X_t, \xi \rangle_{\mathbb{H}} = Y_t$ follows. \square

Remark : For the fractional Brownian motion, it is proved in [3] that

$$0 \leq K_H(t, s) \leq c(t-s)^{H-1/2} s^{-|H-1/2|},$$

hence hypothesis (B) is satisfied with $g(s) = s^{|H-1/2|}$. It is a little counter-intuitive that as H increases towards 1, hypothesis (B) requires an increasing value of ν to be fulfilled. It is due to the increasing singularity of $K_H(t, s)$ whereas K_H as a map is more and more regularizing.

A Deterministic fractional calculus

For $f \in \mathcal{L}^1([0, 1])$, the left and right fractional integrals of f are defined by :

$$(I_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1} dt, \quad x \geq 0,$$

$$(I_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \quad x \leq b,$$

where $\alpha > 0$ and $I^0 = Id$. In what follows, T is a real in $(0, 1]$. For any $\alpha \geq 0$, any $f \in \mathcal{L}^p([0, T])$ and $g \in \mathcal{L}^q([0, T])$ where $p^{-1} + q^{-1} \leq \alpha$, we have :

$$\int_0^T f(s)(I_{0+}^\alpha g)(s) ds = \int_0^T (I_{T-}^\alpha f)(s)g(s) ds. \quad (\text{A.15})$$

The Besov space $I_{0+}^\alpha(\mathcal{L}^p([0, T])) = \mathcal{I}_{\alpha,p}([0, T])$ is usually equipped with the norm :

$$\|f\|_{\mathcal{I}_{\alpha,p}} = \|I_{0+}^{-\alpha} f\|_{\mathcal{L}^p([0, T])}.$$

We then have the following continuity results (see [4, 13]) :

Proposition A.1

For each $0 < T \leq 1$,

1. If $0 < \alpha < 1$, $1 < p < 1/\alpha$, then I_{0+}^α is a bounded operator from $\mathcal{L}^p([0, T])$ into $\mathcal{L}^q([0, T])$ with $q = p(1 - \alpha p)^{-1}$.
2. For any $0 < \alpha < 1$ and any $p \geq 1$, $\mathcal{I}_{\alpha,p}([0, T])$ is continuously embedded in $\mathcal{H}_{\alpha-1/p}([0, T])$ provided that $\alpha - 1/p > 0$. $\mathcal{H}_\nu([0, T])$ denotes the space of Hölder-continuous functions, null at time 0, equipped with the usual norm :

$$\|f\|_{\mathcal{H}_\nu([0, T])} = \sup_{0 \leq t \neq s \leq T} \frac{|f(t) - f(s)|}{|t - s|^\nu}.$$

By $I_{0+}^{-\alpha}$, respectively $I_{1-}^{-\alpha}$, we mean the inverse map of I_{0+}^α , respectively I_{1-}^α . When we don't precise the interval $[0, T]$ in the notations of \mathcal{L}^p spaces or of Besov spaces, it is meant that $T = 1$.

B Malliavin calculus

We only give the few results we need, further details can be found in [9, 15]. We work on the standard Wiener space $(\Omega, \mathbb{H}, \mathbb{P})$ where Ω is the Banach space of continuous functions from $[0, 1]$ into \mathbb{R} , null at time 0, equipped with the sup-norm. \mathbb{H} is the Hilbert space of absolutely continuous function with the norm $\|h\|_{\mathbb{H}} = \|\dot{h}\|_{\mathcal{L}^2}$, where \dot{h} is the time derivative of h . A mapping ϕ from Ω into some separable Hilbert space X is called cylindrical if it is of the form $\phi(w) = f(\langle v_1, w \rangle, \dots, \langle v_n, w \rangle)$ where $f \in C_0^\infty(\mathbb{R}^n, X)$ and $v_i \in \Omega^*$ for $i = 1, \dots, n$. For such a function we define $\nabla\phi$ as

$$\nabla\phi(w) = \sum_{i=1}^n \partial_i f(\langle v_1, w \rangle, \dots, \langle v_n, w \rangle) \tilde{v}_i,$$

where \tilde{v}_i is the image of v_i under the injection $\Omega^* \hookrightarrow \mathcal{L}^2$. From the quasi-invariance of the Wiener measure, it follows that ∇ is a closable operator on $L^p(\Omega; X)$, $p \geq 1$, and we will denote its closure with the same notation. The powers of ∇ are defined by iterating this procedure. For $p > 1$, $k \in \mathbb{N}$, we denote by $\mathbb{D}_{p,k}(X)$ the completion of X -valued cylindrical functions under the following norm

$$\|\phi\|_{p,k} = \sum_{i=0}^k \|\nabla^i \phi\|_{L^p(\Omega; X \otimes (\mathcal{L}^2)^{\otimes i})}.$$

Let us denote by δ the formal adjoint of ∇ with respect to Wiener measure, a classical result stands that δ is an extension of the Itô integral thus we have

$$\mathbb{E}\left[\int_0^t u_s dB_s \varphi\right] = \mathbb{E}\left[\int_0^t u_s \nabla_s \varphi ds\right] \quad (\text{B.16})$$

for any u adapted in $L^2(\Omega; \mathcal{L}^2)$ and any $\varphi \in \mathbb{D}_{2,1}$, where $\{B_t = \delta(\mathbf{1}_{[0,t]})\}$, $t \in [0, 1]$ is a standard Brownian motion on (Ω, \mathbb{P}) .

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