

Supplementary Information to "Athermal dynamics of strongly coupled stochastic three-state oscillators"

Bastien Fernandez¹ and Lev S. Tsimring²

¹ *Centre de Physique Théorique CNRS, Universités de Marseille I et II et de Toulon-Var,
Luminy Case 907, 13288 Marseille CEDEX 09, France*

² *Institute for Nonlinear Science, University of California, San Diego, La Jolla, CA 92093-0402*

PACS numbers:

EIGENVALUES OF THE FULL MASTER EQUATION

Here we present the rigorous analytic computation of the main eigenvalues of the full system in the large coupling limit $b \gg 1$.

Let $Q(a, b)$ be the transition rate matrix associated with the globally coupled system. The individual transition rate a (which is set to 1 in the paper) is allowed to take any non-negative value $a \geq 0$. The matrix $Q(a, b)$ is such that its action on the probability vectors $\{p(n_1, n_2, n_3)\}$ is given by

$$\begin{aligned} (Q(a, b)p)(n_1, n_2, n_3) = & \\ & (n_1 + 1)[a + b(n_2 - 1)]p(n_1 + 1, n_2 - 1, n_3) \\ & + (n_2 + 1)[a + b(n_3 - 1)]p(n_1, n_2 + 1, n_3 - 1) \\ & + (n_3 + 1)[a + b(n_1 - 1)]p(n_1 - 1, n_2, n_3 + 1) \\ & - [aN + b(n_1n_2 + n_2n_3 + n_3n_1)]p(n_1, n_2, n_3) \end{aligned}$$

By the Perron-Frobenius theorem, 0 is a single eigenvalue of $Q(a, b)$ ($a > 0$), and all other eigenvalues have negative real part. We want to compute the eigenvalues with largest negative real part in the strong coupling limit ($b \gg 1$). Up to time rescaling, this is equivalent to the assumption that a is small.

For $a = 0$, transitions from any edge node ($n_1n_2n_3 = 0$) to any inner node ($n_1n_2n_3 > 0$) have zero probability, and the Markov process is transient. Moreover, the corners $(0, 0, N)$, $(0, N, 0)$ or $(N, 0, 0)$ are absorbing states. Therefore for $a = 0$ the eigenvalue 0 is triple and all other eigenvalues has negative real part.

Since $Q(a, b)$ is analytic in a , this triple eigenvalue 0 splits for $a > 0$ into 3 eigenvalues, namely 0 and $a\lambda_{\pm} + \mathcal{O}(a^2)$ by using the Puiseux series. By application of a continuity argument to other eigenvalues, the eigenvalues $a\lambda_{\pm} + \mathcal{O}(a^2)$ are certainly the eigenvalues with the largest negative real part when a is sufficiently small.

In order to compute λ_{\pm} , we use the following enumeration $\{i\}_{i=1}^{T_N}$ of all lattice nodes (n_1, n_2, n_3) , where $T_N = \frac{(N+1)(N+2)}{2}$ is the total number of nodes. The indices $\{i\}_{i=1}^{3N}$ correspond to edge nodes, starting from a corner and following the flow around the triangular grid. The indices $\{i\}_{i=3N+1}^{T_N}$ correspond to inner nodes. With this enumeration, the matrix $\lambda \text{Id} - Q(a, b)$ can be written as (the first row and the first column stand for indices)

	1	2	...	N	$N+1$...	$2N$	$3N$...	T_N
1	q_{11}	0	0	$q_{1\ 3N}$		
2	q_{21}	q_{22}	\ddots							0		
\vdots	0	q_{23}	\ddots	\ddots						\vdots		
N	\vdots	\ddots	\ddots	\ddots	\ddots					\vdots		
$N+1$	\vdots		\ddots	$q_{1\ 3N}$	q_{11}	\ddots				\vdots		
\vdots	\vdots			\ddots	q_{21}	\ddots	\ddots			\vdots		
$2N$	\vdots				0	\ddots	\ddots	\ddots		\vdots		
\vdots	\vdots				\vdots	\ddots	\ddots	\ddots	\ddots	\ddots		
\vdots	\vdots				\vdots	\ddots	\ddots	\ddots	\ddots	\ddots	0	
$3N$	\vdots				\vdots			0	\ddots	\ddots		
\vdots	\vdots				\vdots			\vdots		\vdots		
\vdots	\vdots		aQ_1		\vdots	aQ_2		\vdots	aQ_3		Q_4	
T_N	0				0			0				

where

$$q_{11} = \lambda + aN, \quad q_{21} = -aN, \quad q_{22} = \lambda + aN + b(N-1)$$

$$q_{23} = -(N-1)(a+b), \quad \text{and} \quad q_{1\ 3N} = -a + (N-1)b$$

and where blocks Q_i are of order $\mathcal{O}(1)$ in the parameter a .

For $\lambda = aN\lambda_{\pm} + \mathcal{O}(a^2)$, neglecting terms $\mathcal{O}(a^2)$, the determinant can be computed by letting $q_{11} = aN(\lambda_{\pm} + 1)$. Factoring out aN in the three columns $i = 1, N+1$, and $2N+1$, we may assume that $q_{11} = \lambda_{\pm} + 1$ and $q_{21} = -1$. It remains to compute the 0th order term of the remaining determinant, i.e. we may assume that $a = 0$ in the computation.

For $a = 0$, the block Q_4 does not depend on λ_{\pm} and has non-vanishing determinant since inner nodes are transient. For the edge nodes, all columns with $i \neq 1, N+1$ and $i \neq 2N+1$ have only two non-vanishing terms that add to 0. Factoring out once again, we can rewrite the determinant in the form

	1	2	...	N	$N+1$...	$2N$	$3N$
1	$\lambda_{\pm} + 1$	0	-1
2	-1	1	\ddots							0
\vdots	0	-1	\ddots	\ddots						\vdots
N	\vdots	\ddots	\ddots	1	\ddots					\vdots
$N+1$	\vdots		\ddots	-1	$\lambda_{\pm} + 1$	\ddots				\vdots
\vdots	\vdots			\ddots	-1	1	\ddots			\vdots
$2N$	\vdots				\ddots	\ddots	\ddots	\ddots		\vdots
\vdots	\vdots					\ddots	\ddots	\ddots	$\lambda_{\pm} + 1$	\ddots
\vdots	\vdots						\ddots	\ddots	\ddots	0
$3N$	0	0	-1	1

Note that this matrix commutes with the N -fold cyclic permutation $i \mapsto i + N \pmod{3N}$ on edge nodes. It follows that its eigenvectors are of the following form

$$(x_1, \dots, x_N, x_1 e^{4i\pi k/3}, \dots, x_N e^{4i\pi k/3}, x_1 e^{2i\pi k/3}, \dots, x_N e^{2i\pi k/3}), \quad k = 0, 1, 2$$

Moreover these vectors belong to the matrix kernel iff $x_1 = \dots = x_N$ and $\lambda_{\pm} = -1 + e^{2i\pi k/3}$ for $k = 0, 1, 2$. Naturally, the eigenvalue 0 is recovered for $k = 0$. For $k = 1, 2$, we conclude that for small enough a , the eigenvalues of $Q_{a,b}$ with the largest negative real part are

$$a\lambda_k N + \mathcal{O}(a^2)$$

where $\lambda_k = -1 + e^{2i\pi k/3}$ are the eigenvalues of the single oscillator. Note that they do not depend on b . Back in the original notation, for large coupling b the eigenvalues are $\lambda_k N + \mathcal{O}(1/b^2)$.

DYNAMICS OF d -STATE SYSTEMS

A generalization of the three-state oscillators is an ensemble of systems with a cycle of d states with uni-directional transitions from state i to $i + 1 \pmod{d}$, whose rates are proportional to the number of oscillators in the “next” state, $\pi_{i,i+1} = 1 + bn_{i+1}$. By using the corresponding master equation, it is straightforward to see that the stationary probability distribution for globally coupled d -state systems reads

$$p_s(n_1, n_2, \dots, n_d) = C_{b,d} \prod_{i=1}^d \frac{G(b, n_i)}{n_i!} \quad (1)$$

with the same definition of the function $G(x, n)$ as in the main text, and $C_{b,d}$ being the normalization constant.

The distribution (1) for arbitrary d has similar properties to those for $d = 3$. It is invariant under any permutation $\text{Per}(n_1, n_2, \dots, n_d)$ of coordinates. It is convex with a maximum in the center (where all coordinates n_i are equal) when $b < 1$ and is concave with a maximum in the corners (where all coordinates n_i are 0 except for one n_i) for $b > 1$. It is flat for $b = 1$. For zero coupling ($b = 0$) the distribution is multinomial

$$p_s(n_1, n_2, \dots, n_d) = \frac{N!}{d^N \prod_{i=1}^d n_i!}$$

and for large b , it is highly localized in the corners

$$p_s(n_1, n_2, \dots, n_d) = \frac{1}{d} - \frac{2}{db} \sum_{k=1}^{N-1} \frac{1}{k} + O(1/b^2) \quad \text{if } (n_1, n_2, \dots, n_d) = \text{Per}(N, 0, 0 \dots, 0)$$

$$p_s(n_1, n_2, \dots, n_d) = \frac{N}{dbn(N-n)} + O(1/b^2) \quad \text{if } (n_1, n_2, \dots, n_d) = \text{Per}(N-n, n, 0 \dots, 0) \quad (1 \leq n < N)$$

and $p_s(n_1, n_2, \dots, n_d) = O(1/b^2)$ if at least three distinct indices n_i are different from 0. Performing a similar Stirling approximation as in the main text, we obtain the general expression for the order parameter

$$R_d = \frac{b}{b+d} \quad (2)$$

As Figure 1a shows, numerical simulations agree with this prediction very well. However, the case $d = 2$ is fundamentally different from other $d > 2$ because it corresponds not to a cycle system but to a two-state system with reciprocal transitions. In the latter case, the detailed balance condition applies, and the probability current is zero. When strongly coupled, these systems exhibit switching behavior in the large coupling limit, but they *do not* exhibit quasi-regular oscillatory dynamics at the intermediate coupling. Figure 1b shows the coherence parameter as a function of coupling parameter b for $d = 2, 3, 6$, and it is clearly seen that for $d = 2$ the coherence parameter lacks a maximum which is present at $d = 3, 6$.

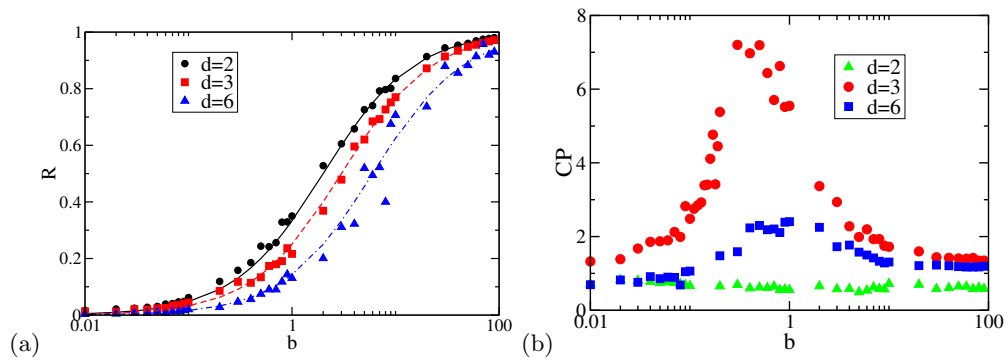


FIG. 1: Order parameter (a) and coherence parameter (b) as a function of b for $d = 2, 3, 6$ and $N = 100$.