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A PROPERTY OF DOMINANCE ON PARTITIONS

MIREILLE BOUSQUET-MÉLOU

Abstract. Given an integer partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and an integer \( k \), denote by \( \lambda^{(k)} \) the sequence of length \( \ell \) obtained by reordering the values \( |\lambda_i - k| \) in non-increasing order. If \( \lambda \) dominates \( \mu \) and has the same weight, then \( \lambda^{(k)} \) dominates \( \mu^{(k)} \).

A partition of an integer \( n \) in \( k \) parts is a non-increasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of positive integers that sum to \( n \). The integer \( n \) is called the weight of \( \lambda \), denoted \(|\lambda|\), while \( k \) is the length of \( \lambda \), denoted \( \ell(\lambda) \). Partitions are partially ordered by dominance:

\[
\lambda \geq \mu \quad \text{if} \quad \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \quad \text{for all} \ i,
\]

where it is understood that \( \lambda_i = 0 \) if \( i > \ell(\lambda) \) (and similarly for \( \mu \)). Note that \( \lambda \geq \mu \) implies in particular that \(|\lambda|\) \geq \(|\mu|\). The definition of dominance can be extended to finite non-increasing sequences of non-negative integers. Observe that adding some zeroes at the end of two such sequences \( \lambda \) and \( \mu \) does not affect their dominance relation.

Definition 1. For an \( \ell \)-tuple \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) of non-negative integers and \( k \in \mathbb{N} \), let \( \lambda^{(k)} \) be the sequence of length \( \ell \) obtained by reordering the values \( |\lambda_i - k| \) in non-increasing order.

For instance, if \( \lambda = (4, 2, 1, 0) \), then

\[
\lambda^{(1)} = (3, 1, 1, 0), \quad \lambda^{(2)} = (2, 2, 1, 0), \quad \lambda^{(3)} = (3, 2, 1, 1), \quad \lambda^{(4)} = (4, 3, 2, 0).
\]

The following result was observed experimentally by Jean Creignou in his work with Hervé Diet on codes in unitary groups and Schur polynomials. More precisely, it was useful when implementing their technique to obtain certain bounds on the size of codes \( \mathcal{B} \).

Proposition 2. Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) and \( \mu = (\mu_1, \ldots, \mu_\ell) \) be two partitions of the same weight, possibly completed with zeroes so that they have the same length. Assume that \( \lambda \) dominates \( \mu \). Then for all positive integer \( k \), \( \lambda^{(k)} \) dominates \( \mu^{(k)} \).

Example. Take \( \lambda = (4, 2, 1, 0) \) as above and \( \mu = (4, 1, 1, 1) \). These two partitions have the same weight, and \( \lambda \geq \mu \). The sequences \( \lambda^{(k)} \) are listed above for \( 1 \leq k \leq 4 \), and

\[
\mu^{(1)} = (3, 0, 0, 0), \quad \mu^{(2)} = (2, 1, 1, 1), \quad \mu^{(3)} = (2, 2, 2, 1), \quad \mu^{(4)} = (3, 3, 3, 0).
\]

It is easy to check that the proposition holds on this example.

Observe that the proposition would be obvious for negative integers \( k = -m \): in the construction of \( \lambda^{(k)} \), every part of \( \lambda \) is simply increased by \( m \), and the order of the parts does not change. For similar reasons, the proposition is clear when \( k \geq \lambda_1 \): the values \( \lambda_i - k \) and \( \mu_i - k \) are non-positive for all \( i \), so that the order of the parts is simply reversed: \( \lambda^{(k)} = (k - \lambda_{\ell}, \ldots, k - \lambda_2, k - \lambda_1) \). The fact that reversion is involved also shows that the result cannot hold for partitions of different weights. Take for instance \( \lambda = (2), \mu = (1) \) and \( k = 2 \).

Proof. Recall that \( \lambda \) covers \( \mu \) if there exist no \( \nu \) such that \( \lambda \geq \nu > \mu \). The covering relations for the dominance order on partitions of the same weight were described by Brylawski. The partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) covers the partition \( \mu = (\mu_1, \mu_2, \ldots) \) if and only if there exists \( i < j \) such that

\[
\lambda_i = \mu_i + 1, \lambda_j = \mu_j - 1, \lambda_p = \mu_p \quad \text{for} \quad p \not\in \{i, j\}, \quad \text{and} \quad \text{either} \quad j = i + 1 \quad \text{or} \quad \mu_i = \mu_j.
\]

Let us prove the proposition when \( \lambda \) covers \( \mu \). The general case then follows by transitivity. Recall that \( \lambda^{(k)} \) is obtained by reordering the multiset \( M_\lambda^{(k)} = \{ |\lambda_i - k|, 1 \leq p \leq \ell \} \). Denote

\[
M_\lambda^{(k)} = \{ |\lambda_i - k|, 1 \leq p \leq \ell \}.
\]
similarly $M^{(k)} = \{|\mu_p - k|, 1 \leq p \leq \ell\}$. Then $M^{(k)}$ is obtained from $M^{(k)}$ by replacing a copy of $|\mu_i - k|$ by $|\mu_i + 1 - k|$, and a copy of $|\mu_j - k|$ by $|\mu_j - 1 - k|$. We study separately 5 cases, depending on the value of $k$. For each of them, we describe how $\lambda^{(k)}$ is obtained from $\mu^{(k)}$. From this description, it should be clear that the dominance relation is preserved.

- If $k < \mu_j$, the first occurrence of $\mu_i - k$ in $\mu^{(k)}$ is replaced by $\mu_i - k + 1$, while the last occurrence of $\mu_j - k$ is replaced by $\mu_j - k - 1$.
- If $k = \mu_j < \mu_i$, the first occurrence of $\mu_i - k$ in $\mu^{(k)}$ is replaced by $\mu_i - k + 1$, while the first occurrence of 0 is replaced by 1.
- If $k = \mu_j = \mu_i$, the first two copies of 0 in $\mu^{(k)}$ are both replaced by 1.
- If $\mu_j < k \leq \mu_i$, then $\mu^{(k)}$ contains entries $\mu_i - k$ and $k - \mu_j$, in some order. To obtain $\lambda^{(k)}$, the first occurrence of $\mu_i - k$ is replaced by $\mu_i - k + 1$, and the first occurrence of $k - \mu_j$ is replaced by $k - \mu_j + 1$.
- If $k > \mu_i$, then the values $k - \mu_j$ occur before the values $k - \mu_i$ in $\mu^{(k)}$. To obtain $\lambda^{(k)}$, the first occurrence of $k - \mu_j$ is replaced by $k - \mu_j + 1$, and the first occurrence of $k - \mu_i$ is replaced by $k - \mu_i - 1$.

References


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